SYMMETRIC DIFFERENTIALS, KÄHLER GROUPS AND BALL QUOTIENTS.

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Abstract. While Margulis' superrigidity theorem completely describes the finite dimensional linear representations of lattices of higher rank simple real Lie groups, almost nothing is known concerning the representation theory of complex hyperbolic lattices. The main result of this paper (theorem 1.3) is a strong rigidity theorem for a certain class of cocompact arithmetic complex hyperbolic lattices. It relies on two ingredients:

- the fact that the representations of the topological fundamental group of a smooth projective complex variety X are controlled by the global symmetric differentials on X.
- an arithmetic vanishing theorem for global symmetric differentials on certain compact ball quotients using automorphic forms, in particular deep results of Clozel on base change.

1. Introduction and results

The main open question concerning lattices of Lie groups is certainly the study of complex hyperbolic lattices and their finite dimensional representations. Let $n > 1$ be an integer. Let h denote the Hermitian form $h(\mathbf{z}, \mathbf{w}) = z_0\overline{w_0} + \cdots + z_{n-1}\overline{w_{n-1}} - z_n\overline{w_n}$ on $\mathbb{C}^{n+1}$. Let us denote by $\mathbf{SU}(n, 1)$ the real algebraic group $\mathbf{SU}(\mathbb{C}^{n+1}, h)$ of complex linear transformations of $\mathbb{C}^{n+1}$ preserving h. Let $\Gamma \overset{i}{\hookrightarrow} \mathbf{SU}(n, 1)(\mathbb{R}) = SU(n, 1)$ be a lattice (i.e. a discrete subgroup of finite co-volume). What are the finite dimensional representations of Γ ? Recall that this problem is completely solved if one replaces $\mathbf{SU}(n, 1)$ by a simple real linear algebraic group $\mathbf{G}$ of real rank larger than 1 (not necessarily of Hermitian type): let $\Gamma \subset \mathbf{G}(\mathbb{R})$ be a lattice, Margulis' superrigidity theorem [26] states that any homomorphism $\rho : \Gamma \longrightarrow \mathbf{G}'(F)$, where $\mathbf{G}'$ denotes a simple algebraic group over a local field F, either has bounded image in $\mathbf{G}'(\mathbb{R})$ or F is Archimedean and ρ extends to a real algebraic morphism $\rho : \mathbf{G} \longrightarrow \mathbf{G}'$.

In [22] I proved a *local rigidity theorem* for the standard representations of any cocompact complex hyperbolic lattice. The proof was geometric (Hodge theoretic). The main result of this paper, on the other hand, proves a *global rigidity result* for representations of certain arithmetic cocompact lattices in $SU(n, 1)$. The proof is arithmetic: it relies on an arithmetic vanishing theorem 1.11 using automorphic forms, in particular deep results of Clozel on base change.

1.1. **Rigidity for Kottwitz lattices.** Recall that arithmetic lattices in $SU(n, 1)$ are obtained as follows. Let F be a totally real number field, F_c a CM-extension of F, D a division algebra

Date: April 27, 2012.

over F_c of degree d dividing $n+1 = d \cdot r$, with an involution of second kind $\varepsilon : D \longrightarrow D$ (thus ε is an antiautomorphism of algebras and the set $F_c^{\varepsilon=1}$ of ε-fixed points on the center F_c of D is exactly F). Let h be a non-degenerate ε-Hermitian form on D^r and $\mathbf{G} = \mathbf{SU}(h)$ be the special unitary algebraic group over F associated to the data $(F, F_c, D, \varepsilon, h)$. We assume that for one real place $v_0 : F \hookrightarrow \mathbb{R}$ the group $\mathbf{G}_{F_{v_0}}$ is isomorphic to $\mathbf{SU}(n,1)$ and that for any other real place $v \neq v_0$ the group $\mathbf{G}_{F_v}$ is isomorphic to the compact form $\mathbf{SU}(n+1)$.

Definition 1.1. *An arithmetic lattice Γ in $SU(n,1)$ is a lattice commensurable with $\mathbf{G}(\mathcal{O}_F)$ for $\mathbf{G}$ a unitary group defined as above and $\mathcal{O}_F$ the ring of integers of F. In this case the group $\mathbf{G}$ associated to Γ is uniquely defined.*

The arithmetic lattice Γ is said moreover to be a congruence subgroup *if*

$$\Gamma = (\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{Q}) \cap K_f \quad ,$$

where $K_f \subset (\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{A}_f)$ denotes a compact open subgroup of the finite adèles $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{A}_f)$.

Remark 1.2. Using Godement's criterium one easily checks that an arithmetic lattice associated to $\mathbf{G}$ is cocompact as soon as $d > 1$, or $d = 1$ and $F \neq \mathbb{Q}$.

It has long been understood that the Abelian representations of such a lattice Γ strongly depend on the arithmetic type of $\mathbf{G}$. While Kazhdan [20] exhibits cocompact congruence lattices Γ associated to matrix algebras (case $d = 1$) with infinite Abelian quotients, Rapoport-Zink [31] for $n = 2$ (cf. also [7]) and Clozel [11] in general show that arithmetic congruence lattices in $SU(n,1)$ associated to division algebras (case $d = n+1$) satisfy $H^1(\Gamma, \mathbb{C}) = 0$. Following the suggestion of Clozel in [11] such lattices will be called *Kottwitz lattices*.

In this paper we show that the situation is similar for the non-Abelian representation theory of Γ: it becomes simpler as the divisor d of $n+1$ tends to $n+1$. Indeed while some of Kazhdan's lattices surject onto a free group hence have a wild non-Abelian representation theory, we expect Kottwitz lattices to be much closer to higher rank lattices, in particular much more rigid. The main result of this paper is the following generalization of Rapoport-Zink's and Clozel's result in the case where $n+1$ is prime:

Theorem 1.3. *Let $\Gamma \subset SU(n,1)$ be a Kottwitz lattice. Assume $n+1$ is prime. Then:*

(i) *any representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n-1, \mathbb{C})$ is rigid (i.e. the affine complex variety $\mathrm{Hom}(\Gamma, \mathbf{GL}(n-1,\mathbb{C}))//\mathbf{GL}(n-1,\mathbb{C})$ is zero dimensional). Moreover ρ is conjugate to a representation $\rho_0 : \Gamma \longrightarrow \mathbf{GL}(n-1, \mathcal{O}_K)$ for $\mathcal{O}_K$ the ring of integers of some number field $K \subset \mathbb{C}$.*

(ii) *if F is a characteristic zero non-Archimedean local field, any representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n-1, F)$ has bounded image (for the topology of $\mathbf{GL}(n-1, F)$ defined by the topology of the local field F).*

Remarks 1.4. (a) As far as I know theorem 1.3 is the first statement concerning a global rigidity result for a large class of lattices in $SU(n,1)$. It would be very interesting to try to extend it to representations into $\mathbf{GL}(n)$ or $\mathbf{GL}(n+1)$. There the answer might depend not only on the fact that Γ is of Kottwitz type, but also on the level of Γ. We hope to come back on this question in the future.

(b) Concerning the condition "$n + 1$ prime", see the remark 1.12(c).

1.2. **Symmetric powers and rigidity of Kähler groups.** The strategy for proving theorem 1.3 comes from a more general question. Let X be a connected compact Kähler manifold. Hodge theory provides a clear link between the topology of X and the cohomology of the sheaves of holomorphic *exterior* differentials $\Lambda^i\Omega^1_X$, $i \in \mathbb{N}$ (where Ω^1_X denotes the holomorphic cotangent sheaf of X):

$$H^p(X, \mathbb{C}) = \bigoplus_{p=i+j} H^j(X, \Lambda^i\Omega^1_X) \ .$$

An interesting question is the following: what is the relation between the topology of X and the sheaves of holomorphic *symmetric* differentials $S^i\Omega^1_X$, in particular their global sections ?

Remark 1.5. The sheaves $S^i\Omega^1_X$ play a fundamental role in Bogomolov's work on stability [8]. They also appear in various vanishing theorems [25]. As far as I know they are not systematically studied in the literature (see however [32]). In these references symmetric differentials come through the following standard construction. Let $\pi : \mathbf{P}((\Omega^1_X)^*) \longrightarrow X$ be the variety of hyperplanes of Ω^1_X and $\mathcal{O}(1)$ be the tautological line bundle on $\mathbf{P}((\Omega^1_X)^*)$. Then $\pi_*\mathcal{O}(i) = S^i\Omega^1_X$. In particular: $H^j(X, S^i\Omega^1_X) \simeq H^j(\mathbf{P}((\Omega^1_X)^*), \mathcal{O}(i))$.

It follows from Simpson's non-Abelian Hodge theory [33], [34], [35] and its partial p-adic version that global holomorphic symmetric differentials on X control the rigidity of finite dimensional linear representations of the topological fundamental group of X, at least when X is a connected smooth projective variety:

Theorem 1.6. *(Arapura, Zuo) Let X be a connected smooth projective variety over $\mathbb{C}$, Γ its topological fundamental group and r a positive integer. Suppose that $H^0(X, S^i\Omega^1_X) = 0$ for $0 < i \leq r$. Then:*

(i) *any representation $\rho : \Gamma \longrightarrow \mathbf{GL}(r, \mathbb{C})$ is rigid.*

(ii) *Let F be a characteristic zero non-Archimedean local field. Then any representation $\rho : \Gamma \longrightarrow \mathbf{GL}(r, F)$ has bounded image. Here ρ is said to have bounded image if $\rho(\Gamma)$ is contained in a compact subgroup of $\mathbf{GL}(r, F)$ for the topology of $\mathbf{GL}(r, F)$ defined by the topology of the local field F.*

Remarks 1.7. (a) Theorem 1.6 should hold for compact Kähler manifolds but some technical ingredients are still missing in the literature.

(b) Theorem 1.6(i) is a nice corollary of Simpson's non-Abelian Hodge theory, proven by Arapura [1, prop.2.4]. Its non-archimedean version theorem 1.6(ii) is proven by Zuo [41, section 4.1.4].

Theorem 1.6 has an interesting arithmetic corollary:

Corollary 1.8. *Let X be a connected smooth projective variety over $\mathbb{C}$, Γ its topological fundamental group and r a positive integer. Suppose that $H^0(X, S^i\Omega_X^1) = 0$ for $0 < i \leq r$. Then any representation $\rho : \Gamma \longrightarrow \mathbf{GL}(r, \mathbb{C})$ is conjugate to a representation $\rho_0 : \Gamma \longrightarrow \mathbf{GL}(r, \mathcal{O}_K)$ for $\mathcal{O}_K$ the ring of integers of some number field $K \subset \mathbb{C}$.*

Proof. Let X, Γ, r and $\rho : \Gamma \longrightarrow \mathbf{GL}(r, \mathbb{C})$ as in the statement of corollary 1.8. By theorem 1.6(i) the representation ρ is locally rigid. As the affine variety $\mathrm{Hom}(\Gamma, \mathbf{GL}(r, \mathbb{C}))/\mathbf{GL}(r, \mathbb{C})$ is defined over $\mathbb{Z}$ and has finitely many irreducible components, the component defined by the class $\{\rho\}$ of ρ is defined over some number field K. This exactly means that up to conjugacy we can assume that ρ takes values in $\mathbf{GL}(r, K)$. For each finite place v of K the representation $\rho_v : \Gamma \longrightarrow \mathbf{GL}(r, K_v)$ obtained from ρ through the embedding $K \hookrightarrow K_v$ has bounded image in $\mathbf{GL}(r, K_v)$ by theorem 1.6(ii). Hence $\rho(\Gamma)$ lies in $\mathbf{GL}(r, \mathcal{O}_K)$. □

Remark 1.9. Simpson conjectured more than the integrality result above: any rigid $\rho : \Gamma \longrightarrow \mathbf{GL}(r, \mathbb{C})$ should be motivic, i.e. a direct factor of the monodromy of a local system $R^i\pi_*\mathbb{Z}$ for some smooth proper connected morphism $\pi : Y \longrightarrow X$ and some positive integer i.

Let X be a smooth projective variety with infinite topological fundamental group. Usually the cotangent sheaf Ω_X^1 will have some "positivity". On the other hand, such a positivity will usually imply that $H^0(X, S^i\Omega_X^1)$ does not vanish for i large. In the extreme case where Ω_X^1 is ample the sheaf $S^i\Omega_X^1$ is generated by its sections for i sufficiently large. We are thus lead to the following interesting delicate problem in order to apply theorem 1.6:

Question 1.10. Given X with Ω_X^1 positive in some sense, can we detect the smallest $i \in \mathbb{N}$ for which $S^i\Omega_X^1$ has non-zero sections?"

1.3. **Symmetric differentials and arithmetic: the case of ball quotients.** Theorem 1.3 follows from theorem 1.6 and the following vanishing theorem, which partially answers question 1.10 in the case of ball quotients. Let $\Gamma \subset SU(n, 1)$ be a torsion-free arithmetic cocompact lattice. Let $X = \Gamma\backslash\mathbf{B}_{\mathbb{C}}^n$ be the corresponding smooth compact ball quotient, where $\mathbf{B}_{\mathbb{C}}^n$ is the Hermitian symmetric domain associated to $\mathbf{SU}(n, 1)$, realized as the complex n-dimensional unit ball with its Bergman metric. The compact Kähler manifold X is naturally a smooth complex projective variety [4]. Kazhdan's examples show that X can have non-zero symmetric differentials of any degree. On the other hand in the Kottwitz' case we prove:

Theorem 1.11. *Let $\Gamma \subset SU(n, 1)$ be a torsion-free Kottwitz lattice and $X = \Gamma\backslash\mathbf{B}_{\mathbb{C}}^n$ the corresponding compact ball quotient.*

Assume $n + 1$ is prime. Then $H^0(X, S^i\Omega_X^1) = 0$ for $0 < i \leq n - 1$.

Remarks 1.12. (a) The ball quotients $X = \Gamma\backslash\mathbf{B}_{\mathbb{C}}^n$ with Γ a Kottwitz lattice are the "simple Shimura varieties" in the étale cohomology of which Harris and Taylor [16] realize the local Langlands correspondence.

(b) Theorem 1.11 is partially similar in the coherent world to the work of Clozel [11] in the Betti one. Clozel obtained a complete description of the Betti cohomology of Kottwitz's lattices.

(c) The condition"$n + 1$ prime" already appeared in [11]. In this case Clozel proved that the Betti cohomology of Γ is much simpler than for a composite $n + 1$. Similarly we expect theorem 1.3 to hold for a smaller range of i's in the case $n + 1$ is composite. More importantly the assumption "$n+1$ prime" seems technically unavoidable for the time being: it is the only case where the Langlands functoriality we need is proven [12].

(d) for a general arithmetic lattice Γ of $SU(n, 1)$ we expect the following : the bigger the ratio $d/(n+1)$, the bigger the maximal integer m_Γ for which $H^0(X, S^i\Omega^1_X) = 0$ for all $i < m_\Gamma$ should be.

The proof of theorem 1.11 follows two steps. First, it is well-known since a series of works by Harris ([14], [15]) that computing the coherent cohomology of the automorphic bundles $S^i\Omega^1_X$ on X is quite similar to the classical problem of computing the Betti cohomology $H^\bullet(\Gamma, \mathcal{E})$ of a lattice $\Gamma \subset G = SU(n, 1)$ with value in a local system $\mathcal{E}$: while the classical Matsushima's formula computes $H^i(\Gamma, \mathcal{E})$ in terms of irreducible unitary representations $\pi_\infty \in \hat{G}$ appearing in $L^2(\Gamma\backslash G)$, a similar formula holds for coherent cohomology of automorphic vector bundles (cf. 3.1). However its study is more involved: while representations $\pi_\infty \in \hat{G}$ contributing to the Betti cohomology of local systems are completely classified (cf. [39]) the description of representations $\pi_\infty \in \hat{G}$ with $\overline{\partial}$-cohomology is still open (cf. [14]).

Our first step in the proof of theorem 1.11 consists in showing that *tempered* representations $\pi_\infty \in \hat{G}$ contribute to $H^0(X, S^i\Omega^1_X)$ only for large i. This step is valid for any cocompact lattice in $SU(n, 1)$. Using results of Mirkovic [28] and Blasius-Harris-Ramakrishnan [6] we show (cf. section 3 for detailed definitions and proposition 3.9 for a much more precise statement):

Theorem 1.13. *Let $n \geq 2$ be an integer. Let $\Gamma \subset G = SU(n, 1)$ be a torsion-free cocompact lattice in $SU(n, 1)$ and $X = \Gamma\backslash\mathbf{B}^n_\mathbb{C}$ be the corresponding smooth compact ball quotient. The only $\pi_\infty \in \hat{G}$ contributing to $H^0(X, S^i\Omega^1_X)$, $1 \leq i \leq n-1$, are non-tempered.*

Remark 1.14. This result was well-known in the case $i = 1$, which is also the only case where the cohomology $H^0(X, S^i\Omega^1_X)$ appears as (part of) the cohomology of a local system (in this case $H^1(X, \mathbb{C})$).

From theorem 1.13 the proof of theorem 1.11 reduces to showing that certain automorphic forms π with prescribed infinitesimal character and *non-tempered* component at infinity π_∞ do not appear in $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_{f,F}))$, when $\mathbf{G}$ is associated to a division algebra (case $d = n+1$). This is the hard part of the paper. Heuristically speaking we are in good shape: in fact Langlands functoriality and the Ramanujan conjecture for $\mathbf{GL}(n)$ predict that non-tempered automorphic representations π should not occur at all in $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_{f,F)})$ for such a $\mathbf{G}$.

Our problem is similar to the problem studied by Clozel in [11] for Betti cohomology but more involved: the π's we have to consider are not known to be attached to Galois representation, hence the Galois techniques developed by Clozel in [10] and [11] are not available to us. Luckily enough the stable trace formula and the base change he proved in [12] are enough to deal with the functoriality we require. As in [12] we moreover know enough on the representations we consider to avoid the Ramanujan conjecture. We prove the following result, which implies theorem 1.11:

Theorem 1.15. *Let $\Gamma \subset SU(n,1)$ be a Kottwitz lattice. Assume that $n+1$ is an odd prime. Let $\pi_\infty \in \widehat{SU(n,1)}$ be a non-tempered representation contributing to $H^0(X, S^\bullet\Omega^1_X)$. Then π_∞ does not appear in $L^2(\Gamma\backslash SU(n,1))$.*

1.4. **Symmetric differentials and geometry: the case of Hermitian locally symmetric spaces.** Before dealing with ball quotients and arithmetic, we investigate in section 2 the question 1.10 for an irreducible Hermitian locally symmetric space $X = \Gamma\backslash D$ of any rank, in a purely geometric way. Here D is an irreducible symmetric bounded domain in $\mathbb{C}^N$ and Γ denotes a torsion-free cocompact lattice in the simple real Lie group $G = \mathrm{Aut}^0(D)$ connected component of the identity of the group of biholomorphisms of D. Let us recall the description of the irreducible classical bounded symmetric domains $D = G/K$:

$$\begin{aligned}
D^I_{p,q} &= \{Z \in M(p,q,\mathbb{C}) \simeq \mathbb{C}^{pq} \ / \ I_q - Z^*Z > 0\} \simeq SU(p,q)/S(U(p)\times U(q)) \ . \\
D^{II}_n &= \{Z \in D^I_{n,n} \ / \ Z^t = -Z\} \simeq SO^*(2n)/U(n) \ . \\
D^{III}_n &= \{Z \in D^I_{n,n} \ / \ Z^t = Z\} \simeq Sp(n,\mathbb{R})/U(n) \ . \\
D^{IV}_n &= \{Z \in SL(2n,\mathbb{C}) \ / \ Z^tZ = I_{2n} \text{ and } Z^*J_nZ = J_n\} \simeq SO^0(n,2)/SO(n)\times SO(2) \ .
\end{aligned}$$

Here Z^* denotes as usual the complex conjugate transpose of the matriz Z, I_n is the identity matrix of size $n \times n$, $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ is the standard symplectic form in $\mathbb{C}^{2n}$, and $SO^0(n,2)$ denotes the connected component of the identity of the group $SO(n,2)$.

For example $D = D^I_{1,n}$ is the complex unit ball $\mathbf{B}^n_{\mathbb{C}}$ with its Bergman metric ; $D = D^{III}_n$ is the Siegel upper half space of degree n.

In this general setting the quotient X is still a smooth projective variety [4]. Moreover it is well-known that Ω^1_X is semi-positive in the sense of Griffiths (equivalently the bisectional holomorphic curvature of X is non-positive) and Ω^1_X is ample if and only if $D = \mathbf{B}^n_{\mathbb{C}}$, $n \geq 1$. Despite the positivity of Ω^1_X Matsushima [27] proved a long time ago that $H^0(X, \Omega^1_X)$ vanishes unless maybe if D is the unit ball $\mathbf{B}^n_{\mathbb{C}}$. We generalize Matsushima's result to the following:

Theorem 1.16. *Let D be a classical irreducible bounded symmetric domain, $\Gamma \subset G = \mathrm{Aut}^0(D)$ a torsion-free cocompact lattice and $X = \Gamma\backslash D$ the corresponding quotient.*

Then $H^0(X, S^i\Omega^1_X) = 0$ for $i < m_D$, where:

- *$m_D = \inf(p,q)$ if $D = D^I_{p,q} = SU(p,q)/S(U(p)\times U(q))$.*
- *$m_D = [n/2]$ if $D = D^{II}_n = SO^*(2n)/U(n)$ or $D = D^{III}_n = Sp(n,\mathbb{R})/U(n)$.*
- *$m_D = 2$ if $D = D^{IV}_n = SO^0(2,n)/SO(2)\times SO(n)$.*

Remark 1.17. These results are essentially sharp: given D one can in most cases find a lattice $\Gamma \in G$ such that $H^0(X, S^{m_D}\Omega^1_X)$ is non-zero.

Remarks 1.18. (a) The proof of theorem 1.16 is geometric. It relies on a deep vanishing theorem of Mok [30] for homogeneous semi-positive bundles which are not strictly positive. One would like to extend the result of theorem 1.16 from the locally symmetric case to a general class of non-positively curved Kähler manifolds with vanishing curvature in some directions.

(b) As we noticed already in section 1.3, the purely geometric theorem 1.16 can say nothing in the rank 1 case ($D = \mathbf{B}^n_{\mathbb{C}}$): there exist ball quotients satisfying $H^0(X, \Omega^1_X) \neq 0$.

(c) Theorem 1.16 and theorem 1.6 together prove a rigidity result for representations of Γ in a low dimensional range. It recovers only a small (still, non-trivial) part of Margulis's purely group theoretic result in the higher rank case.

1.5. **Acknowledgments.** I would like to thank L.Clozel for some useful discussions and his comments on a first version of this paper. The importance of his works [10], [11], [12] for the proof of theorem 1.15 will be obvious to the reader.

After I wrote a first version of this paper, Brunebarbe, Totaro and I continued to work on symmetric differentials (cf.[9]). During this process Totaro discovered that a proof of theorem 1.6 already existed in works of Arapura and Zuo. I thank him heartily for providing me with these references, for his useful comments and his interest in this work.

2. Symmetric differentials for locally Hermitian symmetric spaces

2.1. **The ingredients for the proof of theorem 1.16.** Theorem 1.16 is a combination of a deep vanishing theorem of Mok [30] and classical plethysm. Let us first recall the results we will need.

Let X be a complex manifold, V a holomorphic vector bundle on X, h an Hermitian metric on V, D the Hermitian connection of (V, h). The curvature $\Theta = \sqrt{-1}D^2$ of (V, h) lies in $A^{1,1}(\mathrm{End}\,(V))$. In local holomorphic coordinates z^i, $1 \leq i \leq \dim_{\mathbb{C}} X$ and e^α, $1 \leq \alpha \leq \mathrm{rk}\ V$, one can write

$$\Theta := \sqrt{-1}\,\Theta_\alpha{}^\beta{}_{i\bar{j}}\, e^\alpha \otimes e_\beta\, dz^i \wedge \overline{dz^j} \quad .$$

Identifying $V^\vee$ to $\overline{V}$ we define a Hermitian form P on $V \otimes TX$ by extending the following:

$$P(v \otimes \eta, v' \otimes \eta') := \Theta_{v\overline{v'}\eta\overline{\eta'}} \quad .$$

Definition 2.1. *The Hermitian holomorphic vector bundle (V, h) is said to be semi-negative (resp. negative, resp. semi-positive, resp. positive) in the sense of Griffiths at a point $x \in X$ if:*

$$\forall\, v \in V_x \setminus \{0\},\ \ \forall\, \eta \in T_X \setminus \{0\},\ \ P(v \otimes \eta, v \otimes \eta) \leq 0 \quad \textit{resp.} < 0\,, \geq 0\,, > 0 \quad .$$

Remark 2.2. Notice that (V, h) is semi-negative if and only if $(V^\vee, h^\vee)$ is semi-positive. Notice also that a Hermitian manifold (X, g) has non-positive holomorphic bisectional curvature if and only if (TX, g) is semi-negative in the sense of Griffiths.

Definition 2.3. *The Hermitian holomorphic vector bundle (V, h) is said to be* properly *semi-negative if (V, h) is semi-negative in the sense of Griffiths at every point $x \in X$ and for every $x \in X$, there exists $v \in V_x \setminus \{0\}$ and $\xi \in T_x X \setminus \{0\}$ such that $P(v \otimes \xi, v \otimes \xi) = 0$.*

Suppose now that D is an irreducible bounded symmetric domain. Let $G = \mathrm{Aut}^0(D)$ be connected component of the identity of the group of biholomorphisms of D, hence $D = G/K$ where K is a maximal compact subgroup of G. Let $\Gamma \subset G$ a torsion-free lattice in G and define $X := \Gamma \backslash D$. The smooth variety X is quasi-projective of finite volume with respect to its natural canonical metric.

Definition 2.4. *An irreducible automorphic vector bundle V_κ on X is a Hermitian holomorphic bundle of the form $V = \Gamma \backslash G \times_\kappa V_0$, where $\kappa : K \longrightarrow GL(V_0)$ is an irreducible complex finite dimensional representation of K (equivalently an irreducible module for the complexified group $K_{\mathbb{C}}$), and V_κ is endowed with the natural metric deduced from a K-invariant Hermitian metric on V_0.*

It is easy to identify properly semi-negative irreducible automorphic vector bundles on X in terms of highest weight theory:

Proposition 2.5. [30, prop.2 p.204] *Fix $\mathfrak{h} \subset \mathfrak{k}$ a Cartan subalgebra of the Lie algebra $\mathfrak{k}$ of K and C a positive Weyl chamber in $\mathfrak{h}^*_{\mathbb{R}}$. We denote by $< \cdot, \cdot >$ the natural scalar product on $\mathfrak{h}^*_{\mathbb{R}}$. Let $\mu \in C$ be the highest root of $\mathfrak{g}_{\mathbb{C}}$ and ω the lowest weight of $\kappa : K \longrightarrow GL(V_0)$.*

Then (V_κ, h) is properly semi-negative if and only if $< \omega, \mu >= 0$.

The main vanishing result of Mok is then the following:

Theorem 2.6. [29] [30, p.205 and p.211] *Let (V, h) be an irreducible automorphic vector bundle on a locally Hermitian symmetric space $X = \Gamma \backslash D$. If (V, h) is properly semi-negative then any Hermitian metric of semi-negative curvature on V is proportional to h.*

As a corollary if moreover (V, h) is non-trivial then $H^0(X, V^\vee) = 0$.

In [30, p.204], Mok proves that (TX, g) is properly semi-negative as soon as $\mathrm{rk}\ G \geq 2$. The vanishing statement of theorem 1.16 is a corollary of theorem 2.6 and of the following:

Theorem 2.7. *Let D be a classical irreducible bounded symmetric domain, $\Gamma \subset G = \mathrm{Aut}^0(D)$ a torsion-free lattice and $X = \Gamma \backslash D$ the corresponding quotient. Then for $0 < i \leq m_D$ the automorphic bundle $S^i TX$ is a direct sum of properly seminegative irreducible non-trivial automorphic bundles.*

2.2. **Proof of theorem 2.7.** Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- \quad ,$$

be the decomposition of the Lie algebra $\mathfrak{g}$ into irreducible K-modules. The tangent space TX is the irreducible automorphic bundle associated to the $K_{\mathbb{C}}$-module $\mathfrak{p}_+$, hence $S^i TX$ is the automorphic bundle associated to $S^i \mathfrak{p}_+$. To prove theorem 2.7 we have to compute the lowest weight of all irreducible $K_{\mathbb{C}}$-submodules of $S^i \mathfrak{p}_+$ and apply proposition 2.5.

Recall that the lowest weight of an irreducible $K_{\mathbb{C}}$-module κ is the opposite of the highest weight of the contragredient $\kappa^\vee$. Thus it remains to show that the highest weight τ of any irreducible $K_{\mathbb{C}}$-factor of $S^i\mathfrak{p}_-$ is non-trivial and satisfies $<\tau,\mu>=0$ for $0<i\leq m_D$. Computing the decomposition into irreducible factors of a symmetric power is in general intractable. We are lucky enough that our computations coïncide exactly with classically known explicit plethysm formulas.

We recall standard notations concerning Schur functors (cf. [13], [40]). Let d be a positive integer. To a partition $\lambda=(\lambda_1\geq\cdots\geq\lambda_k\geq 1)$ of d one associates an idempotent $c_\lambda\in\mathbb{C}[S_d]$ in the algebra of the symmetric group as follows. One defines the Young diagram of λ as the diagram with λ_i boxes in the i-th row, the rows of boxes lined up on the left and we number consecutively the boxes from left to right and up to down. The idempotent c_λ is

$$c_\lambda := (\sum_{g\in P_\lambda} e_g)\cdot(\sum_{g\in Q_\lambda}\mathrm{sign}(g)\cdot e_g) \ ,$$

where P_λ denotes the subgroup of S_d preserving each row of the Young diagram of λ and Q_λ the one preserving each column. For example $c_{(1,\cdots,1)}=\sum_{g\in S_d}\mathrm{sign}(g)\cdot e_g$ while $c_{(d)}=\sum_{g\in S_d}e_g$. Let $V\simeq\mathbb{C}^d$ be the standard $GL(d)$-module. We denote

$$\mathbb{S}_\lambda V := \mathrm{Im}(c_{\lambda|V^{\otimes d}})$$

the $GL(V)$-module image of c_λ for the natural action by permutation of S_d on $V^{\otimes d}$. This is an irreducible $GL(V)$-module and any irreducible $GL(V)$-module can be realized this way. The functor $\mathbb{S}_\lambda: V\mapsto\mathbb{S}_\lambda V$ is called the Schur functor associated to λ. Notice that our convention follows [13] while our functor $\mathbb{S}_\lambda$ corresponds to the functor $\mathbb{S}_{\lambda^\vee}$ associated to the conjugate partition $\lambda^\vee$ in [40].

2.2.1. *Case* $D^I_{p,q}$. In this case $K\simeq S(U(p)\times U(q))$, thus $K_{\mathbb{C}}\simeq S(GL(p)\times GL(q))$ and $\mathfrak{p}_-\simeq(\mathbb{C}^p)^\vee\otimes\mathbb{C}^q$ where $\mathbb{C}^k$ denotes the standard $GL(k)$-module. Recall the following Cauchy formula (cf. [40, p.60]):

$$S^m\mathfrak{p}_- = S^m((\mathbb{C}^p)^\vee\otimes\mathbb{C}^q) = \bigoplus_{|\lambda|=m}\mathbb{S}_\lambda((\mathbb{C}^p)^\vee)\otimes\mathbb{S}_\lambda\mathbb{C}^q$$

giving the decomposition of $S^i((\mathbb{C}^p)^\vee\otimes\mathbb{C}^q)$ into irreducible $GL(p)\times GL(q)$-modules.

Choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}(\mathfrak{u}(p)\times\mathfrak{u}(q))$ to consist of purely imaginary diagonal matrices of trace 0. Let E_{ij} denote the $(p+q)\times(p+q)$-matrix with zero entries except the unit at the (i,j)-th entry. Write L_i the linear form on $\mathfrak{u}_{\mathbb{C}}$ taking the value 1 at E_{ii} and 0 at E_{jj}, $j\neq i$, $1\leq j\leq p+q$. We choose the positive Weyl chamber $C\subset\mathfrak{h}^*_{\mathbb{R}}$ as the set of the $\sum_i a_iL_i$, $a_1\geq\cdots\geq a_i$. The scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ is given by $<\sum_i a_iL_i,\sum_j a_jL_j>=\sum_i a_ib_i$.

The highest weight $\tau_{\mathbb{S}_\lambda((\mathbb{C}^p)^\vee)\otimes\mathbb{S}_\lambda\mathbb{C}^q}$ is given by:

$$\begin{aligned}\tau_{\mathbb{S}_\lambda((\mathbb{C}^p)^\vee)\otimes\mathbb{S}_\lambda\mathbb{C}^q} &= \tau_{\mathbb{S}_\lambda((\mathbb{C}^p)^\vee)}+\tau_{\mathbb{S}_\lambda\mathbb{C}^q}\\ s &= -(\lambda_1L_p+\lambda_2L_{p-1}+\cdots+\lambda_mL_{p-m+1})+(\lambda_1L_{p+1}+\cdots+\lambda_mL_{p+m})\\ &= \sum_{i=1}^m\lambda_i(L_{p+i}-L_{p-i+1}) \ ,\end{aligned} \tag{1}$$

with the convention that $\lambda_i = 0$ for $i > \max(p, q)$.

On the other hand the highest root of $\mathfrak{g}_{\mathbb{C}}$ is $\mu = L_1 - L_{p+q}$. Hence

$$(2) \qquad < \tau_{\mathbb{S}_\lambda((\mathbb{C}^p)^\vee)\otimes\mathbb{S}_\lambda\mathbb{C}^q}, \mu >= \lambda_q - \lambda_p \quad .$$

For $|\lambda| = m$ all the $< \tau_{\mathbb{S}_\lambda((\mathbb{C}^p)^\vee)\otimes\mathbb{S}_\lambda\mathbb{C}^q}, \mu >$ vanish as soon as $m < m_{D^I_{p,q}} = \inf(p, q)$. Hence the result in this case.

2.2.2. *Case D^{II}_n.* This time $K \simeq U(n)$, $K_{\mathbb{C}} \simeq GL(n, \mathbb{C})$ and $\mathfrak{p}_+ \simeq \Lambda^2\mathbb{C}^n$ with the natural action of $GL(n, \mathbb{C})$ on skew-symmetric matrices: an element $g \in GL(n, \mathbb{C})$ acts by mapping a skew-symmetric matrix Z to the skew-symmetric matrix $g \cdot Z \cdot g^t$. One has the following decomposition into irreducible $K_{\mathbb{C}}$-modules [40, prop.2.3.8 p.63]:

$$S^m\mathfrak{p}_- = S^m(\Lambda^2(\mathbb{C}^n)^\vee) = \bigoplus_{\substack{|\lambda|=2m \\ \lambda_i^\vee \text{ even for all } i}} \mathbb{S}_\lambda((\mathbb{C}^n)^\vee) \quad .$$

The highest weight of $\mathbb{S}_\lambda((\mathbb{C}^n)^\vee)$ is

$$\tau_{\mathbb{S}_\lambda((\mathbb{C}^n)^\vee)} = -\sum_{i=1}^{2m} \lambda_i L_i \quad .$$

Hence $< \tau_{\mathbb{S}_\lambda((\mathbb{C}^n)^\vee)}, \mu >= -\lambda_n$ equals zero if $2m < n$ i.e. if $m < m_{D^{II}_n}$.

2.2.3. *Case D^{III}_n.* This case is similar to the previous one. One still has $K \simeq U(n)$, $K_{\mathbb{C}} \simeq GL(n, \mathbb{C})$ but this time $\mathfrak{p}_+ \simeq S^2\mathbb{C}^n$ with the natural action of $GL(n, \mathbb{C})$ on symmetric matrices: an element $g \in GL(n, \mathbb{C})$ maps a symmetric matrix Z to the symmetric matrix $g \cdot Z \cdot g^t$. One has the following decomposition into irreducible $K_{\mathbb{C}}$-modules [40, prop.2.3.8 p.63]:

$$S^m\mathfrak{p}_- = S^m(S^2((\mathbb{C}^n)^\vee)) = \bigoplus_{\substack{|\lambda|=2m \\ \lambda_i \text{ even for all } i}} \mathbb{S}_\lambda((\mathbb{C}^n)^\vee) \quad .$$

One concludes as in case D^{II}_n.

2.2.4. *Case D^{IV}_n.* In this case $K \simeq SO(n) \times SO(2)$, $K_{\mathbb{C}} \simeq SO(n, \mathbb{C}) \times \mathbb{C}^*$ and $\mathfrak{p}_+ \simeq \mathbb{C}^n \otimes \chi^{-1}$ where $\mathbb{C}^n$ denotes the standard $SO(n, \mathbb{C})$-module and $\chi : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is the identity character. Thus $S^2\mathfrak{p}_- \simeq S^2((\mathbb{C}^n)^\vee) \otimes \chi^2$ contains the $K_{\mathbb{C}}$-stable line χ^2 corresponding to the quadratic form $\phi \in S^2((\mathbb{C}^n)^\vee)$ fixed by $SO(n, \mathbb{C})$.

In terms of sheaves: the automorphic sheaf $S^2\Omega^1_X$ contains as a direct factor the automorphic line bundle $\mathcal{L}^2$, where $\mathcal{L}$ denotes the automorphic line bundle associated to the character χ of $K_{\mathbb{C}}$.

Notice that $\Lambda^n\mathfrak{p}_- = \chi^n$, hence $\mathcal{L}^n$ is nothing else than the canonical sheaf K_X. By [4] the canonical sheaf K_X is ample, hence $\mathcal{L}$ (thus $\mathcal{L}^2$) is also ample. The bundle $\mathcal{L}^2$ will have sections if we choose Γ sufficiently small.

This finishes the proof of theorem 2.7, hence of theorem 1.16. □

3. The case of ball quotients

3.1. **Coherent Matsushima formula.** Let $D = G/K$ be an irreducible bounded symmetric domain with G a connected Lie group locally isomorphic to $\mathrm{Aut}^0(D)$, Γ a torsion-free cocompact lattice in G and $X = \Gamma \backslash D$ the corresponding smooth projective variety. Let $\kappa : K_{\mathbb{C}} \longrightarrow GL(V_0)$ be an irreducible K-module and V_κ the corresponding automorphic vector bundle on X. The link between $H^\bullet(X, V_\kappa)$ and $(\mathfrak{q}, K_{\mathbb{C}})$-cohomology (like the one between Betti cohomology and $(\mathfrak{g}, K)$-cohomology) is classical. We refer to [15] and [14, section 4] for details on this section.

Recall that given a $(\mathfrak{q}, K_{\mathbb{C}})$-module λ, the cohomology $H^\bullet(\mathfrak{q}, K_{\mathbb{C}}; \lambda)$ is the cohomology of the complex $C^\bullet := \mathrm{Hom}_{K_{\mathbb{C}}}(\wedge^\bullet \mathfrak{p}_-, \lambda)$ with the differential

$$df(x_0, \cdots, x_q) = \sum_i (-1)^i x_i \cdot f(x_0, \cdots, \widehat{x_i}, \cdots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \cdots, \widehat{x_i}, \cdots, \widehat{x_j}, \cdots, x_q) \ .$$

We denote by $\hat{G}$ the unitary dual of G i.e. the set of isomorphism classes of irreducible unitary representations of G. Given π an admissible G-module we will also denote by π its associated $(\mathfrak{g}K)$-module. Then:

$$\begin{aligned} H^\bullet(X, V_\kappa) &= H^\bullet(\mathfrak{q}, K_{\mathbb{C}}; \mathcal{C}^\infty(\Gamma\backslash G) \otimes_{\mathbb{C}} \kappa) \\ &= \bigoplus_{\pi \in \hat{G}} m_\pi(\Gamma)\ H^\bullet(\mathfrak{q}, K_{\mathbb{C}}; \pi \otimes_{\mathbb{C}} \kappa) \ , \end{aligned} \tag{3}$$

where $m_\pi(\Gamma) := \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$.

Definition 3.1. *We say that $\pi \in \hat{G}$ has $\overline{\partial}$-cohomology in degree r with coefficients in κ (or contributes to $H^r(X, V_\kappa)$) if $H^r(\mathfrak{q}, K_{\mathbb{C}}; \pi \otimes_{\mathbb{C}} \kappa) \neq 0$. We say that π has $\overline{\partial}$-cohomology if π has $\overline{\partial}$-cohomology in degree r with coefficients in κ for some representation κ of K and some natural integer r.*

Let $\pi \in \hat{G}$. By Schur's lemma the center $\mathbf{Z}(\mathcal{U}_{\mathfrak{g}_{\mathbb{C}}})$ of the envelopping algebra $\mathcal{U}_{\mathfrak{g}_{\mathbb{C}}}$ acts on π via a character $\chi_\pi : \mathbf{Z}(\mathcal{U}_{\mathfrak{g}_{\mathbb{C}}}) \longrightarrow \mathbb{C}$, called the infinitesimal character of π. Given $\mathfrak{h} \subset \mathfrak{k}$ a Cartan subalgebra, we denote $\theta : \mathbf{Z}(\mathcal{U}_{\mathfrak{g}_{\mathbb{C}}}) \xrightarrow{\sim} S(\mathfrak{h}_{\mathbb{C}})^W$ the Harish-Chandra isomorphism, where $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the Weyl group. Any $\tau \in \mathfrak{h}_{\mathbb{C}}^*$ defines a homomorphism $e_\tau : S(\mathfrak{h}_{\mathbb{C}}) \longrightarrow \mathbb{C}$ hence an algebra homomorphism $\chi_\tau = e_\tau \circ \theta : \mathbf{Z}(\mathcal{U}_{\mathfrak{g}_{\mathbb{C}}}) \longrightarrow \mathbb{C}$. For any $\pi \in \hat{G}$ it is known that $\chi_\pi = \chi_\tau$ for some $\tau \in \mathfrak{h}_{\mathbb{C}}^*$ uniquely determined modulo the action of W. Let $C_{\mathfrak{g}} \in \mathbf{Z}(\mathcal{U}_{\mathfrak{g}_{\mathbb{C}}})$ be the Casimir operator. Harmonic theory shows that necessarily

$$\chi_\pi = \chi_{-\tau_\kappa - \delta_G} \quad \text{and} \quad \chi_\pi(C_{\mathfrak{g}}) = <\tau_\kappa, \tau_\kappa + 2\delta_G>$$

if $H^\bullet(\mathfrak{q}, K_{\mathbb{C}}; \pi \otimes_{\mathbb{C}} \kappa) \neq 0$, where τ_κ is the highest weight of κ with respect to $\mathfrak{h}$ and $2\delta_G$ is the sum of the positive roots of $\mathfrak{g}$. Hence:

Proposition 3.2.

$$H^\bullet(X, S^i\Omega^1_X) = \bigoplus_{\substack{\pi\in\hat{G} \\ \chi_\pi=\chi_{-\tau_{S^i\mathfrak{p}_-}-\delta_G} \\ \chi_\pi(C_\mathfrak{g})=<\tau_{S^i\mathfrak{p}_-},\tau_{S^i\mathfrak{p}_-}+2\delta_G>}} m_\pi(\Gamma)\, H^\bullet(\mathfrak{q}, K_\mathbb{C}; \pi\otimes_\mathbb{C} S^i\mathfrak{p}_-) \ .$$

Remark 3.3. A classical result of Harish-Chandra ensures that the number of irreducible $(\mathfrak{g}, K)$-modules with given infinitesimal character is finite, hence the sum in the right hand side of (3.2) is a finite sum.

3.2. **Langlands classification.** In this section we describe the π's in $\hat{G}$ contributing to $H^\bullet(X, S^i\Omega^1_X)$ in the case $G = SU(n,1)$, $K = U(n)$, $D = \mathbf{B}^n_\mathbb{C}$, $X = \Gamma\backslash\mathbf{B}^n_\mathbb{C}$ for $\Gamma \subset SU(n,1)$ a torsion-free cocompact lattice.

We adopt the notations of section 2.2.1 for $p = n$, $q = 1$. In particular $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ and $\mathfrak{q} := \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_-$. We moreover define:

(i) $\mathfrak{a} := \mathbb{R}H_0$ a Cartan subspace in $\mathfrak{p}$ with corresponding Cartan subgroup $A = (\exp(tH_0))_{t\in\mathbb{R}}$, where $H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Let $\alpha \in \mathfrak{a}^*$ be defined by $\alpha(tH_0) = t$. Then $R(\mathfrak{g}, \mathfrak{a}) = \{\pm\alpha, \pm 2\alpha\}$ is a restricted root system for $(\mathfrak{g}, \mathfrak{a})$. We identify $\mathfrak{a}^*_\mathbb{C} \simeq \mathbb{C}$ by $s\alpha \mapsto s$.

(ii) $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{g}$ the subalgebra generated by the eigenspaces of the positive roots α and 2α ; let $N \subset G$ be the corresponding subgroup.

(iii) M the centralizer of A in K.

(iv) $P = MAN$ the usual minimal parabolic subgroup of G.

(v) $\mathfrak{t}_\mathbb{C} \subset \mathfrak{m}_\mathbb{C}$ the Cartan subalgebra of diagonal matrices. Hence $\mathfrak{a}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$.

Let us recall the Langlands classification of admissible representations of $G = SU(n,1)$. We follow [5, chapitre 4] (although [5, chapitre 4] deals with $U(n,1)$ rather than $SU(n,1)$ its notations, statements and proofs immediately adapt to $SU(n,1)$).

Definition 3.4. *Let $P = MAN$ be the usual minimal parabolic subgroup of G. Given $\sigma \in \hat{M}$ and $s \in \mathbb{C} \simeq \mathfrak{a}^*_\mathbb{C}$ we denote by $I_{\sigma,s} = \mathrm{ind}_P^G(\sigma\otimes e^s \otimes 1)$ the admissible representation of G obtained from the representation $\sigma\otimes e^s\otimes 1$ of P by unitary induction.*

Theorem 3.5 (Langlands, Knapp and Zuckerman)**.** *Given $\sigma \in \hat{M}$ and $s \in \mathbb{C} \simeq \mathfrak{a}^*$ with positive real part, the representation $I_{\sigma,s}$ admits a unique irreducible quotient (its Langlands quotient) denoted $\pi_{\sigma,s}$.*

Every irreducible admissible representation π of G is either:

(i) *a discrete series,*

(ii) *a non-degenerate limit of discrete series,*

(iii) *$I_{\sigma,s}$ for $s \in i\mathbb{R}$, or*

(iv) *$\pi_{\sigma,s}$ for $s \in \mathbb{C}$ with positive real part.*

These representations are pairwise non-isomorphic.

Any π occurring in cases (i), (ii) and (iii) are obviously unitary, in fact tempered. On the other hand the π's occurring in case (iv) are never tempered. It is not so easy to describe which among these $\pi_{\sigma,s}$ are unitary but we won't need the general result here.

3.3. **The tempered case: proof of theorem 1.13.** The tempered representations with $\overline{\partial}$-cohomology are described by the following result of Mirkovic [28] (generalized by Soergel [36]):

Theorem 3.6 (Mirkovic). *Suppose $\pi \in \hat{G}$ is a tempered representation with $\overline{\partial}$-cohomology. Then π is a discrete series or non-degenerate limit of discrete series.*

Remarks 3.7.
(i) This result of Mirkovic illustrates in the tempered world why coherent cohomology is richer than Betti cohomology. Indeed it is well-known that among tempered representations only discrete series do potentially contribute to the cohomology of local systems on X when rk $G =$ rk K.
(ii) In fact Mirkovic's theorem is not needed if we are interested only in $H^0(X, S^i\Omega^1_X)$: proposition 3.12, to be proven in the next section, implies that if $\pi = \pi_{\sigma,s}$ is not a discrete series or a limit of discrete series and contributes to $H^0(X, S^i\Omega^1_X)$ then $s \in \mathbb{R}^*$, hence $\pi_{\sigma,s}$ is not tempered.

Theorem 1.13 then follows from the following result on discrete series and their non-degenerate limits:

Proposition 3.8. *Let i be a positive integer. Let $\pi \in \hat{G}$ contributing to $H^0(X, S^i\Omega^1_X)$.*
(a) *If π is a discrete series then $i \geq n+1$.*
(b) *If π is a non-degenerate limit of discrete series then $i \in \{n, n+1\}$.*

It is worth to prove the following more precise result:

Proposition 3.9. *Let i be a positive integer.*
(a) *If $i = 1$ or $i \geq n+1$, there exists a unique tempered $\pi \in \hat{G}$ contributing to $H^\bullet(X, S^i\Omega^1_X)$; it is a discrete series.*
If $i \geq n+1$ then $H^\bullet(\mathfrak{q}, K; \pi \otimes S^i\mathfrak{p}_-) = H^0(\mathfrak{q}, K; \pi \otimes S^i\mathfrak{p}_-) \simeq \mathbb{C}$.
If $i = 1$ then $H^\bullet(\mathfrak{q}, K; \pi \otimes \mathfrak{p}_-) = H^{n-1}(\mathfrak{q}, K; \pi \otimes \mathfrak{p}_-) \simeq \mathbb{C}$.
(b) *If $2 \leq i \leq n$ there are exactly two distinct tempered $\pi, \pi' \in \hat{G}$ contributing to $H^\bullet(X, S^i\Omega^1_X)$; both are non-degenerate limits of discrete series.*
Moreover

$$H^\bullet(\mathfrak{q}, K; \pi \otimes S^i\mathfrak{p}_-) = H^{n-i}(\mathfrak{q}, K, \pi \otimes S^i\mathfrak{p}_-) \simeq \mathbb{C}$$

while

$$H^\bullet(\mathfrak{q}, K; \pi' \otimes S^i\mathfrak{p}_-) = H^{n+1-i}(\mathfrak{q}, K, \pi' \otimes S^i\mathfrak{p}_-) \simeq \mathbb{C}\ .$$

3.3.1. *Proof of proposition 3.9.* Proposition 3.9 will follow from results of [6]. We use the notations of section 2.2.1 defining $\mathfrak{h}$, C, L_i $(1 \leq i \leq n+1)$, once more up to the obvious modifications needed for passing from $U(n,1)$ to $SU(n,1)$. Let R^+ be the set of positive roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ defined by C. Let R_c^+ (resp. R_n^+) be the subset of R^+ of compact (resp. non-compact) roots. Hence R_n^+ is the set of roots on $\mathfrak{p}_+$. Let $\delta_G = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. Let $H \subset K$ be the maximal torus with lie algebra $\mathfrak{h}$. Let $\mathcal{F} \subset \mathfrak{h}_\mathbb{C}^*$ denote the set of differentials of algebraic characters of the torus H and $<,>$ be the bilinear form on $\mathfrak{h}_\mathbb{C}$ induced by the Killing form.

Definition 3.10. *Let $\Lambda \in \mathcal{F}$ and define $\lambda = \Lambda + \delta_G$. A system Ψ of positive roots for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is said to be adapted to Λ if:*

(a) $\Psi \supset R_c^+$,
(b) *λ is dominant with respect to Ψ,*
(c) *$< \lambda, \alpha >> 0$ for all $\alpha \in R_c^+$ such that α is simple with respect to Ψ.*

If Ψ is adapted to Λ we define $q_{\lambda,\Psi} := \#(\Psi \cap R_n^+)$

Given $\Lambda \in \mathcal{F}$ and Ψ a system adapted to Λ one may define the non-degenerate *limit of discrete series* $\pi(\lambda, \Psi)$ as in [23, XII, paragraph 7]. If λ is non-singular for R then there exists a unique system Ψ adapted to Λ and $\pi(\lambda, \Psi)$ is the discrete series π_λ with Harish-Chandra parameter λ.

Theorem 3.11 (Blasius-Harris-Ramakrishnan)**.** [15, theor. 3.4] *Let $\Lambda \in \mathcal{F}$ and Ψ a system adapted to Λ. Let $\tau \in \mathfrak{h}_\mathbb{C}^*$ be an R_c^+-dominant integral weight and $\sigma \in \hat{K}$ the corresponding finite dimensional irreducible representation of K with highest weight τ. Then:*

(i) *$H^q(\mathfrak{q}, K; (\pi(\lambda, \Psi))^* \otimes \sigma) = 0$ unless $q = q_{\lambda,\Psi}$ and $\tau = \Lambda$.*
(ii) *If $\tau = \Lambda$ then $\dim H^{q_{\lambda,\Psi}}(\mathfrak{q}, K; (\pi(\lambda, \Psi))^* \otimes \sigma) = 1$.*

Let us apply this result in our case. Fix i a positive integer. We are looking for $\Lambda \in \mathcal{F}$ and Ψ adapted to Λ such that $H^0(\mathfrak{q}, K; (\pi(\lambda, \Psi))^* \otimes S^i\mathfrak{p}_-) \neq 0$. As

$$\tau_{S^i\mathfrak{p}_-} = -i(L_n - L_{n+1})$$

and $R_c^+ = \{L_j - L_k, 1 \leq j < k \leq n\}$, $\tau_{S^j\mathfrak{p}_-}$ is R_c^+-dominant integral. As

$$\delta_G = \frac{1}{2}\sum_{r=1}^{n+1}(n - 2(r-1))L_r,$$

theorem 3.11 implies that necessarily

$$\lambda = \tau_{S^i\mathfrak{p}_-} + \delta_G = \frac{1}{2}[\sum_{r=1}^{n-1}(n - 2(r-1))L_r - (n + 2i - 2)L_n + (2i - n)L_{n+1}] \ . \tag{4}$$

Hence

$$< \lambda, L_j - L_k >= \begin{cases} k - j & \text{if } 1 \leq j < k \leq n \ , \\ n + 1 - (i + j) & \text{if } 1 \leq j \leq n - 1 < k = n + 1 \ , \\ 1 - 2i & \text{if } j = n \text{ and } k = n + 1 \ . \end{cases} \tag{5}$$

Formula (5) implies that λ is R-regular if $i \geq n+1$ or $i = 1$. Hence the only tempered $\pi \in \hat{G}$ contributing to $H^\bullet(X, S^i\Omega^1_X)$ are discrete series $(\pi_\lambda)^*$. The same formula immediately implies that $q_\lambda = 0$ for $i \geq n+1$ and $q_\lambda = n-1$ for $i = 1$.

For $2 \leq i \leq n$ then λ belongs to the singular hyperplane $< x, L_{n+1-i} - L_{n+1} >= 0$ but to no others. Hence there are exactly two non-degenerate limits of discretes series $\pi := \pi(\lambda, \Psi)$ and $\pi' := \pi(\lambda, \Psi')$ contributing to $H^\bullet(X, S^i\Omega^1_X)$: they correspond to $\Psi = R_c^+ \cup \{L_i - L_{n+1},\ 1 \leq i \leq n-i\}$.

3.4. **The non-tempered case: proof of theorem 1.15.**

3.4.1. *Possible σ and s.* From now on we restrict ourselves to cohomology in degree 0, in which case proposition 3.2 becomes:

$$H^0(X, S^i\Omega^1_X) = \bigoplus_{\substack{\pi \in \hat{G} \\ \chi_\pi = \chi_{-\tau_{S^i\mathfrak{p}_-} - \delta_G} \\ \chi_\pi(C_\mathfrak{g}) = <\tau_{S^i\mathfrak{p}_-}, \tau_{S^i\mathfrak{p}_-} + 2\delta_G>}} m_\pi(\Gamma)\ \mathrm{Hom}_{K_\mathbb{C}}(S^i\mathfrak{p}_+, \pi) \ .$$

Proposition 3.12. *Let i be a positive integer and let $\pi = \pi_{\sigma,s} \in \hat{G}$ be a non-tempered representation contributing to $H^0(X, S^i\Omega^1_X)$. Then:*

(a) *there exists an integer k, $0 \leq k \leq i$, such that the M-module $\sigma = \sigma_k$ has highest weight*
$$\tau_k = k(L_2 - \frac{L_1 + L_{n+1}}{2}) \ .$$
(b) *For $\sigma = \sigma_k$ one has $s = s_k > 0$ with $s_k^2 = n^2 + k^2 + 4(i-k)(i+k-1)$.*

(c) *the infinitesimal character of π_{σ_k,s_k} with respect to $\mathfrak{h}$ is*
$$\chi_{\pi_{\sigma_k,s_k}} = (s_k - \frac{k}{2}, k + \frac{n-2}{2}, \frac{n-4}{2}, \cdots, -\frac{n-4}{2}, -\frac{n-2}{2}, -s_k - \frac{k}{2}) \ .$$

Proof. The computations are similar to those in [5, chapitre 4]. To simplify the notations we work with $U(n,1)$ rather than $SU(n,1)$. Hence $K = U(n) \times U(1)$ and

$$M = U(n-1) \times U(1) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & x \end{pmatrix},\ x \in S^1, A \in U(n-1) \right\} \ .$$

For (a): notice that $I_{\sigma,s} = \mathrm{ind}_M^K \sigma$ as a K-module. By Frobenius reciprocity one has:

$$\mathrm{Hom}_K(S^i\mathfrak{p}_+, I_{\sigma,s}) = \mathrm{Hom}_M(S^i\mathfrak{p}_+, \sigma) \ .$$

We are thus reduced to compute the M-types of the K-module $S^i\mathfrak{p}_+$. Denote by $\chi : U(1) \longrightarrow S^1$ the identity character. We denote by $\mathrm{St}_{U(m)}$ the standard action of the unitary group $U(m)$ on $\mathbb{C}^m$. Hence $\mathfrak{p}_+ = \mathrm{St}_{U(n)} \otimes \chi^{-1}$ as a K-module. As an M-module one has $\mathrm{St}_{U(n)} = \chi \oplus \mathrm{St}_{U(n-1)}$. Finally $\mathfrak{p}_+ = (\mathrm{St}_{U(n-1)} \otimes \chi^{-1}) \oplus 1$ as an M-module. By taking symmetric powers we obtain that the M-types of $S^i\mathfrak{p}_+$ are the $S^k\mathrm{St}_{U(n-1)} \otimes \chi^{-k}$ for $0 \leq k \leq i$. The highest weight of the M-module $S^k\mathrm{St}_{U(n-1)} \otimes \chi^{-k}$ is $\tau_k = k(L_2 - \frac{L_1 + L_{n+1}}{2})$: this proves (a).

For (c): let $2\delta_G$ (resp. $2\delta_M$) be the sum of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ (resp. for $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$). Hence

$$2\delta_G = \sum_{j=1}^{n+1} (n+2-2j)L_j \quad \text{and} \quad 2\delta_M = \sum_{j=2}^{n} (n+2-2j)L_j \ .$$

It follows from [23, prop.8.22] that $\chi_{\pi_{\sigma,s}} = \tau_\sigma + \delta_M + s$ with respect to the Cartan subalgebra $\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}$. Statement (c) then follows from (a).

For (b): we use the constraint

$$\chi_{\pi_{\sigma,s}}(C_{\mathfrak{g}}) =< \tau_{S^i\mathfrak{p}_-}, \tau_{S^i\mathfrak{p}_-} + 2\delta_G > \ . \tag{6}$$

On the one hand: from [23, lemma 12.28] one has:

$$\pi_{\sigma,s}(C_{\mathfrak{g}}) = (< \tau_\sigma + \delta_M + s, \tau_\sigma + \delta_M + s > - < \delta_G, \delta_G >) \cdot \mathrm{Id} \ .$$

Here $<,>$ is the scalar product deduced from the Killing form normalized by $(X,Y) \mapsto \mathrm{Tr}\ (XY)/2$ on $\mathfrak{g}$ (in particular $< L_i, L_j >= 2\delta_{i,j}$). Hence

$$\pi_{\sigma,s}(C_{\mathfrak{g}}) = s^2 - n^2 + < \tau_\sigma, \tau_\sigma + 2\delta_M > \ . \tag{7}$$

An easy computation gives:

$$< \tau_{\sigma_k}, \tau_{\sigma_k} + 2\delta_M >=< k(L_2 - \frac{L_1 + L_{n+1}}{2}), k(L_2 - \frac{L_1 + L_{n+1}}{2}) + (n-2)L_2 >= 3k^2 + 2k(n-2) \ . \tag{8}$$

On the other hand:

$$\begin{aligned} < \tau_{S^i\mathfrak{p}_-}, \tau_{S^i\mathfrak{p}_-} + 2\delta_G > &=< i(L_{n+1} - L_n), i(L_{n+1} - L_n) + \{(2-n)L_n - nL_{n+1}\} > \\ &= 4i^2 - 2i(2-n) - 2in = 4i(i-1) \ . \end{aligned} \tag{9}$$

From equations (6), (7), (8) and (9) one deduces (c). □

3.4.2. *The adelic language.* From now on F is a totally real number field, F_c a CM-extension of F, D a division algebra over F_c of degree $n+1$ prime with an involution of second kind $\varepsilon : D \longrightarrow D$, h a non-degenerate ε-Hermitian form on D and $\mathbf{G} = \mathbf{SU}(h)$ the special unitary algebraic group over F associated to the data $(F, F_c, D, \varepsilon, h)$. We assume that for one real place $v_0 : F \hookrightarrow \mathbb{R}$ one has $\mathbf{G}(F_{v_0}) \simeq SU(n,1)$ and that for any other real place $v \neq v_0$ then $\mathbf{G}(F_v) \simeq SU(n+1)$.

We denote by $\mathbf{SU}$ the restriction of scalars $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$, hence $\mathbf{SU}(\mathbb{Q}) = \mathbf{G}(F)$ and $\mathbf{SU}(\mathbb{R}) \simeq SU(n+1)^{[F:\mathbb{Q}]-1} \times SU(n,1)$. Let $K_f \subset \mathbf{SU}(\mathbb{A}_f)$ be a compact open subgroup and denote by $\Gamma = \mathbf{G}(F) \cap K_f$ the corresponding Kottwitz lattice. The compact-open subgroup K_f is supposed to be neat so that Γ is torsion-free. We denote by $X = \Gamma\backslash\mathbf{B}^n_{\mathbb{C}}$ the corresponding compact ball quotient. By strong approximation

$$X = \mathrm{Sh}_{K_f}(\mathbf{SU}, \mathbf{B}^n_{\mathbb{C}}) := \mathbf{SU}(\mathbb{Q})\backslash\mathbf{SU}(\mathbb{A})/K_f K_\infty = \mathbf{G}(F)\backslash(\mathbf{B}^n_{\mathbb{C}} \times \mathbf{G}(\mathbb{A}_{F,f})/K_f) \ ,$$

where $K_\infty \subset \mathbf{SU}(\mathbb{R})$ denotes a maximal compact subgroup.

The coherent Matsushima formula 3.2 then gives:

$$H^\bullet(X, S^i\Omega^1_X) = \bigoplus_{\substack{\pi \text{ irreducible summand of } \mathcal{A}_{\mathbf{G}} \\ \chi_{\pi_\infty}(C_{\mathfrak{g}}) = <\Lambda_{S^i\mathfrak{p}_-}, \Lambda_{S^i\mathfrak{p}_-}+2\delta_G>}} m(\pi)\cdot \pi_f^{K_f} \otimes H^\bullet(\mathfrak{q}, K_{\mathbb{C}}; \pi_\infty \otimes_{\mathbb{C}} S^i\mathfrak{p}_-) \ ,$$

where $\mathcal{A}_{\mathbf{G}} = L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_{F,f}))$, $m(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}(\pi, \mathcal{A}_{\mathbf{G}})$ and irreducible summands of $\mathcal{A}_{\mathbf{G}}$ are automorphic representations of $\mathbf{G}(\mathbb{A}_F)$. We are reduced to showing that for $1 \leq i \leq n-1$ there is no automorphic representation π of $\mathbf{G}(\mathbb{A}_F)$ such that π_∞ is non-tempered and $\chi_{\pi_\infty}(C_{\mathfrak{g}}) =< \Lambda_{S^i\mathfrak{p}_-}, \Lambda_{S^i\mathfrak{p}_-} + 2\delta_G >$.

3.4.3. *Functoriality.* Let us give the heuristic argument concluding the proof of theorem 1.15. For simplicity we denote by $\mathbf{G}$, for the time of this heuristic, the unitary group associated to (D, h) rather than the special unitary one. We want to associate to an automorphic representation π of $\mathbf{G}(\mathbb{A}_F)$ an automorphic representation τ_c of $\mathbf{GL}(n+1, \mathbb{A}_{F_c})$ through the following diagram of functorialities:

$$\begin{tikzcd}
\mathbf{GL}_1(D)/F_c \arrow[r, dash] \arrow[d, dash] & \mathbf{GL}(n+1)/F_c \\
\mathbf{G}/F &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_c \arrow[r, dash] \arrow[d, dash] & \tau_c \\
\pi &
\end{tikzcd}$$

The vertical arrow represents base change ; the horizontal one is the Jacquet-Langlands correspondance. The vertical arrow has no *a priori* reason to exist in terms of the general principles of Langlands functoriality (cf. [24]). It does exist, however, as shown in [12], thanks to the very strong stability properties of the trace formula (for $\mathbf{G}$) and the twisted trace formula (for $\mathbf{GL}_1(D)$) discovered by Kottwitz. As $\mathbf{G}$ is F-anisotropic the representation π is cuspidal. Unless π is an Abelian character the proof of [12] (for $n+1$ prime !) implies that τ_c is cuspidal too. The Ramanujan conjecture for $\mathbf{GL}(n+1)/F_c$ then implies that τ_c, hence also π, is tempered. Hence $\mathcal{A}_{\mathbf{G}}$ does not contain any non-tempered automorphic representation.

Let us formalize the argument. We essentially follow Clozel's proof of property τ, cf. [12, p.325-326].

The special unitary group $\mathbf{G}$ is the derived group of the reductive F-group $\tilde{\mathbf{G}}$ of unitary similitudes defined by setting for R any F-algebra:

$$\tilde{\mathbf{G}}(R) = \{d \in (D \otimes_F R)^*, \ / \ dd^\varepsilon \in R^*\} \ .$$

The center $\tilde{\mathbf{Z}}$ of $\tilde{\mathbf{G}}$ is $\mathrm{Res}_{F^c/F}\mathbf{G}_m$. Let $A_{\tilde{\mathbf{G}}} \simeq \mathbb{R}_+^*$ be the neutral component of the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}}\tilde{\mathbf{Z}}$. The group $A_{\tilde{\mathbf{G}}}$ is a closed subgroup of $\tilde{\mathbf{G}}(\mathbb{A}_{F,f})$. We denote by $\mathcal{A}_{\tilde{\mathbf{G}}}$ the space $L^2(A_{\tilde{\mathbf{G}}}\tilde{\mathbf{G}}(F)\backslash\tilde{\mathbf{G}}(\mathbb{A}_{F,f}))$, its irreducible components are the automorphic representations of $\tilde{\mathbf{G}}(\mathbb{A}_F)$.

Lemma 3.13. [12, lemma 3.4] *Let π be an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$. There exists an automorphic representation $\tilde{\pi}$ of $\tilde{\mathbf{G}}(\mathbb{A}_F)$ (with central character trivial on $A_{\tilde{\mathbf{G}}}$) such that $\tilde{\pi}_{|\mathbf{G}(\mathbb{A}_F)}$ is a discrete sum of automorphic representations of $\mathbf{G}(\mathbb{A}_F)$ containing π.*

We hope for the following diagram of functorialities:

$$\begin{tikzcd}
\tilde{\mathbf{G}}/F_c \simeq (\mathbf{GL}_1(D) \times \mathbf{G}_m)/F_c \arrow[rr, dash] \arrow[dd, dash] & & (\mathbf{GL}(n+1) \times \mathbf{G}_m)/F_c \\
& \Pi \arrow[r, dash] \arrow[dd, dash] & R \\
\tilde{\mathbf{G}}/F & & \\
& \tilde{\pi} &
\end{tikzcd}$$

The main result of [12] is the following base-change result:

Theorem 3.14. [12, theor.2.13] *Suppose $n+1$ is an odd prime. Let $\tilde{\pi}$ be an automorphic representation of $\tilde{\mathbf{G}}(\mathbb{A}_F)$. There exist a unique automorphic representation Π of $(\mathbf{GL}_1(D) \times \mathbf{G}_m)(\mathbb{A}_{F_c})$ such that:*

(i) *for any finite place v of F such that $\tilde{\pi}_v$ and $\tilde{\mathbf{G}}$ are non-ramified at v, and for any place $w|v$ of F_c, the representation Π_w is unramified and associated to π_v by local base-change.*

(ii) *for any Archimedean place v of F and any $w|v$ the infinitesimal characters of π_v and Π_w are associated:*
$$\chi_{\Pi_w} = \chi_{\pi_v} \circ N \ \ ,$$
where N denotes the norm between the centers of the corresponding envelopping algebras.

One has $(F_c)_{v_0} = F_c \otimes F_{v_0} \simeq \mathbb{C}$ and $\tilde{\mathbf{G}}((F_c)_{v_0}) \simeq \mathbf{GL}(n+1, \mathbb{C}) \times \mathbb{C}^*$. Let us write $\Pi = (\Pi^+, \chi)$ where Π^+ is an automorphic representation of $\mathbf{GL}_1(D)(\mathbb{A}_{F_c})$ and χ is a character of $\mathbb{A}^*_{F_c}/F_c^*$. Let N be the norm between the center of the envelopping algebras of $\mathbf{SL}(n+1, \mathbb{C})$ and $\mathbf{G}(F_{v_0}) \simeq SU(n,1)$. By theorem 3.14 one has

$$\chi_{\Pi^+_{v_0,|\mathbf{SL}(n+1,\mathbb{C})}} = \chi_{\pi_{v_0}} \circ N \ \ , \tag{10}$$

where $\Pi^+_{v_0,|\mathbf{SL}(n+1,\mathbb{C})}$ denotes the restriction of the representation $\Pi^+_{v_0}$ of $\mathbf{GL}(n+1, \mathbb{C})$ to $\mathbf{SL}(n+1, \mathbb{C})$.

From now on suppose that π_{v_0} is non-tempered and contributes to $H^0(X, S^i\Omega^1_X)$. Hence by proposition 3.12 there exists an integer k, $0 \leq k \leq i$, such that $\pi_{v_0} = \pi_{\sigma_k, s_k}$. The Lie algebra of a maximal $\mathbb{R}$-split torus of the real algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{SL}(n+1)$ is canonically isomorphic to $\mathfrak{h}\times\mathfrak{h}$ (still with the notations of section 2.2.1). An infinitesimal character for $\mathbf{SL}(n+1,\mathbb{C})$ is parametrized by an element of $\mathfrak{h}^*_{\mathbb{C}}\times\mathfrak{h}^*_{\mathbb{C}}$ modulo the action of $S_{n+1}\times S_{n+1}$. Condition (ii) in theorem 3.14 and proposition 3.12(c) implies that

$$\chi_{\Pi^+_{v_0,|\mathbf{SL}(n+1,\mathbb{C})}} = (Y,Y) \mod S_{n+1}\times S_{n+1}, \tag{11}$$
$$\text{with } Y = (s_k-\frac{k}{2}, k+\frac{n-2}{2}, \frac{n-4}{2}, \cdots, -\frac{n-4}{2}, -\frac{n-2}{2}, -s_k-\frac{k}{2}) \ .$$

As $n+1$ is prime, it follows from [37], [2], [3] that Π^+ is associated to an automorphic representation R^+ of $\mathbf{GL}(n+1,\mathbb{A}_{F_c})$; moreover R^+ is cuspidal or an Abelian character ; and $R^+_{v_0}\simeq\Pi^+_{v_0}$.

If R^+ is an Abelian character then the infinitesimal character of $R^+_{v_0,|\mathbf{SL}(n,\mathbb{C})}$ is (X,X) mod $S_{n+1}\times S_{n+1}$ with $X=(\frac{n}{2},\frac{n-2}{2},\cdots,-\frac{n}{2})$. Hence $k=0$ and $s_0=\frac{n}{2}$. But $s_0^2=n^2+4i(i-1)$ by proposition 3.12(b), contradiction.

If R^+ is cuspidal then it is well-known that $R^+_{v_0}$ is generic. By a theorem of Vogan [38] one has

$$R^+_{v_0} = \mathrm{ind}_B^G(\varepsilon_1|\cdot|^{t_1},\cdots,\varepsilon_{n+1}|\cdot|^{t_{n+1}}) \ ,$$

where the characters ε_i of $\mathbb{C}^*$ are unitary and $t_i\in]-\frac{1}{2},\frac{1}{2}[$. The infinitesimal character of $R^+_{v_0}$ is (p,q) mod $S_{n+1}\times S_{n+1}$, where $p=(p_1,\cdots,p_{n+1})$ and $q=(q_1,\cdots,q_{n+1})$, $p_i-q_i\in\mathbb{Z}$ and $\frac{1}{2}\Re(p_i+q_i)\in]-\frac{1}{2},\frac{1}{2}[$. From equation (11) one can assume that:

$$(p_1,\cdots,p_{n+1}) = (s_k-\frac{k}{2}, k+\frac{n-2}{2}, \frac{n-4}{2}, \cdots, -\frac{n-4}{2}, -\frac{n-2}{2}, -s_k-\frac{k}{2}) \ , \tag{12}$$
$$(q_1,\cdots,q_{n+1}) = (s_k-\frac{k}{2}, k+\frac{n-2}{2}, \frac{n-4}{2}, \cdots, -\frac{n-4}{2}, -\frac{n-2}{2}, -s_k-\frac{k}{2}) \mod S_{n+1} \ .$$

Let us determine the value of q_2. By proposition 3.12(b) the number s_k is real hence all the q_i's are real. As $p_2=k+\frac{n-2}{2}$ and q_2 has to satisfy the condition $p_2+q_2\in]-1,1[$, there are only three possibilities:

(i) either $q_2\in\{k+\frac{n-2}{2},\frac{n-4}{2},\cdots,-\frac{n-4}{2},-\frac{n-2}{2}\}$. In this case the sum p_2+q_2 is integral hence $p_2+q_2=0$ as $p_2+q_2\in]-1,1[$. As $k\geq 0$ necessarily $q_2=-\frac{n-2}{2}$ and $k=0$. It then follows that

$$(p_1,\cdots,p_{n+1}) = (q_{n+1},q_n,\cdots,q_1) = (s_0,\frac{n-2}{2},\frac{n-4}{2},\cdots,-\frac{n-4}{2},-\frac{n-2}{2},-s_0) \ .$$

Hence $\pi_{v_0}=\pi_{1,s_0}$ is unramified. But for $\pi_{\sigma,s}$ unitary and unramified one necessarily has $|s|\leq\frac{n-1}{2}$. This contradicts the inequality $s_0\geq n$ forced by proposition 3.12(b).

(ii) or $q_2=s_k-\frac{k}{2}$. Hence $p_2+q_2=s_k+\frac{k}{2}+\frac{n-2}{2}$ has to belong to $]-1,1[$. This is impossible as $s_k\geq n$ by proposition 3.12(b) and $n\geq 2$.

(iii) or $q_2=-s_k-\frac{k}{2}$. The condition $p_2+q_2\in]-1,1[$ reads this time $s_k\in]\frac{k+n}{2}-2,\frac{k+n}{2}[$. This implies $s_k^2<(\frac{k+n}{2})^2$ as $k+n\geq 2$. On the other hand proposition 3.12(b) gives

$$s_k^2 = k^2+n^2+4(i-k)(i+k-1)\geq k^2+n^2$$

as $0 \leq k \leq i$. Hence necessarily $k^2 + n^2 < (\frac{k+n}{2})^2 < \frac{1}{2}(k+n)^2$ i.e. $\frac{1}{2}(k-n)^2 < 0$ which is impossible.

This concludes the proof of theorem 1.15.

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