Contents

Algebraic Geometry

0. Motivation

Algebraic Geometry = study of solutions of polynomial eq^{ns}

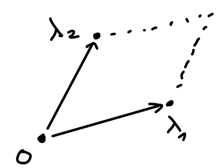
Why? — analysis, algebra, geometry & topology, number theory

Example: Elliptic curves

$L \subset \mathbb{C}$ lattice,

ie $L = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$

w/ λ_1, λ_2 an $\mathbb{R}$-basis of $\mathbb{C}$



Weierstrass function:

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in L \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

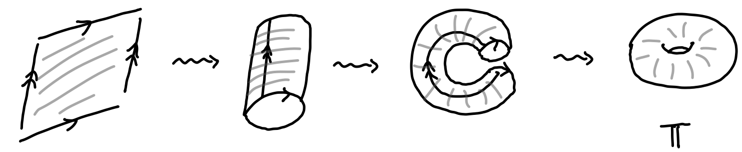
meromorphic fct^n w/ poles only in L

and periodic:

$\wp(z+\lambda) = \wp(z)$ for all $z \in \mathbb{C}$

$\Rightarrow$ fct^n $\wp : \mathbb{C}/L \longrightarrow \mathbb{C} \cup \{\infty\}$

As a topological space $\mathbb{T} := \mathbb{C}/L$ is a torus:



Can also view $\mathbb{T}$ as a manifold,

in fact a Riemann surface (= 1 dim cplex mfd),

and $\wp$ as meromorphic fct on $\mathbb{T}$.

Other meromorphic fct^s on $\mathbb{T}$?

e.g. $\wp'(z) = -\sum_{\lambda \in L} \frac{2}{(z-\lambda)^3}$

For any $F(x,y) \in \mathbb{C}[x,y]$

get a meromorphic fct^n $F(\wp(z), \wp'(z))$.

$\Rightarrow$ ring homom.

$$\mathbb{C}[x,y] \longrightarrow \mathcal{M}(\mathbb{T}) := \{\text{merom. fct}^s \text{ on } \mathbb{T}\}$$

$$F \longmapsto F(\wp, \wp').$$

Not injective: $\exists\, a, b \in \mathbb{C}$:

$$(\wp'(z))^2 = 4\,\wp(z)^3 + a\cdot\wp(z) + b$$

(Weierstrass diff eq^n)

Get ring homom.

$$R := \mathbb{C}[x,y]/(y^2-4x^3-ax-b) \longrightarrow \mathcal{M}(\mathbb{T})$$

<u>Thm</u> This induces an iso $\mathrm{Quot}(R) \xrightarrow{\sim} \mathcal{M}(\mathbb{T})$.

In particular, every merom. fct^n on $\mathbb{T}$ is a rational function in $\wp$ and $\wp'$.

Geometric picture:

Get ~~bijection~~ "isomorphism"

"compactify"

$$\mathbb{T} \xrightarrow{\sim} \{(x,y)\in\mathbb{C}^2 \mid y^2 = 4x^3-ax-b\} \cup \{\infty\} =: E(\mathbb{C})$$

$$z \longmapsto (\wp(z), \wp'(z))$$

"elliptic curve"

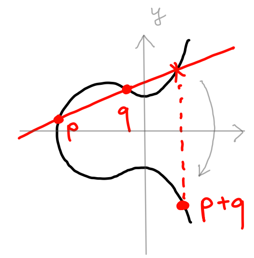
cplex analysis $\longleftrightarrow$ algebra

$E(\mathbb{R}) \subset E(\mathbb{C})$
(can draw only $\mathbb{R}^2 \subset \mathbb{C}^2$)

<u>Rem</u> $\mathbb{T} = \mathbb{C}/L$ is an abelian gp, a quotient of $(\mathbb{C},+)$ by the subgroup $L \subset \mathbb{C}$.

One can see this gp structure in $E(\mathbb{C})$ as follows:

$p+q$ = "3rd point of $E(\mathbb{C})$ on the line through p and q, reflected at the x-axis"



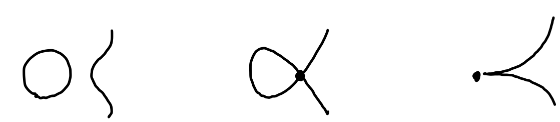
Neutral element: ∞

Now forget cplex analysis!

Given any cubic polynomial $f(x) \in \mathbb{C}[x]$, consider

$$E(\mathbb{C}) := \{(x,y)\in\mathbb{C}^2 \mid y^2 = f(x)\} \cup \{\infty\}$$

<u>Caution:</u> Should assume f has no multiple zero



$y^2 = x(x^2+1)$ smooth

$y^2 = x^2(x+1)$ node

$y^2 = x^3$ cusp

If moreover $f(x) \in \mathbb{Z}[x]$,

consider

$$E(R) := \{(x,y) \in R^2 \mid y^2 = f(x)\} \cup \{\infty\}$$

for any commutative ring R, eg $R = \mathbb{Q}, \mathbb{Z}, \mathbb{F}_p \ldots$

$\longrightarrow$ arithmetic ... For instance:

Thm (Mordell-Weil) $E(\mathbb{Q})$ is a fin. gen. abelian gp.

$\Longrightarrow$ $E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus$ finite ab. gp.

We call $r = \mathrm{rk}\, E$ the rank of the elliptic curve.

It enters in the famous BSD conjecture, which

predicts (among other things)

$$\underset{\substack{\uparrow \\ \text{arithmetic}}}{\mathrm{rk}\, E} = \mathrm{ord}_{s=1} \underset{\substack{\uparrow \\ \text{analysis}}}{(L(E,s))}$$

Here $L(E,s)$ is a meromorphic fct^n defined in terms of $\# E(\mathbb{F}_{p^n})$ for all primes p & $n \in \mathbb{N}$.

It also enters in Wiles-Taylor's proof of FLT ...

Upshot

- Polynomial equations are very interesting
- Want to work over arbitrary comm rings (e.g. $\mathbb{Q}, \mathbb{Z}, \mathbb{F}_p, \ldots$)
- Want to understand local geometry (e.g. smoothness) via algebra (e.g. $f(x)$ no multiple roots)
- Want to work also globally (eg compactify by ∞)

Outline (tentative)

I. Affine schemes — the local models

II. Schemes — Global constructions

III. Geometric properties — Dimension, finiteness properties, ...

IV. Fiber products & base change

V. Coherent sheaves

VI. Line bundles & divisors

VII. Differentials

VIII. Flat & smooth morphisms

⋮ ⋮

I. Affine Schemes

1. Warm-Up: Affine varieties

Let k be a field.

Def For $S \subset k[x_1, \dots, x_n]$ the vanishing locus

$$V(S) := \{ p \in k^n \mid \forall f \in S : f(p) = 0 \}$$

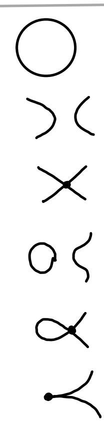
is called an <u>algebraic subset of k^n</u> or an <u>affine algebraic set over k</u>.

For $S = \{f_1, \dots, f_m\}$ finite we put

$$V(f_1, \dots, f_m) := V(\{f_1, \dots, f_m\}).$$

Ex Curves in the affine plane:

$f \in \mathbb{R}[x,y]$	$V(f) \subset \mathbb{R}^2$
x^2+y^2-1	circle
x^2-y^2-1	hyperbola
x^2-y^2	union of lines
y^2-x^3-x	elliptic curve
$y^2-x^3-x^2$	nodal cubic
y^2-x^3	cuspidal cubic
$\vdots$	$\vdots$



Prop The collection of algebraic subsets in k^n is stable under

a) arbitrary intersections:

$$\bigcap_{i \in I} V(S_i) = V(S) \quad \text{w/ } S := \bigcup_{i \in I} S_i$$

b) finite unions:

$$\bigcup_{i=1}^{m} V(S_i) = V(S_1 \cdots S_m)$$

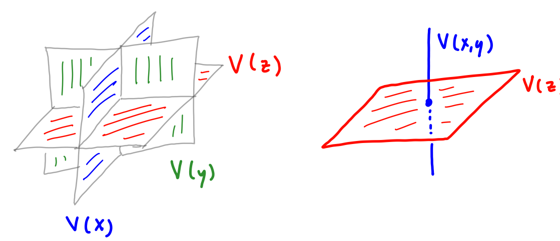
$$\text{w/ } S_1 \cdots S_m := \{ f_1 \cdots f_m \mid \forall i : f_i \in S_i \}$$

Ex Points: $\{p\} = V(x_1 - p_1, \dots, x_n - p_n) \subset k^n$

$\Longrightarrow$ any finite subset of k^n is algebraic

Ex $V(xyz) = V(x) \cup V(y) \cup V(z) \subset k^3$

$V(xz, yz) = V(x,y) \cup V(z) \subset k^3$



Pf of prop. a) Clear by definition.

b) Wlog $m = 2$. Want: $V(S_1) \cup V(S_2) = V(S_1 S_2)$.

$V(S_1) \cup V(S_2) \subseteq V(S_1 S_2)$ clear from def^n

For "$\supseteq$" let $p \in V(S_1 S_2) \setminus V(S_1)$

$\Rightarrow \exists f_1 \in S_1 : f_1(p) \neq 0$

$\forall f_2 \in S_2 : f_1(p) f_2(p) = (f_1 f_2)(p) = 0$

$\Rightarrow \forall f_2 \in S_2 : f_2(p) = 0 \Rightarrow p \in V(S_2)$ $\square$

Cor $\exists!$ topology on k^n whose closed subsets are the algebraic subsets $V(S) \subseteq k^n$ w/ $S \subseteq k[x_1, .., x_n]$.

Pf. Check axioms for closed subsets of a topology:

- $\emptyset = V(1)$, $k^n = V(0)$ are closed
- intersections & finite unions of closed sets are closed by the previous proposition. $\square$

Def This topology is called the Zariski topology. The affine n-space is the topological space $\mathbb{A}^n(k) := k^n$ w/ Zariski topology. We endow algebraic subsets $V(S) \subset k^n$ w/ the topology induced from $\mathbb{A}^n(k)$.

Equations are not intrinsic:

e.g.

$V(f, g) = V(f + gh, g)$ for any $h \in k[x_1, .., x_n]$ ...

Lemma For $S \subseteq k[x_1, .., x_n]$, consider the ideal $J := \langle S \rangle \trianglelefteq k[x_1, .., x_n]$ gen^{td} by S.

Then $V(S) = V(J)$.

Pf. • $V(S) \supseteq V(J)$ clear since $S \subseteq J$

• $V(S) \subseteq V(J)$: Let $f \in J$.

$\Rightarrow \exists f_i \in S, a_i \in k[x_1, .., x_n] : f = \sum_{i=1}^m a_i f_i$.

$\Rightarrow \forall p \in V(S) : f(p) = \sum_{i=1}^m a_i(p) \underbrace{f_i(p)}_{=0} = 0$

since $f_i \in S$ and $p \in V(S)$

$\Rightarrow V(S) \subseteq V(f)$

$\underset{f \in J \text{ arbitrary}}{\Rightarrow} V(S) \subseteq V(J)$ $\square$

Cor For any $S \subset k[x_1, .., x_n]$, $\exists m \in \mathbb{N}$ $\exists f_1, .., f_m \in S : V(S) = V(f_1, .., f_m)$.

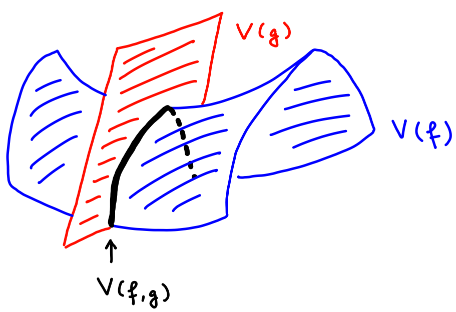
Pf. k is Noetherian since k is a field

$\Rightarrow$ by Hilbert's basis thm, $k[x_1,..,x_n]$ is Noetherian

$\Rightarrow$ $J := \langle S \rangle \trianglelefteq k[x_1,..,x_n]$ is fin. generated, say $J = \langle f_1, \dots, f_m \rangle$

$$\Rightarrow \quad V(S) \underset{\substack{\uparrow \\ \text{lemma}}}{=} V(J) \underset{\substack{\uparrow \\ J = \langle f_1,..,f_m \rangle}}{=} V(f_1, \dots, f_m) \qquad \square$$

<u>Slogan</u>: Every algebraic subset in k^n is an intersection of finitely many hypersurfaces.



How close is the relation between

- algebra: ideals $J \trianglelefteq k[x_1,..,x_n]$,
- geometry: closed subsets of $\mathbb{A}^n(k)$?

<u>Ex</u> For $m \in \mathbb{N}$ put $I_m := (x^m) \trianglelefteq k[x]$

$\Rightarrow$ $V(I_m) = \{0\}$ for all m,

even though the ideals I_m are all distinct!

But they all have the same radical:

<u>Def</u> Let R be a ring (commutative with 1).

The <u>radical</u> of an ideal $I \trianglelefteq R$ is the ideal

$$\sqrt{I} := \{ a \in R \mid \exists n \in \mathbb{N} : a^n \in I \} \trianglelefteq I.$$

We call I a <u>radical ideal</u> if $\sqrt{I} = I$.

<u>Rem</u> Let $I \trianglelefteq R$ be an ideal.

a) $I \subseteq \sqrt{I}$, and $\sqrt{I} \trianglelefteq R$ is a radical ideal.

b) Put $A := R/I$. Then $A_{red} := A/\sqrt{0} \simeq R/\sqrt{I}$ is a <u>reduced</u> ring, ie it has no nilpotents $\neq 0$.

c) Any ring homom $\varphi : A \to B$ to a reduced ring B factors uniquely as

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr] & & B \\ & A_{red} \arrow[ur, dashed, "\exists ! \bar{\varphi}"'] & \end{tikzcd}$$

For the description of algebraic subsets in $\mathbb{A}^n(k)$ it is enough to consider radical ideals:

<u>Lemma</u> For any $I \trianglelefteq k[x_1, \dots, x_n]$, we have

$$V(I) = V(\sqrt{I}).$$

Pf. $I \subseteq \sqrt{I} \Rightarrow V(\sqrt{I}) \subseteq V(I)$

Conversely, let $p \in V(I)$. Then:

$$f \in \sqrt{I} \Rightarrow \exists m: f^m \in I$$
$$\Rightarrow (f(p))^m = 0 \quad \text{since } p \in V(I)$$
$$\Rightarrow f(p) = 0 \quad \text{since } k \text{ has no nilpotents} \neq 0$$

This holds for all $f \in \sqrt{I}$, so $p \in V(\sqrt{I})$. $\square$

Thus the map

$$V: \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } k[x_1, \dots, x_n] \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{algebraic} \\ \text{subsets of } \mathbb{A}^n(k) \end{matrix} \right\}$$
$$I \longmapsto V(I)$$

is surjective. It needn't be injective:

<u>Ex</u> For $k = \mathbb{R}$, consider $I_1 = (1)$, $I_2 = (x^2+1)$

$\Rightarrow$ both are radical ideals in $\mathbb{R}[x]$, they are distinct but $V(I_1) = V(I_2) = \emptyset$.

This problem is resolved if we replace $\mathbb{R}$ by $\mathbb{C}$, then $\emptyset = V(I_1) \neq \{\pm i\} = V(I_2)$.

<u>Thm</u> (Hilbert's Nullstellensatz)

Let $k = \bar{k}$ be an algebraically closed field.

Then the map

$$V: \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } k[x_1, \dots, x_n] \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{algebraic} \\ \text{subsets of } \mathbb{A}^n(k) \end{matrix} \right\}$$
$$I \longmapsto Z := V(I)$$

is bijective, with inverse

$$Z \longmapsto I := \operatorname{Ann}(Z) := \{ f \in k[x_1, \dots, x_n] \mid \forall p \in Z : f(p) = 0 \}.$$

<u>Cor</u> For any proper ideal $I \trianglelefteq_{\neq} k[x_1, \dots, x_n]$, $\exists\, p \in \bar{k}^n$ w/ $f(p) = 0$ for all $f \in I$.

(hence the name Nullstellensatz; cf. previous example)

Pf of cor. Wlog $k = \bar{k}$ (exercise)

$I \subsetneq k[x_1, \ldots, x_n]$ proper ideal

$\Longrightarrow 1 \notin I$

$\Longrightarrow 1 \notin \sqrt{I}$

$\Longrightarrow \sqrt{I} \neq (1)$

Nullstellensatz

$\Longrightarrow V(I) = V(\sqrt{I}) \neq V(1) = \emptyset$ $\square$

Pf of the Nullstellensatz.

Surjectivity of V: Done already

Injectivity:

Enough to show

$$\mathrm{Ann}(V(I)) = \sqrt{I} \quad \text{for all } I \trianglelefteq R.$$

"$\supseteq$": $f \in \sqrt{I} \Longrightarrow \forall p \in V(I): f(p) = 0$
(using that $V(I) = V(\sqrt{I})$)

$\Longrightarrow f \in \mathrm{Ann}(V(I))$

"$\subseteq$": $f \notin \sqrt{I} \Longrightarrow f^m \notin I$ for all $m \in \mathbb{N}$

$\Longrightarrow \bar{f} := (f \bmod I) \in R := k[x_1, \ldots, x_n]/I$
satisfies $\bar{f}^m \neq 0$ for all $m \in \mathbb{N}$

$\Longrightarrow$ The localization $A := R_{\bar{f}}$ is $\neq \{0\}$

<u>Fact</u>: For any fin. gen. k-algebra $A \neq \{0\}$,
$\exists$ homom. of k-algebras $\varphi : A \to K$
to a field $K \supseteq k$ w/ $[K : k] < \infty$
(see below).

Since k is alg. closed, we have $K = k$.

Put $p := (\varphi(\bar{x}_1), \ldots, \varphi(\bar{x}_n)) \in k^n$ w/ $\bar{x}_i := x_i \bmod I$

- $f(p) = f(\varphi(\bar{x}_1), \ldots, \varphi(\bar{x}_n)) = \varphi(\bar{f}) \in \varphi(A^*) \subseteq K^*$

  $\Rightarrow f(p) \neq 0$

- for any $g \in I$, $g(p) = \varphi(g(\bar{x}_1, \ldots, \bar{x}_n)) = \varphi(0) = 0$

  $g(x_1, \ldots, x_n) \in I$

  $\Rightarrow p \in V(I) \subseteq K^n$

Thus $p \in V(I) \setminus V(f)$, ie $f \notin \mathrm{Ann}(V(I))$ $\square$

We have used:

<u>Prop</u> For any fin. gen. k-algebra $A \neq \{0\}$,
$\exists$ homom. of k-algebras $\varphi: A \to K$
to a field $K \supseteq k$ w/ $[K:k] < \infty$

Pf. Pick a max. ideal $m \trianglelefteq A$ (Zorn's lemma)

Get $A \to C := A/m$.

Noether normalization:

C fin gen k-algebra $\Rightarrow$ $\exists m \in \mathbb{N}$ $\exists$ embedding of k-algebras
$B := k[y_1, \dots, y_m] \hookrightarrow C$
which is a finite ring extension
($\Rightarrow$ integral)

In our case $C = A/m$ is a field.

For $B \hookrightarrow C$ integral then also B is a field.*)

Since $B = k[y_1, \dots, y_m]$, it follows that $m = 0$

$\Rightarrow B = k$

$\Rightarrow k \hookrightarrow C$ finite & we can take $K := C$. $\square$

*) $b \in B \setminus \{0\} \Rightarrow \exists c := b^{-1} \in C$ since C is a field
$\Rightarrow \exists$ relat^n $c^d + b_{d-1} c^{d-1} + \dots + b_0 = 0$ w/ $b_i \in B$ since $B \hookrightarrow C$ is integral
$\Rightarrow$ multiply by b^{d-1} to get $b^{-1} = \dots \in B$

For the rest of this section assume k is alg. closed, so the Nullstellensatz gives a bijection

$$\left\{ \begin{array}{l} \text{radical ideals} \\ \text{in } k[x_1, \dots, x_n] \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{algebraic} \\ \text{subsets of } \mathbb{A}^n(k) \end{array} \right\}$$

algebra $\longleftrightarrow$ geometry

In geometry we study "spaces" (top. spaces, mfds, ...) together with "functions" on them (continuous, smooth, ...):

<u>Def</u> A <u>regular function</u> on an algebraic set $Z \subset \mathbb{A}^n(k)$ is a function $f|_Z : Z \subset k^n \to k$, $p \mapsto f(p)$ given by evaluation of some $f \in k[x_1, \dots, x_n]$.

<u>Note</u>: The regular function $f|_Z$ determines f only modulo $\operatorname{Ann}(Z) \trianglelefteq k[x_1, \dots, x_n]$.

We call

$$\mathcal{O}(Z) := k[x_1, \dots, x_n] / \operatorname{Ann}(Z)$$

the <u>algebra of regular functions on Z</u> or the <u>coordinate ring of Z</u>.

Rem $\mathcal{O}(Z)$ is a fin. gen. k-algebra, and it is reduced (ie has no nilpotents) since $\mathrm{Ann}(Z) \trianglelefteq k[x_1, \dots, x_n]$ is a radical ideal.

Def An <u>affine k-algebra</u> is a fin. gen. reduced k-algebra.

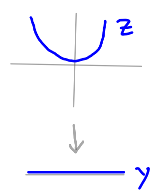
Def A <u>morphism</u> between algebraic sets $Z \subset \mathbb{A}^n(k)$ and $Y \subset \mathbb{A}^m(k)$ is a map

$$f = (f_1, \dots, f_m) : Z \longrightarrow Y \subset \mathbb{A}^n(k)$$
$$p \longmapsto (f_1(p), \dots, f_m(p))$$

given by regular functions $f_1, \dots, f_m \in \mathcal{O}(Z)$.

It is called an <u>isomorphism</u> if there exists a morphism $g : Y \to Z$ w/ $\begin{cases} g \circ f = \mathrm{id}_Y, \\ f \circ g = \mathrm{id}_Z. \end{cases}$

We then also write $f : Z \xrightarrow{\sim} Y$.



Ex Let $Z = V(y - x^2) \subset \mathbb{A}^2(k)$.

$\Longrightarrow$ $pr_1 : Z \xrightarrow{\sim} \mathbb{A}^1(k)$, $(x, y) \mapsto x$ iso

(so we can talk intrinsically about algebraic sets without reference to an ambient affine space ...)

Ex $Z = \mathbb{A}^1(k) \longrightarrow Y = V(y^2 - x^3) \subset \mathbb{A}^2(k)$

$$a \longmapsto (a^2, a^3)$$

is a morphism which is bijective but NOT an iso, since there does not exist a polynomial $g \in k[x, y]$ with $g(a^2, a^3) = a$ for all $a \in k$ (exercise).

Rem a) If $f : Z \to Y$ and $g : Y \to X$ are morphisms of algebraic sets, then so is $g \circ f : Z \to X$.

$\Longrightarrow$ algebraic sets over k form a category

We denote it by $\mathrm{Aff\,Var}(k)$ and call it the category of <u>affine varieties over k</u>.

Recall: A <u>category</u> $\mathcal{C}$ consists of a class $\mathrm{ob}(\mathcal{C})$ of objects and for all $Z, Y \in \mathrm{ob}(\mathcal{C})$ a class $\mathrm{Hom}_{\mathcal{C}}(Z, Y)$ of morphisms w/ composition maps

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Z, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z, X)$$

sth

1) $\forall Y \in \mathrm{ob}(\mathcal{C})$ $\exists\, \mathrm{id}_Y \in \mathrm{Hom}_{\mathcal{C}}(Y, Y)$:

$\mathrm{id}_Y \circ f = f$ for all f

$g \circ \mathrm{id}_Y = g$ for all g

2) $(h \circ g) \circ f = h \circ (g \circ f)$ for all f, g, h.

b) We have $\mathcal{O}(X) = \mathrm{Hom}(X, \mathbb{A}^1(k))$,

so any $\varphi \in \mathrm{Hom}(Y, X)$ induces a k-algebra homom. $\varphi^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$, $f \mapsto f \circ \varphi$.

Since $\mathrm{id}^*(f) = f$,

$$(\psi \circ \varphi)^*(f) = \varphi^*(\psi^*(f)),$$

we get a functor

$$F : \mathrm{AffVar}(k)^{op} \longrightarrow \mathrm{AffAlg}(k) :=$$ category of affine k-algebras (w/ morphisms the k-algebra hom.)

$$Z \mapsto F(Z) := \mathcal{O}(Z)$$

$$(\varphi : Y \to Z) \mapsto F(\varphi) := \varphi^*$$

Recall:

- A (covariant) functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of maps $ob(\mathcal{C}) \longrightarrow ob(\mathcal{D})$, $X \longmapsto F(X)$

  $\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \longmapsto F(f)$

  s.th. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$ for all f, g.

- A contravariant functor is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ where $\mathcal{C}^{op}$ denotes the opposite category defined by $ob(\mathcal{C}^{op}) := ob(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Thm $F : \mathrm{AffVar}(k)^{op} \xrightarrow{\sim} \mathrm{AffAlg}(k)$

is an equivalence of categories.

Recall: A functor $F : \mathcal{C} \to \mathcal{D}$ is called

- essentially surjective if

  $\forall D \in ob(\mathcal{D})\ \exists C \in ob(\mathcal{C})\ \exists$ iso $D \xrightarrow{\sim} F(C)$ in $\mathcal{D}$

- full / faithful / fully faithful if $\forall X, Y \in ob(\mathcal{C})$ the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is surjective / injective / bijective.

- an equivalence of cat's if it is fully faithful and essentially surjective.

Pf of the thm.

- Essential surjectivity:

  Any fin. gen. k-algebra is a quotient $A \simeq k[x_1, .., x_n]/I$

  If A is reduced, then $I = \sqrt{I}$ is a radical ideal and then by the Nullstellensatz $A \simeq F(V(I))$.

- F fully faithful: Exercise. $\square$

Next: From affine varieties to affine schemes:

Replace affine k-algebras by arbitrary rings (comm. with 1)

2. The spectrum of a ring

Recall: For alg. closed fields $k = \bar{k}$ have

$$\mathrm{AffVar}(k)^{op} \xrightarrow{\sim} \mathrm{AffAlg}(k), \quad Z \mapsto \mathcal{O}(Z).$$

Goal: Replace affine k-algebras by arbitrary rings!
(always assumed commutative with unit)

What should be the points of the arising top. spaces?

Ex Let k be an alg. closed field and $I \trianglelefteq_{\neq} k[x_1, \dots, x_n]$.

For $p \in \mathbb{A}^n(k)$ consider the evaluation

$$ev_p : k[x_1, \dots, x_n] \longrightarrow k, \quad f \mapsto f(p).$$

We have:

$$\begin{aligned} p \in Z := V(I) &\iff \forall f \in I : f(p) = 0 \\ &\iff I \subseteq \ker(ev_p) \\ &\iff ev_p \text{ factors over a } k\text{-alg. hom.} \\ &\qquad \mathcal{O}(Z) = k[x_1, \dots, x_n]/I \longrightarrow k \end{aligned}$$

Every k-algebra hom $\varphi : \mathcal{O}(Z) \to k$ arises like this for $p = (p_1, \dots, p_n)$ w/ $p_i := \varphi(x_i|_Z)$

$\Rightarrow$ get bijection $Z \simeq \mathrm{Hom}_{k\text{-alg.}}(\mathcal{O}(Z), k)$

Alternative description of $\mathrm{Hom}_{k\text{-alg}}(-, k)$:

Rem Let $A \neq \{0\}$ be a k-algebra.

a) For any $\varphi \in \mathrm{Hom}_{k\text{-alg}}(A, k)$, $\ker(\varphi) \trianglelefteq A$ is a maximal ideal and φ induces an iso $\bar{\varphi} : A/\ker\varphi \xrightarrow{\sim} k$.

b) If k is alg. closed & A is fin.gen $/k$, the map

$$\mathrm{Hom}_{k\text{-alg}}(A, k) \longrightarrow \left\{ \begin{matrix} \text{maximal} \\ \text{ideals of } A \end{matrix} \right\}$$ is bijective.

Pf. a) φ induces an injective homom. $\bar{\varphi} : A/\ker\varphi \hookrightarrow k$. As a homom. of k-algebras, it is also surjective. So $A/\ker\varphi \simeq k$ is a field, and $\ker\varphi \trianglelefteq A$ maximal.

b) Surjectivity: $\mathfrak{m} \trianglelefteq A$ max ideal $\Rightarrow A/\mathfrak{m}$ is a field

$A/\mathfrak{m}$ fin gen k-alg $\Rightarrow \exists$ hom. of k-alg $\bar{\varphi} : A/\mathfrak{m} \longrightarrow K$ to a field $K \supseteq k$ w/ $[K:k] < \infty$

k alg. closed $\Rightarrow K = k \Rightarrow$ done since $\mathfrak{m} = \ker(A \to A/\mathfrak{m} \xrightarrow{\bar{\varphi}} K)$.

Injectivity: For any $\mathfrak{m}$, $\exists!$ k-algebra hom. $A/\mathfrak{m} \longrightarrow k$. $\square$

Def The <u>maximal spectrum</u> of a ring A is the set

$$\mathrm{Spm}(A) := \{ \text{maximal ideals } \mathfrak{m} \trianglelefteq R \}.$$

Rem If $Z \subset \mathbb{A}^n(k)$ is an alg subset and $k = \bar{k}$, the above gives a bijection

$$\begin{array}{ccl} Z & \xrightarrow{\sim} & \mathrm{Spm}(\mathcal{O}(Z)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ p & \longmapsto & \mathfrak{m}_p := \ker(\mathrm{ev}_p) = \{ f \in \mathcal{O}(Z) \mid f(p) = 0 \}. \end{array}$$

For a morphism $\varphi : \mathcal{O}(Y) \to \mathcal{O}(Z)$ of affine k-alg, the corresponding $f : Z \to Y$ with $\varphi = f^*$ is given by

$$(f^*(g))(p) = (\varphi(g))(p)$$

$$\begin{aligned} \mathfrak{m}_{f(p)} &= \{ g \in \mathcal{O}(Y) \mid g(f(p)) = 0 \} \\ &= \{ g \in \mathcal{O}(Y) \mid \varphi(g) \in \mathfrak{m}_p \} \\ &= \varphi^{-1}(\mathfrak{m}_p), \end{aligned}$$

i.e. we have a comm. diagram:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\wr"'] & Y \arrow[d, "\wr"'] \\ \mathrm{Spm}\,\mathcal{O}(Z) \arrow[r, "\varphi^{-1}"'] & \mathrm{Spm}\,\mathcal{O}(Y) \end{tikzcd}$$

Good: Def^n of $\mathrm{Spm}(A)$ works for any ring A, no need to restrict to affine algebras over k

Bad: $\mathrm{Spm}(A)$ is NOT functorial in A, e.g. for the ring hom. $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$ the max. ideal $\mathfrak{m} := (0) \in \mathrm{Spm}(\mathbb{Q})$ has $\varphi^{-1}(\mathfrak{m}) = (0) \notin \mathrm{Spm}(\mathbb{Z})$ $\lightning$

Way out: Enlarge our definition!

Def The <u>spectrum</u> of a ring A is the set

$$\mathrm{Spec}(A) := \{ \text{prime ideals } \mathfrak{m} \trianglelefteq R \}.$$

Rem Still have $\mathrm{Spm}(A) \subseteq \mathrm{Spec}(A)$ but there may be "new" points coming from non-maximal prime ideals, e.g.

- $\mathrm{Spec}\,\mathbb{Z} = \underbrace{\{(2),(3),(5),(7),\dots\}}_{= \mathrm{Spm}\,\mathbb{Z}} \cup \underbrace{\{(0)\}}_{= \mathrm{Spec}\,\mathbb{Q}}$

- $\mathrm{Spec}\,\mathbb{C}[x] = \underbrace{\{(x-a) \mid a \in \mathbb{C}\}}_{= \mathrm{Spm}\,\mathbb{C}[x] = \mathbb{A}^1(\mathbb{C})} \cup \underbrace{\{(0)\}}_{= \mathrm{Spec}\,\mathbb{C}(x)}$

These "new" points will become very useful later ...

Lemma Any ring homom. $\varphi : A \to B$ induces a map

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A, \quad \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}).$$

Pf. For $\mathfrak{p} \trianglelefteq B$ prime, also $\varphi^{-1}(\mathfrak{p}) \trianglelefteq A$ is prime:

$$\begin{aligned} ab \in \varphi^{-1}(\mathfrak{p}) &\Rightarrow \varphi(ab) \in \mathfrak{p} \\ &\Rightarrow \varphi(a) \in \mathfrak{p} \text{ or } \varphi(b) \in \mathfrak{p} \\ &\Rightarrow a \in \varphi^{-1}(\mathfrak{p}) \text{ or } b \in \varphi^{-1}(\mathfrak{p}) \qquad \square \end{aligned}$$

This gives a functor

$$\begin{tikzcd} \mathrm{Spec} : (\text{Rings})^{op} \arrow[r] \arrow[dr, dashed, "\text{Zariski topology (see next section)!}"'] & (\text{Sets}) \\ & (\text{Topol. spaces}) \arrow[u, "\text{forget}"'] \end{tikzcd}$$

3. The Zariski topology

Recall: We defined Zariski closed subsets of $\mathbb{A}^n(k)$ as the zero loci of subsets $S \subseteq k[x_1, \dots, x_n]$,

$$V(S) := \{ p \in \mathbb{A}^n(k) \mid \forall f \in S : f(p) = 0 \}$$

$$= \{ p \in \mathbb{A}^n(k) \mid S \subseteq m_p \}$$

for the maximal ideals

$$m_p := \{ f \in k[x_1, \dots, x_n] \mid f(p) = 0 \}$$

Def For a ring A, the zero locus of a subset $S \subseteq R$ is

$$V(S) := \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid S \subseteq \mathfrak{p} \} \subseteq \mathrm{Spec}(A).$$

Ex For $A = \mathbb{Z}$, have $V(60) = \{ (2), (3), (5) \}$.

Prop Zero loci in Spec(A) are stable under

a) arbitrary intersections:

$$\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i) \qquad \text{"gcd"}$$

b) finite unions:

$$\bigcup_{i=1}^{m} V(S_i) = V(S_1 \cdots S_m) \qquad \text{"lcm"}$$

Pf.

a) $$\bigcap_{i \in I} V(S_i) = \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid \forall i \in I : S_i \subseteq \mathfrak{p} \}$$
$$= \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid \bigcup_{i \in I} S_i \subseteq \mathfrak{p} \}$$
$$= V(\bigcup_{i \in I} S_i)$$

b) $$\bigcup_{i=1}^{m} V(S_i) = \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid \exists i : S_i \subseteq \mathfrak{p} \}$$
$$= \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid S_1 \cdots S_m \subseteq \mathfrak{p} \}$$
$$= V(S_1 \cdots S_m)$$

($\supseteq$ uses that $\mathfrak{p}$ is prime)

$\square$

Cor $\exists!$ topology on Spec(A), called the Zariski topology whose closed subsets are the subsets $V(S)$, $S \subseteq A$. Moreover, for any ring homomorphism $\varphi : A \to B$ the map $\mathrm{Spec}(\varphi) : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is continuous for the Zariski topologies.

Pf. Continuity of $\mathrm{Spec}(\varphi)$ follows from

$$\mathrm{Spec}(\varphi)^{-1}(V(S)) = \{ \mathfrak{p} \in \mathrm{Spec}\, B \mid \varphi^{-1}(\mathfrak{p}) \in V(S) \}$$
$$= \{ \mathfrak{p} \in \mathrm{Spec}\, B \mid S \subseteq \varphi^{-1}(\mathfrak{p}) \}$$
$$= V(T) \text{ for } T := \varphi(S). \qquad \square$$

$\rightsquigarrow$ Have functor $\mathrm{Spec} : (\text{Rings})^{op} \to (\text{topol. spaces})$

Basic properties of Zariski closed subsets:

<u>Prop</u> a) For any subset $S \subseteq A$ we have

$$V(S) = V(I) \text{ for the ideal } I := \langle S \rangle \trianglelefteq A.$$

b) For any $I \trianglelefteq A$, the quotient hom. $\varphi: A \twoheadrightarrow A/I$ induces a homeomorphism

$$\mathrm{Spec}(A/I) \xrightarrow{\sim} V(I) \subseteq \mathrm{Spec}(A)$$

c) We have $V(I) = V(\sqrt{I})$, and the map

$$\left\{ \begin{matrix} \text{radical} \\ \text{ideals of } A \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Zariski closed} \\ \text{subsets of Spec } A \end{matrix} \right\}$$

$$I \longmapsto Z = V(I)$$

is a bijection with inverse

$$Z \mapsto I = \mathrm{Ann}(Z) := \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$$

<u>Note</u> Part c) restores the Nullstellensatz:

If $A = k[x_1, \dots, x_n]$ with k not alg. closed, then $\mathrm{Spm}(A)$ has more points than $\mathbb{A}^n(k)$,

e.g. $V(x^2+1) = \emptyset$ in $\mathbb{A}^1(\mathbb{R})$,

$V(x^2+1) = \{(x^2+1)\}$ in $\mathrm{Spm}(\mathbb{R}[x])$.

a max. ideal

Pf of the prop.

a) $S \subseteq \mathfrak{p} \iff \langle S \rangle \subseteq \mathfrak{p}$ for any ideal $\mathfrak{p} \trianglelefteq A$.

b) The map $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$, $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$

- is continuous by the previous corollary,
- is injective since $\mathfrak{p} = \varphi(\varphi^{-1}(\mathfrak{p}))$ for any $\mathfrak{p} \in \mathrm{Spec}(A/I)$,
- has image

$$\varphi^{-1}(\mathrm{Spec}(A/I)) = \{ \varphi^{-1}(\mathfrak{p}) \trianglelefteq A \mid \mathfrak{p} \in \mathrm{Spec}(A/I) \}$$
$$= \{ \mathfrak{q} \in \mathrm{Spec}\, A \mid \mathfrak{q} \supseteq I \}$$
$$= V(I).$$

$\Longrightarrow$ continuous bijection $\varphi^{-1}: \mathrm{Spec}(A/I) \to V(I)$.

To show this is a homeomorphism, we need to check it maps closed subsets to closed subsets:

$Z = V(J) \subseteq \mathrm{Spec}(A/I)$ closed, $J \trianglelefteq A/I$

$$\Longrightarrow \varphi^{-1}(Z) = \{ \varphi^{-1}(\mathfrak{p}) \trianglelefteq A \mid \mathfrak{p} \in \mathrm{Spec}(A/I),\ J \subseteq \mathfrak{p} \}$$
$$= \{ \mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} \supseteq \varphi^{-1}(J) \}$$
$$= V(\varphi^{-1}(J))$$

$\Longrightarrow \varphi^{-1}(Z) \subseteq \mathrm{Spec}(A)$ closed as required.

c) $V(I) = \{ \wp \in \mathrm{Spec}(A) \mid I \subseteq \wp \}$

$= \{ \wp \in \mathrm{Spec}(A) \mid \sqrt{I} \subseteq \wp \}$ (using $\wp$ is prime)

$= V(\sqrt{I})$

Hence the map

$$\left\{ \begin{matrix} \text{radical} \\ \text{ideals of } A \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Zariski closed} \\ \text{subsets of Spec } A \end{matrix} \right\}$$

$$I \longmapsto Z = V(I)$$

is surjective. So it suffices to show for all $I \trianglelefteq A$ that $\mathrm{Ann}(V(I)) = \sqrt{I}$:

"$\supseteq$": $I \subseteq \wp$ for all $\wp \in V(I)$ by def^n

$\Rightarrow \sqrt{I} \subseteq \wp$ for all such $\wp$ (since $\wp$ is prime)

$\Rightarrow \sqrt{I} \subseteq \bigcap_{\wp \in V(I)} \wp =: \mathrm{Ann}(V(I))$

"$\subseteq$": $f \notin \sqrt{I} \Rightarrow \bar{f} := (f \bmod I) \in R := A/I$ is not nilpotent, so $R_{\bar{f}} \not\cong \{0\}$

$\Rightarrow \exists$ max. ideal $\mathfrak{m} \in \mathrm{Spm}(R_{\bar{f}}) \subseteq \mathrm{Spec}(R_{\bar{f}})$

$\Rightarrow$ For the natural hom. $\varphi: A \to R_{\bar{f}}$ we have $\wp := \varphi^{-1}(\mathfrak{m}) \in V(I)$ and $f \notin \wp$

$\Rightarrow f \notin \mathrm{Ann}(V(I))$ as required. $\square$

<u>Cor</u> a) $V(f) = \emptyset \iff f \in A^*$

b) $V(f) = \mathrm{Spec}\, A \iff f$ nilpotent

c) $\mathrm{Spec}\, A = \emptyset \iff A = 0$

Pf. a) $V(f) = \emptyset = V(1) \overset{\text{by previous prop.}}{\iff} \sqrt{(f)} = \sqrt{(1)}$

$\iff 1 \in \sqrt{(f)}$

$\iff f \in A^*$

b) $V(f) = \mathrm{Spec}\, A = V(0) \iff \sqrt{(f)} = \sqrt{(0)}$

$\iff f \in \sqrt{(0)}$

$\iff f$ nilpotent

c) $V(0) = \mathrm{Spec}\, A = \emptyset \overset{a)}{\iff} 0 \in A^*$

$\iff A = 0$ $\square$

<u>Rem</u> More generally, for any subset $Z \subseteq \mathrm{Spec}(A)$ (not necessarily Zariski closed) put

$$\mathrm{Ann}(Z) := \bigcap_{\wp \in Z} \wp = \{ f \in A \mid Z \subseteq V(f) \},$$

then the closure of $Z \subseteq \mathrm{Spec}(A)$ in the Zariski topology is

$$\bar{Z} = V(\mathrm{Ann}(Z)).$$

Pf. $V(\mathrm{Ann}(Z)) \subseteq \mathrm{Spec}(A)$ is Zariski closed, and for any other Zariski closed subset $V(I) \subseteq \mathrm{Spec}(A)$ we have:

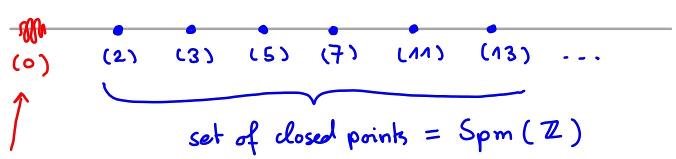
$$
\begin{aligned}
Z \subseteq V(I) &\iff \forall \mathfrak{p} \in Z: \ \mathfrak{p} \in V(I) \\
&\iff \forall \mathfrak{p} \in Z: \ I \subseteq \mathfrak{p} \\
&\iff I \subseteq \bigcap_{\mathfrak{p} \in Z} \mathfrak{p} =: \mathrm{Ann}(Z) \\
&\iff V(I) \supseteq V(\mathrm{Ann}(Z)). \qquad \square
\end{aligned}
$$

Ex The Zariski closure of a point $\mathfrak{p} \in \mathrm{Spec}(A)$ is the set

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{ \mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{p} \subseteq \mathfrak{q} \}$$

$\Longrightarrow$ The closed pt^s of $\mathrm{Spec}(A)$ are only the maximal ideals: All other prime ideals are non-closed points!

A cartoon of $\mathrm{Spec}(\mathbb{Z})$:

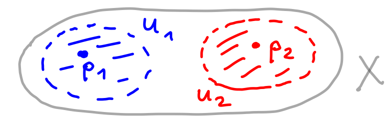


this point is dense, its Zariski closure is all of $\mathrm{Spec}(\mathbb{Z})$!

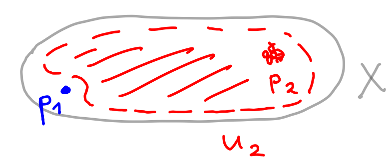
The Zariski topology is usually not Hausdorff. Recall:

Def Let X be a topological space.

a) X is Hausdorff if for all $p_1 \neq p_2 \in X$, $\exists\, U_1, U_2 \subset X$ open with $U_1 \cap U_2 = \emptyset$, $U_1 \ni p_1$, $U_2 \ni p_2$



b) X is a T_0-space if for all $p_1 \neq p_2 \in X$, $\exists\, U \subset X$ open containing either p_1 or p_2 but not both:



c) X is irreducible if any non-empty open subsets $U_1, U_2 \subseteq X$ have $U_1 \cap U_2 \neq \emptyset$.

d) A point $\eta \in X$ is called a generic point if its closure is $\overline{\{\eta\}} = X$.

Rem a) X irreducible $\iff$ any non-empty open subset of X is dense

$\iff$ X is not a union $X = X_1 \cup X_2$ of proper closed subsets $X_i \overset{\neq}{\hookrightarrow} X$

b) $\{\text{generic points of } X\} = \bigcap_{\substack{U \subseteq X \\ \text{open} \neq \emptyset}} U$

c) If X has a generic point, then X is irreducible.

d) If X is T_0, then it has at most one generic point.

e) If X is irreducible with ≥ 2 points, it is NOT Hausdorff.

Pf. a) The first equivalence holds by definition of "dense". The second is clear since

$$X = X_1 \cup X_2 \iff U_1 \cap U_2 = \emptyset \text{ for } U_i := X \setminus X_i.$$

consider $U := X \setminus Z$

b) $\eta \in \bigcap_{\substack{U \subseteq X \\ \text{open} \neq \emptyset}} U \iff \forall \text{ closed } Z \overset{\neq}{\hookrightarrow} X : \ \eta \notin Z$

$\iff \overline{\{\eta\}} = X$

c) $X = \overline{\{\eta\}} \overset{b)}{\Longrightarrow}$ every nonempty open $U \subseteq X$ contains η & hence is dense in X

$\overset{a)}{\Longrightarrow}$ X irreducible

d), e): Similar (exercise). $\square$

Prop Let A be a ring.

a) $\operatorname{Spec}(A)$ is a T_0-space.

b) $\operatorname{Spec}(A)$ is irreducible iff $A_{red} := A/\sqrt{(0)}$ is an integral domain. In that case, it has a unique generic point: $\eta = \sqrt{(0)} \in \operatorname{Spec}(A)$.

c) More generally, a closed subset $Z \subseteq \operatorname{Spec}(A)$ is irreducible iff $\mathfrak{p} := \operatorname{Ann}(Z) \trianglelefteq A$ is a prime ideal, in which case $Z = \overline{\{\mathfrak{p}\}}$.

Pf. a) Let $\mathfrak{p} \neq \mathfrak{q} \in \operatorname{Spec}(A)$.
Wlog $\exists f \in \mathfrak{p} \setminus \mathfrak{q}$. Then $\mathfrak{p} \in V(f)$ but $\mathfrak{q} \notin V(f)$, hence $\mathfrak{q} \in U$ but $\mathfrak{p} \notin U$ for the open $U = \operatorname{Spec}(A) \setminus V(f)$

b) A_{red} integral domain $\Rightarrow$ $\eta := \sqrt{(0)}$ is a prime ideal and the point $\eta \in \operatorname{Spec}(A)$ has $\overline{\{\eta\}} = \operatorname{Spec}(A)$

$\Rightarrow$ $\operatorname{Spec}(A)$ irreducible w/ generic pt η

A_{red} not integral domain $\Rightarrow \exists \bar{f}, \bar{g} \in A_{red} \setminus \{0\} : \bar{f}\bar{g} = 0$

$\Rightarrow \exists f, g \in A \setminus \sqrt{(0)} : fg \in \sqrt{(0)}$

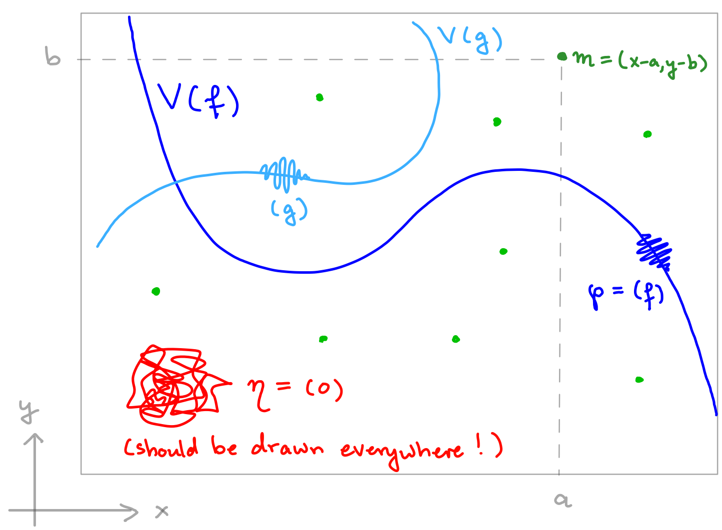
$\Rightarrow \operatorname{Spec}(A) = V(fg) = V(f) \cup V(g)$ not irreducible

c) Follows from b) via $V(I) \simeq \operatorname{Spec}(A/I)$. $\square$

Ex A picture of the affine plane $\mathbb{A}^2_k := \operatorname{Spec} k[x,y]$, for $k = \bar{k}$ an alg. closed field:

Its points (= prime ideals of $k[x,y]$) are

- Closed points $\mathfrak{m} = (x-a, y-b) \in \operatorname{Spm} k[x,y]$ for each $(a,b) \in k^2$ (these are all the closed pts by Hilbert's Nullstellensatz)
- A non-closed point $\mathfrak{p} = (f) \in \operatorname{Spec} k[x,y]$ for each irreducible polynomial $f \in k[x,y]$, with Zariski closure the "curve" $V(f) \subset \mathbb{A}^2_k$.
- A unique generic point $\eta = (0)$

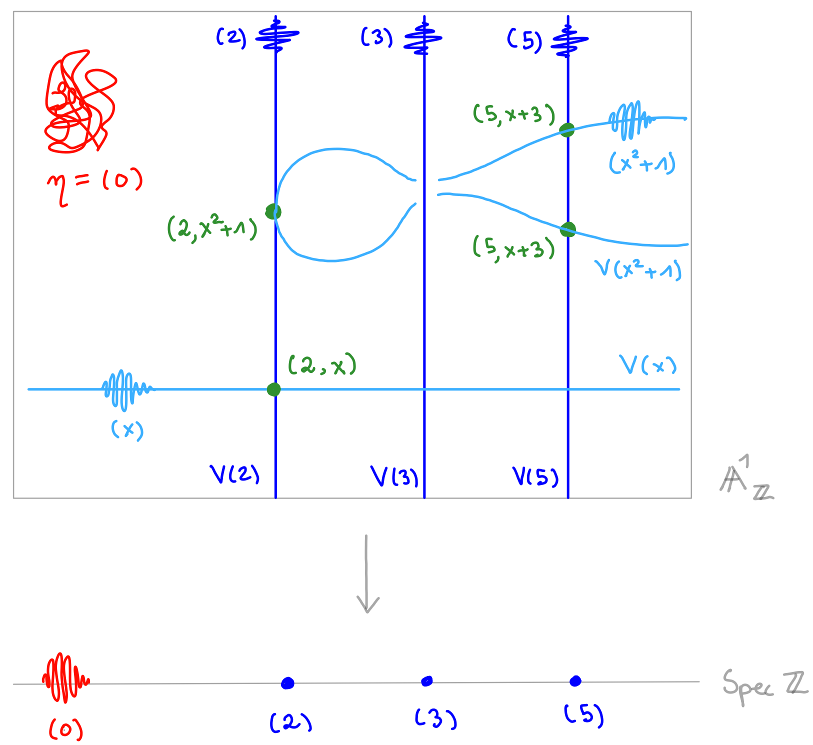


These are all the points of $\mathbb{A}^2_k$, since $k[x,y]$ has Krull dimension $= 2$!

Ex A picture of $\mathbb{A}^1_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x]$:

Its points ($\hat{=}$ prime ideals of $\mathbb{Z}[x]$) are

- closed points $\mathfrak{m} = (p, f) \in \operatorname{Spm}(\mathbb{Z}[x])$ with $p \in \mathbb{Z}$ prime and $f \in \mathbb{Z}[x]$ monic & irreducible mod p
- non-closed points (p) with $p \in \mathbb{Z}$ prime; non-closed points (f) with $f \in \mathbb{Z}[x]$ irreducible
- A unique generic point $\eta = (0)$.



So far we've mostly used Zariski closed sets.
Let's finally take a look at open subsets:

__Recall__ A topological space X is called __quasicompact__ if for every cover $X = \bigcup_i U_i$ by open subsets $U_i \subseteq U$,

$\exists$ finitely many $i_1, \dots, i_n$ with $X = U_{i_1} \cup \dots \cup U_{i_n}$

(we use "compact" only for "quasicompact + Hausdorff")

__Lemma__ For any ring A, $\operatorname{Spec}(A)$ is quasicompact.

Pf. Let $\operatorname{Spec}(A) = \bigcup_i U_i$ w/ $U_i = \operatorname{Spec}(A) \setminus V(I_i)$

$\Rightarrow \emptyset = \bigcap_i V(I_i) = V(\bigcup_i I_i) = V(J)$

for the ideal $J = \langle I_i \mid \text{all } i \rangle \trianglelefteq A$

$\Rightarrow J = (1)$, i.e.

$\exists i_1, \dots, i_n \; \exists f_i \in I_i$ for $i \in \{i_1, \dots, i_n\}$: $1 = \sum_{k=1}^n f_{i_k}$

$\Rightarrow I_{i_1} + \dots + I_{i_n} = (1)$

$\Rightarrow \emptyset = V(\bigcup_{k=1}^n I_{i_k}) = \bigcap_{k=1}^n V(I_{i_k})$

$\Rightarrow \operatorname{Spec}(A) = \bigcup_{k=1}^n U_{i_k}$ $\square$

By def^n any Zariski open subset of $\operatorname{Spec}(A)$ has the form $U = \operatorname{Spec}(A) \setminus V(I)$ for some $I \trianglelefteq A$. Those with $I = (f)$ principal are particularly useful:

__Def__ A __distinguished (or principal) open subset__ is a subset

$$D(f) := \operatorname{Spec}(A) \setminus V(f) \subseteq \operatorname{Spec}(A) \text{ w/ } f \in A.$$

"complement of a hypersurface $V(f)$"

__Prop__ Let $f \in A$.

a) The localization $A \to A_f = A[\frac{1}{f}]$ induces a homeomorphism

$$\operatorname{Spec}(A_f) \xrightarrow{\sim} D(f) \subseteq \operatorname{Spec}(A).$$

b) For any ring homomorphism $\varphi: A \to B$, the map $\operatorname{Spec}(\varphi): \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ has its image contained in $D(f) \subseteq \operatorname{Spec}(A)$ iff φ factors over A_f:

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[dr, dashed, "\exists"'] & \operatorname{Spec} A \\ & D(f) \arrow[u, phantom, "\cup"] \end{tikzcd} \iff \begin{tikzcd} B & A \arrow[l, "\varphi"'] \arrow[d] \\ & A_f \arrow[ul, dashed] \end{tikzcd}$$

c) Hence: $D(f) = D(g)$ iff $A_f \simeq A_g$ as A-algebras.

Pf. a) The localization map $\lambda : A \to A_f$ induces

$$\mathrm{Spec}(A_f) = \{ \mathfrak{p} \cdot A_f \mid \mathfrak{p} \in \mathrm{Spec}(A),\ f \notin \mathfrak{p} \} \longrightarrow \mathrm{Spec}(A)$$

$$\mathfrak{p} \cdot A_f \longmapsto \lambda^{-1}(\mathfrak{p} \cdot A_f) = \mathfrak{p}$$

(exercise in algebra)

which is continuous & injective with image

$$\{ \mathfrak{p} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p} \} = D(f).$$

To show it is a homeomorphism onto $D(f)$, we must show it maps closed subsets to closed subsets:

For $I \trianglelefteq A_f$,

$$\begin{aligned} (\mathrm{Spec}(\lambda))(V(I)) &= \{ \lambda^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in V(I) \} \\ &= \{ \lambda^{-1}(\mathfrak{p}) \mid \mathfrak{p} \trianglelefteq A_f \text{ prime}, I \subseteq \mathfrak{p} \} \\ &= \{ \mathfrak{q} \in \mathrm{Spec}(A) \mid \lambda^{-1}(I) \subseteq \mathfrak{q}, f \notin \mathfrak{q} \} \\ &= V(\lambda^{-1}(I)) \cap D(f) \quad \text{closed in } D(f) \end{aligned}$$

b) $(\mathrm{Spec}(\varphi))(\mathrm{Spec}(B)) = \{ \varphi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(B) \}$

This is $\subseteq D(f) \iff \forall \mathfrak{p} \in \mathrm{Spec}(B) : f \notin \varphi^{-1}(\mathfrak{p})$

$\iff \varphi(f)$ not element of any $\mathfrak{p} \in \mathrm{Spec}(B)$

$\iff \varphi(f) \in B^* \iff \varphi$ factors over A_f

c) follows from b) by symmetry. $\square$

Prop a) The set $\mathcal{B} := \{ D(f) \mid f \in A \}$ forms a basis of open subsets for the Zariski top on $\mathrm{Spec}(A)$.

(ie any Zariski open is a union of $D(f)$'s)

b) We have $D(f) \cap D(g) = D(fg)$, and the $D(f)$ are quasicompact.

Pf. a) $U \subseteq \mathrm{Spec}(A)$ Zariski open

$\Rightarrow \exists I \trianglelefteq A : U = \mathrm{Spec}(A) \setminus V(I)$.

$\Rightarrow U = \bigcup_{f \in I} D(f)$ since $V(I) = \bigcap_{f \in I} V(f)$.

b)
$$\begin{aligned} D(f) \cap D(g) &= (\mathrm{Spec}(A) \setminus V(f)) \cap (\mathrm{Spec}(A) \setminus V(g)) \\ &= \mathrm{Spec}(A) \setminus (V(f) \cup V(g)) \\ &= \mathrm{Spec}(A) \setminus V(fg) \\ &= D(fg) \end{aligned}$$

Quasicompactness of $D(f)$ is clear as $D(f) \overset{\text{homeom.}}{\simeq} \mathrm{Spec}(A_f)$. $\square$

Rem A theorem of Hochster ('69) says that for any topological space X, the following are equivalent:

a) $X \simeq \operatorname{Spec} A$ for a ring A

b) X is quasicompact,
every irred. closed subset of X has a unique generic pt,
and $\exists$ basis of quasicompact open sets stable under finite intersections.

c) $X = \varprojlim X_i$ is an inverse limit of finite T_0-spaces X_i.

4. Digression: Sheaves

Motivation Study functions on a space by restriction to open subsets ...

Def Let X be a top. space,

$Op(X) :=$ the category of open subsets $U \subseteq X$, w/ morphisms given by inclusion.

A *presheaf* on X w/ values in a category $\mathcal{C}$ (e.g. $\mathcal{C}$ = Sets, AbGps, Rings, R-Modules ...) is a functor

$$\mathcal{F} : (Op(X))^{op} \longrightarrow \mathcal{C},$$

ie.

- an object $\mathcal{F}(U) \in ob(\mathcal{C})$ for each open $U \subseteq X$,
- restriction morphisms

$$res^U_V \in Hom_{\mathcal{C}}(\mathcal{F}(U), \mathcal{F}(V)) \quad \text{for all } V \subseteq U,$$

sth $\quad res^U_U = id_{\mathcal{F}(U)}$,

$$res^U_W = res^V_W \circ res^U_V \quad \text{for all } W \subseteq V \subseteq U.$$

Notation: $s|_V := res^U_V(s)$ for $s \in \mathcal{F}(U)$.

Ex $\mathcal{F}(U) := \{\text{continuous maps } f : U \to \mathbb{R}\}$

Functions are determined locally & can be glued ...

Def A *sheaf of sets* is a presheaf $\mathcal{F}$ of sets sth

a) Separation: For any $s, t \in \mathcal{F}(U)$,

$$s|_{U_i} = t|_{U_i} \text{ for all } i \implies s = t$$

b) Gluing: Given $s_i \in \mathcal{F}(U_i)$ for each $i \in I$

sth $s_i|_{U_i \cap U_j} = s_j|_{U_{ij}}$ for all $i, j \in I$,

$\exists\, s \in \mathcal{F}(U) : \; s_i = s|_{U_i}$ for all $i \in I$.

↳ unique by axiom a)

Rem a) & b) are equivalent to:

$$(*) \left\{ \begin{array}{l} \mathcal{F}(U) \text{ is a limit of the diagram} \\ \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_{ij}) \quad \text{w/ } U_{ij} := U_i \cap U_j. \end{array} \right.$$

Def Let $\mathcal{C}$ be a category with products.
A sheaf w/ values in $\mathcal{C}$ is a presheaf

$\mathcal{F} : (Op(X))^{op} \longrightarrow \mathcal{C}$ satisfying $(*)$

for all open covers $U = \bigcup_{i \in I} U_i$.

Rem Let $\mathcal{C}$ be a "category of sets w/ extra structure"
ie. $\exists$ faithful functor forget : $\mathcal{C} \to$ Sets
which commutes w/ limits and is conservative

(ie φ iso in $\mathcal{C}$ iff forget(φ) bijective)

e.g. $\mathcal{C}$ = AbGps, Rings, R-Mod, ...
Then:

$\mathcal{F} : (Op(X))^{op} \longrightarrow \mathcal{C}$ is a sheaf

iff forget $\circ \mathcal{F}$ is a sheaf of sets.

(exercise)

$\Longrightarrow$ Will argue set-theoretically if needed...

Examples

- $\mathcal{F} : (Op(X))^{op} \longrightarrow$ Rings,
  $U \longmapsto \{$continuous fct^s $f : U \to \mathbb{R}\}$

  Same for smooth fct^s on a mfd X, etc.

- For any sheaf $\mathcal{F}$ we have $\mathcal{F}(\emptyset) = \{0\}$ (terminal object in $\mathcal{C}$)
  (the empty set $U = \emptyset$ has the empty open cover w/ index set $I = \emptyset$. Since $\prod_{i \in \emptyset} \cdots = \{0\}$, $\mathcal{F}(\emptyset)$ is then the limit of $\{0\} \rightrightarrows \{0\} \ldots$)

- For a closed pt $p \in X$, the skyscraper sheaf at p w/ value $A \in ob(\mathcal{C})$ is
  $\mathcal{F} : (Op(X))^{op} \longrightarrow$ Rings, $U \longmapsto \begin{cases} A & \text{if } p \in U \\ \{0\} & \text{else} \end{cases}$

- The constant presheaf w/ values in $A \in ob(\mathcal{C})$,
  $(Op(X))^{op} \to \mathcal{C}$, $U \longmapsto A$, is usually not a sheaf:

  Gluing fails if $\exists\, U, V \subseteq X$ open w/ $U \cap V = \emptyset$.
  But the constant sheaf

  $A_X : (Op(X))^{op} \to \mathcal{C}$, $U \longmapsto \prod_{\pi_0(U)} A$ is a sheaf.

  $\pi_0(U)$ = connected cpt^s of U

- For any sheaf $\mathcal{F}$ on X, its restriction to an open $U \subseteq X$ is the sheaf $\mathcal{F}|_U := \mathcal{F} \circ \iota$ where $\iota : Op(U) \hookrightarrow Op(X)$.

Non-examples

- If X has a nontrivial open cover, the presheaf

  $$\mathcal{F} : (\mathrm{Op}(X))^{op} \longrightarrow \mathrm{Sets}$$

  $$U \longmapsto \begin{cases} \{0,1\} & \text{for } U = X \\ \{0\} & \text{for } U \neq X \end{cases}$$

  satisfies gluing but not separation

- For $X = \mathbb{C}^*$ the presheaf

  $$\mathcal{F} : (\mathrm{Op}(X))^{op} \longrightarrow \mathrm{Sets}$$

  $$U \longmapsto \{ f : U \to \mathbb{C} \mid \exists \text{ holom. } g : U \to \mathbb{C} \text{ w/ } f(z) = (g(z))^2 \}$$

  satisfies the separation axiom but not glueing

- For X not quasicompact,

  $$\mathcal{F} : (\mathrm{Op}(X))^{op} \longrightarrow \mathrm{Rings}$$

  $$U \longmapsto \{ \text{bounded fct}^s\ f : U \to \mathbb{R} \}$$

  satisfies glueing for finite covers, but not for infinite covers!

Next: Look at sheaves locally in arbitrarily small nbhoods of a given point $x \in X$ ...

From now on, assume $\mathcal{C}$ has filtered colimits.

Def The stalk of a presheaf $\mathcal{F}$ at a point $x \in U$ is the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

over all open nbhoods $U \subseteq X$ of x.

For $\mathcal{C} = \mathrm{Sets}$ this means:

$$\mathcal{F}_x \simeq \coprod_{U \ni x} \mathcal{F}(U) \Big/ \sim$$

where $\sim$ is the equivalence relation defined by

$$s \sim t \quad \text{for } s \in \mathcal{F}(U),\ t \in \mathcal{G}(V)$$

$$:\Longleftrightarrow \exists\, W \subseteq U \cap V \text{ with } x \in W : \quad s|_W = t|_W .$$

Rem For $U \subseteq X$ open w/ $x \in U$, have a natural morphism $\mathcal{F}(U) \longrightarrow \mathcal{F}_x$

For $\mathcal{C} = \mathrm{Sets}$ & $s \in \mathcal{F}(U)$, we call the image $s_x \in \mathcal{F}_x$ the germ of s at x.

Ex • The constant presheaf & sheaf have stalks

$$A^{pre}_{X,x} \simeq A_{X,x} \simeq A .$$

• A skyscraper sheaf $\mathcal{F}$ at a closed pt $x_0 \in X$ w/ value A has

$$\mathcal{F}_x \simeq \begin{cases} A & \text{for } x = x_0 \\ \{0\} & \text{for } x \neq x_0 \end{cases}$$

• The sheaf $\mathcal{O}^{an}_{\mathbb{C}}$ of holom. fct^s on $X = \mathbb{C}$ has

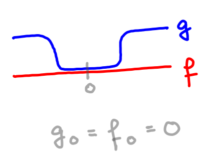
$\mathcal{O}^{an}_{\mathbb{C},0}$ = { power series convergent in a nbhood of 0 } $\subset \mathbb{C}[\![x]\!]$.

For $f \in \mathcal{O}^{an}_{\mathbb{C}}(U)$ w/ $U \subseteq \mathbb{C}$ open, $0 \in U$, the germ of f is its Taylor series:

$$f_x = \sum_{n \geq 0} \frac{1}{n!} f^{(n)}(0)\, x^n \in \mathbb{C}[\![x]\!]$$

Here $\mathcal{O}^{an}_{\mathbb{C}}(U) \hookrightarrow \mathcal{O}^{an}_{\mathbb{C},0}$ is injective for U connected.

• For the sheaf $\mathcal{C}^\infty_{\mathbb{R}}$ of smooth fct^{ns} on $X = \mathbb{R}$, $\mathcal{C}^\infty_{\mathbb{R}}(U) \to \mathcal{C}^\infty_{\mathbb{R},x}$ is not injective:



But it separates e.g. $h(x) = e^{-1/x^2}$ from $f(x) = 0$, so sees more than Taylor series!

For sheaves, stalks see everything:

Lemma Let $\mathcal{F}$ be a sheaf of sets on X

$\Longrightarrow$ for any open $U \subseteq X$, the natural map

$$\iota : \mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x \quad \text{is injective}$$

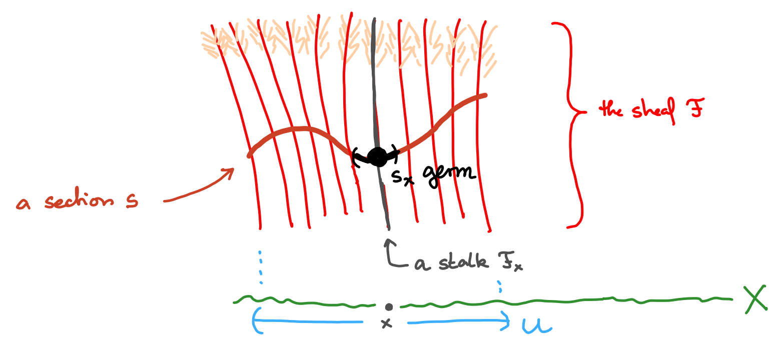
Pf. Let $s, t \in \mathcal{F}(U)$

$\iota \circ s = \iota \circ t \Longrightarrow \forall x \in U: \ s_x = t_x : M \to \mathcal{F}_x$

$\Longrightarrow \forall x \in U \ \exists U_x \subseteq U$ w/ $x \in U_x$:

$$s|_{U_x} = t|_{U_x} : M \to \mathcal{F}(U_x)$$

Separation axiom for sheaves $\Longrightarrow s = t$ $\square$



Def A <u>morphism of (pre-)sheaves</u> $\mathcal{F}, \mathcal{G} : (\mathrm{Op}(X))^{op} \to \mathcal{C}$ is a natural transformation of functors $\varphi : \mathcal{F} \to \mathcal{G}$

ie $\forall\, U \subseteq X$ open a $\varphi(U) \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(U), \mathcal{G}(U))$ sth for $V \subseteq U$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

Rem $Sh_{\mathcal{C}}(X) :=$ category of sheaves w/ values in $\mathcal{C}$ is by def^n a full subcategory of $PSh_{\mathcal{C}}(X) :=$ category of presheaves w/ values in $\mathcal{C}$,

ie $\mathrm{Hom}_{Sh}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{PSh}(\mathcal{F}, \mathcal{G})$ $\forall\, \mathcal{F}, \mathcal{G} \in Sh(X)$

<u>Q</u>: What about mono- / epi- / isomorphisms?

For $U \subseteq X$ open, $x \in X$, consider the functors

- $PSh(X) \to \mathcal{C}$, $\mathcal{F} \mapsto \mathcal{F}(U)$,
- $PSh(X) \to \mathcal{C}$, $\mathcal{F} \mapsto \mathcal{F}_x$.

Def Let $\mathcal{F}, \mathcal{G} \in PSh(X)$ be presheaves of sets.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is

- <u>injective</u> if $\forall\, U \subseteq X$, $\varphi(U)$ is injective.
- <u>bijective</u> if $\forall\, U \subseteq X$, $\varphi(U)$ is bijective.
- <u>surjective</u> if $\forall\, U \subseteq X$, $\varphi(U)$ is surjective.
- <u>locally surjective</u> if $\forall\, U \subseteq X$ $\forall\, g \in \mathcal{G}(U)$ $\exists$ open cover $U = \bigcup_i U_i$ $\exists\, f_i \in \mathcal{F}(U_i)$: $g|_{U_i} = \varphi(f_i)$.

<u>Ex</u> For $X = \mathbb{C}$ define $\mathcal{F} \in Sh(X)$ by $\mathcal{F}(U) := \{\text{holom. fct}^s\ f : U \to \mathbb{C} \setminus \{0\}\}$

$\Rightarrow$ $\varphi : \mathcal{F} \to \mathcal{F}$, $f \mapsto f^2$ locally surjective but not surjective on $U = \mathbb{C}^*$!

<u>Prop</u> For $\mathcal{F} \in Sh(X)$, $\mathcal{G} \in PSh(X)$, $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$,

a) φ injective $\iff$ $\forall\, x \in X$: φ_x injective

b) φ bijective $\iff$ $\forall\, x \in X$: φ_x bijective

c) φ locally surj $\iff$ $\forall\, x \in X$: φ_x surjective

Pf. a) If $\exists x \in X$: φ_x not injective,

then $\exists\, s_x \neq t_x \in \mathcal{F}_x$ w/ $\varphi_x(s_x) = \varphi_x(t_x)$

$\Rightarrow \exists\, U \subseteq X$ open, $s, t \in \mathcal{F}(U)$ representing s_x, t_x

w/ $\varphi(s) = \varphi(t)$ on U

$\Rightarrow \varphi(U)$ not injective

Conversely, if φ_x is injective for all $x \in X$, then $\varphi(U)$ is injective:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathcal{G}(U) \arrow[d] \\
\prod_{x \in U} \mathcal{F}_x \arrow[r, hook, "\prod \varphi_x"'] & \prod_{x \in U} \mathcal{G}_x
\end{tikzcd}$$

c) Clear by definition of germs

b) φ bijective $\Rightarrow$ φ_x bijective by a) & c)

Conversely: If φ_x bijective for all $x \in X$,

then φ is injective by a)

φ is locally surjective by b)

But inj + loc surj $\Rightarrow$ surjective

(local preimages glue by uniqueness of preimages) $\square$

Exercise Let $\mathcal{F}, \mathcal{G} \in Sh(X)$, $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

a) φ mono- / epi- / isomorphism in $PSh(X)$

$\Leftrightarrow$ φ injective / surjective / bijective

b) φ mono- / epi- / isomorphism in $Sh(X)$

$\Leftrightarrow$ φ_x injective / surjective / bijective

Def For a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ in $PSh(X)$,

define its <u>image presheaf</u> $\mathrm{im}_{PSh}(\varphi) \subseteq \mathcal{G}$ "sub-presheaf"

by

$$(\mathrm{im}_{PSh}(\varphi))(U) := \mathrm{im}(\mathcal{F}(U) \xrightarrow{\varphi} \mathcal{G}(U)).$$

Rem φ epi of presheaves $\Leftrightarrow$ $\mathrm{im}(\varphi) = \mathcal{G}$.

Two problems:

- analogue for epi's of sheaves?
- even if $\mathcal{F}, \mathcal{G}$ are sheaves, usually $\mathrm{im}_{PSh}(\varphi)$ is only a presheaf! (think of example of locally surj $\not\Rightarrow$ surj)

But any presheaf has a "best approximation by a sheaf":

Prop For any $\mathcal{F} \in PSh(X)$,

$\exists$ morphism $\mathcal{F} \xrightarrow{can} \mathcal{F}^{sh}$ to a sheaf $\mathcal{F}^{sh} \in Sh(X)$ over which any other morphism $\varphi : \mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G} \in Sh(X)$ factors uniquely:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "can"'] & \mathcal{G} \\ & \mathcal{F}^{sh} \arrow[u, dashed, "\exists! \varphi^{sh}"'] \end{tikzcd}$$

presheaf

sheaves

By this property the pair $(\mathcal{F}^{sh}, can)$ is determined uniquely up to iso. We get a functor

$$(-)^{sh} : PSh(X) \longrightarrow Sh(X)$$

left adjoint to the inclusion $\imath : Sh(X) \hookrightarrow PSh(X)$:

$$Hom_{PSh}(\mathcal{F}, \imath(\mathcal{G})) = Hom_{Sh}(\mathcal{F}^{sh}, \mathcal{G})$$

for all $\mathcal{F} \in PSh(X)$, $\mathcal{G} \in Sh(X)$.

Pf. For $U \subseteq X$ open, put

$$\mathcal{F}^{sh}(U) := \{ (s(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \mid (*) \text{ holds} \}$$

$(*)$ $\forall x_0 \in U$ $\exists$ nbhood $U_0 \subseteq U$ of x_0 $\exists s \in \mathcal{F}(U_0)$:

$$s(x) = s_x \text{ for all } x \in U_0.$$

Then image $(\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x) \subseteq \mathcal{F}^{sh}(U)$

$\Rightarrow$ $\exists$ natural morphism of presheaves $\mathcal{F} \to \mathcal{F}^{sh}$

Exercise: $\mathcal{F}^{sh}$ is a sheaf

Universal property:

If $\mathcal{G}$ is a sheaf then $\mathcal{G}(U) \xrightarrow{\sim} \mathcal{G}^{sh}(U)$ is an iso by the sheaf axioms applied to $(*)$.

$\Longrightarrow$ any $\varphi \in Hom(\mathcal{F}, \mathcal{G})$ induces $\varphi^{sh} \in Hom(\mathcal{F}^{sh}, \mathcal{G})$ via

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d] \\ \tilde{\mathcal{F}} \arrow[r, dashed, "\varphi^{sh}"'] \arrow[ur, dashed, "\exists \varphi^{sh}"] & \tilde{\mathcal{G}} \end{tikzcd}$$

iso by the above

$\square$

Def We call $\mathcal{F}^{sh}$ the <u>sheaf associated to</u> $\mathcal{F}$ or the <u>sheafification</u> of the presheaf $\mathcal{F}$.

Def For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves we define its <u>image sheaf</u> as

$$\mathrm{im}(\varphi) := (\mathrm{im}_{PSh}(\varphi))^{sh} \quad (\text{a subsheaf of } \mathcal{G}).$$

Exercise φ epi in $Sh(X) \iff \mathrm{im}(\varphi) = \mathcal{G}$

Functoriality for continuous maps?

Def For $f : X \to Y$ continuous, have a functor $f^{-1} : Op(Y) \to Op(X)$.
Define the <u>direct image functor</u> by

$$f_* : PSh(X) \to PSh(Y), \quad \mathcal{F} \mapsto \mathcal{F} \circ f^{-1}$$

i.e. $\quad (f_* \mathcal{F})(V) := \mathcal{F}(f^{-1}(V)).$

Exercise Direct images of sheaves are sheaves, so we get $f_* : Sh(X) \to Sh(Y)$.

Ex For $f : \{p\} \hookrightarrow Y$ a closed pt, $f_*(A) =$ skyscraper A at p
For $f : X \to \{p\}$, $f_*(\mathcal{F}) = \mathcal{F}(X)$ (as sheaf on a pt)

Def We define

$$f^{-1}_{pre} : PSh(Y) \to PSh(X)$$

by $\quad (f^{-1}_{pre} \mathcal{F})(U) := \mathcal{F}(f(U))$

<u>Two issues</u>: 1) $f(U) \subseteq Y$ needn't be open!
We define

$$\mathcal{F}(f(U)) := \varinjlim_{\substack{V \supseteq f(U) \\ V \in Op(Y)}} \mathcal{F}(V)$$

2) The functor f^{-1}_{pre} doesn't send sheaves to sheaves in general. For instance, for $Y = \{pt\}$ we have $\quad f^{-1}_{pre}(A) =$ constant presheaf w/ value A.

In general, we define the <u>inverse image functor</u>

$$f^{-1} : Sh(Y) \to Sh(X)$$

by $\quad f^{-1}\mathcal{F} := (f^{-1}_{pre} \mathcal{F})^{sh}$

Ex For $f : \{p\} \hookrightarrow Y$ we have $f^{-1}\mathcal{F} \simeq \mathcal{F}_p$ (stalk at p)
For $f : X \to \{p\}$ we have $f^{-1}A \simeq A_X$ (constant sheaf)

Lemma f^{-1} is left adjoint to f_*:

$\exists$ natural iso

$$\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{Sh(Y)}(\mathcal{F}, f_*\mathcal{G})$$

Pf. For presheaves $\mathcal{F} \in PSh(Y)$, $\mathcal{G} \in PSh(X)$:

$\varphi \in \mathrm{Hom}(f^{-1}_{pre}\mathcal{F}, \mathcal{G}) \rightsquigarrow \forall U \subseteq X$ open,

$$\varphi(U) : (f^{-1}_{pre}\mathcal{F})(U) \to \mathcal{G}(U)$$

$$\| $$

$$\varinjlim_{W \supseteq f(U)} \mathcal{F}(W)$$

$\rightsquigarrow \forall W \subseteq Y$ open,

taking $U := f^{-1}W \subseteq X$

gives a morphism

$$\psi(W) : \mathcal{F}(W) \to \mathcal{G}(f^{-1}W) = (f_*\mathcal{G})(W)$$

$\rightsquigarrow \psi \in \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$

Exercise: This gives a bijection

$$\mathrm{Hom}(f^{-1}_{pre}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$\|$$

$\mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G})$ if $\mathcal{F}, \mathcal{G}$ are sheaves $\square$

Ex For any presheaf $\mathcal{F} \in PSh(Y)$, the morphism $\mathcal{F} \to \mathcal{F}^{sh}$ induces on stalks an iso

$$\mathcal{F}_p \xrightarrow{\sim} (\mathcal{F}^{sh})_p .$$

Pf. Consider $i : \{p\} \hookrightarrow Y$. For any test object A,

$$\begin{aligned}
\mathrm{Hom}(\mathcal{F}_p, A) &= \mathrm{Hom}_{PSh}(i^{-1}_{pre}\mathcal{F}, A) \\
&= \mathrm{Hom}_{PSh}(\mathcal{F}, i_* A) \\
&= \mathrm{Hom}_{Sh}(\tilde{\mathcal{F}}, i_* A) \\
&= \mathrm{Hom}_{Sh}(i^{-1}\tilde{\mathcal{F}}, A) \\
&= \mathrm{Hom}((\tilde{\mathcal{F}})_p, A).
\end{aligned}$$

$\square$

5. Affine schemes

Goal: Define a "sheaf $\mathcal{O}_X$ of functions" on $X = \operatorname{Spec} A$ w/ global sections $\mathcal{O}_X(X) = A$.

Recall from exercises: X top space
$\mathcal{B}$ basis of open sets w/ $U \cap V \in \mathcal{B}$ for all $U, V \in \mathcal{B}$

$$\Rightarrow Sh(X) \xrightarrow{\sim} Sh(\mathcal{B}) \qquad (*)$$

where $Sh(\mathcal{B})$ is the category of functors $\mathcal{F}: \mathcal{B}^{op} \to \text{Sets}$ sth

$$\mathcal{F}(U) = \lim \Big(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \Big)$$

for all $U \in \mathcal{B}$ & all open covers $U = \bigcup_{i \in I} U_i$ w/ $U_i \in \mathcal{B}$.

For $X = \operatorname{Spec} A$,
take $\mathcal{B} = \{ D(f) \mid f \in A \}$
w/ $D(f) := X \setminus V(f) \simeq \operatorname{Spec} A_f$.

Define a sheaf of rings $\mathcal{O}_X \in Sh(X)$ via $(*)$ by

$$\mathcal{O}_X(D(f)) := A_f.$$

Restriction maps:

$$D(g) \subseteq D(f) \iff \begin{tikzcd} \operatorname{Spec} A_g \arrow[r, dashed, "\exists"] \arrow[dr] & \operatorname{Spec} A_f \arrow[d, phantom, "\cap"] \\ & \operatorname{Spec} A \end{tikzcd}$$

$$\iff \begin{tikzcd} A \arrow[r] \arrow[rr, bend right] & A_f \arrow[r, dashed, "\exists \varrho"] & A_g \end{tikzcd}$$

Define

$$\operatorname{res}^{D(f)}_{D(g)} := \varrho : \underset{= A_f}{\mathcal{O}_X(D(f))} \longrightarrow \underset{= A_g}{\mathcal{O}_X(D(g))}$$

Thm The functor $\mathcal{O}_X : \mathcal{B}^{op} = \{ D(f) \mid f \in A \}^{op} \to \text{Rings}$ satisfies the sheaf conditions on the basis $\mathcal{B}$, hence defines a sheaf of rings $\mathcal{O}_X \in Sh(X)$.

Pf. Let $U = D(f) = \bigcup_{i \in I} \underbrace{D(f_i)}_{=: U_i} \subseteq X = \operatorname{Spec}(A)$.

Want:

a) $s, t \in \mathcal{O}(U)$ w/ $s|_{U_i} = t|_{U_i}$ for all i $\Rightarrow s = t$

b) $s_i \in \mathcal{O}(U_i)$ w/ $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j

$$\Rightarrow \exists s \in \mathcal{O}(U) : s_i = s|_{U_i}$$

Replace A by A_f $\rightsquigarrow$ wlog $f = 1$.

U quasicompact $\rightsquigarrow$ $\exists$ finite subcover $\rightsquigarrow$ wlog I finite

a) $s, t \in A$

$\frac{s}{1} = \frac{t}{1} \in A_{f_i}$ for all $i \in I$

$\Rightarrow \exists m \in \mathbb{N}: \quad f_i^m \cdot (s-t) = 0 \in A$

But $X = \bigcup_i D(f_i) = \bigcup_i D(f_i^m)$

$\Rightarrow \exists a_i \in A: \quad \sum_i a_i f_i^m = 1 \quad \leftarrow$ "partition of 1"

$$\Rightarrow \quad s - t = 1 \cdot (s-t)$$

$$= \sum_i a_i \underbrace{f_i^m (s-t)}_{=0} = 0 \quad \text{in } A$$

b) $s_i = \frac{c_i}{f_i^m} \in A_{f_i}$ sth $s_i = s_j$ in $A_{f_i f_j}$

$\Rightarrow \exists r: \quad (f_i f_j)^r \cdot (c_i f_j^m - c_j f_i^m) = 0$ in A $\quad (*)$

But again $\exists a_i \in A: \quad \sum_i a_i \cdot f_i^{m+r} = 1 \quad (**)$

$\Rightarrow s := \sum_j a_j c_j f_j^r \in A$ satisfies

$$f_i^{m+r} \cdot s = \sum_j a_j \cdot c_j f_i^m \cdot (f_i f_j)^r$$

$$\underset{(*)}{=} \sum_j a_j \cdot c_i f_j^m \cdot (f_i f_j)^r \underset{(**)}{=} c_i f_i^r$$

$\Rightarrow \quad s = c_i / f_i^m = s_i$ in A_{f_i} $\quad \square$

Stalks are easy to describe:

Prop The stalk of $\mathcal{O}_X$ at a pt $x = \mathfrak{p} \in X = \operatorname{Spec} A$ is the local ring

$$\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}}.$$

Pf. $\mathcal{O}_{X,x} := \varinjlim_{\substack{U \ni x \\ \text{open}}} \mathcal{O}_X(U) \simeq \varinjlim_{D(f) \ni x} \mathcal{O}_X(D(f))$

$$= \varinjlim_{f \in S} A_f \quad \text{w/ } S = A \setminus \mathfrak{p}$$

We have a natural epi $\varinjlim_{f \in S} A_f \twoheadrightarrow A_{\mathfrak{p}} = S^{-1} A$.

This is an iso:

Say $\frac{a}{f^n} \in A_f$ maps to zero in $A_{\mathfrak{p}}$

$\Rightarrow \exists s \in S: \quad s \cdot a = 0$ in A

$\Rightarrow \frac{a}{f^n} = 0$ in A_g for $g := sf \in S$ $\quad \square$

Def $\mathfrak{m}_x := \mathfrak{p} A_{\mathfrak{p}} \trianglelefteq \mathcal{O}_{X,x} \simeq A_{\mathfrak{p}}$ max. ideal at $x \in X$

$\kappa(x) := \mathcal{O}_{X,x} / \mathfrak{m}_x$ residue field at $x \in X$

$\rightsquigarrow ev_x: \mathcal{O}_{X,x} \longrightarrow \kappa(x)$, "$f \mapsto f(x)$" evaluation at x

w/ kernel $\ker(ev_x) = \mathfrak{m}_x$.

Back to the "global" picture:

Def A <u>ringed space</u> is a pair $(X, \mathcal{O}_X)$ w/ X a top space, $\mathcal{O}_X$ a sheaf of rings on X.

Pull-back of "functions":

(Idea: $\varphi : X \to Y$ continuous & $U \subseteq Y$ open
$f \in \mathcal{O}_Y(U)$ "function" $\rightsquigarrow$ "$f \circ \varphi$" $\in \mathcal{O}_X(\varphi^{-1}U) = (\varphi_* \mathcal{O}_X)(U)$)

Def A <u>morphism</u> between ringed spaces $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ is a pair $(\varphi, \varphi^\#)$ consisting of

- a continuous map $\varphi : X \to Y$
- a morphism of sheaves of rings

$$\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$$

$$(\text{equivalently: } \varphi^\flat : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X)$$

Given another morphism $(\psi, \psi^\#) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$

define

$$\begin{tikzcd}
\mathcal{O}_Z \arrow[r, "\psi^\#"] \arrow[drr, "(\psi\circ\varphi)^\#"'] & \psi_* \mathcal{O}_Y \arrow[r, "\psi_*(\varphi^\#)"] & \psi_* \varphi_* \mathcal{O}_X \arrow[d, equal] \\
 & & (\psi \circ \varphi)_* \mathcal{O}_X
\end{tikzcd}$$

$\rightsquigarrow$ category of ringed spaces

Ringed spaces see more than top. spaces:

Lemma We get a faithful functor

$$\mathrm{Spec} : (\mathrm{Rings})^{op} \to (\mathrm{Ringed\ Spaces}).$$

Pf. Let $h : A \to B$ be a ring homom.

$\varphi = {}^a h : X = \mathrm{Spec}\, B \to Y = \mathrm{Spec}\, A$, $\mathfrak{p} \mapsto h^{-1}(\mathfrak{p})$

map associated with h: Shorthand for $\mathrm{Spec}(h)$

By the equivalence of categories $Sh(Y) \xrightarrow{\sim} Sh(\mathcal{B})$ w/ $\mathcal{B} := \{ D(f) \subseteq Y \mid f \in A \}$, suffices to define $\varphi^\flat$ on the basic open subsets $D(f)$. We put

$$\begin{tikzcd}
\varphi^\#(D(f)) : \mathcal{O}_Y(D(f)) \arrow[r] \arrow[dd, equal] & (\varphi_* \mathcal{O}_X)(D(f)) \arrow[d, equal] \\
 & \mathcal{O}_X(\varphi^{-1} D(f)) \arrow[d, equal] \\
A_f \arrow[r, "h"] & B_{h(f)}
\end{tikzcd}$$

$\varphi^{-1}(D(f)) = D(h(f))$

Functoriality: ${}^a\mathrm{id}^\# = \mathrm{id}$, ${}^a(g \circ h)^\# = {}^a g^\# \circ {}^a h^\#$ (exercise)

Faithfulness: Can recover h from $\varphi^\# = {}^a h^\#$ as

$$h = \varphi^\#(Y) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X). \qquad \square$$

Caution: NOT fully faithful! E.g. take $p>0$ prime,

$$\begin{array}{ccc} X = \operatorname{Spec} \mathbb{Q} & \xrightarrow{f} & Y = \operatorname{Spec} \mathbb{Z}, \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ (0) & \longmapsto & (p) \end{array} \quad f^b : \begin{array}{ccc} f^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \\ \| & & \| \\ \mathbb{Z}_{(p)} & \hookrightarrow & \mathbb{Q} \end{array}$$

$\Rightarrow (f, f^b)$ NOT induced by a ring hom $\mathbb{Z} \to \mathbb{Q}$!

Which morphisms of ringed spaces come from ring homomorphisms?

Recall: $h : B \longrightarrow A$ ring hom.

$$\rightsquigarrow ({}^a h, {}^a h^\#) : \begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} B \\ \| & & \| \\ (X, \mathcal{O}_X) & & (Y, \mathcal{O}_Y) \end{array}$$

On stalks at $x = \mathfrak{p} \in X$, $y = {}^a h(x) =: \mathfrak{q} \in Y$:

$$\begin{tikzcd} \mathcal{O}_{Y,y} \arrow[r] \arrow[rr, bend left, "\varphi_x^\#"] \arrow[d, equal] & ({}^a h_* \mathcal{O}_X)_y \arrow[r] & \mathcal{O}_{X,x} \arrow[d, equal] \\ B_\mathfrak{q} \arrow[rr, "h_\mathfrak{p}"'] & & A_\mathfrak{p} \end{tikzcd}$$

Note $h_\mathfrak{p}$ induced by $h : B \to A$ w/ $\mathfrak{q} = h^{-1}(\mathfrak{p})$

$\Rightarrow h_\mathfrak{p}^{-1}(\mathfrak{p} A_\mathfrak{p}) = \mathfrak{q} B_\mathfrak{q}$ (unlike for $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$)!

Def A local ring is a ring A w/ a unique max. ideal $\mathfrak{m}_A$.

A local homomorphism is a homom. $\varphi : B \to A$ of local rings w/ $\mathfrak{m}_B = \varphi^{-1}(\mathfrak{m}_A)$ $(\Leftrightarrow \varphi(\mathfrak{m}_B) \subseteq \mathfrak{m}_A)$.

Def A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ for which the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$.

A morphism of locally ringed spaces is a morphism $(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ s.th. for all $x \in X$,

$\varphi_x^\# : \mathcal{O}_{Y, \varphi(x)} \longrightarrow \mathcal{O}_{X,x}$ is a local homomorphism.

We get a category

$$\mathrm{LRS} := (\text{Locally ringed spaces}).$$

Def An affine scheme is a locally ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for a ring A.

These form a full subcategory $\mathrm{AffSch} \subseteq \mathrm{LRS}$.

Convention From now on we put

$$\operatorname{Spec} A := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \in \mathrm{AffSch}.$$

Now <u>all</u> morphisms are controlled by ring homomorphisms:

<u>Thm</u> (morphisms to affine schemes)

For any $(X, \mathcal{O}_X) \in LRS$ & any ring A,

$\exists$ natural bijection

$$\begin{array}{ccc} \mathrm{Hom}_{LRS}((X,\mathcal{O}_X), \mathrm{Spec}\, A) & \xrightarrow{\sim} & \mathrm{Hom}_{Rings}(A, \Gamma(\mathcal{O}_X)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\varphi, \varphi^\#) & \longmapsto & \Gamma(\varphi^\#) \end{array}$$

induced by $\Gamma : Sh(X) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}(X)$.

"functor of global sections"

<u>Cor</u> The functor $\mathrm{Spec} : \mathrm{Rings} \xrightarrow{\sim} \mathrm{AffSch}$

is an equivalence of categories.

Pf of cor. Essentially surjective by def^n of AffSch.

Fully faithful: Take $(X, \mathcal{O}_X) = \mathrm{Spec}\, B$ in the theorem

$$\Rightarrow \mathrm{Hom}_{AffSch}(\mathrm{Spec}\, B, \mathrm{Spec}\, A) \xrightarrow{\sim} \mathrm{Hom}_{Rings}(A, B),$$

$$(\varphi, \varphi^\#) \mapsto \Gamma(\varphi^\#) = h$$

for $(\varphi, \varphi^\#) = ({}^a h, {}^a h^\#)$. $\square$

Pf of thm.

a) Injectivity:

Let $(\varphi_i, \varphi_i^\#) \in \mathrm{Hom}_{LRS}((X, \mathcal{O}_X), \mathrm{Spec}\, A)$, $i = 1, 2$,

w/ $h := \Gamma(\varphi_1^\#) = \Gamma(\varphi_2^\#) \in \mathrm{Hom}_{Rings}(A, \Gamma(\mathcal{O}_X))$.

Let $x \in X$ and $\mathfrak{p}_i := y_i := \varphi_i(x) \in Y = \mathrm{Spec}\, A$.

Taking stalks gives:

$$\begin{tikzcd} A \arrow[r, "\lambda_{y_i}"] \arrow[d, equal] & A_{\mathfrak{p}_i} \arrow[d, equal] \\ \Gamma(\mathcal{O}_Y) \arrow[d, "h"'] & \mathcal{O}_{Y,\varphi_i(x)} \arrow[d, "\varphi_{i,x}^\#"] \\ \Gamma(\mathcal{O}_X) \arrow[r, "\lambda_x"] & \mathcal{O}_{X,x} \end{tikzcd}$$

$$\Rightarrow \mathfrak{p}_i = \lambda_i^{-1}(\underbrace{\mathfrak{m}_{\varphi_i(x)}}_{=(\varphi_{i,x}^\#)^{-1}(\mathfrak{m}_x)}) = (\varphi_{i,x}^\# \circ \lambda_{y_i})^{-1}(\mathfrak{m}_x) = (\lambda_x \circ h)^{-1}(\mathfrak{m}_x)$$

since $\varphi_{i,x}^\#$ is a local homom.

$\Rightarrow \mathfrak{p}_1 = \mathfrak{p}_2$, i.e. $\varphi_1(x) = \varphi_2(x)$.

The above diagram then also shows $\varphi_{1,x}^\# = \varphi_{2,x}^\#$.

Since this holds for all $x \in X$, we get:

$$(\varphi_1, \varphi_1^\#) = (\varphi_2, \varphi_2^\#).$$

b) Surjectivity:

For $h \in \mathrm{Hom}_{\mathrm{Rings}}(A, \Gamma(X))$, define $\varphi : X \to \operatorname{Spec} A$

by

$$\varphi(x) := \text{preimage of } \mathfrak{m}_x \trianglelefteq \mathcal{O}_{X,x} \text{ under the ring homomorphism } A \xrightarrow{h} \Gamma(X) \xrightarrow{\lambda_x} \mathcal{O}_{X,x}$$

Need to check φ is continuous:

Consider a basic open $U = D(f) \subset \operatorname{Spec} A$.

$$\Rightarrow \varphi^{-1}(U) = \{x \in X \mid \varphi(x) \in D(f)\}$$

$$= \{x \in X \mid f \notin (\lambda_x \circ h)^{-1}(\mathfrak{m}_x)\}$$

$$= \{x \in X \mid \lambda_x(\underbrace{h(f)}_{=: g \in \Gamma(\mathcal{O}_X)}) \notin \mathfrak{m}_x\}$$

$$= \{x \in X \mid g_x \in \mathcal{O}_{X,x}^*\}$$

$$= \{x \in X \mid \exists U \ni x \text{ open w/ } g|_U \in (\mathcal{O}_X(U))^*\}$$

$$=: D(g) \subseteq X \text{ open}$$

$\Rightarrow \varphi : X \to \operatorname{Spec} A$ continuous

Now define $\varphi^\# : \mathcal{O}_{\operatorname{Spec} B} \longrightarrow \varphi_* \mathcal{O}_X$ on basic open subsets $D(f) \subseteq \operatorname{Spec}(A)$ by

$$\begin{array}{cccc}
\varphi^\#(D(f)) : & \mathcal{O}_{\operatorname{Spec} B}(D(f)) & \longrightarrow & (\varphi_* \mathcal{O}_X)(D(f)) \\
& \| & & \| \\
& A_f & \dashrightarrow & \mathcal{O}_X(D(g)) \\
& \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
& f^{-n} \cdot a & \longmapsto & g^{-n} \cdot h(a)|_{D(g)}
\end{array}$$

w/ $g := h(f) \in (\mathcal{O}_X(D(g)))^*$

$a \in A \rightsquigarrow h(a) \in \Gamma(\mathcal{O}_X)$

$\rightsquigarrow h(a)|_{D(g)} \in \mathcal{O}_X(D(g))$

These are compatible w/ restriction to smaller open sets

$\Rightarrow$ morphism $\varphi^\# : \mathcal{O}_{\operatorname{Spec} B} \to \varphi_* \mathcal{O}_X$ of sheaves

$\Rightarrow (\varphi, \varphi^\#) \in \mathrm{Hom}_{\mathrm{LRS}}((X, \mathcal{O}_X), \operatorname{Spec} B)$

& by construction $\Gamma(\varphi^\#) = h$. $\square$

II. Schemes: Basic notions & examples

1. Schemes and morphisms

Recall: An affine scheme is a locally ringed space which is isomorphic as locally ringed space to Spec (A) for a ring A.

Affine schemes see much more than the underlying top. spaces: Have equivalence of categories

$$\mathrm{Spec}: (\text{Rings}) \xrightarrow{\sim} (\text{affine schemes})$$

Def $|X| :=$ underlying top space of a ringed space X.

Ex K number field

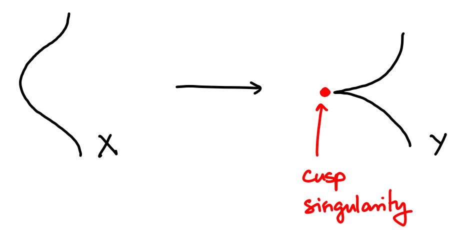
$\Rightarrow$ $X = \mathrm{Spec}\, K$ has top space $|X| =$ a single pt, but as an affine scheme it remembers the Galois gp:

$$\mathrm{Hom}_{\mathrm{AffSch}}(X, X) \simeq \mathrm{Hom}_{\mathrm{Rings}}(K, K) = \mathrm{Gal}(K/\mathbb{Q})$$

Ex $X = \mathrm{Spec}\, k[t] \longrightarrow Y = \mathrm{Spec}\, k[x,y]/(y^2 - x^3)$

defined by $h: k[x,y]/(y^2-x^3) \longrightarrow k[t]$, $x \mapsto t^2$, $y \mapsto t^3$

$\Rightarrow$ $|h|: |X| \longrightarrow |Y|$ homeomorphism, but $X \not\simeq Y$ as affine schemes: We see the singularity of Y!



Affine schemes are not enough:

- Want to consider in Spec A arbitrary open sets, not only basic sets $D(f) \simeq \mathrm{Spec}\, A_f$ ...
- Want to construct "global schemes" by gluing affine patches, e.g. to projective space $\mathbb{P}^n_k$ ($\rightarrow$ add "missing pts at ∞") etc ...

Def A *scheme* is a locally ringed space $X = (|X|, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme in the following sense:

$\exists$ open cover $|X| = \bigcup_{i \in I} U_i$ & rings A_i

s.th. $(U_i, \mathcal{O}_X|_{U_i}) \simeq \operatorname{Spec} A_i$ as loc. ringed space.

We call $(U_i)_{i \in I}$ an *affine cover / atlas* of X.

Let $\mathrm{Sch} :=$ (full subcategory of schemes) $\subset \mathrm{LRS}$

Ex $X = (|X|, \mathcal{O}_X)$ a scheme

$\Rightarrow$ for any open $U \subseteq |X|$, $(U, \mathcal{O}_X|_U)$ is a scheme.

Indeed: Pick a cover $X = \bigcup_{i \in I} U_i$ as above

$\Rightarrow U = \bigcup_{i \in I} U \cap U_i$ w/ $U \cap U_i$ open in $U_i \simeq \operatorname{Spec} A_i$

$\Rightarrow \exists$ cover $U_i = \bigcup_j U_{ij}$ w/ $U_{ij} := \underbrace{D(f_{ij})}_{\simeq \operatorname{Spec}(A_i)_{f_{ij}}} \subseteq \operatorname{Spec} A_i$

$\Rightarrow (U_{ij})_{i,j}$ an affine open cover of U

Def We write $U := (U, \mathcal{O}_X|_U) \subseteq X$ and call it an *open subscheme* of the scheme X.

Open subschemes of affine schemes needn't be affine:

Ex $X = \mathbb{A}^2_k = \operatorname{Spec} A$ w/ $A = k[u,v]$

$\cup$

$U = X \setminus V(u,v)$ "punctured plane"

$\Rightarrow$ The scheme U is NOT affine:

Suppose $(U, \mathcal{O}_X|_U) \simeq \operatorname{Spec} B$.

We have $U = U_1 \cup U_2$ w/ $U_i := D(x_i) \subset X$.

The sheaf axiom gives

$$B = \lim \Big(\prod_{i=1}^{2} \mathcal{O}_X(U_i) \rightrightarrows \mathcal{O}(U_1 \cap U_2) \Big)$$

$$= \ker \big(A_u \times A_v \longrightarrow A_{uv} \big), \quad (\alpha, \beta) \longmapsto \alpha - \beta$$

$$= \{ (u^{-m} a, v^{-n} b) \mid a, b \in A,\ m, n \in \mathbb{N},\ a v^n = b u^m \}$$

$\Updownarrow$ A is a UFD: $\exists c \in A: a = c u^m,\ b = c v^n$

$$= \operatorname{im}(A \hookrightarrow A_u \times A_v)$$

$\Rightarrow A \xrightarrow{\sim} B$ Iso

$\Rightarrow U = \operatorname{Spec} B \xrightarrow{\sim} X = \operatorname{Spec} A$ Iso (see below)

$\lightning$ $U \hookrightarrow X$ NOT homeomorphism

Here we used:

<u>Prop.</u> (Morphisms to affine schemes)

For any scheme X & any ring A,

$$\mathrm{Hom}_{Sch}(X, \mathrm{Spec}\, A) \simeq \mathrm{Hom}_{Rings}(A, \Gamma(\mathcal{O}_X))$$

Pf. We've seen this more generally for any loc. ringed space X. $\square$

<u>Cor</u> Let X be a scheme and $A = \Gamma(\mathcal{O}_X)$.

a) $\exists$ natural morphism $f : X \to \mathrm{Spec}\, A$.

b) any other morphism $g : X \to \mathrm{Spec}\, B$ to an affine scheme factors uniquely over f:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathrm{Spec}\, B \\ & \mathrm{Spec}\, A \arrow[ur, dashed, "\exists!\, \bar{g}"'] & \end{tikzcd}$$

Pf. Let f correspond to $\mathrm{id} : A \to \Gamma(\mathcal{O}_X) = A$. $\square$

<u>Def.</u> $X^{\mathrm{aff}} := \mathrm{Spec}\, \Gamma(\mathcal{O}_X)$ is called the <u>affinization</u> of X.

<u>Upshot</u> X affine $\iff$ $f : X \xrightarrow{\sim} X^{\mathrm{aff}}$ is an iso

More generally:

<u>Def</u> A <u>scheme over a ring A</u> is a pair (X, p_X) of a scheme X & a morphism $p_X : X \to \mathrm{Spec}\, A$

($\iff$ ring homom. $A \to \Gamma(\mathcal{O}_X)$)

Schemes over A form a category Sch_A with

$$\mathrm{Hom}_{Sch_A}((X, p_X), (Y, p_Y))$$
$$:= \{ f \in \mathrm{Hom}_{Sch}(X, Y) \mid p_Y \circ f = p_X \}$$

ie we impose commutativity of the diagrams

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & \mathrm{Spec}\, A & \end{tikzcd}$$

<u>Ex</u>
- $Sch = Sch_{\mathbb{Z}}$
- $\mathbb{A}^n_A := \mathrm{Spec}\, A[x_1, \dots, x_n] \in Sch_A$

  For $A = \mathbb{Z}$ we put $\mathbb{A}^n := \mathrm{Spec}\, \mathbb{Z}[x_1, \dots, x_n]$
- $$\begin{tikzcd} A\text{-algebras} \arrow[r, "\sim"] \arrow[d, phantom, "\cap" description] & AffSch_A \arrow[r, hook] \arrow[d, phantom, "\cap" description] & Sch_A \arrow[d, phantom, "\cap" description] \\ Rings \arrow[r, "\sim"] & AffSch \arrow[r, hook] & Sch \end{tikzcd}$$

This was about morphisms to affine schemes. Morphisms from affine schemes are "points w/ coordinates in R":

Def Let $X \in Sch$ and R a ring.

We call $X(R) := \mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, X)$

the set of <u>R-valued points</u> of the scheme X.

(more about the "functor of points" later...)

Ex • $X = \mathbb{A}^n := \mathrm{Spec}\, \mathbb{Z}[x_1, .., x_n]$

$\Rightarrow \mathbb{A}^n(R) = \mathrm{Hom}_{Rings}(\mathbb{Z}[x_1, .., x_n], R) = R^n$

(a <u>set</u>, not to be confused w/

the <u>scheme</u> $\mathbb{A}^n_R := \mathrm{Spec}\, R[x_1, .., x_n]$)

• $X = \mathrm{Spec}\, \mathbb{Z}[x_1, .., x_n]/(f_1, \ldots, f_m) \in Sch$

$\Rightarrow X(R) = \{ p \in R^n \mid f_1(p) = \cdots = f_m(p) = 0 \}$

Lemma. $X \in Sch$ & k a field

$$\Rightarrow X(k) = \left\{ (p, \iota) \;\middle|\; \begin{array}{l} p \in |X| \\ \iota : \kappa(p) \hookrightarrow k \end{array} \right\}$$

Pf. Given $(f, f^\#) \in X(k) = \mathrm{Hom}_{Sch}(\underbrace{\mathrm{Spec}\, k}_{=:Y}, X)$,

let $q := (0) \in \mathrm{Spec}\, k$.

Put $p := f(q) \in |X|$ & define ι by

$$\begin{tikzcd} \mathcal{O}_{X,p} \arrow[r, "f_p^\#"] \arrow[d] & \mathcal{O}_{Y,q} = k \\ \mathcal{O}_{X,p}/m_p = \kappa(p) \arrow[ur, dashed, "\exists ! \iota"'] & \end{tikzcd}$$

Conversely, given $p \in |X|$ & $\iota : \kappa(p) \hookrightarrow k$,

define
$$f : |Y| \to |X|, \quad q \mapsto p$$

$$\begin{tikzcd} f^b : f^{-1}\mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_Y = k \\ \mathcal{O}_{X,p} \arrow[r, two heads] & \mathcal{O}_{X,p}/m_p = \kappa(p) \arrow[u, hook, "\iota"] \end{tikzcd}$$

$\square$

Ex $X = \mathrm{Spec}\, k$

$\Rightarrow X(k) = \mathrm{Aut}(k)$

2. Gluing schemes

Goal: Construct schemes by gluing together open sets!

Ex Let U_1, U_2 be schemes

w/ open subschemes $U_{12} \subseteq U_1$

$U_{21} \subseteq U_2$

& an iso $\varphi : U_{12} \xrightarrow{\sim} U_{21}$.

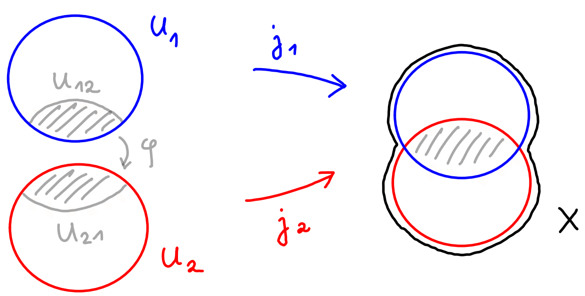
Consider the top. space

$$|X| := |U_1| \sqcup |U_2| \,/\sim$$

quotient by the equivalence relation $\sim$

given by $\varphi_{12}(u) \sim u$ for $u \in |U_{12}|$

endowed w/ quotient topology



Glue structure sheaves as follows:

Let $j_\alpha : |U_\alpha| \hookrightarrow |X|$ be the natural inclusions,

then

by def of quotient topology

$$W \subseteq |X| \text{ open} \iff W_\alpha := j_\alpha^{-1}(W) \subseteq |U_\alpha| \text{ open}$$

for $\alpha = 1, 2$

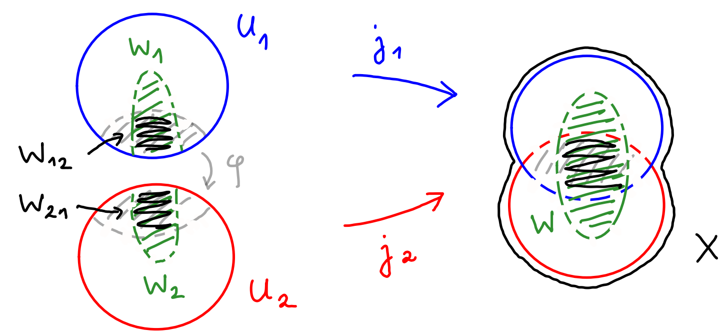
Put $W_{\alpha\beta} := U_{\alpha\beta} \cap W_\alpha$

Use $\varphi^b : \mathcal{O}_{U_2}(W_{21}) \xrightarrow{\sim} \mathcal{O}_{U_1}(W_{12})$

to define

$$\mathcal{O}_X(W) := \ker \Big(\prod_{\alpha=1}^{2} \mathcal{O}_{U_\alpha}(W_\alpha) \longrightarrow \mathcal{O}_{U_1}(W_{12}) \Big)$$

$$(f, g) \longmapsto f|_{W_{12}} - \varphi^b(g|_{W_{21}})$$



Claim $X := U_1 \cup_{\varphi} U_2 := (|X|, \mathcal{O}_X)$ is a scheme

w/ an open cover $X = V_1 \cup V_2$

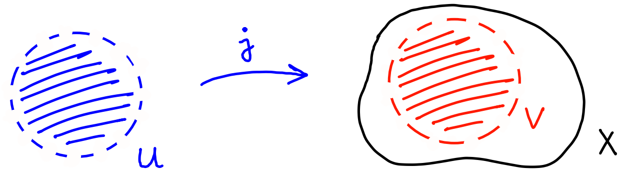
w/ iso of schemes $j_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha \subseteq X$

sth $\varphi = j_2^{-1} \circ j_1|_{U_{12}}$.

(Pf: Exercise, see more general proposition below)

Def A morphism $j : U \to X$ of schemes is called an <u>open embedding</u> or <u>open immersion</u> if it factors over an isomorphism onto an open subscheme $V \subseteq X$. Explicitly, for $j = (\varphi, \varphi^\#)$ this means:

- $\varphi : |U| \underset{\text{homeomorphism}}{\xrightarrow{\sim}} V \underset{\text{open}}{\subseteq} |X|$
- $\varphi^\# : \varphi^{-1}\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_U$ iso of sheaves



More "symmetric" reformulation of gluing:

Let U_1, U_2, U be schemes w/ open immersions $j_\alpha : U \hookrightarrow U_\alpha$ for $\alpha = 1, 2$

Define

- $|X| := |U_1| \sqcup |U_2| \,/\, (j_1(u) \sim j_2(u),\ u \in |U|)$

  $\rightsquigarrow$ $\exists$ homeomorphisms $j_\alpha : |U_\alpha| \xrightarrow{\sim} V_\alpha \underset{\text{open}}{\subseteq} |X|$

  $j : |U| \xrightarrow{\sim} V_1 \cap V_2 \underset{\text{open}}{\subseteq} |X|$

- for $W \subseteq |X|$ open,

$$\mathcal{O}_X(W) := \lim \left(\prod_{\alpha=1}^{2} \mathcal{O}_{U_\alpha}(j_\alpha^{-1} W) \rightrightarrows \mathcal{O}_U(j^{-1} W) \right)$$

Claim $X := U_1 \cup_U U_2 := (|X|, \mathcal{O}_X)$ is a scheme

$= \operatorname{colim}\left(U \overset{j_1}{\underset{j_2}{\rightrightarrows}} U_1 \sqcup U_2 \right)$ in the category of schemes

ie. for any scheme Z we have

$$\mathrm{Hom}_{Sch}(X, Z) = \left\{ (f_1, f_2) \in \prod_{\alpha=1}^{2} \mathrm{Hom}_{Sch}(U_\alpha, Z) \;\middle|\; f_1 \circ j_1 = f_2 \circ j_2 \right\}$$

Basic example: $U_\alpha = \operatorname{Spec} R_\alpha \quad (\alpha = 1,2)$

$$U = \operatorname{Spec} S \xrightarrow[j_\alpha]{\sim} D(f_\alpha) \subseteq U_\alpha$$

We have

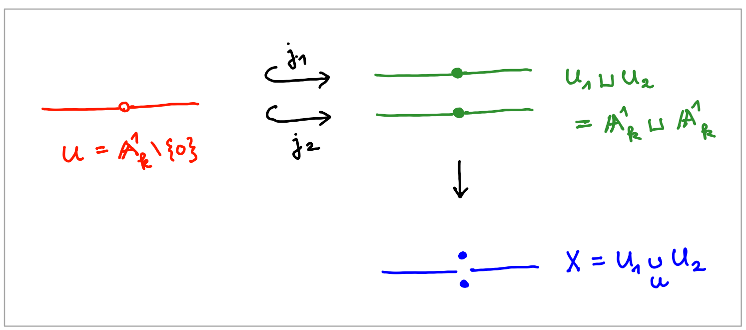
$$\begin{tikzcd} \Gamma(j_\alpha^\#) : R_\alpha \arrow[rr] \arrow[dr] & & S \\ & R_{\alpha, f_\alpha} \arrow[ur, dashed, "\sim", "\exists \text{ iso } h_\alpha"'] & \end{tikzcd}$$

So $X := U_1 \underset{U}{\cup} U_2$ has

$$\Gamma(\mathcal{O}_X) = \{ (r_1, r_2) \in R_1 \times R_2 \mid h_1(r_1) = h_2(r_2) \}$$

Ex The affine line with doubled origin:

Take $S = k[t, t^{-1}] \supset R_1 = R_2 = k[t]$

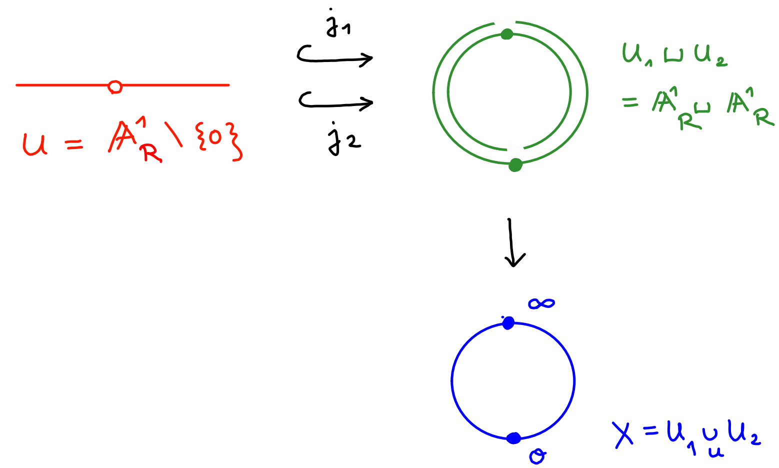


$\Rightarrow \Gamma(\mathcal{O}_X) = \{ (r_1, r_2) \in k[t] \mid r_1 = r_2 \} \simeq k[t]$

$\Rightarrow X$ not affine (since $X \not\cong X^{\mathrm{aff}} = \mathbb{A}^1_k$)

Ex The projective line over a ring R:

Take $S = R[t, t^{-1}] \supset R_1 = R[t]$
$\cup$
$R_2 = R[t^{-1}]$



We put $\mathbb{P}^1_R := U_1 \underset{U}{\cup} U_2$

$0 := (\text{image of } 0 \in U_1) \in \mathbb{P}^1_R$

$\infty := (\text{image of } 0 \in U_2) \in \mathbb{P}^1_R$

$\Longrightarrow \quad \mathbb{P}^1_R \setminus \{\infty\} = U_1 \simeq \mathbb{A}^1_R$

$\mathbb{P}^1_R$ is not affine, as we have $\Gamma(\mathcal{O}_{\mathbb{P}^1_R}) \simeq R$.

For $R = \mathbb{Z}$ we put $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{Z}}$.

Rem The transition between the two charts of $\mathbb{P}^1_R$ is described on the overlap by

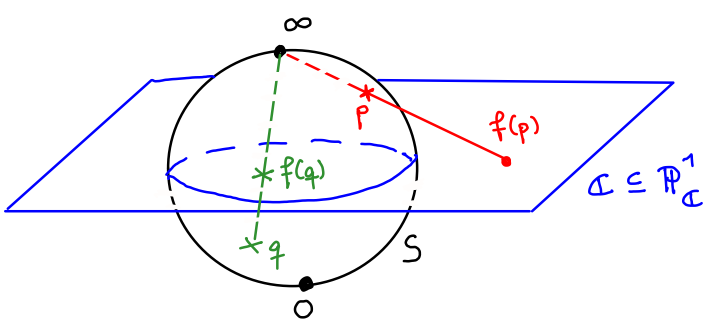
$$\begin{array}{c} \operatorname{Spec} R[t]_t = D(t) \subset U_1 = \operatorname{Spec} R[t] \\ \varphi \downarrow \wr \\ \operatorname{Spec} R[s]_s = D(s) \subset U_2 = \operatorname{Spec} R[s] \end{array}$$

previously called t^{-1}

with

$$\varphi^\# : R[t]_t = R[t, t^{-1}] \xrightarrow{\sim} R[s, s^{-1}]$$
$$t \longmapsto s^{-1}$$

For $R = \mathbb{C}$ we can then identify closed pt^s of $|\mathbb{P}^1_{\mathbb{C}}|$ w/ the Riemann sphere S by stereographic projection (where $\varphi^\# \mathrel{\hat{=}}$ "antipodal map"):



For any field k, points in $\mathbb{P}^1(k)$ parametrize lines:

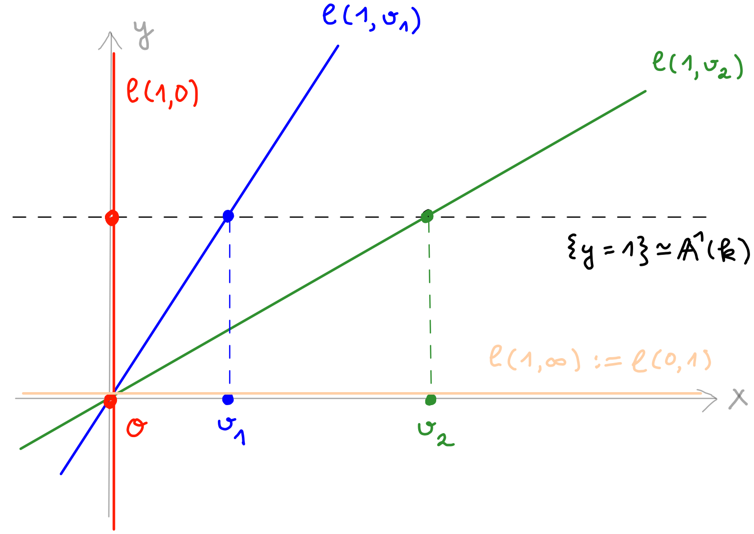
For $(u,v) \in k^2 \setminus \{(0,0)\}$ consider the line

$$\ell(u,v) := \{(x,y) \in k^2 \mid ux = vy\}$$

Clearly

$$\ell_{(u,v)} = \ell_{(u',v')} \iff \exists c \in k^* : (u,v) = c \cdot (u',v')$$

So $\{\text{lines in } k^2\} \xrightarrow{\sim} \mathbb{A}^1(k) \cup \{\infty\}$, $\ell_{(u,v)} \longmapsto \frac{v}{u}$:



Lemma. For any field k, we have a bijection

$$p : (\mathbb{A}^2(k) \setminus \{(0,0)\}) / k^* \xrightarrow{\sim} \mathbb{P}^1(k)$$

caution: This uses that k is a field!

Pf. $\mathbb{P}^1(k) = U_1(k) \sqcup U_2(k) / \sim$

w/ $U_1(k) = U_2(k) = \mathbb{A}^1(k)$ & $\underset{U_1(k)}{\overset{\cap}{t}} \sim \underset{U_2(k)}{\overset{\cap}{s}}$ iff $s = t^{-1}$

Define

$$p((u,v) \bmod k^*) := \begin{cases} t = \frac{v}{u} \in U_1(k) & \text{if } u \neq 0 \\ s = \frac{u}{v} \in U_2(k) & \text{if } v \neq 0 \end{cases} \qquad \square$$

Notation We put $[u:v] := p(u,v)$

"homogenous coordinates"

ie $[cu : cv] = [u:v]$ for all $c \in k^*$

To do the same in higher dimension we need more charts. Here we need a compatibility condition on triple overlaps:

Def A gluing datum for schemes consists of a set I

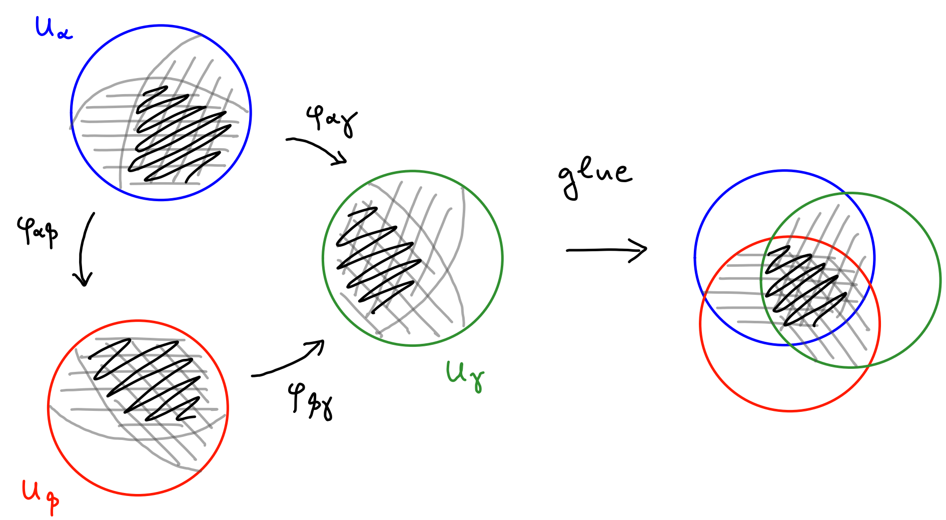
w/
- schemes U_α $(\alpha \in I)$
- open subschemes $U_{\alpha\beta} \subseteq U_\alpha$

  and isomorphisms $\varphi_{\alpha\beta} : U_{\alpha\beta} \xrightarrow{\sim} U_{\beta\alpha}$ $(\alpha,\beta \in I)$

satisfying the cocycle conditions

- $U_{\alpha\alpha} = U_\alpha$ & $\varphi_{\alpha\alpha} = \mathrm{id}$
- $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta}$ on $U_{\alpha\beta\gamma} := U_{\alpha\beta} \cap U_{\alpha\gamma}$

  for all $\alpha, \beta, \gamma \in I$



Prop For any such gluing datum, $\exists$ scheme X

w/ open immersions $j_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha \subseteq X$ sth

- $j_\alpha = j_\beta \circ \varphi_{\alpha\beta}$ on $U_{\alpha\beta}$
- $j_\alpha(U_{\alpha\beta}) = V_\alpha \cap V_\beta$
- $X = \bigcup_{\alpha \in I} V_\alpha$

These properties determine X uniquely up to iso.

Pf. Existence:

Define $|X| := \coprod_{\alpha \in I} |U_\alpha| \,/\, \sim$

disjoint union

= coproduct in cat of top spaces

where $\sim$ is the equivalence relation generated by

$$u \sim \varphi_{\alpha\beta}(u) \quad \text{for } u \in U_{\alpha\beta}$$

& we endow $|X|$ with the quotient topology, ie.

$U \subseteq |X|$ open $:\Longleftrightarrow$ $\forall \alpha$: $j_\alpha^{-1}(U) \subseteq |U_\alpha|$ open

for the natural map $j_\alpha : |U_\alpha| \to |X|$

Put $j_{\alpha\beta} := j_\alpha|_{U_{\alpha\beta}} : |U_{\alpha\beta}| \hookrightarrow |X|$

Have restriction morphisms in $Sh(|X|)$:

$$\begin{tikzcd} & j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}} \arrow[dd, "\varphi_{\beta\alpha}^{\#}"] \\ j_{\alpha *}\mathcal{O}_{U_\alpha} \arrow[ur, "\rho_{\alpha\beta}"] \arrow[dr, "\exists!\, \rho_{\beta\alpha}"'] & \\ & j_{\beta\alpha *}\mathcal{O}_{U_{\beta\alpha}} \end{tikzcd}$$

Define

$$\mathcal{O}_X := \lim \Big(\prod_\alpha j_{\alpha *}\mathcal{O}_{U_\alpha} \underset{\rho_{\beta\alpha}}{\overset{\rho_{\alpha\beta}}{\rightrightarrows}} \prod_{\alpha,\beta} j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}} \Big)$$

$$= \ker \Big(\prod_\alpha j_{\alpha *}\mathcal{O}_{U_\alpha} \xrightarrow[\rho_{\alpha\beta} - \rho_{\beta\alpha}]{} \prod_{\alpha,\beta} j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}} \Big)$$

ie for $W \subseteq X$ open,

$$\mathcal{O}_X(W) = \Big\{ (f_\alpha)_{\alpha \in I} \in \prod_\alpha \mathcal{O}_{U_\alpha}(W_\alpha) \;\Big|\; \forall \alpha,\beta:\ f_\alpha|_{W_{\alpha\beta}} = \varphi_{\beta\alpha}^{\#}(f_\beta|_{W_{\beta\alpha}}) \Big\}$$

where $W_\alpha := j_\alpha^{-1}(W)$

$W_{\alpha\beta} := j_{\alpha\beta}^{-1}(W)$

Note For any family of sheaves $\mathcal{F}_\alpha$ $(\alpha \in I)$ the presheaf $\prod_\alpha \mathcal{F}_\alpha : W \mapsto \prod_\alpha \mathcal{F}_\alpha(W)$ is a sheaf (exercise).

$$\Rightarrow \quad \mathcal{O}_X = \ker\Big(\prod_\alpha j_{\alpha *} \mathcal{O}_{U_\alpha} \longrightarrow \cdots \Big) \text{ is a sheaf.}$$

For any $\alpha_0 \in I$, $\quad pr_{\alpha_0} : \prod_\alpha j_{\alpha *} \mathcal{O}_{U_\alpha} \twoheadrightarrow j_{\alpha_0 *} \mathcal{O}_{U_{\alpha_0}}$ induces an iso

$$j_{\alpha_0}^{-1}(\mathcal{O}_X) \xrightarrow{\sim} \mathcal{O}_{U_{\alpha_0}}$$

$\Rightarrow \quad X := (|X|, \mathcal{O}_X)$ a scheme

& $j_\alpha : U_\alpha \hookrightarrow X$ open immersions

Uniqueness: Check that

$$X = \operatorname{colim}\Big(\coprod_{\alpha,\beta} U_{\alpha\beta} \rightrightarrows \coprod_\alpha U_\alpha \Big)$$

colimit in category of schemes (exercise) $\square$

Ex Projective space over a ring R:

Let $U_\alpha := \operatorname{Spec} R\left[\frac{u_0}{u_\alpha}, \dots, \frac{u_n}{u_\alpha}\right] \simeq \mathbb{A}^n_R$

$\cup$

$U_{\alpha\beta} := D\left(\frac{u_\beta}{u_\alpha}\right) \qquad$ for $\alpha, \beta = 0, 1, \dots, n$,

w/ $\varphi_{\alpha\beta} : U_{\alpha\beta} \xrightarrow{\sim} U_{\beta\alpha}$ given by

$$R\left[\frac{u_0}{u_\beta}, \dots, \frac{u_n}{u_\beta}\right]\left[\left(\frac{u_\alpha}{u_\beta}\right)^{-1}\right] \xrightarrow{id} R\left[\frac{u_0}{u_\alpha}, \dots, \frac{u_n}{u_\alpha}\right]\left[\left(\frac{u_\beta}{u_\alpha}\right)^{-1}\right]$$

We put $\mathbb{P}^n_R := \coprod_\alpha U_\alpha / \text{gluing} \in Sch_R$.

For $R = \mathbb{Z}$ we write $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{Z}}$.

Rem
- $\Gamma(\mathcal{O}_{\mathbb{P}^n_R}) \simeq R \Rightarrow \mathbb{P}^n_R$ not affine
- Most "global" schemes in algebraic geometry are projective, ie. closed subschemes of $\mathbb{P}^n$

↳ see next section ...

Lemma For any field k, have a bijection

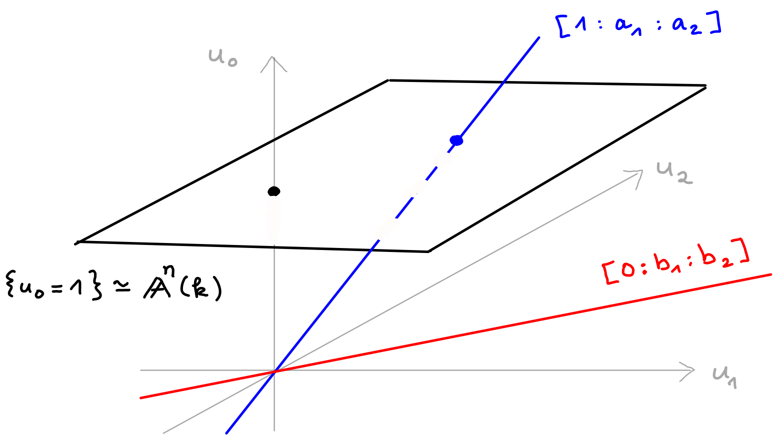
$$\phi : (\mathbb{A}^{n+1}(k) \setminus \{0\}) / k^* \xrightarrow{\sim} \mathbb{P}^n(k)$$

Pf. As in the case $n = 1$. $\square$

Notation. $[u_0 : \cdots : u_n] := \phi(u_0, \dots, u_n) \in \mathbb{P}^n(k)$

"homogenous coordinates"

n = 2:



$$\mathbb{P}^n(k) = \underset{[1 : a_1 : a_2]}{\mathbb{A}^n(k)} \cup \underset{[0 : b_1 : b_2]}{\mathbb{P}^{n-1}(k)}$$

3. Immersions

Recall A morphism $j: U \to X$ of schemes is called an open immersion if it factors over an isomorphism onto an open subscheme of X, i.e.

- $j: |U| \xrightarrow{\sim} V \subseteq |X|$ homeom (open)
- $j^b: j^{-1}\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_U$ iso

Similar notion of closed immersion / subscheme?

Ex $X = \operatorname{Spec} A$ affine

$V(I) \subseteq |X|$ closed, $I \trianglelefteq A$

Have morphism

$i: Z = \operatorname{Spec}(A/I) \longrightarrow X = \operatorname{Spec}(A)$

sth

- $i: |Z| \xrightarrow{\sim} V(I) \subseteq |X|$ homeom. (closed)
- $i^b: i^{-1}\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ epi

$$\left(\text{on stalks: } \begin{array}{ccc} A_{\mathfrak{p}} & \twoheadrightarrow & (A/I)_{\mathfrak{q}} \\ \| & & \| \\ \mathcal{O}_{X,i(z)} & & \mathcal{O}_{Z,z} \end{array} \quad \text{for } \begin{array}{ccc} \mathfrak{q} & \mapsto & \mathfrak{p} \\ \| & & \| \\ z & & i(z) \end{array}\right)$$

Def A morphism $i: Z \to X$ of schemes is called a closed immersion if

- $i: |Z| \xrightarrow{\sim} Y \subseteq |X|$ homeom. (closed)
- $i^b: i^{-1}\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ epi

$\Leftrightarrow \quad i^{\#}: \mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z$

$\Leftrightarrow \quad \mathcal{O}_X/\mathcal{J} \xrightarrow{\sim} i_*\mathcal{O}_Z \quad$ w/ $\mathcal{J} := \ker(i^{\#})$

Def A sheaf of ideals is a subsheaf $\mathcal{J} \subseteq \mathcal{O}_X$ sth for all open $U \subseteq |X|$,

$\mathcal{J}(U) \trianglelefteq \mathcal{O}_X(U)$ is an ideal.

We then also write $\mathcal{J} \trianglelefteq \mathcal{O}_X$.

Def A closed subscheme of X is a scheme $Y = (|Y|, \mathcal{O}_Y)$ w/

$|Y| \subseteq |X|$ a closed subspace

$\mathcal{O}_Y = i^{-1}(\mathcal{O}_X/\mathcal{J})$ for some $\mathcal{J} \trianglelefteq \mathcal{O}_X$

$(\Leftrightarrow \; i_*\mathcal{O}_Y \simeq \mathcal{O}_X/\mathcal{J}$ for some $\mathcal{J} \trianglelefteq \mathcal{O}_X)$

Rem For any sheaf of ideals $\mathcal{J} \trianglelefteq \mathcal{O}_X$ we have the ringed space

$$V(\mathcal{J}) := \left(\operatorname{Supp}(\mathcal{O}_X/\mathcal{J}),\ \operatorname{incl}^{-1}(\mathcal{O}_X/\mathcal{J})\right)$$

$\Rightarrow$ Any closed immersion $i : Z \to X$ factors over an iso

$i : Z \xrightarrow{\sim} V(\mathcal{J})$ w/ $\mathcal{J} := \ker(i^{\#} : \mathcal{O}_X \twoheadrightarrow i_* \mathcal{O}_Z)$.

Q: Which ideals $\mathcal{J} \trianglelefteq \mathcal{O}_X$ arise like this?

Ex Let $X = \mathbb{A}^1$

$\mathcal{J} := j_! \mathcal{O}_U$ w/ $j : U = \mathbb{A}^1 \setminus \{0\} \hookrightarrow X$

ie $\quad \mathcal{J}(W) := \begin{cases} \mathcal{O}_X(W) & \text{if } 0 \notin W \\ 0 & \text{if } 0 \in W \end{cases}$

$\Rightarrow V(\mathcal{J}) = (\{0\}, \mathcal{O}_{X,0})$ is NOT a scheme!

(schemes with only one point are affine, being covered by open affines — but $\operatorname{Spec} \mathcal{O}_{X,0}$ has two points ...)

Note: $X = \operatorname{Spec} A$ determined by $A = \Gamma(\mathcal{O}_X)$,
but $\mathcal{J} \neq 0$ whereas $I := \Gamma(\mathcal{J}) = 0 \trianglelefteq A$.

Prop For any affine scheme $X = \operatorname{Spec} A$, $\exists$ natural bijection

$$\{\text{ideals } I \trianglelefteq A\} \xrightarrow{\sim} \{\text{closed subschemes of } X\}$$

Pf. 1) Given $I \trianglelefteq A$,
have closed immersion $\operatorname{Spec}(A/I) \to X = \operatorname{Spec} A$ giving an iso onto a closed subscheme $Y = V(I) \subseteq X$.

Can recover I from this subscheme as

$$\begin{aligned} I &= \ker(A \twoheadrightarrow A/I) \\ &= \ker(\Gamma(\mathcal{O}_X) \twoheadrightarrow \Gamma(\mathcal{O}_Y)). \end{aligned}$$

2) Conversely, given a closed subscheme Y of X consider

$$I := \ker(\underbrace{\Gamma(\mathcal{O}_X)}_{A} \to \Gamma(\mathcal{O}_Y)) \trianglelefteq A$$

We claim that $Y = V(I)$.

Indeed:

- $\mathrm{Hom}_{Sch}(Y, X) = \mathrm{Hom}_{Rings}(A, \Gamma(\mathcal{O}_Y))$ ← $X = \mathrm{Spec}\, A$ affine

  $\cup\!|$ $\qquad\qquad$ $\cup\!|$

  $\mathrm{Hom}_{Sch}(Y, V(I)) = \mathrm{Hom}_{Rings}(A/I, \Gamma(\mathcal{O}_Y))$

  $\Longrightarrow$ the embedding of Y in X factors over a morphism $i : Y \to V(I) \subseteq \mathrm{Spec}\, A$.

  Need to check this is an isomorphism.

- Replacing A by A/I can assume $I = (0)$, ie wlog $A = \Gamma(\mathcal{O}_X) \xhookrightarrow{i^\#} \Gamma(\mathcal{O}_Y)$ injective.

  Need to show that then $i : Y \to X$ is an iso.

- First check $i : |Y| \hookrightarrow |X|$ is a homeom: (closed)

  As a Zariski closed subset

  $$|Y| = |V(J)| \text{ for some } J \trianglelefteq A,$$

  and we must show $J \subseteq \mathrm{Rad}(A) = \sqrt{0}$.

Let $f \in J$.

$\Longrightarrow$ $|Y| \subseteq V(f)$ inside $|X| = |\mathrm{Spec}\, A|$

$\Longrightarrow$ for any open affine $U = \mathrm{Spec}\, R \subseteq Y$, the restriction $g := (i^\#(f))|_U \in R$ has $V(g) = |U| = |\mathrm{Spec}\, R|$

$\Longrightarrow$ $g \in \mathrm{Rad}(R)$,
i.e. $\exists n \in \mathbb{N} : g^n = 0$ in $R = \mathcal{O}_Y(U)$

Can cover $|Y|$ by finitely many such charts (since it is quasicompact, being closed in the quasicompact space $|X| = |\mathrm{Spec}\, A|$).

$\Longrightarrow$ $\exists n \in \mathbb{N} : (i^\#(f))^n = 0$ in $\Gamma(\mathcal{O}_Y)$.

But $i^\# : A \hookrightarrow \Gamma(\mathcal{O}_Y)$ injective

$\Longrightarrow$ $f^n = 0$ in A, i.e. $f \in \mathrm{Rad}(A)$

Hence $J \subseteq \mathrm{Rad}(A)$, ie $|Y| = |X|$.

- Remains to show $i^\#: \mathcal{O}_X \to i_* \mathcal{O}_Y$ iso.

Surjective by def^n of closed subschemes.

Injectivity: Enough to check on stalks.

Let $x = \mathfrak{p} \in X = \operatorname{Spec} A$,

$$s = \frac{a}{1} \in \ker \left(A_\mathfrak{p} = \mathcal{O}_{X,x} \xrightarrow{i_x^\#} \mathcal{O}_{Y,x} \right)$$

identify $|Y| = |X|$

We must show

$$(*) \qquad \exists n \in \mathbb{N}\ \exists f \in A \setminus \mathfrak{p} : \ f^n \cdot a = 0 \quad \text{in } A$$

$$\left(\Longleftrightarrow \ i^\#(f^n \cdot a) = 0 \ \text{ in } B = \Gamma(\mathcal{O}_Y) \right)$$

(since $i^\#: A \to B$ injective)

Pick a finite open cover $|Y| = \bigcup_{\alpha=0}^{m} U_\alpha$

sth

- $(U_i, \mathcal{O}_Y|_{U_i}) \simeq \operatorname{Spec} B_\alpha$ affine
- $x \in U_0$ and $i^\#(a)|_{U_0} = 0 \in \mathcal{O}_Y(U_0)$.

Pick $f \in A$ with $\underset{x}{\underset{\cup}{D(f)}} \subseteq U_0$ (inside $|\operatorname{Spec} A|$).

Check $(*)$:

$$\begin{aligned} D(f)_\alpha &:= D(i^\#(f)|_{U_\alpha}) \qquad (\text{in } |\operatorname{Spec} B_\alpha|) \\ &= D(f) \cap U_\alpha \qquad (\text{in } |\operatorname{Spec} A|) \\ &\subseteq U_0 \cap U_\alpha \end{aligned}$$

$$\Longrightarrow \ i^\#(a)\big|_{D(f)_\alpha} = 0 \quad \text{since } i^\#(a)|_{U_0} = 0$$

But $D(f)_\alpha \simeq \operatorname{Spec}(B_{\alpha, i^\#(f)})$

$$\Longrightarrow \ \exists n_\alpha : \ f^{n_\alpha} \cdot i^\#(a)|_{U_\alpha} = 0 \ \text{ in } B_\alpha$$

$$\Longrightarrow \ i^\#(f^n \cdot a) = 0 \ \text{ in } B = \Gamma(\mathcal{O}_Y)$$

for $n \geq \max\{n_i\}$ $\square$

<u>Cor</u> Any closed subscheme of an affine scheme $X = \operatorname{Spec} A$ is again affine, of the form

$\operatorname{Spec}(A/I) \xrightarrow{\sim} V(I) \subseteq X$ for some $I \trianglelefteq A$.

Rem One similarly shows that for any closed subscheme

$Z = V(I) \xhookrightarrow{i} X = \mathrm{Spec}(A)$, the sheaf

of ideals $\mathcal{J} := \ker(i^{\#} : \mathcal{O}_X \to i_* \mathcal{O}_Z)$ is

given on basic open sets by

$$\left.\begin{array}{l} \mathcal{J}(D(f)) = I \cdot A_f \trianglelefteq \mathcal{O}_X(D(f)) = A_f \\ \text{for } I := \mathcal{J}(\mathrm{Spec}\, A) \text{ and all } f \in A \end{array}\right\} (**)$$

Cor For any scheme X and $\mathcal{J} \trianglelefteq \mathcal{O}_X$, TFAE:

a) $V(\mathcal{J}) := (\mathrm{Supp}(\mathcal{O}_X/\mathcal{J}),\ \mathrm{incl}^{-1}(\mathcal{O}_X/\mathcal{J}))$

is a closed subscheme of X

b) $\forall$ affine open $U = \mathrm{Spec}(A) \subseteq X$,

the ideal sheaf $\mathcal{J}|_U \trianglelefteq \mathcal{O}_U$ satisfies $(**)$

c) $\exists$ cover of X by affine open sets $U = \mathrm{Spec}\, A$

on which $(**)$ holds.

Such ideal sheaves are called <u>quasi-coherent</u>.

(more about quasicoherent sheaves later)

Let's go back to geometry:

Note Closed subschemes are not determined by

the underlying topological subspace!

Ex Let $X = \mathrm{Spec}\, k[t]$

$I_n = (t^n)$ for $n \in \mathbb{N}$

$\Rightarrow Y_n := V(I_n) = \mathrm{Spec}\, k[t]/(t^n)$

$= (\{0\},\ k[t]/(t^n))$ "fat point"

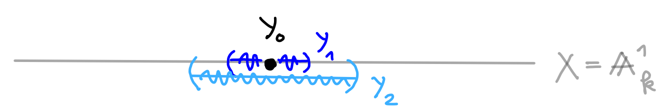
We have

$\Gamma(\mathcal{O}_{Y_n}) = k[t]/(t^n)$

$\Rightarrow$ For $f(t) = \sum_{i \geq 0} a_i t^i \in \Gamma(\mathcal{O}_X)$,

$f|_{Y_n} = \sum_{i=0}^{n} a_i t^n$ "n-th Taylor approximation"

We call Y_n the "n-th order thickening of the origin"



Nilpotents are useful :

Def Let X be a scheme.

The <u>scheme-theoretic intersection</u> of two closed subschemes $Y_\alpha = V(\mathcal{J}_\alpha) \subseteq X$ w/ $\mathcal{J}_\alpha \trianglelefteq \mathcal{O}_X$ $(\alpha = 1,2)$

is the closed subscheme

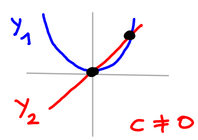
$$Y_1 \cap Y_2 := V(\mathcal{J}) \subseteq X$$

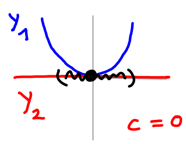
defined by $\mathcal{J} := \mathcal{J}_1 + \mathcal{J}_2 \trianglelefteq \mathcal{O}_X$.

Ex $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$

$Y_1 = V(y - x^2)$

$Y_2 = V(y - cx)$ with $c \in k$





$$\Rightarrow \quad Y_1 \cap Y_2 = V(y - cx,\ x(x-c))$$

$$= \begin{cases} V(x,y) \sqcup V(x-c,\ y-c^2), & c \neq 0 \\ V(y, x^2), & c = 0 \end{cases}$$

fat point sees tangency !

Non-affine examples of closed subschemes :

Ex $S = R[x_0, \dots, x_n] \ni f_i$ homogeneous polynomials $(i \in I)$

$\mathbb{P}^n_R = \bigcup_{\alpha=0}^{n} U_\alpha$ w/ $U_\alpha = \operatorname{Spec} R\left[\frac{x_0}{x_\alpha}, \dots, \frac{x_n}{x_\alpha}\right]$

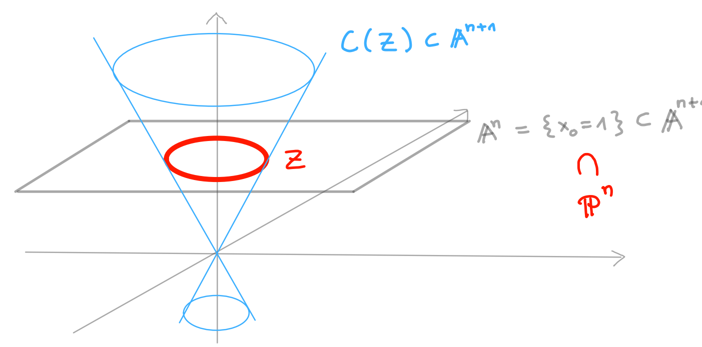
$f_{i,\alpha} := f_i\left(\frac{x_0}{x_\alpha}, \dots, \frac{x_n}{x_\alpha}\right) = x_\alpha^{-\deg(f_i)} \cdot f_i(x_0, \dots, x_n)$

$Z_\alpha := V(f_{i,\alpha} \mid i \in I) \subseteq U_\alpha$

$\rightsquigarrow$ Glue to $Z \subseteq \mathbb{P}^n_R$ w/ $Z \cap U_\alpha = Z_\alpha$ for all α

Def $V_+(f_i \mid i \in I) := Z \subseteq \mathbb{P}^n_R$ a <u>projective scheme over R</u>.

$C(Z) := V(f_i \mid i \in I) \subseteq \mathbb{A}^{n+1}_R$ the <u>affine cone over Z</u>.



Def A scheme is called <u>quasiprojective</u> if it is isomorphic to an open subscheme of a projective scheme.

Can combine open & closed immersions:

<u>Def</u> A morphism $f : Z \to X$ is an <u>immersion</u> if

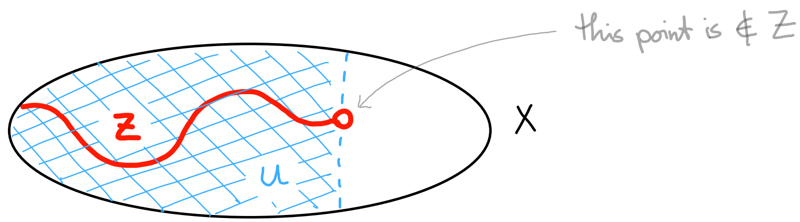
$$f = j \circ i : Z \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$$

w/ i an open immersion, j a closed immersion.

Equivalently:

locally closed

- $f : |Z| \xrightarrow{\sim} f(|Z|) \subseteq |X|$
- $f^\#_z : \mathcal{O}_{X,f(z)} \twoheadrightarrow \mathcal{O}_{Z,z}$ epi for all $z \in |Z|$.



<u>Rem</u> If $f : Z \to X$ is quasicompact,

i.e. $f^{-1}(U)$ is qc for all qc open $U \subseteq X$,

then we can also factor the immersion

as $f = i \circ j$ w $j : Z \hookrightarrow Y$ open
$i : Y \hookrightarrow X$ closed.

(look at scheme-theoretic images, see below – exercise)

<u>Exercise</u> $f : Z \to X$ qc morphism of schemes,

$$\mathcal{J} := \ker(f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Z),$$

then

a) $V(\mathcal{J}) := (\operatorname{Supp}(\mathcal{O}_X/\mathcal{J}), \operatorname{incl}^{-1}(\mathcal{O}_X/\mathcal{J}))$ is a scheme

b) $\exists$ factorizat^n

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, dashed] & & X \\ & V(\mathcal{J}) \arrow[ur, hook, "i"] & \end{tikzcd}$$

i closed imm.

and for any other closed subscheme $W \subseteq X$ over which f factors, we have $V(\mathcal{J}) \subseteq W$ (as closed subscheme)

<u>Def</u> $\overline{\operatorname{im}(f)} := V(\mathcal{J})$ "<u>scheme-theoretic image of f</u>"

<u>Ex</u> $Z = \operatorname{Spec} k[t]/(t^2) \xrightarrow{f} X = \operatorname{Spec} k[s]$

$$\Rightarrow \overline{\operatorname{im}(f)} = \begin{cases} V(s^2) & \text{for } f^\#(s) = t \\ V(s) & \text{for } f^\#(s) = 0 \end{cases}$$

<u>Ex</u> $f : U \hookrightarrow X$ open immersion, X irred $\Rightarrow \overline{\operatorname{im}(f)} = X$

4. Reduced & integral schemes

A closed subset of a scheme can have many different scheme structures:

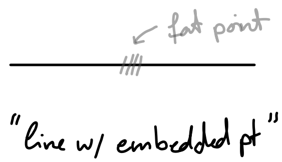
<u>Ex</u> Inside $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$,

consider the closed subschemes

$Z_1 = V(y)$

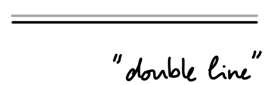
$\downarrow$

$Z_2 = V(xy, y^2)$



$\downarrow$

$Z_3 = V(y^2)$



$\Longrightarrow |Z_1| = |Z_2| = |Z_3|$

Smallest closed subscheme w/ given underlying space?

<u>Def</u> A scheme X is called <u>reduced</u> if for all open $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is reduced,

ie $\operatorname{Rad}(\mathcal{O}_X(U)) = (0)$.

<u>Lemma</u> For any scheme X, the following are equivalent:

a) X is reduced

b) $\exists$ cover $X = \bigcup_{\alpha \in I} U_\alpha$ w/ open affine $U_\alpha = \operatorname{Spec} A_\alpha$ sth A_α is a reduced ring.

c) The stalk $\mathcal{O}_{X,p}$ is a reduced ring for all points $p \in |X|$.

<u>Rem</u> For X quasicompact, it suffices to check c) only on closed points $p \in |X|$ (see exercises).

Pf of lemma.

$\neg a) \Rightarrow \neg c)$:

X not reduced

$\Rightarrow \exists U \subseteq X$ open $\exists f \in \mathcal{O}_X(U)$:

$f \neq 0$ but $f^n = 0$ for some $n \in \mathbb{N}$

$\Rightarrow \exists p \in |U|: f_p \neq 0$ but $f_p^n = 0$ in $\mathcal{O}_{X,p}$

$\Rightarrow \mathcal{O}_{X,p}$ not reduced

$\neg c) \Rightarrow \neg b)$: "Localizations of reduced rings are reduced"

$\mathcal{O}_{X,p}$ not reduced for some $p \in |X|$

$\Rightarrow \exists\, 0 \neq f_p \in \mathcal{O}_{X,p}\ \exists n \in \mathbb{N}: f_p^n = 0$

Let $U = \operatorname{Spec} A \subseteq X$ open affine w/ $p = \mathfrak{p} \in U = \operatorname{Spec} A$

& write $f_p = \frac{a}{b} \in A_\mathfrak{p}$ w/ $b \in A \setminus \mathfrak{p}$

Then $a \neq 0$ in $A_\mathfrak{p} \Rightarrow \forall s \in A \setminus \mathfrak{p}: s \cdot a \neq 0$ in A

but $a^n = 0$ in $A_\mathfrak{p} \Rightarrow \exists s \in A \setminus \mathfrak{p}: s \cdot a^n = 0$ in A

hence $(s \cdot a)^n = 0$

$\Rightarrow s \cdot a \in A \setminus \{0\}$ nilpotent

$\neg b) \Rightarrow \neg a)$: trivial. $\square$

<u>Prop</u> Let X be a scheme, $S \subseteq |X|$ a closed subset.

$\Rightarrow \exists!$ closed subscheme $i: Z \hookrightarrow X$

sth Z is reduced and $|Z| = S$.

Pf. For $U = \operatorname{Spec} A \subseteq X$ affine open,

put $I := \operatorname{Ann}(S \cap U) \trianglelefteq \mathcal{O}_X(U) = A$.

For any closed subscheme $Z_U = \operatorname{Spec} A/J \subseteq U$, have

a) $|Z_U| = S \cap U \iff \sqrt{J} = I$

b) Z_U reduced $\iff \sqrt{J} = J$

Thus $\exists!$ closed subscheme $Z_U \subseteq U$ w/ a) & b), namely $Z_U = \operatorname{Spec} A/I$.

On $U' = D(f) \subseteq U = \operatorname{Spec} A$ for $f \in A$, have $I' := \operatorname{Ann}(S \cap U') = I_f \trianglelefteq \mathcal{O}_X(D(f)) = A_f$

$\Rightarrow$ The $Z_U \subseteq U$ glue to a closed subscheme $Z \subseteq X$. $\square$

Def We say that $S_{red} := Z \subseteq X$ is "S with the reduced induced scheme structure".
For $S = |X|$ we call $X_{red} \subseteq X$ the <u>reduced subscheme underlying X</u>.

Lemma Let X, Y be schemes, with Y reduced.

a) For any closed subscheme $Z \subseteq X$, a morphism $f : Y \to X$ factors over $Z \subseteq X$ iff $f(|Y|) \subseteq |Z|$.

b) So $i : X_{red} \hookrightarrow X$ induces a bijection

$$\mathrm{Hom}_{Sch}(Y, X_{red}) \xrightarrow{\sim} \mathrm{Hom}_{Sch}(Y, X)$$
$$f \longmapsto i \circ f.$$

Pf. a) $Z \subseteq X$ subscheme

$\Rightarrow$ If $f : Y \to X$ factors over $g : Y \to Z$ then g is determined uniquely by f

$\Rightarrow$ Enough to show existence of g locally for $f|_U : \underset{\text{open}}{U \subseteq f^{-1}(V)} \to \underset{\text{open}}{V \subseteq X}$ & glue

So wlog

$$\begin{tikzcd} Y = \operatorname{Spec} B \arrow[r, "f"] \arrow[dr, dashed, "\exists ?"'] & X = \operatorname{Spec} A \\ & Z = \operatorname{Spec} A/I \arrow[u, hook] \end{tikzcd}$$

Let $h = \Gamma(f^\#) : A \to B$.

Then:

f factors over $Z \iff h$ factors over A/I

$\iff I \subseteq \ker(h) = h^{-1}(0)$

$\left(\begin{array}{l} B = \Gamma(\mathcal{O}_Y) \text{ reduced} \\ \Rightarrow \bigcap_{\mathfrak{p} \in \operatorname{Spec} B} \mathfrak{p} = 0 \end{array} \right) \iff \forall \mathfrak{p} \in \operatorname{Spec} B : \; I \subseteq h^{-1}(\mathfrak{p})$

$\iff \forall p \in Y : \; f(p) \in V(I)$

$\iff f(|Y|) \subseteq |Z|$

b) Special case of a) where $Z = X_{red} \subseteq X$. $\square$

Cor $(-)_{red} : Sch \to Sch$ is a functor:

Any $f : S \to T$ induces $f_{red} : S_{red} \to T_{red}$.

Pf. Take $X = T \supseteq Z = T_{red}$ & $Y = S_{red}$. $\square$

Def A scheme X is called

a) <u>irreducible</u> if $|X|$ is irreducible

b) <u>integral</u> if it is reduced and irreducible.

Lemma A scheme X is integral iff $\forall\, U \subseteq X$ open, the ring $\mathcal{O}_X(U)$ is an integral domain.

Pf. X non-reduced $\Rightarrow \exists\, U \subseteq X$ open w/ $\mathcal{O}_X(U)$ non-reduced, hence not a domain.

X reducible $\Rightarrow \exists$ open $\emptyset \neq U_1, U_2 \subseteq X$ w/ $U_1 \cap U_2 = \emptyset$

$\Rightarrow$ For $U := U_1 \cup U_2$,

$\mathcal{O}_X(U) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ not a domain

$U \subseteq X$ open w/ $\mathcal{O}_X(U)$ not a domain

$\Rightarrow \exists\, f_1, f_2 \in \mathcal{O}_X(U) \setminus \{0\}: \; f_1 f_2 = 0$

$\Rightarrow U = Z_1 \cup Z_2$ w/ $Z_\alpha := V(f_\alpha) \subseteq U$

If $Z_1 \neq Z_2$: U reducible $\Rightarrow X$ reducible

If $Z_1 = Z_2$: $\sqrt{(f_1)} = \sqrt{(f_2)} \Rightarrow \exists n: f_\alpha^n = 0$

$\Rightarrow X$ non-reduced $\square$

Lemma Let X be integral w/ generic pt η.

a) For U affine we have

$$\mathrm{Quot}(\mathcal{O}_X(U)) \xrightarrow{\sim} \mathcal{O}_{X,\eta}.$$

b) For any open $V \subseteq X$ & $p \in V$ the homom.

$$\mathcal{O}_X(V) \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow \mathcal{O}_{X,\eta}$$

are injective w/

$$\mathcal{O}_X(V) = \bigcap_{p \in V} \mathcal{O}_{X,p} \subseteq \mathcal{O}_{X,\eta}.$$

Pf. a) Write $U = \mathrm{Spec}\, A \ni \eta = (0)$ (A is a domain!)

b) For $p, q \in V$ w/ $p \in \overline{\{q\}}$, the restriction gives natural homom. $\mathcal{O}_X(V) \to \mathcal{O}_{X,p} \to \mathcal{O}_{X,q}$.

Injectivity: Clear from a) applied to open affine cover.

If $h \in \bigcap_{p \in V} \mathcal{O}_{X,p} \subseteq \mathcal{O}_{X,\eta}$, then

$\forall p \in V\; \exists$ open $\underset{p}{\overset{\cup\!\!\!\!\!\!\!\;\,\shortmid}{U_p}} \subseteq V \;\exists\, h_{U_p} \in \mathcal{O}_X(U_p)$ w/ $h_{U_p,\eta} = h_\eta$ in $\mathcal{O}_{X,\eta}$

By injectivity these h_{U_p} agree on overlaps

$\Rightarrow$ glue to a unique section in $\mathcal{O}_X(V)$. $\square$

Def We call $K(X) := \mathcal{O}_{X,\eta}$ the *function field* of the integral scheme X (if $X \in Sch_k$ for a field k, we also denote the function field by $k(X)$). Its elements are called *rational functions* on X. If $f \in \mathcal{O}_{X,p} \subseteq K(X)$, we say that the rational function is *defined at* p.

Ex $K(\mathbb{A}^1) \simeq \mathbb{Q}(t)$ rat^l fction field in t

Rem For $f \in K(X)$, the subset
$$U := \{ p \in X \mid f \in \mathcal{O}_{X,p} \} \subseteq X$$
is open. We say f is *defined on* $V \subseteq X$ if $V \subseteq U$, and call $Z := X \setminus U \subseteq X$ the *indeterminacy locus* of f.

Caution Even if $X = \operatorname{Spec} A$ is affine, a rational function $f \in \mathcal{O}_X(U) \subseteq K(X)$ need NOT be globally of the form $f = \frac{a}{b}$ w/ $a, b \in A$, $b|_U \in \mathcal{O}_X(U)^*$ (see exercises)

If $f \in K(X)$ is defined on $V \subseteq X$, we get a morphism $f_V : V \longrightarrow \mathbb{A}^1$ via
$$\mathcal{O}_X(V) = \operatorname{Hom}_{Rings}(\mathbb{Z}[t], \mathcal{O}_X(V)) = \operatorname{Hom}_{Sch}(V, \mathbb{A}^1).$$

If f is also defined on $W \subseteq X$, then
$$f_V|_U = f_W|_U \text{ on } U = V \cap W.$$

Def Let X, Y be schemes & $V, W \subseteq X$ open dense. We say $g : V \to Y$ & $h : W \to Y$ are *equivalent* if $g|_U = h|_U$ for some open dense subset $U \subseteq V \cap W$.

A *rational map* $f : X \dashrightarrow Y$ is an equivalence class of morphisms $V \to Y$ for this equivalence relation.

Cor For X integral we have
$$K(X) = \{ \text{rational maps } f : X \dashrightarrow \mathbb{A}^1 \}$$

(More about rational maps in the exercises ...)

5. Finiteness conditions

<u>Recall</u> A top space X is

a) <u>quasi-compact</u> if $\forall$ open cover $X = \bigcup_{i \in I} U_i$, $\exists$ finite subset $I_0 \subseteq I$ w/ $X = \bigcup_{i \in I_0} U_i$.

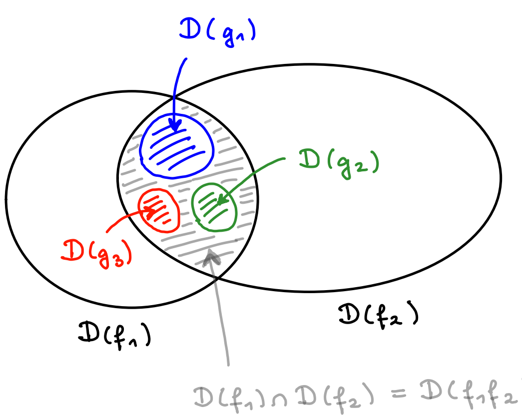
b) <u>quasi-separated</u> if $U_1 \cap U_2$ is quasi-compact for all quasi-compact open $U_1, U_2 \subseteq X$.

A scheme X is called <u>qc / qs</u> if $|X|$ is.

<u>Ex</u> Affine schemes are qcqs := quasicpt + quasisep :

qc by $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i) \iff (f_i \mid i \in I) = A$

qs since $D(f_1) \cap D(f_2) = D(f_1 f_2)$



<u>Rem</u> A scheme X is qcqs iff $\exists$ finite cover $X = \bigcup_{i=1}^{n} U_i$ w/

- $U_i \subseteq X$ open affine, and
- $U_i \cap U_j$ = finite union of open affines.

Most finiteness conditions: "qc + an affine-local property"

<u>Def</u> A property P of affine open subschemes of X is <u>affine-local</u> if

a) $P(\operatorname{Spec} A) \Rightarrow P(\operatorname{Spec} A_f)$ for all $f \in A$

b) If $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ (ie $(f_i \mid i \in I) = A$) then

$P(\operatorname{Spec} A_{f_i})$ for all $i \Rightarrow P(\operatorname{Spec} A)$.

<u>Ex</u>

- $P(U)$:= "U is reduced" is an affine-local property
- $P(U)$:= "U is integral" is NOT an affine-local property

Lemma Let $\mathcal{P}$ be an affine-local property.

If X is a scheme & $\mathcal{P}(U_i)$ holds for all i on some open affine cover $X = \bigcup_{i\in I} U_i$, then $\mathcal{P}(U)$ holds for all open affine $U \subseteq X$.

Pf. Let $U = \operatorname{Spec} A \subseteq X$ open affine

$$\Longrightarrow U = \bigcup_{i\in I} U \cap U_i$$

Claim: Can cover $U \cap U_i$ by open subsets U_{ij} that are basic both in U and in U_i:

$$U_{ij} = D(f_{ij}) \text{ in } U = \operatorname{Spec} A \qquad (f_{ij} \in A)$$

$$= D(g_{ij}) \text{ in } U_i = \operatorname{Spec} B_i \qquad (g_{ij} \in B_i)$$

Then:

$\mathcal{P}(U_i) \Rightarrow \mathcal{P}(U_{ij})$ since U_{ij} basic open in U_i (affine-local condition a))

$\Rightarrow \mathcal{P}(U)$ since U_{ij} basic open in $U = \bigcup_{i,j} U_{ij}$ (affine-local condition b))

Claim follows from:

$U = \operatorname{Spec} A$, $V = \operatorname{Spec} B \subseteq X$ open affine

$\Rightarrow$ any $p \in U \cap V$ has an open nbhood which is basic in both open charts:

$$p \in W = D(f) \text{ in } U \quad (f \in A)$$
$$= D(g) \text{ in } V \quad (g \in B) \qquad (*)$$

To show $(*)$:

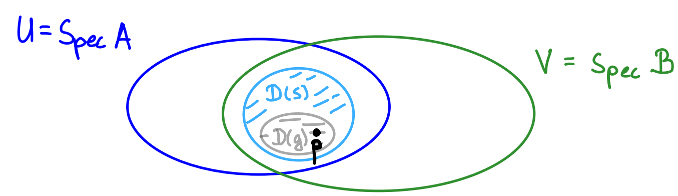
$\operatorname{Spec}(A_s) \subseteq U = \operatorname{Spec}(A)$

Pick $s \in A$ w/ $D(s) \subseteq U \cap V$

$$D(s) \ni p \in D(g)$$

Pick $g \in B$ w/ $D(g) \subseteq D(s)$

$\operatorname{Spec}(B_g) \subseteq V = \operatorname{Spec}(B)$



Let

$$g|_{D(s)} = \frac{h}{s^n} \in \mathcal{O}_X(D(s)) = A_s$$

$$\Rightarrow \underset{\substack{\parallel \\ D(g)}}{\operatorname{Spec} B_g} \simeq \operatorname{Spec}(A_s)_{g|_{D(s)}} = \underset{\substack{\parallel \\ D(f)}}{\operatorname{Spec} A_f} \text{ w/ } f := sh$$

$\square$

Def Let $\mathcal{P}$ be an affine-local property.

We say that a scheme X is

a) *locally* $\mathcal{P}$ if $\mathcal{P}(U)$ holds $\forall$ open affine $U \subseteq X$

($\Leftrightarrow$ $\exists$ open affine cover $X = \bigcup_{i \in I} U_i$ sth $\forall i: \mathcal{P}(U_i)$)

b) $\mathcal{P}$ if X is qc and locally $\mathcal{P}$.

($\Leftrightarrow$ $\exists$ finite open affine cover $X = \bigcup_{i=1}^{n} U_i$ sth $\forall i: \mathcal{P}(U_i)$)

Ex A scheme X is called

a) *locally Noetherian* if it has an open cover $X = \bigcup_{i \in I} U_i$ w/ $U_i = \operatorname{Spec} A_i$ where the A_i are Noetherian rings.

b) *Noetherian* if it has a finite such cover.

The same then holds for every open affine in X:

Lemma $\mathcal{P}(\operatorname{Spec} A) :=$ "the ring A is Noetherian" is an affine-local property.

Pf.

a) A Noetherian $\Rightarrow$ A_f Noetherian for any $f \in A$:

φ $I_1 \subsetneq I_2 \subsetneq \dots \trianglelefteq A_f$ is a strictly ascending chain of ideals, then so is the chain of ideals $J_\alpha = \{a \in A \mid \frac{a}{1} \in I_\alpha\} \trianglelefteq A$.

b) $\left.\begin{array}{l} f_i \in A \text{ w/ } (f_i \mid i \in S) = (1) \\ \& \text{ all } A_{f_i} \text{ Noetherian} \end{array}\right\} \Rightarrow A$ Noetherian:

Wlog $S = \{1, \dots, n\}$ finite (as $\operatorname{Spec} A$ is qc)

Let $I_1 \subsetneq I_2 \subsetneq \dots \trianglelefteq A$ strictly ascending chain of ideals $\quad (*)$

$\Rightarrow$ $I_1 \cdot A_{f_i} \subseteq I_2 A_{f_i} \subseteq \dots \trianglelefteq A_{f_i}$ has to stabilize, since A_{f_i} Noetherian

$\Rightarrow$ $\exists k_0 \in \mathbb{N}: \quad I_k \cdot A_{f_i} = I_{k+1} \cdot A_{f_i} \quad \forall i \in S \quad \forall k \geq k_0$

__Claim:__ $I_k = I_{k+1} \quad \forall k \geq k_0$ ($\Rightarrow A$ Noetherian):

If not, let $k \geq k_0$ and $a \in I_{k+1} \setminus I_k$.

Know: $\frac{a}{1} \in I_{k+1} A_{f_i} \overset{!}{=} I_k \cdot A_{f_i}$ for all $i \in S$

$\underset{S \text{ finite}}{\Longrightarrow} \exists m \in \mathbb{N} : f_i^m \cdot a \in I_k$ for all $i \in S$

But $(f_i \mid i \in S) = (1)$

$$\Rightarrow \operatorname{Spec} A = \bigcup_i D(f_i) = \bigcup_i D(f_i^m)$$

$\Rightarrow (f_i^m \mid i \in S) = (1)$

ie. $\exists c_i \in A : \sum_i c_i f_i^m = 1$ "partition of 1"

$$\Rightarrow a = a \cdot 1 = \sum_i c \cdot \underbrace{a f_i^m}_{\in I_k} \in I_k \quad ⚡$$

$\square$

__Def__ A top space Y is called __Noetherian__ if every descending chain $Y \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$ of closed subsets $Z_i \subseteq Y$ stabilizes.

__Lemma__ Let Y be a top space.

a) Y Noetherian $\Rightarrow$ Y quasicompact and every subspace $W \subseteq Y$ is Noetherian

needn't be closed!

b) $\exists!$ decomposition

$$Y = Y_1 \cup \cdots \cup Y_m$$

w/ $Y_i \subseteq Y$ closed irreducible sth $Y_i \not\subseteq Y_j$ for all $i \neq j$

We call the Y_i __irreducible components__ of Y.

Pf. If $Y = \bigcup_{i \in I} U_i$ is an open cover which has no finite subcover, we find an infinite chain

$$U_{i_1} \subsetneq U_{i_1} \cap U_{i_2} \subsetneq U_{i_1} \cap U_{i_2} \cap U_{i_3} \subsetneq \cdots$$

$\Rightarrow$ complements give chain of closed subsets that doesn't stabilize, so Y not Noetherian

The other claims: Exercise. $\square$

Rem Let X be a Noetherian scheme. Then:

a) $|X|$ is a Noetherian top space

b) $\forall p \in |X|$ the local ring $\mathcal{O}_{X,p}$ is Noetherian.

(check on affine charts ...)

Converse is NOT true, even for $X = \operatorname{Spec} A$ affine:

a) $A = k[x_1, x_2, \dots] / (x_1^2, x_2^2, \dots)$ (k a field)

$\Rightarrow$ $|\operatorname{Spec} A| = \{pt\}$ Noetherian space,

but $\operatorname{Spec} A$ is not Noetherian

since A is not a Noetherian ring!

b) $A = k[x, y_1, y_2, \dots] / (y_i^2, (x - a_i) y_{i+1} - y_i \mid i \in \mathbb{N})$

w/ $a_1, a_2, \dots \in k^*$ pairwise distinct

$\Rightarrow$ $X = \operatorname{Spec} A$ not Noetherian

but $\mathcal{O}_{X,p} = A_p$ Noetherian for all $p \in X$

(enough to check on max. ideals — exercise)

Other useful finiteness properties:

Ex A scheme $X \in \mathrm{Sch}_R$ is

a) <u>locally of finite type over R</u> if it has an open cover $X = \bigcup_{i \in I} U_i$ w/ $U_i = \operatorname{Spec} A_i$ with A_i finitely generated as an R-algebra.

b) <u>of finite type over R</u> if $\exists$ a finite such cover.

We call $X \in \mathrm{Sch}$ <u>lft / ft</u> if it is so over $R = \mathbb{Z}$.

Lemma Let $X \in \mathrm{Sch}_R$. For $U = \operatorname{Spec} A \subseteq X$ open, the property

$P(U) :=$ "A is finitely generated over R"

is an affine-local property.

Pf. a) A fin.gen./R, ie $A = R[a_1, \dots, a_n]$ w/ $a_i \in A$

$\Rightarrow$ $A_f = R[a_1, \dots, a_n, \frac{1}{f}]$ fin.gen/R ($f \in A$)

b) Let $f_i \in A$ w/ $(f_1, \dots, f_k) = (1)$ & each A_{f_i} fin.gen/R

We need to show A is fin.gen/R.

Let $c_1, \dots, c_k \in A$ w/ $\sum_{i=1}^{k} c_i f_i = 1$.

Let $a_1, \dots, a_n \in A$ w/ $A_{f_i} = R[a_1, \dots, a_n, 1/f_i]$ for $i = 1, \dots, k$.

Put

$$B := R[a_1, \dots, a_n, c_1, \dots, c_k, f_1, \dots, f_k] \subseteq A$$

<u>Claim</u>: $A = B$ (hence A is fingen / R)

Indeed, given any $a \in A$, write

$$\frac{a}{1} = \frac{b_i}{f_i^{N_i}} \text{ in } A_{f_i} \text{ w/ } b_i \in R[a_1, \dots, a_n],\ N_i \in \mathbb{N}.$$

Since $1 = \sum_i c_i f_i$,

we have $1 = \sum_i \tilde{c}_i f_i^{N_i}$ w/ $\tilde{c}_i \in R[c_1, \dots, c_k, f_1, \dots, f_k]$

$$\Rightarrow a = a \cdot 1 = \sum_i \tilde{c}_i \cdot \underbrace{a f_i^{N}}_{= b_i} \in B$$

$\square$

<u>Ex</u> a) $\mathbb{P}^n_R$ is of finite type / R

b) $X \in Sch_R$ (locally) fin. type / R

$\Rightarrow$ every locally closed subscheme of X is so

c) Hence every quasiprojective scheme over R is also of finite type over R.

d) R Noetherian ring

$X \in Sch_R$ (locally) fin. type / R

$\Rightarrow X$ is (locally) Noetherian

(Converse is NOT true: $X = \operatorname{Spec} \mathbb{Q}$ not fin. type / $R = \mathbb{Z}$)

More generally:

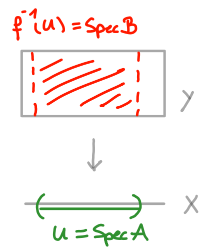
<u>Def</u> A morphism $f: X \to Y$ is ft / lft if for all open affine $V = \operatorname{Spec} R \subseteq Y$, $U := f^{-1}(V) \in Sch_R$ is ft / lft over R.

<u>Exercise</u> It suffices to check this on all $V = V_i$ among a given open affine cover $Y = \bigcup_i V_i$.

6. Affine & finite morphisms

Sometimes working locally on the target is as good as working locally on the source:

Def A morphism $f : Y \to X$ is affine if for all open affine $U \subseteq X$, the preimage $f^{-1}(U) \subseteq Y$ is affine.



We only need to check this on one open affine cover:

Prop For any morphism $f : Y \to X$ of schemes,

$$\mathcal{P}(U) := \text{"} f^{-1}(U) \text{ is affine"}$$

is an affine-local property of $U \subseteq X$.

Pf. a) Let $U = \operatorname{Spec} A \subseteq X$

Assume $f^{-1}(U) = \operatorname{Spec} B$ affine

$\Rightarrow \forall s \in A$, $f^{-1}(\operatorname{Spec} A_s) = \operatorname{Spec} B_{h(s)}$ affine

$\hookrightarrow h := f^{\#}(U) : A \to B$

b) Let $U = \operatorname{Spec} A = \bigcup_{i=1}^{n} U_i$ w/ $U_i := D(s_i)$, $s_i \in A$

Assume $W_i := f^{-1}(D(s_i)) = \operatorname{Spec} B_i$ is affine for all i

Claim: Then $W := f^{-1}(U)$ is affine.

Indeed, let $B := \mathcal{O}_Y(W)$

Consider the affinization

$$\begin{tikzcd} W \arrow[r, "a"] \arrow[dr, "f"'] & W^{\mathrm{aff}} = \operatorname{Spec} B \arrow[d, "\exists! g"] \\ & U = \operatorname{Spec} A \end{tikzcd}$$

Want to show $a : W \xrightarrow{\sim} W^{\mathrm{aff}}$ is an iso.

Enough to do so over each $U_i \subseteq U$:

$$W_i = f^{-1}(U_i) \xrightarrow{a} g^{-1}(U_i)$$

Iso?

$W_i = \operatorname{Spec} B_i$ by assumption

$g^{-1}(U_i) = \operatorname{Spec} B_{t_i}$ because $g : \operatorname{Spec} B \to \operatorname{Spec} A$ is induced by $h = f^{\#} : A \to B$ & then $g^{-1}(D(s_i)) = D(t_i)$ for $t_i := h(s_i)$

Note: $W_i = \operatorname{Spec} B_i$ is affine!

$\Rightarrow$ To show $a: W_i \to g^{-1}(U_i) = \operatorname{Spec} B_{t_i}$ is an iso, it suffices to check on rings:

$$B_{t_i} \xrightarrow{\sim} B_i = \mathcal{O}_Y(W_i).$$

But
$$W_i = f^{-1}(D(s_i)) = \{ p \in W \mid (\underbrace{f^\#(s_i)}_{\substack{=: t_i \\ \in B = \mathcal{O}_Y(W)}})_p \notin m_p \} =: D(t_i)$$

inside W

The claim then follows from the next lemma, applied to W (which is qcqs – exercise). $\square$

Lemma. For any qcqs W & $t \in B := \Gamma(\mathcal{O}_W)$, put $D(t) := \{ p \in W \mid t_p \notin m_p \} \underset{\text{open}}{\subseteq} W$

Then
$$B_t \xrightarrow{\sim} \mathcal{O}_W(D(t)).$$

Pf. W qc $\Rightarrow$ $\exists$ finite open affine cover $W = \bigcup_i W_i$

W qs $\Rightarrow$ $\forall i,j$ $\exists$ finite open affine cover $W_i \cap W_j = \bigcup_k W_{ijk}$

Say $W_i = \operatorname{Spec} B_i$ & $W_{ijk} = \operatorname{Spec} B_{ijk}$

Then

$$B_t = \Gamma(\mathcal{O}_W)_t = \lim \left(\prod_i B_{i,t} \rightrightarrows \prod_{ijk} B_{ijk,t} \right) \qquad (*)$$

localization at t is exact & commutes w/ finite products

But we also have $D(t) = \bigcup_i D_i$

w/
$$D_i := W_i \cap D(t) = \operatorname{Spec} B_{i,t}$$

and $D_i \cap D_j = \bigcup_k D_{ijk}$

w/
$$D_{ijk} := W_{ijk} \cap D(t) = \operatorname{Spec} B_{ijk,t}.$$

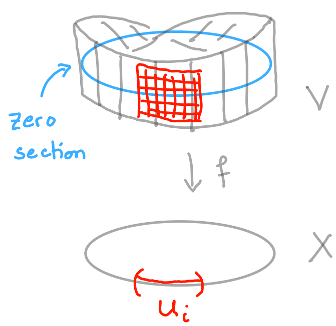
Comparing w/ $(*)$ gives $B_t \simeq \mathcal{O}_W(D(t))$. $\square$

Ex
- Closed immersions are affine morphisms.
- Open immersions needn't be affine, e.g. $\mathbb{A}^2 \setminus 0 \hookrightarrow \mathbb{A}^2$
- Any morphism between affine schemes is affine.

More generally:

Def A vector bundle over a scheme X is a scheme V w/ a morphism $f : V \to X = \bigcup_i U_i$ and an atlas of charts

$$\begin{array}{ccc} f^{-1}(U_i) & \xrightarrow[\varphi_i]{\sim} & \mathbb{A}^n_{R_i} \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{\sim} & \operatorname{Spec} R_i \end{array}$$



$(\Rightarrow f : V \to X$ affine$)$

whose "transition fct^s are linear":

$\forall$ open affine $\operatorname{Spec} R \subseteq U_i \cap U_j$,

$$\left(\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \mathbb{A}^n_R \xrightarrow{\sim} \mathbb{A}^n_R \right) \in GL_n(R).$$

Next we discuss affine morphisms w/ "finite fibers" ...

Recall Let $A \to B$ be a ring extension.

a) We call it a finite extension if B is fin. gen. as an A-module (not just as A-algebra).

b) An element $b \in B$ is called integral over A if the extension $A \to A[b]$ is finite, ie. $\exists$ monic polynomial $f \in A[x]$ w/ $f(b) = 0$.

c) We call $A \to B$ an integral extension if all $b \in B$ are integral over A.

Def A morphism $f : Y \to X$ is called $\left\langle \begin{array}{l} \text{finite} \\ \text{integral} \end{array} \right.$ if it is affine and for all open affine $U = \operatorname{Spec} A \subseteq X$ & $f^{-1}(U) = \operatorname{Spec} B$, the ring extension $A \to B$ is $\left\langle \begin{array}{l} \text{finite} \\ \text{integral} \end{array} \right.$

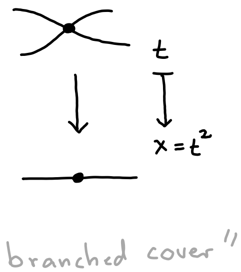
Exercise These properties are affine-local on X!

Ex a) Closed immersions are finite

b) $f : \operatorname{Spec} k[t] \to \operatorname{Spec} k[x]$

w/ $f^\#(x) = p(t) \in k[t]$

a polynomial of $\deg p > 0$



$\Rightarrow$ f is a finite morphism

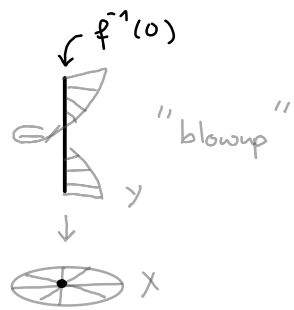
c) $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ is integral but NOT finite.

d) Open immersions are NOT integral (unless iso).

e) Put $Y = V(x - yz) \subset \mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$

$\Rightarrow$ The morphism

$$\begin{array}{ccc} Y & \ni & (x,y,z) \\ f \downarrow & & \downarrow \\ \mathbb{A}^2_k & \ni & (x,y) \end{array}$$



is birational but NOT finite.

Prop Let $f : Y \to X$ be an integral morphism.

a) f is closed.

b) If f is finite, then $\forall p \in |X|$ the set

$f^{-1}(p) = \{q \in |Y| : f(q) = p\}$ is finite.

Pf. a) Let $Z \overset{i}{\hookrightarrow} Y$ be closed

Want: $f(|Z|) \subseteq |X|$ is closed.

But $g := f \circ i : Z \to X$ is still integral

(as a composite of integral morphisms)

$\Rightarrow$ Wlog $Z = Y$ and $g = f$

So we only need to show $f(|Y|) \subseteq |X|$ is closed.

Claim is local on X

$\Rightarrow$ wlog $X = \operatorname{Spec} A$ affine & then $Y = \operatorname{Spec} B$

since f is affine

Say $f = {}^a h$ w/ $h : A \to B$.

Put $I = \ker(h)$, then $f : Y = \operatorname{Spec} B \longrightarrow \underset{\text{closed}}{V(I) \subseteq X}$

Now $A/I \hookrightarrow B$ is injective & integral

We have $f(\mathfrak{q}) = \mathfrak{q} \cap (A/I) \in \operatorname{Spec}(A/I)$

for $\mathfrak{q} \in \operatorname{Spec}(B)$

$\Longrightarrow$ "Lying over":

$|\operatorname{Spec} B| \twoheadrightarrow |\operatorname{Spec} A/I|$ surjective

$\Longrightarrow$ $f(|Y|) = V(I) \subseteq X$ closed

b) As above wlog $f = {}^a h$ w/ $h : A \to B$.

Replace A by A/I w/ $I := \ker(h)$

$\Longrightarrow$ wlog $A \hookrightarrow B$ injective.

If f is finite, then B is finite over A

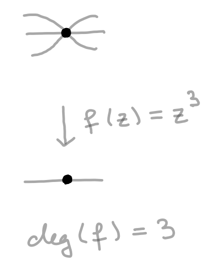
$\Longrightarrow$ $\exists$ only fin. many primes $\mathfrak{q} \in \operatorname{Spec} B$

above a given $\mathfrak{p} \in \operatorname{Spec} A$

(exercise in algebra ...) $\square$

<u>Prop</u> Let $f : Y \to X$ be a finite surjective morphism of integral schemes of finite type over an alg. closed field $k = \bar{k}$. If $k(Y)/k(X)$ is a separable extension, then $\exists$ open dense $U \subseteq X$ sth $\forall$ closed pt^s $x \in |U|$:

$$\# f^{-1}(x) = \underbrace{[k(Y) : k(X)]}_{=: \deg(f)}.$$



Pf. Wlog $f : Y = \operatorname{Spec} B \longrightarrow X = \operatorname{Spec} A$

w/ $A \hookrightarrow B$ (injective as f dominant, see exercises)

Consider $i : k(X) = \underbrace{\operatorname{Quot}(A)}_{=: K} \hookrightarrow k(Y) = \underbrace{\operatorname{Quot}(B)}_{=: L}$.

$B = A[b_1, \ldots, b_n]$ w/ $b_i \in B$ integral over A

$\Longrightarrow$ $L = K(b_1, \ldots, b_n) = K[b_1, \ldots, b_n]$

is a finite extension of K

L/K finite separable

$\Rightarrow L = K[b]$ for some $b \in L$ ("primitive element")

$b \in L$ algebraic over $K = \mathrm{Quot}(A)$

$\Rightarrow$ integral over A_s for some $s \in A$

Wlog $s = 1$ (else replace $f: Y \to X$ by

$f: f^{-1}(U) \to U$ w/ $U = D(s) \subseteq X$)

Can also assume $A \subseteq B \subseteq A[b]$

(localize A further & use $B \subseteq K[b]$ is fin. gen. over A)

$\hookrightarrow K = \mathrm{Quot}(A)$

$b \in L = \mathrm{Quot}(B)$ is integral over A

$\Rightarrow b = \frac{c}{s}$ for some $c \in B$, $s \in A$

$\Rightarrow$ Wlog $b \in B$ (again replacing A by A_s)

$\Rightarrow B = A[b] \simeq A[t]/(p(t))$

for a monic polynomial $p(t) \in A[t]$

$L = K[b] \simeq K[t]/(p(t))$ separable over K

$\Rightarrow p(t) \in K[t]$ separable polynomial

$\Rightarrow$ discriminant $\delta := \mathrm{discr}(p) \in K$ is $\neq 0$

Wlog $\delta \in A$ (else replace A by the localization A_δ).

Put $U := D(\delta) \subseteq X = \mathrm{Spec}(A)$.

$\uparrow$ open dense

Now let $x \in |U|$ be a closed pt,

corresponding to a max. ideal $\mathfrak{m} \trianglelefteq A$.

Let $\bar{A} := \underbrace{A/\mathfrak{m}}_{\simeq k \text{ (since } k = \bar{k} \text{ \& } A \text{ fin type}/k)} \longrightarrow \bar{B} := B/\mathfrak{m}B$

Since $\bar{\delta} :=$ image $(\delta) \in \bar{A}$ is $\neq 0$,

the reduction $\bar{p}(t) \in \bar{A}[t]$ of $p(t) \in A[t]$

splits into distinct linear factors

$$\Rightarrow \bar{B} \simeq \bar{A}[t]/(\bar{p}(t)) \simeq \underbrace{k \times \cdots \times k}_{\deg(p) \text{ factors}}$$

The claim now follows since

$$
\begin{aligned}
f^{-1}(x) &= \{ \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} \cap A = \mathfrak{m} \} \\
&\simeq \operatorname{Spec} \overline{B} \quad (\text{as a set}) \\
&\simeq \operatorname{Spec} (k \times \dots \times k) \\
&= \coprod_{i=1}^{d} \operatorname{Spec}(k) \quad \text{w/ } d = \deg(p) \\
&\qquad\qquad\qquad\qquad = [L:K] \qquad \square
\end{aligned}
$$

<u>Rem</u> a) $Y = \operatorname{Spec} \mathbb{C} \xrightarrow{f} X = \operatorname{Spec} \mathbb{R}$ is finite

w/ $\deg(f) = 2$ but topologically $f^{-1}(pt) = pt$.

b) $k(Y)/k(X)$ separable is also needed:

Let $\operatorname{char} k = p > 0$

$Y = \mathbb{A}^1_k \xrightarrow{f} X = \mathbb{A}^1_k$

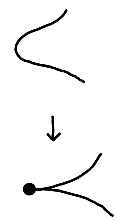
given by $k[t] \longrightarrow k[t]$, $t \mapsto t^p$

$\Rightarrow \deg(f) = p$ but $f : |Y| \xrightarrow{\sim} |X|$ homeom.

7. Normal schemes

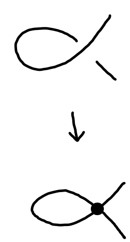
Motivation: Let $f : Y \to X$ be a finite birational morphism of integral schemes. Is f an iso ?

Ex a)
$$\begin{array}{l} Y = \operatorname{Spec} k[t] \\ \quad \downarrow f \\ X = \operatorname{Spec} k[x,y]/(y^2 - x^3) \end{array}$$



$$f^{\#} : k[x,y]/(y^2-x^3) \longrightarrow k[t]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

b)
$$\begin{array}{l} Y = \operatorname{Spec} k[t] \\ \quad \downarrow g \\ X = \operatorname{Spec} k[x,y]/(y^2 - x^2(x+1)) \end{array}$$



$$g^{\#} : k[x,y]/(y^2-x^2(x+1)) \longrightarrow k[t]$$
$$x \longmapsto t^2 - 1$$
$$y \longmapsto t(t^2-1)$$

$\Rightarrow$ f, g finite birational but NOT iso.

f is a homeom. but g is not even bijective !

Note: $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$

- finite $\Rightarrow$ $A \to B$ finite
- birational $\Rightarrow$ $\operatorname{Quot}(A) \xrightarrow{\sim} \operatorname{Quot}(B)$

$\Rightarrow$ $A \underset{\text{integral}}{\hookrightarrow} B \subseteq \operatorname{Quot}(A)$

$\Rightarrow$ If $A \neq B$, then A is NOT integrally closed !

Recall a) The <u>integral closure</u> of A in an extension $A \to C$ is the A-algebra
$$A_C^{\text{int}} := \{ c \in C \mid c \text{ is integral over } A \}.$$

b) The <u>integral closure</u> of a domain A is
$$A^{\text{int}} := A_C^{\text{int}} \text{ for } C := \operatorname{Quot}(A).$$

If $A^{\text{int}} = A$ we call A <u>normal / integrally closed</u>.

Ex
- Any UFD is normal (but not conversely, think of $\mathbb{Z}[\sqrt{-5}]$)
  e.g. $\mathbb{Z}$, $\mathbb{Z}[x_1, \ldots, x_n]$, ...
- Polynomial rings $A[x]$ are normal for A normal
- A normal $\Rightarrow$ Any localization $S^{-1}A$ normal

The last part has a converse:

<u>Lemma</u> For any domain A, TFAE:

a) A is normal.

b) $A_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in \operatorname{Spec} A$.

c) $A_{\mathfrak{m}}$ is normal for all $\mathfrak{m} \in \operatorname{Spm} A$.

Pf. Enough to show c) $\Rightarrow$ a):

Let $a \in K = \operatorname{Quot}(A)$ be integral over A.

Consider $I := \{ s \in A \mid s \cdot a \in A \} \trianglelefteq A$.

Want to show that c) implies $1 \in I$.

If $1 \notin I$, $\exists\, \mathfrak{m} \in \operatorname{Spm}(A)$: $I \subseteq \mathfrak{m}$

But $a \in K = \operatorname{Quot}(A_{\mathfrak{m}})$ integral over $A \subseteq A_{\mathfrak{m}}$

$\Longrightarrow$ By c) we have $a \in A_{\mathfrak{m}}$

$\Longrightarrow$ $\exists\, s \in A \setminus \mathfrak{m}$: $sa \in A$, hence $s \in I$ $\lightning$ $\square$

<u>Def</u> An arbitrary ring A is called <u>normal</u> if $A_{\mathfrak{p}}$ is a normal domain for every $\mathfrak{p} \in \operatorname{Spec}(A)$.

<u>Ex</u> A_1, A_2 normal $\Rightarrow$ $A := A_1 \times A_2$ normal

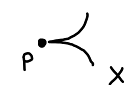
Back to geometry:

<u>Def</u> A scheme X is <u>normal</u> if for all open affine $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is normal.

It is <u>normal at a pt</u> $p \in |X|$ if $\mathcal{O}_{X,p}$ is normal.

<u>Ex</u> a) $X = \operatorname{Spec} A$ w/ $A := k[x,y]/(y^2 - x^3)$ is NOT normal.

$\left(\begin{array}{l} t := \frac{y}{x} \in \operatorname{Quot}(A) \setminus A \text{ satisfies } P(t) = 0 \\ \text{w/ } P(T) := T^2 - x \in A[T] \text{ monic polynomial} \end{array} \right)$



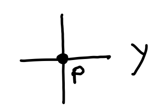
It is NOT normal at $p = (0,0)$ but normal at all other points.

b) $Y = \operatorname{Spec} B$ w/ $B := k[x,y]/(xy)$ is NOT normal:

At $p = (0,0)$ the local ring

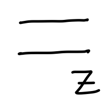
$\mathcal{O}_{X,p} = B_{(x,y)} = k[x,y]_{(x,y)}/(xy)$

is not a domain.



c) $Z = \operatorname{Spec} C$ w/ $C = k[x] \times k[y]$ IS normal

(eg by the criterion below)



Lemma For any scheme X, the following are equivalent:

a) X is normal.

b) $\exists$ affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ w/ $\mathcal{O}_X(U_\alpha)$ normal for all α.

c) X is normal at all points $p \in |X|$.

Pf. a) $\Rightarrow$ b): trivial

A_α normal

b) $\Rightarrow$ c): $p \in U_\alpha = \operatorname{Spec} A_\alpha \Rightarrow \mathcal{O}_{X,p} = A_{\alpha,\mathfrak{p}}$ normal.

c) $\Rightarrow$ a): Let $U = \operatorname{Spec} A \subseteq X$

$\mathcal{O}_{X,p} = A_\mathfrak{p}$ normal for all $\mathfrak{p} \in U \Rightarrow A$ normal $\square$

previous lemma

Add-on If X is quasicompact, it suffices to check c) only on closed points $p \in |X|$.

Pf. X qc $\Rightarrow \forall p \in |X| \; \exists$ closed pt $q \in \overline{\{p\}}$ (see exercises)

If $\mathcal{O}_{X,q}$ is normal, then so is $\mathcal{O}_{X,p}$. $\square$

Ex $\operatorname{Spec} R$ normal $\iff$ R normal

e.g. $\mathbb{A}^n_R$, $\mathbb{P}^n_R$ normal for any normal ring R

Normal schemes can be reducible, but their irreducible cpt^s must be disjoint:

Lemma Let X be a Noetherian scheme. TFAE:

a) X is normal

b) $X \simeq \coprod_{i=1}^n X_i$ w/ X_i integral & normal

finite disjoint union

Pf. b) $\Rightarrow$ a) trivial

(this really belongs in the section about integral schemes...)

a) $\Rightarrow$ b) follows from:

$$\left.\begin{array}{l} X \text{ Noetherian scheme and } \mathcal{O}_{X,p} \text{ a domain for all } p \in |X| \\ \Rightarrow X = \coprod_{i=1}^n X_i \text{ finite disjoint union of irreducible } X_i \end{array}\right\} (*)$$

To show $(*)$, let $X_1, \dots, X_n$ be the irred cpt^s of X

(fin. many since $|X|$ Noetherian)

Goal: $X_i \cap X_j = \emptyset$ for all $i \neq j$.

If not, wlog $\exists \, p \in X_1 \cap \dots \cap X_k$, $p \notin X_{k+1} \cup \dots \cup X_n$

(some $k > 1$)

as it is open in $X_i \subseteq X$

Say $p = \mathfrak{p} \in U = \operatorname{Spec} R \subseteq X$

$$\Rightarrow \quad U = \bigcup_{i=1}^{n} U_i \quad \text{w/} \quad U_i := U \cap X_i \begin{cases} \text{irred cpt of } U \\ \text{or } = \emptyset \end{cases}$$

and $\mathfrak{p} \in U_1 \cap \dots \cap U_k$, $\mathfrak{p} \notin U_{k+1} \cup \dots \cup U_n$

For $U_i \neq \emptyset$, write $U_i = V(\mathfrak{p}_i) \subseteq \operatorname{Spec} R$ w/ $\mathfrak{p}_i \trianglelefteq R$ prime

$\Rightarrow$ the $\mathfrak{p}_i$ are the minimal primes of R

By construction $\mathfrak{p}_i \subseteq \mathfrak{p}$ iff $i \in \{1, 2, \dots, k\}$

$\Rightarrow$ $\overline{\mathfrak{p}}_i := \mathfrak{p}_i \cdot R_{\mathfrak{p}}$ for $i \leq k$ are the minimal primes in $R_{\mathfrak{p}}$

But $R_{\mathfrak{p}} = \mathcal{O}_{X,p}$ is a domain

$$\Rightarrow \quad \overline{\mathfrak{p}}_1 \cap \dots \cap \overline{\mathfrak{p}}_k = \bigcap_{\mathfrak{q} \in \operatorname{Spec} R_{\mathfrak{p}}} \mathfrak{q} = \operatorname{Rad}(R_{\mathfrak{q}}) = (0)$$

$$\Rightarrow \quad \overline{R} \hookrightarrow \prod_{i=1}^{k} R_{\mathfrak{p}} / \overline{\mathfrak{p}}_i \qquad \text{(Chinese remainder thm)}$$

But $\overline{\mathfrak{p}}_i \not\subseteq \overline{\mathfrak{p}}_j$ for $j \neq i$ $\Rightarrow$ $\exists\, x_i \in \overline{\mathfrak{p}}_i \setminus \bigcup_{j \neq i} \overline{\mathfrak{p}}_j$

(prime avoidance)

$\Rightarrow$ $x_1 \cdots x_k = 0$ in $R_{\mathfrak{p}}$ but $x_1, \dots, x_k \neq 0$ in $R_{\mathfrak{p}}$ ↯ $\square$

Any integral scheme has a "best normal approximation":

Prop Let X be an integral scheme.

$\Rightarrow$ $\exists$ dominant $f : Y \longrightarrow X$ sth

a) Y is normal, and

b) any dominant $g : Z \to X$ w/ Z normal & irreducible factors uniquely over f:

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists! \tilde{g}"] \arrow[dr, "g"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

These properties determine Y up to iso over X.

We have:

$\tilde{g}$ iso $\iff$ g integral and birational

Def We call the morphism $X^{\text{norm}} := Y \xrightarrow{f} X$ (and the scheme X^{norm}) the <u>normalization of X</u>.

Pf. Uniqueness is clear from univ. property a), b).

$\Rightarrow$ Enough to show existence locally on X

(uniqueness then allows to glue on overlaps)

$\Rightarrow$ Wlog $X = \operatorname{Spec} A$ affine,

w/ A a domain (since X integral)

integral closure

Put $Y := \operatorname{Spec} A^{int} \xrightarrow{f} X$ w/ $A^{int} \subseteq K(X)$

Let $g : Z \to X$ dominant w/ Z normal & irreducible

Let $U = \operatorname{Spec} B \subseteq Z$ any open affine

$\Rightarrow$ injections

$$\begin{tikzcd}
K(X) \arrow[r, hook] & K(Z) = \operatorname{Quot}(B) \\
A^{int} \arrow[u, phantom, "\cup" description] \arrow[r, hook, "\exists!"] & B^{int} \arrow[u, phantom, "\cup" description] \\
A \arrow[u, phantom, "\cup" description] \arrow[r, hook] & B \arrow[u, phantom, "\|" description]
\end{tikzcd}$$

(x integral over A $\Rightarrow$ x integral over B)

equality since Z is normal

$\Rightarrow \exists!\, \tilde{g}_U : U \to Y$ w/ $g|_U = f \circ \tilde{g}_U$

By uniqueness the $\tilde{g}_U$ for different $U \subseteq Z$ glue

to a unique $\tilde{g} : Z \to Y$ w/ $g = f \circ \tilde{g}$.

By construction f is integral & birat

$\Rightarrow$ If $\tilde{g}$ is an iso, then g is integral & birat.

Conversely, assume $g : Z \to X$ integral & birat.

g integral $\Rightarrow$ g affine

$\Rightarrow$ for $X = \operatorname{Spec} A$ affine,

also $Z = \operatorname{Spec} B$ affine

iso since g birat

Then
$$\begin{tikzcd}
K(X) \arrow[r, "\sim"] & K(Z) \\
A^{int} \arrow[u, phantom, "\cup" description] \arrow[r] & B^{int} \arrow[u, phantom, "\cup" description] \\
A \arrow[u, phantom, "\cup" description] \arrow[r] & B \arrow[u, phantom, "\|" description]
\end{tikzcd}$$

integral as g integral

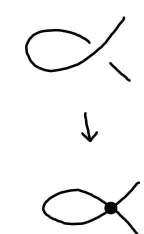
$\Rightarrow A^{int} = B$

$\Rightarrow \tilde{g} : Z \xrightarrow{\sim} Y$ Iso

$\square$

Ex $X = \operatorname{Spec} k[x,y]/(y^2 - x^2(x+1))$

$\Rightarrow X^{norm} = \operatorname{Spec} k[t] \xrightarrow{f} X$



$f^{\#} : k[x,y]/(y^2 - x^2(x+1)) \longrightarrow k[t]$

$x \longmapsto t^2 - 1$

$y \longmapsto t(t^2 - 1)$

Rem $\exists$ Noetherian local domains A w/ $A \hookrightarrow A^{int}$ NOT finite

(Akizuki 1935, see Miles Reid: arXiv: alg-geom/9503017)

$\Rightarrow X^{norm} \longrightarrow X$ is in general NOT a finite morphism,

even if we assume X Noetherian

But:

Prop Let A be a domain of finite type over a field k.

Let $K = \mathrm{Quot}(A)$ & M/K a finite extension field.

$\Rightarrow C := A^{int}_M$ is finite over A.

Cor For any integral scheme X of finite type $/k$, the normalization $X^{norm} \to X$ is finite.

Rem For the corollary we only need the case $M = K$ of the proposition. But the proof of the proposition goes by reduction to $M \neq K$ ...

Pf of the prop.

Noether normalization

$\Rightarrow \exists$ finite injective hom $k[x_1, .., x_n] \hookrightarrow A$.

$\Rightarrow$ wlog $A = k[x_1, \ldots, x_n]$

A Noetherian $\Rightarrow$ any submodule of a fingen A-module is again a fingen A-module

$\Rightarrow$ can replace L by its normal hull inside a given algebraic closure

$\Rightarrow$ wlog L/K normal

Let $G = \mathrm{Aut}(M/K)$ & $L := M^G \subseteq M$.

Have towers

$$\begin{array}{ll} M & \qquad C := A^{int}_M = B^{int}_M \\ \mid \sim \text{Galois} & \qquad \mid \\ L & \qquad B := A^{int}_L \\ \mid \sim \text{purely insep} & \qquad \mid \\ K & \qquad A \end{array}$$

$\Rightarrow$ enough to show both $A \subseteq B$ & $B \subseteq C$ are finite

<u>Step 1</u> B is finite over A:

If $L = K$ that's trivial (recall $A = k[x_1, .., x_n]$)

So let $L \neq K$.

Purely insep $\Rightarrow$ char $k = p > 0$

and $L = K(\sqrt[q]{f_1}, .., \sqrt[q]{f_m})$

for some $f_i \in A$ and $q = p^r$

Enlarge L if needed

$\Rightarrow$ wlog $L = k'(x_1^{1/q}, \dots, x_n^{1/q})$

w/ $k' = k(a_1^{1/q}, \dots, a_s^{1/q})$, some $a_i \in k$.

$\Rightarrow B = A_L^{int} = k'[x_1^{1/q}, \dots, x_n^{1/q}]$ finite over A

<u>Step 2</u> C is finite over B: See next lemma. $\square$

<u>Lemma</u> B Noetherian normal domain

$L = \mathrm{Quot}(B)$

M/L finite separable field extension

$\Rightarrow C := B_M^{int}$ finite $/ B$

Pf. M/L separable

$\Rightarrow tr_{M/L} : M \times M \longrightarrow L$ nondegenerate

Let $e_1, \dots, e_n \in C$ be a basis for M over L

and $e_1^*, \dots, e_n^* \in M$ the dual basis:

$$tr_{M/L}(e_i, e_j^*) = \delta_{ij}$$

For $c \in C$, write

$$c = \sum_{j=1}^{n} \alpha_j e_j^* \quad \text{w/} \quad \alpha_j \in L$$

then

$$\alpha_i = Tr_{M/L}(\underbrace{ce_i}_{\in C}) \in C \cap L = B$$

B normal

$\Rightarrow c \in Be_1^* + \dots + Be_n^*$

$\Rightarrow C \subseteq \underbrace{Be_1^* + \dots + Be_n^*}_{\text{fingen } B\text{-mod}}$

$\Rightarrow C$ fingen B-module (as B Noetherian) $\square$

III. Fiber products & base change

1. Schemes as functors of points

<u>Recall</u> For $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$ we have the algebraic set

$$\{ p \in \mathbb{A}^n(k) \mid f_1(p) = \dots = f_m(p) = 0 \}$$

$$\cong \mathrm{Hom}_{k\text{-alg}}\left(k[x_1, \dots, x_n]/(f_1, \dots, f_m),\ k \right)$$

$$\cong \mathrm{Hom}_{\mathrm{Sch}_k}(\mathrm{Spec}\, k,\ X) =: X(k).$$

If k is not alg closed or X is not assumed to be reduced, then the subscheme $X \hookrightarrow \mathbb{A}^n_k$ is not determined by this set of points.

$\rightsquigarrow$ look at $X(R) := \mathrm{Hom}_{\mathrm{Sch}_k}(\mathrm{Spec}\, R, X)$ for other k-algebras R

$\rightsquigarrow$ functor $\mathrm{Hom}_{\mathrm{Sch}_k}(-, X) : \mathrm{AffSch}_k^{op} \to \mathrm{Sets}$

<u>Def</u> For a category $\mathcal{C}$ we let $\hat{\mathcal{C}} := \mathrm{Func}(\mathcal{C}^{op}, \mathrm{Sets})$ $(= \mathrm{PSh}(\mathcal{C}))$ be the category w/

- objects: functors $F : \mathcal{C}^{op} \to \mathrm{Sets}$
- morphisms: natural transformations $F \xrightarrow{\varphi} G$.

<u>Ex</u> Any $X \in \mathrm{ob}(\mathcal{C})$ gives a functor

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \mathrm{Sets}$$

For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we get a natural trafo $f \circ (-) : h_X \longrightarrow h_Y$.

We get a functor
$$\begin{aligned} h : \mathcal{C} &\longrightarrow \hat{\mathcal{C}} \\ X &\longmapsto h_X \end{aligned}$$

called the <u>Yoneda embedding</u>.

<u>Def</u> A functor $F \in \mathrm{ob}(\hat{\mathcal{C}})$ is <u>representable</u> if $F \simeq h_X$ for some $X \in \mathrm{ob}(\mathcal{C})$.

The representing object X (if it exists) is determined uniquely up to isomorphism:

<u>Yoneda lemma</u> a) The functor

$$h : \mathcal{C} \to \hat{\mathcal{C}} \quad \text{is fully faithful}$$

i.e. $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, h_Y)$

for all $X, Y \in \mathrm{ob}(\mathcal{C})$.

b) More generally

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F)$$

for all $X \in \mathrm{ob}(\mathcal{C})$, $F \in \mathrm{ob}(\hat{\mathcal{C}})$.

Pf. b) Any $z \in F(X)$ gives rise to $\varphi_z \in \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F)$ via

$$\begin{array}{ccc} h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X) & \xrightarrow{\varphi_z} & F(Y) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & (F(f))(z) \end{array}$$

Any $\varphi \in \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F)$ arises as $\varphi = \varphi_z$ for a unique $z \in F(X)$, namely

$$z = \varphi(\mathrm{id}_X) \quad \text{w/} \quad \mathrm{id}_X \in h_X(X).$$

a) Apply b) to $F = h_Y$. $\square$

<u>Def</u> Any scheme $X \in \mathrm{Sch}_S$ is determined up to iso by the functor

$$\mathrm{Hom}_{\mathrm{Sch}}(-, X) : \mathrm{Sch}_S^{op} \to \mathrm{Sets}.$$

We write $X(Z) := \mathrm{Hom}_{\mathrm{Sch}_S}(Z, X)$.

$$\parallel$$

$X(R)$ if $Z = \mathrm{Spec}\, R$ is affine

<u>Caution</u>: S must be clear from the context:

$$\mathbb{A}^1_k(R) = \mathrm{Hom}_{k\text{-alg}}(k, R) = R \quad \text{for } S = k$$

$$\text{but} \quad \mathbb{A}^1_k(R) = \mathrm{Hom}_{\mathrm{Rings}}(k, R) \quad \text{for } S = \mathbb{Z}.$$

The functorial view is useful to "add structure":

<u>Def</u> A <u>group scheme over S</u> is a functor $\mathrm{Sch}_S^{op} \to \mathrm{Groups}$ that is representable by a scheme over S.

<u>Ex</u> a) The <u>additive group</u> over $S = \mathrm{Spec}\, R$ is defined by $\mathbb{G}_{a,R}(Z) := (\Gamma(\mathcal{O}_Z), +)$.

It is represented by $\mathbb{G}_{a,R} = \mathrm{Spec}\, R[t] = \mathbb{A}^1_R$.

b) The multiplicative group over $S = \operatorname{Spec} R$ is defined by $\mathbb{G}_{m,R}(Z) := (\Gamma(\mathcal{O}_Z))^*$.

It is represented by $\mathbb{G}_{m,R} = \operatorname{Spec} R[t, t^{-1}]$.

c) For $n \in \mathbb{N}$ the group of n-th roots of 1 is $\mu_{n,R}(Z) := \{ a \in \Gamma(\mathcal{O}_Z) \mid a^n = 1 \}$.

It is represented by $\mu_{n,R} = \operatorname{Spec} R[t,t^{-1}]/(t^n - 1)$.

d) The general linear group over R is

$$GL_n(Z) := \{ M \in \mathrm{Mat}_{n\times n}(\Gamma(\mathcal{O}_Z)) \mid \exists N \in \mathrm{Mat}_{n\times n}(\Gamma(\mathcal{O}_Z)) : MN = NM = 1 \},$$

represented by $\operatorname{Spec} R[x_{ij} \mid 1 \le i,j \le n]_{\det(x_{ij})}$

localize at the determinant (as a polynomial in the x_{ij})

To recover a scheme $X \in \mathrm{Sch}_S$ we only need to know its functor of points on the subcategory of affine S-schemes

S needn't be affine

$$\mathrm{AffSch}_S := \{ Z \in \mathrm{Sch}_S \mid Z \to S \text{ affine} \} \subset \mathrm{Sch}_S :$$

Prop $h: \mathrm{Sch}_S \longrightarrow \mathrm{Func}(\mathrm{AffSch}_S, \mathrm{Sets})$ is fully faithful.

Pf. Any $Z \in \mathrm{Sch}_S$ has an open cover $Z = \bigcup_i U_i$ w/ $U_i \in \mathrm{AffSch}_S$.

Faithfulness:

$f = g$ in $\mathrm{Hom}_{\mathrm{Sch}_S}(Y, X)$

$\Leftrightarrow$ $f \circ (-) = g \circ (-) : \mathrm{Sch}_S^{op} \longrightarrow \mathrm{Sets}$

$\Leftrightarrow$ $\forall Z \in \mathrm{Sch}_S \ \forall h \in \mathrm{Hom}_{\mathrm{Sch}_S}(Z, Y) : f \circ h = g \circ h$

$\Leftrightarrow$ $\ldots \ \forall i : f \circ h|_{U_i} = g \circ h|_{U_i}$

$\Leftrightarrow$ $f \circ (-) = g \circ (-) : \mathrm{AffSch}_S^{op} \to \mathrm{Sets}$.

Fullness: Gluing of morphisms

$\square$

2. Fiber products

Recall Let $\mathcal{C}$ be a category and $X_1, X_2, S \in \mathcal{C}$ w/ morphisms $f_i : X \to S$ $(i = 1, 2)$. If the functor

$$\mathcal{C}^{op} \longrightarrow Sets,$$

$$Z \longmapsto \left\{ (g_1, g_2) \;\middle|\; \begin{array}{l} g_i \in Hom_{\mathcal{C}}(Z, X_i) \\ \text{w/ } f_1 \circ g_1 = f_2 \circ g_2 \end{array} \right\}$$

is representable by an object $Y \in \mathcal{C}$, we call

$$X_1 \underset{S}{\times} X_2 := Y$$

the <u>fiber product</u> of X_1 and X_2 over S.

In other words

$$X_1 \underset{S}{\times} X_2 := \lim \left(\begin{array}{c} X_2 \xrightarrow{f_2} \\ X_1 \xrightarrow[f_1]{} \end{array} S \right)$$

(if this limit exists in $\mathcal{C}$).

The map

$$\begin{array}{ccc} Hom_{\mathcal{C}}(Z, X_1 \underset{S}{\times} X_2) & \longrightarrow & Hom_{\mathcal{C}}(Z, X_i) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g_1, g_2) & \longmapsto & g_i \end{array}$$

defines by Yoneda a morphism $pr_i : X_1 \underset{S}{\times} X_2 \to X_i$ & the universal property is summarized by:

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "g_2"'] & & \\ & Y \arrow[r, "pr_1"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\lrcorner"] & X_1 \arrow[d, "f_1"] \\ & X_2 \arrow[r, "f_2"'] & S \end{tikzcd}$$

(for all g_1, g_2 w/ $g_1 \circ f_1 = g_2 \circ f_2$)

We also say the square is <u>Cartesian</u> / a <u>fiber square</u> and indicate this by a the symbol $\lrcorner$ as above.

<u>Ex</u> a) For $\mathcal{C}$ = Sets all fiber products exist, here

$$X_1 \underset{S}{\times} X_2 = \{ (p_1, p_2) \in X_1 \times X_2 \mid f_1(p_1) = f_2(p_2) \}$$

b) For $\mathcal{C} = \{\text{smooth manifolds}\}$, not all fiber products exist (exercise ...)!

<u>Thm</u> The category of schemes has all fiber products.

Pf. Let $f_i \in \mathrm{Hom}_{sch}(X_i, S)$ for $i = 1, 2$.

① If $X_i = \mathrm{Spec}\, A_i$ & $S = \mathrm{Spec}\, R$ are affine,

then $X_1 \underset{S}{\times} X_2 := \mathrm{Spec}\,(A_1 \otimes_R A_2)$ works:

Indeed, for any scheme Z (needn't be affine), we have

$$\mathrm{Hom}_{sch}(Z, \mathrm{Spec}\,(A_1 \otimes_R A_2))$$

$$\simeq \mathrm{Hom}_{Rings}(A_1 \otimes_R A_2, \Gamma(\mathcal{O}_Z))$$

$$\simeq \left\{ (\varphi_1, \varphi_2) \in \prod_{i=1}^{2} \mathrm{Hom}_{Rings}(A_i, \Gamma(\mathcal{O}_Z)) \;\middle|\; \varphi_1 \circ f_1^\# = \varphi_2 \circ f_2^\# \text{ in } \mathrm{Hom}_{Rings}(R, \Gamma(\mathcal{O}_Z)) \right\}$$

$$\simeq \left\{ (g_1, g_2) \in \prod_{i=1}^{2} \mathrm{Hom}_{sch}(Z, \mathrm{Spec}\, A_i) \;\middle|\; f_1 \circ g_1 = f_2 \circ g_2 \right\}$$

$\Longrightarrow$ claim ①

② "Fiber products localize": If $X_1 \times_S X_2$ exists, then for any open $V \subseteq S$ & $U_i \subseteq f_i^{-1}(V)$ also $U_1 \underset{V}{\times} U_2$ exists and is given explicitly by

$$U_1 \underset{V}{\times} U_2 = pr_1^{-1}(U_1) \cap pr_2^{-1}(U_2) \subseteq X_1 \underset{S}{\times} X_2$$

(since the universal property is inherited)

③ Let $S = \bigcup_{i \in I} V_i$ be an open cover & $U_{\alpha i} = f_\alpha^{-1}(V_i)$.

If $\forall i \in I$ $\exists$ fiber product $Y_i := U_{1i} \underset{V_i}{\times} U_{2i}$

then so does $X_1 \underset{S}{\times} X_2$.

Indeed, put $V_{ij} := V_i \cap V_j$ & $U_{\alpha ij} := f_\alpha^{-1}(V_{ij})$.

By step 2 we have

$$\begin{tikzcd} & Y_i \\ Y_{ij} := U_{1ij} \underset{V_{ij}}{\times} U_{2ij} \arrow[ur, hook, "\text{open}"'] \arrow[dr, hook, "\text{open}"] & \\ & Y_j \end{tikzcd}$$

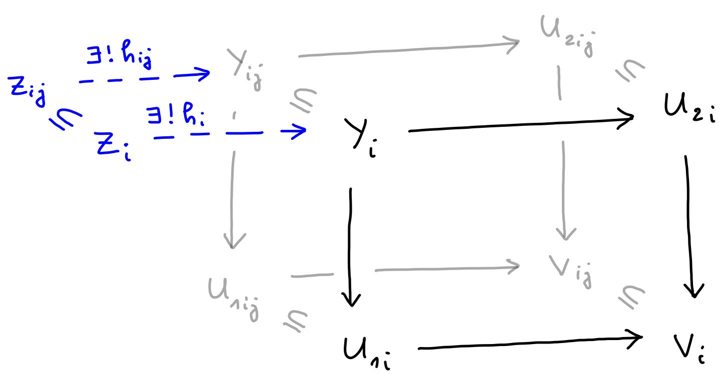
Gluing along these Y_{ij} gives a candidate $Y := \bigcup_{i \in I} Y_i$.

We check that $Y = X_1 \times_S X_2$ in this case:

Let $Z \in Sch$ & $g_\alpha \in \mathrm{Hom}_{Sch}(Z, X_\alpha)$ w/ $f_1 \circ g_1 = f_2 \circ g_2$

Put
$$\begin{array}{ccc} Z_i := g_1^{-1}(U_{1i}) = g_2^{-1}(U_{2i}) & \xrightarrow{g_{\alpha i}} & U_{\alpha i} \subseteq X_\alpha \\ \cup\!\vert & & \cup\!\vert \\ Z_{ij} := g_1^{-1}(U_{1ij}) = g_2^{-1}(U_{2ij}) & \xrightarrow[g_{\alpha ij}]{} & U_{\alpha ij} \end{array}$$

The universal property of Y_i & Y_{ij} gives morphisms $h_i := (g_{1i}, g_{2i})$ & h_{ij} as below:



These glue to a unique $h: Z \to Y$ w/ $pr_i \circ h = g_i$

$\Rightarrow$ claim ③

④ So wlog $S = \mathrm{Spec}\, R$ affine (by step 3).
Similarly use open affine cover of X_1, X_2 and step 2 to reduce to X_1, X_2, S affine $\Rightarrow$ done by step 1 $\square$

<u>Notation</u> We put $X \times_R Y := X \times_S Y$ for $S = \mathrm{Spec}\, R$,

$X \times Y := X \times_{\mathbb{Z}} Y$ over $S = \mathrm{Spec}\, \mathbb{Z}$.

<u>Ex</u> $\mathbb{A}^m_S \times_S \mathbb{A}^n_S \simeq \mathbb{A}^{m+n}_S \simeq \mathbb{A}^{m+n} \times S$

<u>Rem</u> The $pr_i : X_1 \times_S X_2 \to X_i$ w/ $f_1 \circ pr_1 = f_2 \circ pr_2$ induce maps

$g_i : |X_1 \times_S X_2| \to |X_i|$ w/ $|f_1| \circ g_1 = |f_2| \circ g_2$,

hence a map

$$(g_1, g_2) : |X_1 \times_S X_2| \to \underbrace{|X_1| \times_{|S|} |X_2|}_{\text{fiber product in Sets (or in top. spaces)}}$$

This map is usually NOT injective!

<u>Ex</u> $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ via $z \otimes w \mapsto (zw, \bar{z}w)$

$\Rightarrow$ For $X = \mathrm{Spec}\, \mathbb{C} \to S = \mathrm{Spec}\, \mathbb{R}$,

$|X \times_S X| = |\mathrm{Spec}(\mathbb{C} \times \mathbb{C})| = 2$ points,

$|X| \times_{|S|} |X| = pt \times_{pt} pt = 1$ point.

Lemma For any S-schemes X, Y, the map

$$|X \times_S Y| \twoheadrightarrow |X| \times_{|S|} |Y| \text{ is surjective.}$$

Pf. Let $(x,y) \in |X| \times_{|S|} |Y|$.

Pick affine open nbhoods:

$$\begin{array}{ccccc} x = \mathfrak{p}_x & \in & \operatorname{Spec} A & \subseteq & X \\ y = \mathfrak{p}_y & \in & \operatorname{Spec} B & \subseteq & Y \\ \downarrow & & \downarrow & & \downarrow \\ s = \mathfrak{p}_s & \in & \operatorname{Spec} R & \subseteq & S \end{array}$$

We get

$$\begin{tikzcd} A \arrow[r] & A/\mathfrak{p}_x \arrow[r] & \kappa(x) \arrow[dr] & \\ R \arrow[u] \arrow[d] \arrow[r] & R/\mathfrak{p}_s \arrow[u] \arrow[d] \arrow[r] & \kappa(s) \arrow[u] \arrow[d] \arrow[r] & \left(\kappa(x) \otimes_{\kappa(s)} \kappa(y)\right)/\mathfrak{m} =: K \\ B \arrow[r] & B/\mathfrak{p}_y \arrow[r] & \kappa(y) \arrow[ur] & \end{tikzcd}$$

any max. ideal

Apply $\operatorname{Spec}(-)$ to get

$$\begin{tikzcd} & \operatorname{Spec} \kappa(x) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(K) \arrow[ur] \arrow[r] \arrow[dr] & \operatorname{Spec} \kappa(s) \arrow[r] & S \\ & \operatorname{Spec} \kappa(y) \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

$\rightsquigarrow$ image of $|\operatorname{Spec} K|$ gives a point in $|X \times_S Y|$ mapping to (x,y).

$\square$

Fiber products are everywhere in alg geometry!

Def Any morphism of schemes $f : T \to S$ gives a base change functor

$$\begin{aligned} (-)_T : \mathrm{Sch}_S &\longrightarrow \mathrm{Sch}_T \\ X &\longmapsto X_T := X \times_S T \\ f &\longmapsto f_T := (f, \mathrm{id}_T) \end{aligned}$$

$$\begin{tikzcd} Y_T \arrow[r, "f_T"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d] \\ Y \arrow[r, "f"'] & X \arrow[r] & S \end{tikzcd}$$

Ex $\mathbb{A}^1_T$ = base change of $\mathbb{A}^1$ via $T \to S = \operatorname{Spec} \mathbb{Z}$.

Def We say a class of morphisms of schemes is stable under base change if $\forall f : X \to S$ in that class and any morphism $T \to S$, $f_T : X_T \to T$ is again in the same class.

Ex The following classes are stable under base change:

a) open / closed immersions

b) morphisms of finite type

c) affine / integral / finite morphisms

Pf. Let $g \in \mathrm{Hom}_{Sch}(T, S)$.

a) $j : U \hookrightarrow S$ open

$\Rightarrow \quad U_T = U \times_S T = g^{-1}(U) \subseteq T$ open

by our construction of fiber products

$i : X \hookrightarrow S$ closed

$\Longrightarrow$ wlog $X = V(I) \subseteq S = \mathrm{Spec}\, A$ & $T = \mathrm{Spec}\, B$

$\Longrightarrow \quad X_T = \mathrm{Spec}(B \otimes_A A/I) \subseteq \mathrm{Spec}(B)$ closed

(since $B \twoheadrightarrow B \otimes_A A/I$ epi)

$\square$

b), c) similar.

Ex Being injective is NOT stable under base change:

Take $X = \mathrm{Spec}\, \mathbb{C} \to S = \mathrm{Spec}\, \mathbb{R}$ & $T = \mathrm{Spec}\, \mathbb{C}$ ...

Ex Being surjective IS stable under base change:

Def The <u>scheme-theoretic fiber</u> of $f : X \to S$ over a point $p \in S$ ($\hat{=}$ morphism $\mathrm{Spec}\, \kappa(p) \to X$) is the scheme $X_p := X \times_S \mathrm{Spec}\, \kappa(p)$

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \mathrm{Spec}\, \kappa(p) \arrow[r] & S \end{tikzcd}$$

Lemma a) The morphism $X_p \to X$ induces a homeomorphism $|X_p| \xrightarrow{\sim} f^{-1}(p) \subseteq |X|$.

b) For $X = \mathrm{Spec}\, B \to S = \mathrm{Spec}\, A$ affine we have $X_p \simeq \mathrm{Spec}(B/\mathfrak{p}B)_S$ w/ $p \hat{=} \mathfrak{p} \trianglelefteq A$, $S = A \setminus \mathfrak{p}$

c) For any $g : T \to S$ and any $q \in T$ w/ $p = g(q)$, we have $(X \times_S T)_q \xrightarrow{\sim} X_p \times_{\kappa(p)} \mathrm{Spec}\, \kappa(q)$

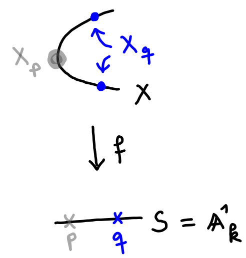
($\Rightarrow$ surjectivity is stable under base change ...)

Pf. b) $X_p = \mathrm{Spec}\, B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \mathrm{Spec}\, S^{-1} B/\mathfrak{p}B$

a) follows from b) by reduction to affine case.

c) clear by transitivity $(X \times_S T) \times_T W \simeq X \times_S W$ $\square$

Ex $f : X = \operatorname{Spec} k[x,y]/(y^2 - x) \to S = \operatorname{Spec} k[x]$



For $p \in k$ consider the closed pt $m_p = (x-p) \in |S|$ w/ residue field $\kappa(p) = k[x]/(x-p) \simeq k$

$$\Rightarrow X_p = \operatorname{Spec}\left(k[y]/(y^2 - p)\right)$$

$$= \begin{cases} \operatorname{Spec} k(\sqrt{p}) & \text{if } \sqrt{p} \notin k \\ \operatorname{Spec} k[y]/(y-\sqrt{p})(y+\sqrt{p}) & \text{if } \sqrt{p} \in k \end{cases}$$

Thus for k perfect:

a) $\operatorname{char} k \neq 2 \Rightarrow X_p \simeq \begin{cases} \operatorname{Spec} k(\sqrt{p}) & (\sqrt{p} \notin k) \\ \operatorname{Spec} k \sqcup \operatorname{Spec} k & (\sqrt{p} \in k^*) \\ \operatorname{Spec} k[y]/(y^2) & (p = 0) \end{cases}$

b) $\operatorname{char} k = 2 \Rightarrow X_p \simeq \operatorname{Spec} k[y]/(y-\sqrt{p})^2$ for all $p \in k$

$\Rightarrow$ fibers over all closed points are non-reduced though generic fiber $X_\eta \simeq \operatorname{Spec} k(\sqrt{x})$ is reduced!

$\uparrow$ inseparable extension of $k(x)$

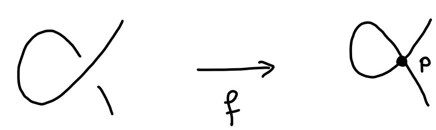
Prop $f : X \to S$ finite surjective morphism of finite type between integral schemes sth $K(X)/K(S)$ is a separable extension of degree d.

$\Rightarrow \exists$ open dense $U \subseteq S$ $\forall p \in S$:

$$d = \sum_{q \in |X_p|} [\kappa(q) : \kappa(p)].$$

Pf. As in section about finite morphisms, but replace naive fibers by scheme-theoretic fibers. $\square$

Rem In general we cannot take $U = S$:



$\deg(f) = 1$ but $f^{-1}(p) =$ two points ...

3. Separated morphisms

Zariski topology is VERY coarse: Almost never Hausdorff!

Goal Replace "Hausdorff" by a more suitable notion ...

Rem A top space X is Hausdorff iff the diagonal

$\Delta \subset X \times X$ is closed (wrt the product topology).

Pf. $p \neq q \in X \iff (p,q) \in X \times X \setminus \Delta$

$\exists U_p \ni p,\ \exists U_q \ni q$ open w/ $U_p \cap U_q = \emptyset$ $\iff$ $\exists U = U_p \times U_q \ni (p,q)$ open w/ $U \cap \Delta = \emptyset$ $\square$

(products $U_p \times U_q$ form a nbhood basis for the product topology on $X \times X$)

The second property has an analog for schemes:

Def For a scheme $X \in Sch_S$ consider

$$\Delta_{X/S} := (id_X, id_X) : X \longrightarrow X \times_S X.$$

We call the morphism $X \to S$ separated or say that X is separated over S if $\Delta_{X/S}$ is a closed immersion.

All affine schemes are separated:

Lemma X affine (over S) $\Rightarrow$ X separated (over S)

Pf. Wlog $S = \operatorname{Spec} A$ affine & $X = \operatorname{Spec} B$.

Then $\Delta_{X/S} : \operatorname{Spec} B \longrightarrow \operatorname{Spec} B \otimes_A B$

is induced by

$$\begin{array}{ccc} B \otimes_A B & \longrightarrow & B \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sum_i b_i' \otimes b_i'' & \longmapsto & \sum_i b_i' b_i'' \end{array}$$

which is surjective $\Rightarrow$ $\Delta_{X/S}$ closed immersion. $\square$

Cor For any $X \in Sch_S$,

$\Delta_{X/S} : X \to X \times_S X$ is a locally closed immersion.

Pf. Pick an open cover $X = \bigcup_i U_i$ w/ U_i affine over S

$\Rightarrow$ $\Delta_{X/S}$ factors over the open $V := \bigcup_i \underbrace{U_i \times_S U_i}_{=: V_i} \subseteq X \times_S X$:

$$\begin{tikzcd} X \arrow[rr, "\Delta_{X/S}"] \arrow[dr, "\exists!\, \delta"'] & & X \times_S X \\ & V \arrow[ur, hook, "\text{open}"'] & \end{tikzcd}$$

But $\delta^{-1}(V_i) = U_i \to V_i = U_i \times_S U_i$ closed immersion by the lemma, and $V = \bigcup_i V_i \Rightarrow \delta$ closed imm. $\square$

Non-examples come from bad gluing of charts:

$\underline{\text{Ex}}$ $X = U_1 \cup_U U_2$ w/ $U_1 = U_2 = \mathbb{A}^1_k \supseteq U = \mathbb{A}^1_k \setminus \{0\}$

affine line w/ doubled origin over a field k

$\Rightarrow$ X is NOT separated over $S = \operatorname{Spec} k$:

Pf. On the open $V := U_1 \times_S U_2 \subseteq Y := X \times_S X$

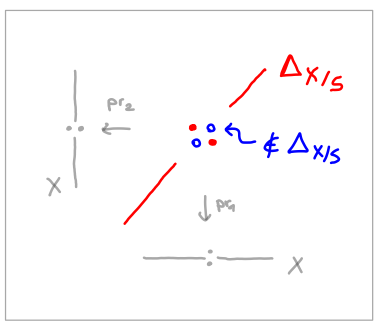
we have:

$$\begin{array}{l} \Delta^{-1}_{X/S}(V) = \mathbb{A}^1_k \setminus \{0\} \\ \quad \Big\downarrow \ \hookleftarrow \text{not closed} \\ U_i = \mathbb{A}^1_k \qquad (i = 1, 2) \end{array}$$

$\Rightarrow$ $\Delta_{X/S}$ is not a closed immersion! $\square$

$\underline{\text{NB}}$. $X \times_S X$ has 4 points above the 2 origins in X, only two are in $\Delta_{X/S}$ but two others are in $U_1 \times_S U_2$ resp $U_2 \times_S U_1$

"off-diagonal charts"



More generally: Let R be a ring.

$\underline{\text{Prop}}$ (separatedness via chart overlaps) TFAE for $X \in Sch_R$:

a) X is separated over $S = \operatorname{Spec} R$

b) $\forall$ open $U, V \subseteq X$ w/ U, V affine:

$U \cap V$ affine and

$\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \twoheadrightarrow \mathcal{O}_X(U \cap V)$ epi.

c) $\exists$ open cover $X = \bigcup_{i \in I} U_i$ w/ U_i affine

sth $\forall i, j \in I$:

$U_{ij} := U_i \cap U_j$ affine and

$\mathcal{O}_X(U_i) \otimes_R \mathcal{O}_X(U_j) \twoheadrightarrow \mathcal{O}_X(U_{ij})$ epi.

Pf. a) $\Rightarrow$ b):

$$U \cap V = \underbrace{\Delta_{X/S}(X)}_{\substack{\text{closed in} \\ X \times_S X \text{ by a)}}} \cap \underbrace{U \times_S V}_{\substack{\text{affine open} \\ \text{in } X \times_S X}} \Rightarrow U \cap V \hookrightarrow U \times_S V$$

closed in an affine

$$\Rightarrow U \cap V \text{ also affine } \& \ \underbrace{\mathcal{O}_{X \times_S X}(U \times_S V)}_{= \ \mathcal{O}_X(U) \otimes_R \mathcal{O}_X(V)} \twoheadrightarrow \mathcal{O}_X(U \cap V)$$

b) $\Rightarrow$ c): trivial c) $\Rightarrow$ a):

$U_{ij} := \Delta_{X/S}^{-1}(U_i \times_S U_j) \longrightarrow U_i \times_S U_j$

c) $\Rightarrow$ this is a closed immersion for all i,j

$\Rightarrow$ $\Delta_{X/S}$ is a closed immersion $\Rightarrow$ a) $\square$

<u>Rem</u> To apply this for S not affine, note that TFAE:

a) $f : X \to S$ separated

b) $\exists$ open affine cover $S = \bigcup_i S_i$

w/ $X_i := f^{-1}(S_i) \to S_i$ separated for all i.

Pf. Use $X \times_S X = \bigcup_i X_i \times_{S_i} X_i$. $\square$

<u>Cor</u> $\mathbb{P}^n_S$ is separated over S.

Pf. Wlog $S = \operatorname{Spec} R$ affine

$\mathbb{P}^n_S = \bigcup_{i=0}^n U_i$ w/ $U_i \simeq \mathbb{A}^n_R \supseteq U_{ij} = D(\frac{x_j}{x_i})$

$\Rightarrow$ previous proposition applies. $\square$

Then all quasiprojective schemes over S are separated, in fact almost all schemes & morphisms in practice:

<u>Prop</u> a) If $X \in Sch_S$ is separated over S, then so is any locally closed subscheme $Y \hookrightarrow X$.

b) Separated morphisms are stable under

i) Composition:

$f : X \to S$ separated, $g : Y \to X$ separated $\Rightarrow$ $f \circ g : Y \to S$ separated

ii) base change:

$f : X \to S$ separated, $T \in Sch_S$

$\Rightarrow$ $f_T : X \times_S T \to T$ separated

Pf. b.i) Consider

closed since $\Delta_{X/S}$ is closed

$$\begin{tikzcd}
Y \arrow[r, hook, "\Delta_{Y/X}"'] & Y \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \times_S Y \arrow[d] \\
& X \arrow[r, hook, "\Delta_{X/S}"'] & X \times_S X
\end{tikzcd}$$

$\Rightarrow$ $Y \hookrightarrow Y \times_S Y$ closed immersion, ie $f \circ g$ separated

a) Closed immersions are affine, hence separated.

Open immersions are separated (trivially).

$\Rightarrow$ claim follows from b.i)

b.ii) Consider

closed since $\Delta_{X/S}$ is closed

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d] & X_T \times_T X_T = (X \times_S X) \times_S T \arrow[d] \\
X \arrow[r, hook, "\Delta_{X/S}"'] & X \times_S X
\end{tikzcd}$$

$\square$

Morphisms from reduced to separated schemes are determined by their restriction to open dense subsets:

Prop Let $f_1, f_2 \in \mathrm{Hom}_{Sch_S}(Y, X)$

w/ Y reduced & X separated over S.

If $\exists$ open dense $U \subseteq Y$ w/ $f_1|_U = f_2|_U$

then

$$f_1 = f_2.$$

Pf. Define a closed subscheme $Z \hookrightarrow Y$ by base change of $\Delta_{X/S}$ via (f_1, f_2):

$$\begin{tikzcd}
 & & U \arrow[d, "\text{open}"] \arrow[dll, dashed, "\exists!"'] \\
Z \arrow[rr, hook, "\text{closed}"] \arrow[d] & & Y \arrow[d, "(f_1,f_2)"] \\
X \arrow[rr, hook, "\Delta_{X/S}"'] & & X \times_S X
\end{tikzcd}$$

closed since X separated $/S$

Since $f_1|_U = f_2|_U$,

the inclusion $U \hookrightarrow Y$

factors over $Z \hookrightarrow Y$.

$\Rightarrow$ $|Z| \subseteq |Y|$ closed

and contains the

dense open $|U| \subseteq |Y|$

$\Rightarrow$ $|Z| = |Y|$

$\Rightarrow$ writing $Z = V(\mathcal{J}) \hookrightarrow Y$ w/ $\mathcal{J} \trianglelefteq \mathcal{O}_Y$,

we have $\mathcal{J} \subseteq \mathrm{Rad}(\mathcal{O}_Y) \underset{\substack{\uparrow \\ Y \text{ reduced}}}{=} (0)$

$\Rightarrow$ $Z = Y$ $\square$

Ex a) $f_1, f_2 : Y := \mathbb{A}^1_k \xrightarrow{\text{chart inclusions}} X :=$ affine line w/ double origin

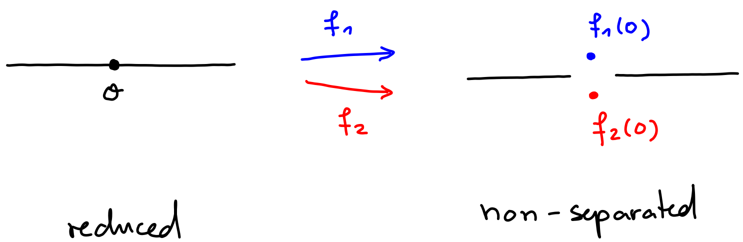
b) $f_1, f_2 : Y = \mathrm{Spec}\, k[x,y]/(xy, y^2) \to X := \mathrm{Spec}\, k[t]$

w/ $f_1^{\#}(t) := x$, $f_2^{\#}(t) := x + y$

$\Rightarrow$ In both cases $f_1 \neq f_2$ but $f_1|_U = f_2|_U$ on a dense open U

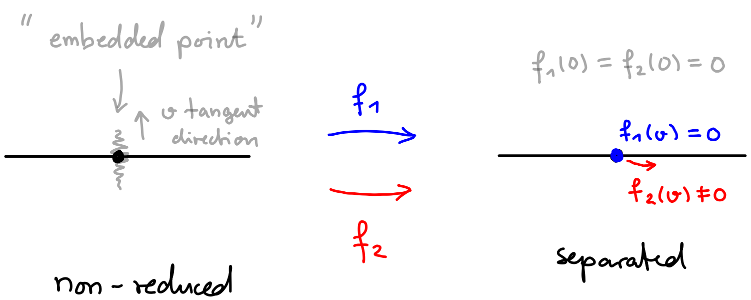
Cartoon:

a)



0
f_1
f_2
$f_1(0)$
$f_2(0)$
reduced
non-separated

b)



"embedded point"
v tangent direction
f_1
f_2
$f_1(0) = f_2(0) = 0$
$f_1(v) = 0$
$f_2(v) \neq 0$
non-reduced
separated

4. Proper morphisms

Zariski topology is VERY coarse: All affine opens are qc!

<u>Goal</u> Replace "qc" by a more suitable notion ...

<u>Rem</u> A locally cpact Hausdorff space X is compact iff for all locally cpact Hausdorff spaces T the projection $\pi: X \times T \to T$ is a closed map.

Pf. "$\Rightarrow$" Assume X cpact. Let $Z \subseteq X \times T$ closed.

T locally cpact $\Rightarrow$ enough to show $\pi(Z) \cap K \subseteq K$ is closed for all compact $K \subseteq T$

But $\pi(Z) \cap K = \pi(\underset{\text{closed}}{Z} \cap \underbrace{\pi^{-1}(K)}_{= X \times K \text{ cpact}}) = \pi(\text{cpact})$ is cpact

$\Rightarrow \pi(Z) \cap K$ closed

"$\Leftarrow$" Assume X not cpact, $T := X \cup \{\infty\}$ 1-pt cpactificat[n] (recall X loc cpt Hausdorff)

Take $Z := \text{graph}(X \hookrightarrow T) \subseteq X \times T$

$\Rightarrow Z \subseteq X \times T$ closed, $\pi(Z) = X \hookrightarrow T$ not closed. $\square$

The second property has an analog for schemes:

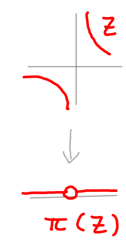
<u>Def</u> A morphism $f: X \to S$ of schemes is called <u>universally closed</u> if for all $T \in Sch_S$ the morphism $f_T: X_T = X \times_S T \to T$ is closed.

A morphism is called <u>proper</u> if it is of finite type, separated & universally closed.

<u>Ex</u> $X = \mathbb{A}^1_S \to S$ is of fin. type & separated but NOT proper:

e.g. $Z = V(xy-1) \subseteq X \times_S Y$ for $Y = \mathbb{A}^1_S$

has $\pi(Z) = Y \setminus \{0\} \subseteq Y$ not closed.



<u>Ex</u> Any integral morphism $f: X \to S$ is separated & universally closed (since integral morphisms are closed and stable under base change).

It is proper iff it is finite

(since finite type + integral = finite).

Prop a) If $X \in Sch_S$ is proper over S, then so is any closed subscheme $Y \hookrightarrow X$.

b) Proper morphisms are stable under

i) composition

ii) base change

c) Being proper is local on the target:

For $f: X \to S$ & an open cover $S = \bigcup_i U_i$,

f proper $\iff \forall i:\ f^{-1}(U_i) \to U_i$ proper

d) For $h = g \circ f$ w/ $f: X \to S$, $g: S \to T$ we have:

h proper & g separated $\Rightarrow$ f proper

Pf. Exercise. For d) show:

g separated $\Rightarrow (id, f): X \to X \times_T S$ closed immersion

Then: h proper $\underset{bii)}{\Rightarrow} X \times_T S \to S$ proper $\underset{bi),\,a)}{\Rightarrow} f = pr \circ (id, f)$ proper $\square$

We'll see $\mathbb{P}^n_S$ & hence all closed $X \hookrightarrow \mathbb{P}^n_S$ are proper over S

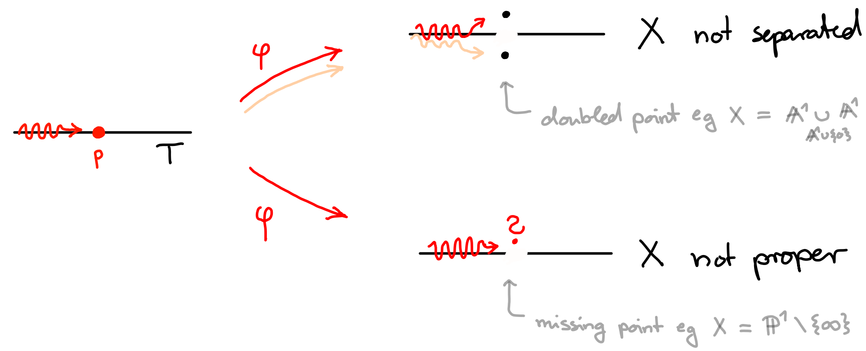
Slogan: Projective schemes are proper!

Intuitively:

X separated $\iff \forall$ "test scheme" $\varphi: U = T \setminus \{p\} \to X$

$\exists$ at most one "$\lim_{q \to p} \varphi(q)$" $\in X$

X proper $\iff \cdots \exists$ unique "$\lim_{q \to p} \varphi(q)$" $\in X$



Roughly speaking:

"test scheme": $\varphi: U \longrightarrow X$, $U \subset T = \operatorname{Spec}(\text{valuation ring})$

"limit": Extension to a morphism $\tilde{\varphi}: T \to X$

To make this work we recall some basic notions:

Def A domain R is a <u>valuation ring</u> if the following equivalent conditions hold, where $K := \mathrm{Quot}(R)$:

a) $\forall a \in K^*$ we have $a \in R$ or $a^{-1} \in R$

b) We have $R = \{ a \in K^* \mid v(a) \geq 0 \} \cup \{0\}$ for some <u>valuation</u> $v : K^* \longrightarrow G$, ie a gp homom. to a totally ordered abelian gp G sth $\forall a, b \in K^*$ w/ $a + b \neq 0$,

$$v(a+b) \geq \min\{ v(a), v(b) \}.$$

[a) $\Rightarrow$ b): take $G := K^*/R^*$ w/ $a \geq b :\Leftrightarrow \frac{a}{b} \in R$

b) $\Rightarrow$ a): use $v(a^{-1}) = -v(a)$]

Ex Any DVR, e.g. $R = k[[t]],\ k[t]_{(t)},\ \mathbb{Z}_{(p)}, \ldots$

Rem a) Any valuation ring R is local w/ max ideal $m_R = \{ a \in R \mid v(a) > 0 \}$

b) Let K be a field, $\mathcal{L} := \{ \text{local rings } S \subsetneq K \}$.
For $R, S \in \mathcal{L}$ we say that S <u>dominates</u> R if $R \subseteq S$ & $m_R = m_S \cap R$.

$\Rightarrow$ partial order on $\mathcal{L}$ whose maximal elements are the valuation rings R w/ $\mathrm{Quot}(R) = K$.

$\exists$ many morphisms from "test schemes":

Lemma For any local ring R & any scheme X,

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, X) = \{ (p, \varphi) \mid p \in |X|,\ \varphi : \mathcal{O}_{X,p} \to R \text{ local hom.} \}$$

Pf. Clear for X affine, so only need to see any $f : \mathrm{Spec}\, R \to X$ factors over some open affine $U \subseteq X$. Pick any $U \ni f(\text{closed pt})$

$\Rightarrow f^{-1}(U) \subseteq \mathrm{Spec}\, R$ open & contains the closed pt

$\Rightarrow f^{-1}(U) = \mathrm{Spec}\, R$, hence f factors over U. $\square$

Cor a) For any $x, y \in |X|$ w/ $y \in \overline{\{x\}}$, $\exists$ valuation ring R & morphism $\mathrm{Spec}\, R \to X$

$$\begin{array}{ccc} \mathrm{Spec}\, R & \longrightarrow & X \\ \text{closed pt} & \longmapsto & y \\ \text{generic pt} & \longmapsto & x \end{array}$$

b) More generally:

$f : X \to S$
$x \in |X|,\ y \in \overline{\{f(x)\}} \subseteq |S|$

$\Rightarrow \exists$ valuation ring R w/ $K := \mathrm{Quot}\, R$ fitting in a diagram:

$$\begin{array}{ccc} * & \longmapsto & x \\ \cap & & \cap \\ \mathrm{Spec}\, K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & S \\ \cup & & \cup \\ \text{closed pt} & \longmapsto & y \end{array}$$

Pf. a) follows from b) with $f = id : X \to S = X$

b) Replace X by $\overline{\{x\}}_{red}$ $\Rightarrow$ wlog S, X integral
S by $\overline{\{f(x)\}}_{red}$ & $f : X \to S$ dominant

$f : X \to S$ gives $f^* : K(S) \hookrightarrow K(X) = \kappa(x)$

$$\begin{tikzcd} K(S) \arrow[r, hook] & K(X) \\ \mathcal{O}_{S,y} \arrow[u, phantom, "\cup"] \arrow[ur, hook] & \end{tikzcd}$$

Let R be a valuation ring of $K(X)$ dominating $\mathcal{O}_{S,y}$

$\Rightarrow$ morphism

$$\begin{tikzcd} \text{closed pt} \arrow[r, mapsto] & y \\ \mathrm{Spec}\, R \arrow[u, phantom, "\cap"] \arrow[r] & S \arrow[u, phantom, "\cap"] \\ \mathrm{Spec}\, K \arrow[u, phantom, "\cup"] \arrow[r] & X \arrow[u] \\ * \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & x \arrow[u, phantom, "\in"] \end{tikzcd}$$

w/ $K = \kappa(x)$ $\square$

<u>Def</u> We call $s \in |S|$ a <u>specialization</u> of $t \in |S|$ if $s \in \overline{\{t\}}$.

A subset $Z \subseteq |S|$ is <u>stable under specialization</u> if for all $t \in Z$ we have $\overline{\{t\}} \subseteq Z$.

<u>Exercise</u> For any qc morphism $f : X \to Y$ of schemes, $f(|X|) \subseteq |Y|$ is closed iff it is stable under specializat^n

<u>Thm</u> (Valuative criterion) Let $f : X \to S$ be a finite type morphism w/ $\Delta_{X/S} : X \to X \times_S X$ qc (e.g. finite type morphism w/ S Noetherian).

a) f is separated iff the following holds:

$\forall$ valuation rings R & $K = \mathrm{Quot}(R)$,

$\forall \varphi : \mathrm{Spec}\, K \to X$ & $\mathrm{Spec}\, R \to S$

making the square

$$\begin{tikzcd} U = \mathrm{Spec}\, K \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ T = \mathrm{Spec}\, R \arrow[r] \arrow[ur, dashed, "\exists ? \tilde{\varphi}"] & S \end{tikzcd}$$

commute, $\exists$ at most one $\tilde{\varphi} : \mathrm{Spec}\, R \to X$ such that the entire diagram commutes.

b) f is proper iff in the above situation there always $\exists$ unique such $\tilde{\varphi}$.

<u>Rem</u> i) For points over S this says $X(R) \to X(K)$ is $<$ injective / surjective

ii) In Noetherian case, taking R a DVR is enough (see exercises)

Before proving the thm, let's see how to use it:

<u>Cor</u> $\mathbb{P}^n_S \to S$ is proper.

Pf. Properness stable under b.ch. $\Rightarrow$ wlog $S = \operatorname{Spec} \mathbb{Z}$.

Let R be a valuation ring & $K = \operatorname{Quot}(R)$.

Let $\varphi \in \mathbb{P}^n(K)$: $\quad \underset{\mathfrak{m}}{*} \longmapsto \underset{\mathfrak{m}}{p}$

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\varphi"] \arrow[d] & \mathbb{P}^n \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists ?\, \tilde{\varphi}"] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

Goal: $\exists\, \tilde{\varphi}$ as shown.

Recall $\mathbb{P}^n = \bigcup_{j=0}^{n} U_j$ w/ $U_j = \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}]$

If $p \in \mathbb{P}^n \setminus U_j$ for some j: Done by induction on n

since $\mathbb{P}^n \setminus U_j \simeq \mathbb{P}^{n-1}$

So wlog $p \in \bigcap_{j=0}^{n} U_j$

$\Longrightarrow \frac{x_i}{x_j} \in \mathcal{O}^*_{U_j,p} = \mathcal{O}^*_{\mathbb{P}^n,p}$ for all i,j

$\Longrightarrow f_{ij} := \varphi^*(\frac{x_i}{x_j}) \in K^*$ for all i,j

Note: $f_{ij} \cdot f_{jk} = f_{ik} \quad \forall\, i,j,k$

Let $v : K^* \to G$ be a valuation w/ valuat^n ring R.

Pick j sth $v(f_{j0}) = \min\{ v(f_{i0}) \mid i = 0, \dots, n\}$

$\Rightarrow \forall i: \; v(f_{ij}) = v(\frac{f_{i0}}{f_{j0}}) = v(f_{i0}) - v(f_{j0}) \geq 0$

$\Rightarrow f_{ij} \in R$ for all i (and this chosen j)

$\Rightarrow$ get a point $\tilde{\varphi} : \operatorname{Spec} R \to U_j \subseteq \mathbb{P}^n$

via $\quad \tilde{\varphi}^\#(\frac{x_i}{x_j}) := f_{ij} \in R$ $\quad\square$

Closedness of $\mathbb{P}^n_S \to S$ is nontrivial:

<u>Ex</u> Take $S = \operatorname{Spec} k[a_0,..,a_m,b_0,..,b_n]$, say $k = \bar{k}$

$$Z := V_+\Big(\sum_i a_i x_0^i x_1^{m-i}, \; \sum_j b_j x_0^j x_1^{n-j}\Big) \xhookrightarrow[\text{closed}]{} \mathbb{P}^1_S \xrightarrow[\text{proper}]{\pi} S$$

$\Longrightarrow W := \pi(Z(k)) \subseteq S(k)$ closed

$= \{ (f,g) \in k[t]_{\leq m} \times k[t]_{\leq n} \mid f,g \text{ have a common zero} \}$

Explicitly: $W = V_+(R)$ w/ $R :=$ "Resultant(f,g)" $\in k[a_0,..,a_m,b_0,..,b_n]$

Pf of valuative criterion. a1) Suppose f separated.

Let $\tilde\varphi_1, \tilde\varphi_2 : T \to X$ be two extensions of φ

$\Rightarrow \tilde\varphi_1|_U = \tilde\varphi_2|_U$ for the dense $U \subseteq T$

$\Rightarrow \tilde\varphi_1 = \tilde\varphi_2$ (since X separated over S & T reduced)

a2) Conversely assume uniqueness in a).

$\Delta_{X/S} : X \to X \times_S X$ quasicompact

$\Rightarrow$ enough to show that $\Delta := \Delta_{X/S}(|X|)$ is stable under specialization (exercise on p. 7)

Let $t \in \Delta$, $s \in \overline{\{t\}}$, and $Z := \overline{\{t\}}_{red} \subseteq X \times_S X$

$\Rightarrow \mathcal{O}_{Z,s}$ local domain w/ $K := \mathrm{Quot}(\mathcal{O}_{Z,s}) = \kappa(t)$

$\Rightarrow \exists$ valuation ring $R \subseteq K$ dominating $\mathcal{O}_{Z,s}$

$\Rightarrow$ morphism $g : \operatorname{Spec} R \longrightarrow Z \subseteq X \times_S X \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} X$

Uniqueness in a) gives $\tilde\varphi_1 = \tilde\varphi_2$ for $\tilde\varphi_i := pr_i \circ g$

$\Rightarrow g$ factors over $\Delta_{X/S}$

$\Rightarrow |Z| \subseteq \Delta$, hence $s \in \Delta$ $\Rightarrow$ done

b1) Suppose f proper and

$$\begin{tikzcd} U = \operatorname{Spec} K \arrow[r, "\varphi"] \arrow[d, hook, "j"'] & X \arrow[d, "f"] \\ T = \operatorname{Spec} R \arrow[r] & S \end{tikzcd}$$

Consider $(j, \varphi) : U \longrightarrow X_T := T \times_S X$

Let $\eta \in |X_T|$ be the image of the unique pt $\xi \in |U|$.

Put $Z := \overline{\{\eta\}}_{red} \subseteq X_T$

f proper $\Rightarrow f_T : X_T \to T$ closed

$\Rightarrow f_T(|Z|) \subseteq |T|$ closed

But by construction $\xi = f_T(\eta) \in f_T(|Z|)$

& ξ is the generic point of $T = \operatorname{Spec} R$

$\Rightarrow f_T(|Z|) = |T|$

$\Rightarrow \exists\, s \in |Z|$ with $f_T(s)$ = the closed pt of T

$\Rightarrow$ $\begin{array}{ccccc} U & \to & Z & \to & T \\ \| & & \cup\!\!\!\text{ı} & & \cup\!\!\!\text{ı} \\ \operatorname{Spec} K & & s & \longmapsto & \text{closed pt} \end{array}$ gives $\underbrace{R}_{\text{valuation ring w/ Quot } R = K} \underbrace{\to}_{\text{local hom.}} \mathcal{O}_{Z,s} \hookrightarrow K$

$\Rightarrow R = \mathcal{O}_{Z,s}$ & we get $T = \operatorname{Spec} R \to X_T$

Composing w/ $X_T \to X$ gives $\tilde\varphi : T \to X$ $\Rightarrow$ done.

b2) Conversely, suppose existence in b).

Want: $f : X \to S$ is universally closed.

Existence in b) is stable by base change

$\Longrightarrow$ enough to show $f : X \to S$ is closed

$\Longrightarrow$ enough to show $f(|X|)$ stable under specializat^{n}

(since f is finite type, hence quasicompact)

Let $x \in |X|$ and $s \in \overline{\{f(x)\}}$.

Pick a "test diagram"

$$\begin{tikzcd}
* \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & x \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\operatorname{Spec} K \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists \tilde{\varphi}"] & S \\
\text{closed pt} \arrow[r, mapsto] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & s \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

By assumption $\exists\, \tilde{\varphi}$

$\Longrightarrow$ $y := \tilde{\varphi}(\text{closed pt})$ satisfies $f(y) = s$

$\Longrightarrow$ $s \in f(|X|)$ as required $\square$

5. Chow's lemma

Why proper schemes & not just projective ones?

- properness is often easier to verify (e.g. in moduli theory)
- in dim ≥ 3 $\exists$ proper schemes that are NOT projective (Hironaka)

But:

Thm (Chow's lemma) Let $f : X \to S$ be separated of finite type w/ S Noetherian.

a) $\exists$ diagram

$$\begin{tikzcd} X' \arrow[r, hook, "\imath"] \arrow[d, "\pi"'] & \mathbb{P}^n_S \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd} \quad \text{in } Sch_S$$

w/ $\imath$ an immersion, π proper surjective,

and $\exists\, U \subseteq X$ open dense w/ $\pi^{-1}(U) \xrightarrow{\sim} U$ iso.

b) f proper $\iff$ $\imath$ closed immersion

Slogan: Proper schemes can be made projective by a birational proper modification π!

Pf. ① Reduction to X irreducible:

Let $X_1, \dots, X_m \subseteq X$ be the irred cpt^s of X

(X is Noetherian since it is finite type over a Noetherian S)

w/ scheme structure defined by

$X_i :=$ scheme-theoretic closure of V_i in X;

$:= \underset{\substack{\uparrow \\ \text{scheme-theoretic image}}}{\mathrm{im}}(V_i \hookrightarrow X) \underset{\substack{\uparrow \\ \text{closed subscheme}}}{\subseteq} X$

for an open affine $V_i \subseteq X \setminus \bigcup_{j \neq i} X_j$

(the scheme structure may depend on V_i)

Suppose $\forall i$ $\exists$ $\begin{tikzcd} X_i' \arrow[r, hook, "\imath_i"] \arrow[d, "\pi_i"] & \mathbb{P}^{n_i}_S \\ X_i \end{tikzcd}$ w/ π_i proper surjective & iso over a dense open $U_i \subseteq X_i$

Then

- $X' := \bigsqcup_i X_i' \overset{\exists}{\hookrightarrow} \mathbb{P}^n_S$ (some n, exercise)
- $\pi := \bigsqcup_i \pi_i : X' \to X$ is proper surjective & restricts to an iso over $U := \bigcup_i U_i \cap V_i$.

② Construction of $\pi : X' \to X$:

Wlog X irreducible (step 1).

$f : X \to S$ fin. type

$\Rightarrow \exists$ open affine cover $X = \bigcup_{i=1}^{r} X_i$ w/

- $X_i \subseteq f^{-1}(S_i)$ w/ $S_i \subseteq S$ open affine
- $\exists\, \varphi_i : X_i \underset{\text{closed}}{\overset{\exists}{\hookrightarrow}} \mathbb{P}^{n_i}_{S_i} \underset{\text{open}}{\subseteq} \mathbb{P}^{n_i}_{S}$ over S

Let $\mathbb{P}_i :=$ scheme-theoretic image of $X_i \hookrightarrow \mathbb{P}^{n_i}_S$

On the overlap, have

$$U := \bigcap_i X_i \xhookrightarrow[\varphi = (\varphi_1, \dots, \varphi_r)]{} \mathbb{P} := \mathbb{P}_1 \times_S \cdots \times_S \mathbb{P}_r$$

Put

$X' :=$ scheme-theoretic image of $U \xhookrightarrow[(\text{incl}, \varphi)]{} X \times_S \mathbb{P}$

$\pi' := pr_1|_{X'} : \; X' \subseteq X \times_S \mathbb{P} \to X$

③ Properties of π :

i) π is proper since $\pi : X' \underset{\text{closed}}{\hookrightarrow} X \times_S \mathbb{P} \xrightarrow[\text{proper}]{} X$

ii) $\pi^{-1}(U) \xrightarrow{\sim} U$ iso :

$g = (id, \varphi) : U \hookrightarrow U \times_S \mathbb{P}$ closed immersion

(being the graph of a separated morphism),

so $g(U) =$ scheme-theoretic image of g

$= \pi^{-1}(U)$

iii) π is surjective :

By i) $\pi(X') \subseteq X$ is closed

By ii) $\pi(X')$ contains $\underset{\text{open dense}}{U \subseteq X}$

$\Rightarrow \pi(X') = X$

④ $\iota := pr_2|_{X'} : X' \to \mathbb{P}$ is an immersion

$$\left(\Rightarrow X' \hookrightarrow \mathbb{P} \hookrightarrow \mathbb{P}_1 \times_S \cdots \times_S \mathbb{P}_r \xrightarrow[\text{Segre}]{} \mathbb{P}^n_S \right) :$$

Recall $X = \bigcup_i X_i$ w/ $X_i \overset{\text{open}}{\hookrightarrow} \mathbb{P}_i$

Define V_i, W_i by the Cartesian diagram

$$\begin{tikzcd}
V_i \arrow[r, "\iota_i"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & W_i \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & X_i \arrow[d, hook] \\
X' \arrow[r, "\iota"'] & \mathbb{P} \arrow[r, "pr_i"'] & \mathbb{P}_i
\end{tikzcd}$$

Consider

$$h : W_i \xhookrightarrow[\text{closed}]{} X_i \times_S W_i \xhookrightarrow[\text{open}]{} X_i \times_S \mathbb{P} \xhookrightarrow[\text{open}]{} \overset{\overset{X'}{\cap|}}{X \times_S \mathbb{P}}$$

↳ as graph of the separated morphism $W_i \to X_i$

$\Rightarrow$ h: immersion & $h \circ \iota_i = \text{incl} : V_i \hookrightarrow X'$

$\Rightarrow$ ι_i immersion for all i

$\Rightarrow$ ι immersion since $\iota(X') \subseteq \bigcup_i W_i$

(exercise)

⑤ f proper $\iff$ ι closed immersion:

This follows from the equivalences

$f : X \to S$ proper

$\overset{(*)}{\iff}$ $f \circ \pi : X' \to S$ proper

$\quad ||$

$\quad pr \circ \iota : X' \to \mathbb{P}^n_S \xrightarrow[\text{separated}]{} S$

$\iff$ ι proper by "cancellation property" (see exercises),

$\iff$ ι closed immersion (being proper & an immersion)

For $(*)$:

"$\Rightarrow$" as $\pi : X' \to X$ is proper

"$\Leftarrow$" as π is surjective & f separated (see exercises)

$\square$

6. More about projective schemes

<u>Recall</u>: $\mathbb{A}^{n+1}_k \setminus \{0\} = \bigcup_{i=0}^{n} D(x_i)$ w/ $D(x_i) = \operatorname{Spec} k[x_0,\dots,x_n]_{x_i}$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$\mathbb{P}^n_k = \bigcup_{i=0}^{n} U_i$ w/ $U_i = \operatorname{Spec} k[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}]$

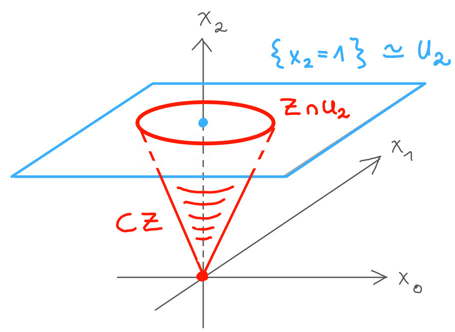
For $f_1,\dots,f_m \in k[x_0,\dots,x_n]$ homogenous

get a cone

$$CZ := V(f_1,\dots,f_m) \subset \mathbb{A}^{n+1}_k$$

over

$$Z := V_+(f_1,\dots,f_m) \subset \mathbb{P}^n_k$$



In charts:

$$Z \cap U_i = \operatorname{Spec} k[\tfrac{x_0}{x_i},\dots,\tfrac{x_n}{x_i}]/(f_1^{(i)},\dots,f_m^{(i)})$$

w/ $f^{(i)} := f(\frac{x_0}{x_i},\dots,\frac{x_n}{x_i})$ for $f \in k[x_0,\dots,x_n]_d$

$= x_i^{-d}\cdot f(x_0,\dots,x_n)$ homogenous of degree d

More generally:

<u>Def</u> A <u>graded ring</u> is a ring S w/ decomposition $S = \bigoplus_{d\in\mathbb{N}_0} S_d$ as abelian gp sth

$$S_d \cdot S_e \subseteq S_{d+e} \quad \forall\, d,e \in \mathbb{N}_0$$

($\Rightarrow$ $S_0 \subseteq S$ subring & all S_d are S_0-modules).

Elements of S_d are called <u>homogenous</u> of degree d.

An ideal $J \trianglelefteq S$ is called <u>homogenous</u> if the following equivalent conditions hold:

i) J is generated by homogenous elements

ii) $J = \bigoplus_{d\geq 0} J_d$ as S_0-module, w/ $J_d := J \cap S_d$.

($\Rightarrow$ $S/J = \bigoplus_{d\geq 0} S_d/J_d$ graded ring).

<u>Ex</u> $S = k[x_0,\dots,x_n]$ w/

$S_d := \{ f \in S$ homog. pol. of $\deg f = d \}$

$J \trianglelefteq S$ homogenous $\Leftrightarrow$ $V(J) \subseteq \mathbb{A}^{n+1}_k$ a cone

$\sqrt{J} \not\supseteq (x_0,\dots,x_n) \Leftrightarrow V(J) \not\subseteq \{0\}$

Def For $S = \bigoplus_{d \geq 0} S_d$ graded ring, let

- $S_+ := \bigoplus_{d > 0} S_d$ "irrelevant ideal"
- $\mathrm{Proj}(S) := \{ \mathfrak{p} \trianglelefteq S$ homogenous prime ideals w/ $S_+ \not\subseteq \mathfrak{p} \}$

For $J \trianglelefteq S$ homogenous let

$$V_+(J) := \{ \mathfrak{p} \in \mathrm{Proj}(S) \mid J \subseteq \mathfrak{p} \}.$$

Rem a) $V_+(J) \cup V_+(\mathcal{J}) = V_+(J \cdot \mathcal{J})$

b) $\bigcap_i V_+(J_i) = V_+\left(\sum_i J_i\right)$

c) $V_+((1)) = \emptyset$ & $V_+((0)) = \mathrm{Proj}(S)$

$\Rightarrow \exists!$ topology on $\mathrm{Proj}(S)$ w/ closed sets $V_+(J)$

For $f \in S$ homogenous put

$$D_+(f) := \{ \mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p} \}$$
$$= \mathrm{Proj}(S) \setminus V_+((f))$$

"distinguished open set in $\mathrm{Proj}(S)$"

Lemma a) The $D_+(f)$ w/ $f \in S_+$ homogenous form a basis of open sets in $\mathrm{Proj}\, S$.

b) For $f, g \in S_+$ we have:

$$D_+(g) \subseteq D_+(f) \iff g \in \sqrt{(f)}$$

Pf. a) Let $\mathfrak{p} \in U = \mathrm{Proj}(S) \setminus V_+(J) \subseteq \mathrm{Proj}(S)$.

$\Rightarrow J \not\subseteq \mathfrak{p}$, hence $\exists f \in J : f \notin \mathfrak{p}$

$\Rightarrow \exists f \in J_d,\ d > 0 : f \notin \mathfrak{p}$ (J homogenous & $S_+ \not\subseteq \mathfrak{p}$)

$\Rightarrow \mathfrak{p} \in D_+(f) \subseteq U$ ($D_+(f) \cap V_+(J) = \emptyset$ for $f \in J$)

b) $g \in \sqrt{(f)} \Rightarrow D_+(g) \subseteq D_+(f)$: clear

$D_+(g) \subseteq D_+(f) \Rightarrow g \in \sqrt{(f)} = \bigcap_{\mathfrak{p} \in V(f)} \mathfrak{p}$:

$\mathfrak{p} \in V(f) \Rightarrow \mathfrak{p}^h := \bigoplus_{d \geq 0} \mathfrak{p} \cap S_d$ homogenous w/ $\mathfrak{p}^h \subseteq \mathfrak{p}$

Case 1: $S_+ \subseteq \mathfrak{p}^h \Rightarrow g \in \mathfrak{p}^h$ as $g \in S_+$

Case 2: $S_+ \not\subseteq \mathfrak{p}^h \Rightarrow \mathfrak{p}^h \in V_+(f) \subseteq V_+(g) \Rightarrow g \in \mathfrak{p}^h$

$\square$

Thm $\exists$ natural sheaf of rings $\mathcal{O}_X$ on X sth $\forall f \in S_d,\ d > 0$:

$$(D_+(f), \mathcal{O}_X|_{D_+(f)}) \xrightarrow{\sim} \operatorname{Spec}(S_{f,0})$$

$$\text{w/} \quad S_{f,0} := \left\{ \tfrac{g}{f^k} \in S_f = S[\tfrac{1}{f}] : g \in S_{kd} \right\}.$$

In particular $\operatorname{Proj}(S)$ is a scheme.

Pf. By construction $\operatorname{Proj}(S) \subseteq \operatorname{Spec}(S)$ w/ subspace topology

(since $\operatorname{Proj}(S) \cap D(f) = \bigcup_d D_+(f_i)$ for $f = \sum_i f_i$ w/ $f_i \in S_i$)

Let $f \in S_d$ w/ $d > 0$. Define a continuous map φ by:

(induced by $S_{f,0} \hookrightarrow S_f$)

$$\begin{tikzcd}
\operatorname{Proj}(S) \arrow[r, hook] & \operatorname{Spec}(S) & & \\
D_+(f) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, hook] \arrow[rrr, bend right, "\varphi"'] & D(f) \simeq \operatorname{Spec}(S_f) \arrow[u, phantom, "\cup" rotate=90] \arrow[rr] & & \operatorname{Spec}(S_{f,0})
\end{tikzcd}$$

Claim 1: φ is surjective:

$\mathfrak{q} \in \operatorname{Spec}(S_{f,0})$

$\Longrightarrow \sqrt{\mathfrak{q} S_f} \trianglelefteq S_f$ homogenous prime ideal:

homogenous because $\mathfrak{q} S_f \trianglelefteq S_f$ is homogenous;

prime: say $ab \in \mathfrak{q} S_f$ for $a, b \in S_f$, wlog

homogenous of degree $\deg a = m$, $\deg b = n$

$\Longrightarrow a' := f^{-m} \cdot a^d,\ b' := f^{-m} \cdot b^d \in S_{f,0}$ for $d = \deg f$

satisfy $a' \cdot b' \in S_{f,0} \cap \mathfrak{q} S_f = \mathfrak{q}$

$\Longrightarrow a' \in \mathfrak{q}$ or $b' \in \mathfrak{q}$

$\Longrightarrow a^d \in \mathfrak{q} S_f$ or $b^d \in \mathfrak{q} S_f$

$\Longrightarrow$ For the localization map $\lambda : S \to S_f$

then

$$\mathfrak{p} := \lambda^{-1}(\sqrt{\mathfrak{q} S_f}) \in D_+(f)$$

and $\varphi(\mathfrak{p}) = \mathfrak{q}$ (exercise).

Claim 2: φ is injective:

$\mathfrak{p}_1, \mathfrak{p}_2 \in D_+(f)$ w/ $\varphi(\mathfrak{p}_1) = \varphi(\mathfrak{p}_2)$

$$\begin{array}{cc} \varphi(\mathfrak{p}_1) & \varphi(\mathfrak{p}_2) \\ \| & \| \\ (\mathfrak{p}_1 S_f)_0 & (\mathfrak{p}_2 S_f)_0 \end{array}$$

$\Longrightarrow$ For any homogenous $b \in \mathfrak{p}_1$, say $\deg b = m$,

$$f^{-m} \cdot b^d \in (\mathfrak{p}_1 S_f)_0 = (\mathfrak{p}_2 S_f)_0 \subseteq \mathfrak{p}_2 S_f$$

and hence $b \in S \cap (\mathfrak{p}_2 S_f) = \mathfrak{p}_2$

$\Longrightarrow$ $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$

$\Longrightarrow$ $\mathfrak{p}_1 = \mathfrak{p}_2$ by symmetry

Claim 3: φ is an open map:

Given $D_+(g) \subseteq D_+(f)$, let $h := g^d \cdot f^{-e} \in S_{f,0}$ ($e = \deg g$)

Then $\varphi(D_+(g)) = D(h) \subseteq \operatorname{Spec}(S_{f,0})$

(for $\mathfrak{p} \in D_+(f)$: $\mathfrak{p} \notin D_+(g) \Leftrightarrow g \in \mathfrak{p} \Leftrightarrow h \in (\mathfrak{p} S_f)_0$
$\Leftrightarrow \varphi(\mathfrak{p}) \notin D(h)$)

Upshot: $\varphi : D_+(f) \xrightarrow{\sim} \operatorname{Spec}(S_{f,0})$ homeom.

Claim 4: These are compatible:

For $D_+(g) \subseteq D_+(f)$ & $h = g^d \cdot f^{-e} \in S_{f,0}$ as above,

$\exists$ natural iso $(S_{f,0})_h \xrightarrow{\sim} S_{g,0}$ making the diagram commute:

$$\begin{tikzcd} D_+(f) \arrow[r, "\sim", "\varphi"'] & \operatorname{Spec} S_{f,0} \\ & D(h) \arrow[u, phantom, "\cup"] \\ D_+(g) \arrow[uu, hook] \arrow[r, "\sim", "\varphi"'] & \operatorname{Spec} S_{g,0} \arrow[u, "\wr"] \end{tikzcd}$$

(exercise: Use $g \in \sqrt{(f)}$ by previous lemma ...)

Now define $\mathcal{O}_X(D_+(f)) := S_{f,0}$

Since the $D_+(f)$ form a basis, we get a sheaf $\mathcal{O}_X$

& by claim 4 we have $\mathcal{O}_X|_{D_+(f)} \simeq \varphi^* \mathcal{O}_{\operatorname{Spec} S_{f,0}}$ $\square$

<u>Ex</u> $\operatorname{Proj} R[x_0, \dots, x_n] = \mathbb{P}^n_R$ (w/ standard grading by degree)

$$\begin{array}{ccc} \operatorname{Proj} R[x_0,\dots,x_n] & = & \mathbb{P}^n_R \\ \cup\!\!\mid & & \cup\!\!\mid \\ D_+(x_i) & = & \operatorname{Spec} R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \end{array}$$

Ex Weighted projective spaces: Say $k = \bar{k}$

For $a_0, \dots, a_n \in \mathbb{N}$, let $S = k[x_0, \dots, x_n]$

be graded by putting $\deg(x_i) := a_i$

$\Rightarrow \mathbb{P}(a_0, \dots, a_n) := \operatorname{Proj} S$ a scheme over k

w/ $\mathbb{P}(a_0, \dots, a_n)(k) = (\mathbb{A}^{n+1}(k) \setminus \{0\}) / \sim$

$(v_0, \dots, v_n) \sim (\lambda^{a_0} v_0, \dots, \lambda^{a_n} v_n) \quad \forall \lambda \in k^*$

Some simple formal properties of $\operatorname{Proj}(-)$:

Lemma Let S be a graded ring & $R := S_0$.

a) $\operatorname{Proj} S$ is a separated scheme over $\operatorname{Spec} R$.

b) S Noetherian $\Rightarrow$ $\operatorname{Proj} S$ Noetherian

S fin. type $/R$ $\Rightarrow$ $\operatorname{Proj} S$ fin. type $/R$

S domain $\Rightarrow$ $\operatorname{Proj} S$ integral

c) For any ring homom $R \to \tilde{R}$,

$$\operatorname{Spec} \tilde{R} \times_{\operatorname{Spec} R} \operatorname{Proj} S \simeq \operatorname{Proj} \underbrace{\tilde{R} \otimes_R S}_{\text{still graded!}}$$

Pf. Exercise. $\square$

Functoriality?

Ex Graded ring homom. $\varphi : S \to S'$ usually

do NOT induce morphisms $\operatorname{Proj} S' \not\to \operatorname{Proj} S$:

$$\begin{array}{ccc} \operatorname{Proj} S' & \not\dashrightarrow & \operatorname{Proj} S \\ \cap & & \cap \\ \operatorname{Spec} S' & \longrightarrow & \operatorname{Spec} S \end{array}$$

e.g. $\varphi = \text{incl}: S = k[x,y] \hookrightarrow S' = k[x,y,z]$

$\mathfrak{p} = (x,y) \in \operatorname{Proj} S' = \mathbb{P}^3$

(a homogeneous prime ideal $\not\supseteq S'_+ = (x,y,z)$)

$\Rightarrow \varphi^{-1}(\mathfrak{p}) = (x,y) \notin \operatorname{Proj} S$

(since $\varphi^{-1}(\mathfrak{p}) \supseteq S_+ = (x,y)$)

($\Leftrightarrow \mathfrak{p} \ni \varphi(f)$ for all $f \in S_+$)

Def For a graded ring homom. $\varphi : S \to S'$

consider the <u>base locus</u>

$$Bs(\varphi) := V_+(J) \subseteq \operatorname{Proj}(S')$$

closed

cut out by

$$J := \langle \varphi(f) \mid f \in S_+ \rangle \trianglelefteq S'$$

Prop Any graded ring homom. $\varphi : S \to S'$ induces a morphism of schemes

$$ {}^a\varphi : \underbrace{U := \operatorname{Proj}(S') \setminus Bs(\varphi)}_{\text{open in } \operatorname{Proj}(S')} \longrightarrow \operatorname{Proj}(S) $$

Pf. $\mathfrak{p} \in U \Rightarrow \varphi^{-1}(\mathfrak{p}) \not\supseteq S_+$

$$\Rightarrow {}^a\varphi(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p}) \in \operatorname{Proj}(S)$$

Get continuous map $h = {}^a\varphi : U \to \operatorname{Proj}(S)$:

$$\begin{tikzcd} \operatorname{Spec} S' \arrow[r, "{}^a\varphi"'] & \operatorname{Spec} S \\ U \arrow[u, hook] \arrow[r, "{}^a\varphi"'] & \operatorname{Proj} S \arrow[u, hook] \end{tikzcd}$$

} subspace topology...

Want to define $h^\# : \mathcal{O}_{\operatorname{Proj}(S)} \to h_* \mathcal{O}_U$

Enough to do so on basic open $D_+(f) \subseteq \operatorname{Proj}(S)$:

$$\begin{array}{ccc} h^\# : \mathcal{O}_{\operatorname{Proj}(S)}(D_+(f)) & \to & \mathcal{O}_U(h^{-1}(D_+(f))) \\ \| & & \| \\ S_{f,0} & \xrightarrow{\varphi} & S'_{\varphi(f),0} \end{array}$$

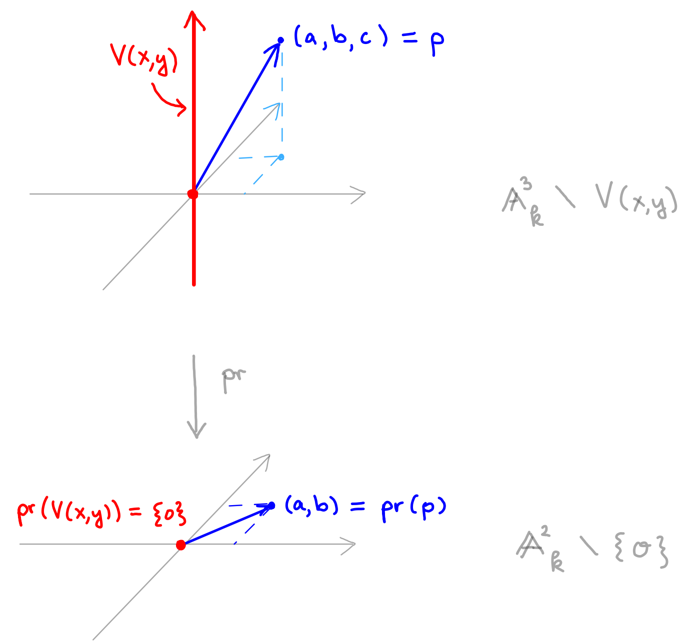
$\square$

Ex "Projection from a point" (here from $[0:0:1]$):

For $\varphi = \text{incl} : k[x,y] \hookrightarrow k[x,y,z]$,

get $Bs(\varphi) = V_+(x,y) = \{[0:0:1]\} \subseteq \mathbb{P}^2_k$

$$ {}^a\varphi : \mathbb{P}^2_k \setminus \{[0:0:1]\} \longrightarrow \mathbb{P}^1$$
$$[x:y:z] \longmapsto [x:y]$$



Coordinate-free description:

V finite-dim vector space / k

$\mathbb{P}V := \mathrm{Proj}(S)$ w/

$S := \mathrm{Sym}(V^*) = \bigoplus_{d\geq 0} \mathrm{Sym}_k^d(V^*)$

For any linear subspace $U \overset{\neq}{\hookrightarrow} V$

& $W := V/U$ get

$\varphi : \mathrm{Sym}(W^*) \hookrightarrow \mathrm{Sym}(V^*)$

w/ $Bs(\varphi) = V_+(\underset{\substack{\uparrow \\ \subseteq \mathrm{Sym}_k^1(V^*)}}{W^*}) \subseteq \mathbb{P}V$

e.g.

$V = k^3$

$V^* = k\cdot x \oplus k\cdot y \oplus k\cdot z$

$S = k[x,y,z]$

$U = \langle (0,0,1) \rangle$

$W^* = \langle x, y \rangle$

$\mathrm{Sym}(W^*) = k[x,y]$

$Bs(\varphi) = V_+(x,y)$

$\Longrightarrow$ Projection

$pr = {}^a\varphi : \mathbb{P}V \setminus Bs(\varphi) \longrightarrow \mathbb{P}W$

Rem Have $\mathbb{P}U \xrightarrow{\sim} Bs(\varphi) \subseteq \mathbb{P}V$

closed immersion

Ex Closed subschemes of $\mathrm{Proj}(S)$:

S graded ring, $J \trianglelefteq S$ homogenous ideal

& $\varphi : S \twoheadrightarrow S' := S/J$ quotient map

$\Longrightarrow Bs(\varphi) = V_+(\varphi(S_+)) = V_+(S_+) = \emptyset$

$\Longrightarrow {}^a\varphi : \mathrm{Proj}(S/J) \longrightarrow \mathrm{Proj}(S)$

closed immersion w/ image $V_+(J)$:

On $D_+(f) \subseteq \mathrm{Proj}(S)$ it is given by

$\mathrm{Spec}(\underbrace{(S/J)_{\bar{f},0}}_{\simeq S_{f,0}/(J\cdot S_f)\cap S_{f,0} \text{ a quotient of } S_{f,0}}) \hookrightarrow \mathrm{Spec}_{f,0}(S_{f,0})$

Concretely:

e.g. $S = k[x_0, \dots, x_n] \trianglerighteq J$ homogenous

$S_{x_i,0} = k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \trianglerighteq J\cdot S_{x_i,0} = (f^{(i)} \mid \exists d \in \mathbb{N} : x_i^d f \in J)$

w/ $f^{(i)} := f(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i})$

"dehomogenization of f"

$Z = V_+(J) \subseteq \mathbb{P}_k^n$

$Z \cap U_i = V(J\cdot S_{x_i,0}) \subseteq U_i = \mathrm{Spec}(S_{x_i,0}) \simeq \mathbb{A}_k^n$

$\underline{Ex}$. Morphisms from affine schemes to $\mathbb{P}^n_A$:

Let B be an A-algebra (not graded)

Morphisms $\underbrace{\operatorname{Spec} B}_{= \mathbb{P}^0_B = \operatorname{Proj} B[t]} \to \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$?

Given $b_0, \dots, b_n \in B$ get graded A-algebra homom.

$$\varphi : A[x_0, \dots, x_n] \longrightarrow B[t]$$
$$x_i \longmapsto b_i t$$

$\Rightarrow [b_0 : \dots : b_n] := {}^a\varphi : \operatorname{Proj}(B[t]) \setminus \operatorname{Bs}(\varphi) \longrightarrow \mathbb{P}^n_A$

$\underline{\text{Lemma}}$

$$\begin{array}{ccc} \operatorname{Proj}(B[t]) & \xrightarrow{\sim} & \operatorname{Spec}(B) \\ \cup\!\mid & & \cup\!\mid \\ \operatorname{Bs}(\varphi) & \xrightarrow{\sim} & V(b_0, \dots, b_n) \end{array}$$

Pf. $\operatorname{Bs}(\varphi) = V_+(\varphi(x_i) \mid i = 0, 1, \dots, n)$

$= V_+(b_i t \mid i = 0, 1, \dots, n)$

& $\operatorname{Proj}(B[t]) \xrightarrow{\sim} \operatorname{Spec}(B)$ via dehomogenization $t \mapsto 1$. $\square$

$\underline{\text{Rem}}$ a) $\operatorname{Bs}(\varphi) = \emptyset \iff (b_0, \dots, b_n) = (1)$

$\iff (b_0 t, \dots, b_n t) = S_+$

the irrelevant ideal of the graded ring $S := B[t]$

b) If $(b_0, \dots, b_n) = (f)$ is a principal ideal, can still extend $[b_0 : \dots : b_n]$ over $\operatorname{Bs}(\varphi)$ via $[b_0' : \dots : b_n'] : \operatorname{Spec}(B) \to \mathbb{P}^n_A$ w/ $b_i' := \frac{b_i}{f}$

What if a), b) fail ?

$\underline{\text{Idea}}$: For $J := (b_0, \dots, b_n) \trianglelefteq B$

consider the graded subalgebra ("Rees algebra")

$$\operatorname{Rees}(J) := \bigoplus_{d \geq 0} J^d \cdot t^d \subseteq S := B[t]$$

condition a) is forced here ...

$\underline{\text{Def}}$ The $\underline{\text{blow-up}}$ of $X = \operatorname{Spec}(B)$ along a closed subscheme $Y = V(J)$ is

$$\operatorname{Bl}_Y(X) := \operatorname{Proj}(\operatorname{Rees}(J)) \longrightarrow X.$$

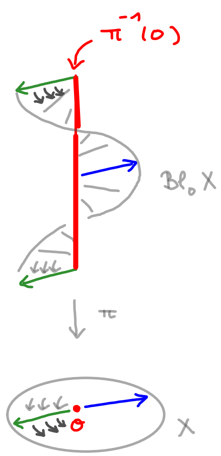
Ex $\quad X = \mathbb{A}^2_k = \operatorname{Spec} \underbrace{k[x,y]}_{=:B} \supset Y = V(x,y)$

$\Rightarrow \quad \tilde{B} := \operatorname{Rees}(x,y) = B[\xi,\eta]$ w/ $\xi := tx,\ \eta := ty$

$\Rightarrow \quad \pi : \tilde{X} = Bl_Y(X) \longrightarrow X$ iso over $X \setminus Y = D(x) \cup D(y)$

$\left(\tilde{B}_x \simeq B_x[t] \Rightarrow \pi^{-1}(D(x)) = \operatorname{Proj} \tilde{B}_x \xrightarrow{\sim} \operatorname{Spec} B_x = D(x),\ \text{same for } D(y) \right)$

but $\quad \pi^{-1}(0) = \operatorname{Proj}\big(\underbrace{\tilde{B} \otimes_B B/(x,y)}_{\simeq k[\xi,\eta]} \big) \simeq \mathbb{P}^1_k .$



Note

- $[x:y] : U \to \mathbb{P}^1_k$ not defined at $0 \in X$ but extends to $[\xi:\eta] : Bl_0 X \longrightarrow \mathbb{P}^1_k$
- $\pi^{-1}(0) \subset Bl_0 X$ locally cut out by ONE equation:
$$\pi^{-1}(0) \cap D_+(\xi) = V\left(\tfrac{\eta}{\xi}\right)$$
$$\pi^{-1}(0) \cap D_+(\eta) = V\left(\tfrac{\xi}{\eta}\right)$$

Def $\quad$ A closed subscheme $E \hookrightarrow W$ is an <u>effective Cartier divisor</u> if $\forall p \in E$ $\exists$ open affine neighborhood $p \in U = \operatorname{Spec} A \subset W$ sth $E \cap U = V(f)$ w/ $f \in A$ not a zero divisor.

Thm $\quad$ Let $X = \operatorname{Spec} \overset{\text{A-algebra}}{B} \supset Y = V(\overset{\text{fin. gen. ideal}}{J})$ w/ $U := X \setminus Y \overset{\text{dense}}{\hookrightarrow} X$

a) For $J = (b_0, \dots, b_n)$ we have $Bl_Y(X) \xhookrightarrow[\text{closed}]{} \mathbb{P}^n_B$

and:

$$\begin{tikzcd} Bl_Y(X) \arrow[r, "\exists"] \arrow[d, "\pi"'] & \mathbb{P}^n_A \\ X \supseteq U \arrow[ur, "{[b_0 : \dots : b_n]}"'] & \end{tikzcd}$$

b) $\pi : Bl_Y(X) \longrightarrow X$ is proper surjective & restricts over $U := X \setminus Y$ to an iso $\pi^{-1}(U) \xrightarrow{\sim} U$.

c) $\pi^{-1}(Y) := Y \times_X Bl_Y(X)$ is an effective Cartier divisor.

We call it the <u>exceptional divisor</u> of the blowup.

d) Any $p : W \longrightarrow X$ w/ $p^{-1}(Y) \subset W$ effective Cartier factors uniquely as:

$$\begin{tikzcd} W \arrow[r, "\exists!"] \arrow[dr, "p"'] & Bl_Y(X) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Rem $\quad Bl_Y(X) \in Sch_X$ is determined uniquely by c), d).

Uniqueness allows to glue also for X not affine.

Pf. a) For $J = (b_0, \dots, b_n) \trianglelefteq B$, have

$$\begin{array}{ccc} B[x_0, \dots, x_n] & \twoheadrightarrow & \mathrm{Rees}(J) := \bigoplus_{d \geq 0} J^d \cdot t^d \subset B[t] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x_i & \longmapsto & b_i t \end{array}$$

$$\begin{tikzcd}
\Rightarrow \quad \mathrm{Bl}_Y(X) := \mathrm{Proj}(\mathrm{Rees}(J)) \arrow[r, hook, "closed"'] \arrow[d, "\pi"'] & \mathbb{P}^n_B \arrow[r] & \mathbb{P}^n_A \\
X \supseteq U := X \setminus V(J) \arrow[rru, bend right, "{[b_0 : \dots : b_n]}"'] & &
\end{tikzcd}$$

b) π proper as $\pi : \mathrm{Bl}_Y(X) \xhookrightarrow[\text{closed}]{} \mathbb{P}^n_B \xrightarrow[\text{proper}]{} X = \mathrm{Spec}\, B$

Now $U = X \setminus V(J) = \bigcup_{f \in J} D(f)$. For any $f \in J$,

$$\pi^{-1}(D(f)) = \mathrm{Proj}(\underbrace{\mathrm{Rees}(J) \otimes_B B_f}_{= B_f[t] \text{ as } J^d \cdot B_f = B_f \text{ for all } d}) = \mathrm{Proj}(B_f[t]) \xrightarrow{\sim} \mathrm{Spec}\, B_f = D(f)$$

$\Rightarrow$ $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$ iso

$\Rightarrow$ π surjective (being closed w/ image $\supseteq U$ dense)

c) On $D_i := D_+(b_i t) = \mathrm{Spec}\, B[\frac{b_0 t}{b_i t}, \dots, \frac{b_n t}{b_i t}]$,

$\pi^{-1}(Y) \cap D_i = V(b_i)$ since $\forall j : b_j = b_i \cdot \frac{b_j t}{b_i t}$

d) Wlog $W = \mathrm{Spec}\, C$ affine (then glue using uniqueness)

$\Rightarrow$ $p = {}^a\varphi$ for some $\varphi \in \mathrm{Hom}_A(B, C)$

$\pi^{-1}(Y) \subseteq W$ effective Cartier divisor

$\Rightarrow$ $\varphi(J) \cdot C = (f)$ w/ $f \in C$ not a zero divisor

Wlog $f = \varphi(b)$ for some $b \in J$.

$$\begin{array}{ccc} \Rightarrow \exists!\ B\text{-algebra hom. } B[\frac{b_0 t}{bt}, \dots, \frac{b_n t}{bt}] & \longrightarrow & C \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \frac{b_i t}{bt} & \longmapsto & c_i \end{array}$$

w/ $c_i \in C$ defined by $\varphi(b_i) = f \cdot c_i$

$\Rightarrow$ unique $W \longrightarrow D_+(bt) \subset \mathrm{Bl}_Y(X)$ as wanted. $\square$

Cor If $Y = V(f) \subset X = \mathrm{Spec}\, A$ w/ $f \in A$ not a zero divisor, then $\mathrm{Bl}_Y(X) \xrightarrow{\sim} X$.

Pf. Clear from universal property c), d). $\square$

How to compute blowups?

<u>Lemma</u> Let $X = \operatorname{Spec} B \supset Y = V(J)$ as above.

a) If X is reduced / integral, then so is $\tilde{X} := Bl_Y(X)$.

b) For $J = (b_0, \dots, b_n)$, consider

$$\tilde{X} \hookrightarrow \mathbb{P}^n_B = \operatorname{Proj} B[\xi_0, \dots, \xi_n] \quad \text{via } \xi_i \mapsto b_i t.$$

Then $\tilde{X} \cap D_+(\xi_i) \simeq \operatorname{Spec} B_i$

w/ $B_i := B[\frac{b_0}{b_i}, \dots, \frac{b_n}{b_i}] \subset B_{b_i}$

c) Have $\tilde{X} \hookrightarrow Z := V_+(b_i \xi_j - b_j \xi_i) \subset \mathbb{P}^n_B$

If $b_0, \dots, b_n$ are a minimal set of generators for $J \trianglelefteq B$ and if Z is integral, then $\tilde{X} = Z$.

Pf. a) B reduced / a domain

$\Leftrightarrow \tilde{B} := \operatorname{Rees}(J) = \bigoplus_{d \geq 0} J^d \cdot t^d$ reduced / a domain

b) Have

$$\begin{tikzcd}
S := B[\xi_0, \dots, \xi_n] \arrow[r, two heads, "\varphi"] \arrow[d] & \tilde{B}, \quad \xi_i \mapsto b_i t \arrow[d] \\
\text{localize: } S_{\xi_i} \arrow[r, two heads, "\varphi_i"] & \tilde{B}_{b_i t} \\
\text{degree } 0: S_{\xi_i, 0} \arrow[u, phantom, "\cup"] \arrow[r, two heads, "\varphi_{i,0}"] & \tilde{B}_{b_i t, 0} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Fix i and put $y_j := \xi_j / \xi_i \in S_{\xi_i, 0}$.

Claim:

$$\begin{tikzcd}
y_j \arrow[rr, mapsto] & & \frac{b_j}{b_i} \\
S[y_0, \dots, y_n] = S_{\xi_i, 0} \arrow[u, phantom, "\cap"] \arrow[rr, two heads, "\psi"] \arrow[dr, two heads, bend right, "\varphi_{i,0}"'] & & B[\frac{b_0}{b_i}, \dots, \frac{b_n}{b_i}] \arrow[u, phantom, "\cap"] \\
& \tilde{B}_{b_i t, 0} \arrow[ur, dashed, "\exists \text{ iso}"'] &
\end{tikzcd}$$

Details are left as exercise. Use:

$$\forall f \in S[y_0, \dots, y_n] \ \exists d \in \mathbb{N} \ \exists b \in B: \qquad (*)$$

$$b_i^d \cdot f = b + \sum_{j \neq i} g_j(y_0, \dots, y_n) \cdot (b_i \cdot y_j - b_j)$$

(use Euclidean division by $y_j - \frac{b_j}{b_i}$ & clear denominators)

c) Clearly $\tilde{X} \hookrightarrow Z$. For equality, enough to show $\tilde{X} \cap D_+(\xi_i) = Z \cap D_+(\xi_i) \ \forall i$.

$b_0, \dots, b_m$ minimal generating system $\Rightarrow f_i \notin \langle f_j \mid j \neq i \rangle$

$\Rightarrow f_i \neq 0$ in $\mathcal{O}_Z(U_i)$, hence not zero divisor if Z integral

$\Rightarrow$ claim follows from $(*)$ $\square$

$\underline{Ex}$ $\quad B = k[x,y]/(y^2 - x^3) \trianglerighteq J = (x,y)$

$\Longrightarrow \tilde{X} = Bl_Y(X) \hookrightarrow \mathbb{P}^1_B = \mathrm{Proj}\, B[\xi,\eta]$

has

$u := \frac{y}{x}$

$\tilde{X} \cap D_+(\xi) = \mathrm{Spec}\, B_1, \quad B_1 = B[u] \subset B_x$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad k[u]$ as $\begin{array}{l} u^2 = x \\ u^3 = y \end{array}$

$v := \frac{x}{y}$

$\tilde{X} \cap D_+(\eta) = \mathrm{Spec}\, B_2, \quad B_2 = B[v] \subset B_y$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad k[v, 1/v]$

$$\Longrightarrow \begin{array}{ccl} \tilde{X} & \simeq & \mathrm{Spec}\, k[u] \\ \downarrow{\scriptstyle \pi} & & \\ X & \simeq & \mathrm{Spec}\, k[x,y]/(y^2 - x^3) \end{array} \qquad \text{w/} \begin{array}{l} x \longmapsto u^2 \\ y \longmapsto u^3 \end{array}$$

IV. Coherent Sheaves

1. Sheaves of modules

Idea: Rings R $\rightsquigarrow$ schemes $X = \operatorname{Spec} R$

R-modules $\rightsquigarrow$ quasicoherent sheaves of $\mathcal{O}_X$-modules

Def Let $X = (|X|, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module is a sheaf $\mathcal{M}$ of ab. gps on X together with maps

$$\mathcal{O}_X(U) \times \mathcal{M}(U) \longrightarrow \mathcal{M}(U), \quad (f, m) \mapsto fm$$

making $\mathcal{M}(U)$ a module over the ring $\mathcal{O}_X(U)$ sth for all open $V \subseteq U \subseteq X$ the diagram below commutes:

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{M}(U) \arrow[r] \arrow[d] & \mathcal{M}(U) \arrow[d] \\
\mathcal{O}_X(V) \times \mathcal{M}(V) \arrow[r] & \mathcal{M}(V)
\end{tikzcd}$$

Def A morphism between $\mathcal{O}_X$-modules $\mathcal{M}, \mathcal{N}$ is a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ of sheaves sth $\forall U$, $\varphi(U) : \mathcal{M}(U) \to \mathcal{N}(U)$ is $\mathcal{O}_X(U)$-linear.

Notation: $\varphi \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$

$\rightsquigarrow$ $\operatorname{Mod}(\mathcal{O}_X) :=$ category of $\mathcal{O}_X$-modules

Rem $\operatorname{Mod}(\mathcal{O}_X)$ is an abelian category

For $\varphi : \mathcal{M} \to \mathcal{N}$:

$\ker(\varphi) = (U \mapsto \ker(\mathcal{M}(U) \to \mathcal{N}(U)))$

$\operatorname{cok}(\varphi) = (U \mapsto \operatorname{cok}(\mathcal{M}(U) \to \mathcal{N}(U)))^{sh}$

Ex
- $\mathcal{M} = \mathcal{J} \trianglelefteq \mathcal{O}_X$ sheaf of ideals
- $\mathcal{M} = \mathcal{O}_X^{\oplus n}$ free $\mathcal{O}_X$-module
- all submodules & quotients of such ...

Further examples arise by functoriality:

Def For $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$, get

Exercise: Usually need to sheafify!

a) the tensor product

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} := \left(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\right)^{sh} \in \mathrm{Mod}(\mathcal{O}_X)$$

b) the internal hom / sheaf hom

Exercise: This defines a sheaf!

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \left(U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)\right) \in \mathrm{Mod}(\mathcal{O}_X)$$

Exercise For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Mod}(\mathcal{O}_X)$,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

Cor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, -)$ right adjoint to $(-) \otimes_{\mathcal{O}_X} \mathcal{G}$.

Exercise
- For any $p \in X$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p \simeq \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{G}_p$
- $(-) \otimes_{\mathcal{O}_X} \mathcal{G}$ is right exact,

  hence $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, -)$ is left exact.
- If $\exists\, \varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^m \to \mathcal{O}_X^n)$ w/ $\mathcal{G} \simeq \mathrm{cok}(\varphi)$,

  then $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))_p \simeq \mathcal{H}om_{\mathcal{O}_{X,p}}(\mathcal{G}_p, \mathcal{H}_p)$

Def Let $f : X \to Y$ morphism of ringed spaces.

a) For $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ get

the direct image / pushforward

$$f_* \mathcal{F} := \left(U \mapsto \underbrace{\mathcal{F}(f^{-1}(U))}_{\text{an } \mathcal{O}_Y(U)\text{-module via } f^\#: \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))}\right) \in \mathrm{Mod}(\mathcal{O}_X)$$

b) For $\mathcal{G} \in \mathrm{Mod}(\mathcal{O}_Y)$ get

the inverse image / pullback

$$f^* \mathcal{G} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} \mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$$

using $f^\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

Exercise $f_* \mathcal{H}om_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F})$.

Cor f^* is right adjoint to f_*:

$$\mathrm{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F}).$$

Exercise
- For any $p \in X$, $(f^* \mathcal{G})_p \simeq \mathcal{G}_{f(p)} \otimes_{\mathcal{O}_{Y,f(p)}} \mathcal{O}_{X,p}$
- f_* left exact & f^* right exact

2. Quasicoherent sheaves

Now let X be a scheme.

Recall: For $\mathcal{J} \trianglelefteq \mathcal{O}_X$ sheaf of ideals, get ringed space

$$V(\mathcal{J}) := \left(\overline{\mathrm{Supp}(\mathcal{O}_X/\mathcal{J})},\ \mathrm{incl}^{-1}(\mathcal{O}_X/\mathcal{J}) \right)$$

This is a closed subscheme of X

iff $\forall$ open affine $U = \mathrm{Spec}\, A \subset X$:

$\mathcal{J}|_U =$ "sheaf associated w/ an ideal $I \trianglelefteq A$" (see II.3)

Def Let $X = \mathrm{Spec}\, A$ affine. For $M \in \mathrm{Mod}(A)$,

define $\tilde{M} \in \mathrm{Mod}(\mathcal{O}_X)$ on basic open sets by

$$\tilde{M}(D(f)) := M_f \simeq A_f \otimes_A M \quad \text{for } f \in A,$$

w/
$$\begin{array}{ccc} \tilde{M}(D(f)) & \longrightarrow & \tilde{M}(D(g)) \\ \| & & \| \\ A_f \otimes_A M & \longrightarrow & A_g \otimes_A M \end{array}$$
for $D(g) \subseteq D(f)$

$\left(\begin{array}{l} \Rightarrow \exists n \in \mathbb{N} : g^n \in (f) \\ \Rightarrow \exists\, A_f \to A_g \end{array} \right)$

Sheaf axioms are checked for $\tilde{M}$ like for $\mathcal{O}_X = \tilde{A}$.

Prop Let $X = \mathrm{Spec}\, A$.

a) For $M \in \mathrm{Mod}(A)$ the stalk of $\tilde{M}$ at $x = \mathfrak{p}$

is $\quad \tilde{M}_x = M_{\mathfrak{p}} \quad (\simeq A_{\mathfrak{p}} \otimes_A A)$

b) The functor $\mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(\mathcal{O}_X)$, $\quad M \mapsto \tilde{M}$

is fully faithful: $\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$

c) It reflects exactness:

$M' \to M \to M''$ exact in $\mathrm{Mod}(A)$

$\iff \tilde{M}' \to \tilde{M} \to \tilde{M}''$ exact in $\mathrm{Mod}(\mathcal{O}_X)$

d) $\exists$ natural iso's $\quad \left(\bigoplus_{i \in I} M_i \right)^{\sim} \xrightarrow{\sim} \bigoplus_{i \in I} \tilde{M}_i$

$$(M \otimes_A N)^{\sim} \xrightarrow{\sim} \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$$

e) For $\varphi : Y = \mathrm{Spec}\, B \longrightarrow X = \mathrm{Spec}\, A$ we have:

$$\begin{array}{ccc} \mathrm{Mod}(B) & \to & \mathrm{Mod}(\mathcal{O}_Y) \\ {\scriptstyle \text{forget}} \downarrow & & \downarrow {\scriptstyle \varphi_*} \\ \mathrm{Mod}(A) & \to & \mathrm{Mod}(\mathcal{O}_X) \end{array} \qquad \begin{array}{ccc} \mathrm{Mod}(B) & \to & \mathrm{Mod}(\mathcal{O}_Y) \\ {\scriptstyle B \otimes_A (-)} \uparrow & & \uparrow {\scriptstyle \varphi^*} \\ \mathrm{Mod}(A) & \to & \mathrm{Mod}(\mathcal{O}_X) \end{array}$$

where the horizontal arrows are given by $M \mapsto \tilde{M}$.

Pf. a) $\tilde{M}_x = \varinjlim_{\substack{U \ni x \\ \text{open}}} \tilde{M}(U)$

$$= \varinjlim_{\substack{f \in A \\ D(f) \ni x}} \tilde{M}(D(f)) = \varinjlim_{f \in A \setminus \mathfrak{p}} M_f \simeq M_{\mathfrak{p}}$$

b) $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$

$\Rightarrow \varphi := \psi(X) \in \mathrm{Hom}_A(M, N)$ since $M = \tilde{M}(X)$, $N = \tilde{N}(X)$, $A = \mathcal{O}_X(X)$

$\Rightarrow$ on basic open $U = D(f) \subseteq X$,

$\psi : \tilde{M}(U) = M_f \longrightarrow \tilde{N}(U) = N_f$

is given by

$$\psi\left(\tfrac{m}{f^k}\right) = \tfrac{1}{f^k} \cdot \psi\left(\tfrac{m}{1}\right) = \tfrac{1}{f^k} \cdot \psi(m|_U) = \tfrac{1}{f^k} \cdot \psi(m)|_U = \tfrac{1}{f^k} \cdot \varphi(m)$$

$\Rightarrow \psi = \tilde{\varphi}$ is induced by $\varphi \in \mathrm{Hom}_A(M, N)$

$\Rightarrow \mathrm{Mod}_A(M, N) \to \mathrm{Mod}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ surjective.

But also injective:

If $\psi = \tilde{\varphi}$ then necessarily $\varphi = \psi(X)$.

c) $M' \xrightarrow{f} M \xrightarrow{g} M''$ exact in $\mathrm{Mod}(A)$

$\Leftrightarrow M'_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} M_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} M''_{\mathfrak{p}}$ exact $\forall \mathfrak{p} \in \mathrm{Spec}(A)$ $(*)$

$\Leftrightarrow \tilde{M}'_x \xrightarrow{\tilde{f}_x} \tilde{M}_x \xrightarrow{\tilde{g}_x} \tilde{M}''_x$ exact $\forall x \in X$ (use a))

$\Leftrightarrow \tilde{M}' \xrightarrow{\tilde{f}} \tilde{M} \xrightarrow{\tilde{g}} \tilde{M}''$ exact in $\mathrm{Mod}(\mathcal{O}_X)$

For $(*)$ consider $Q := \ker(g)/\mathrm{im}(f) \in \mathrm{Mod}(A)$.

Exactness of localization gives $Q_{\mathfrak{p}} \simeq \ker(g_{\mathfrak{p}})/\mathrm{im}(g_{\mathfrak{p}})$,

and for any A-module: $Q \simeq 0 \Leftrightarrow \forall \mathfrak{p} \in \mathrm{Spec}(A)\ Q_{\mathfrak{p}} \simeq 0$.

d) Localization commutes with $\bigoplus_{i \in I}$ for any I.

For $\otimes$ consider the presheaf

$$\tilde{M} \otimes^{\mathrm{pre}}_{\mathcal{O}_X} \tilde{N} := \left(U \mapsto \tilde{M}(U) \otimes_{\mathcal{O}_X(U)} \tilde{N}(U)\right).$$

Have natural morphism $L := \tilde{M} \otimes^{\mathrm{pre}}_{\mathcal{O}_X} \tilde{N} \xrightarrow{\varphi} (M \otimes_{\mathcal{O}_X} N)^{\sim} =: R$

sheafify $\to$ $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ $\exists! \psi$

But on basic open $U = D(f) \subseteq X$

$\varphi(U) : \underset{\substack{\| \\ L(U)}}{M_f \otimes_{A_f} N_f} \xrightarrow{\sim} \underset{\substack{\| \\ R(U)}}{(M \otimes_A N)_f}$ iso

$\Rightarrow \psi$ iso

e1) $M \in \mathrm{Mod}(B)$, $U = D(f) \subseteq X = \mathrm{Spec}\, A$

$$\Rightarrow (\varphi_* \tilde{M})(U) = \tilde{M}(\varphi^{-1}(U)) \quad \text{by def of } \varphi_*$$
$$= \tilde{M}(D(\varphi^\#(f))) \quad \text{as } f^{-1}(D(f)) = D(\varphi^\#(f))$$
$$= M_{\varphi^\#(f)} \quad \text{by def of } \tilde{M}$$
$$= A_f \otimes_A M = (\mathrm{forget}(M))_f$$

$\Rightarrow$ $\varphi_* \tilde{M}$ and $(\mathrm{forget}(M))^\sim$ agree as sheaves on $\mathcal{B} := \{\text{basic open sets}\} \subset \mathrm{Op}(X)$

$\Rightarrow$ $\varphi_* \tilde{M} \simeq (\mathrm{forget}(M))^\sim$ in $\mathrm{Sh}(X)$, hence in $\mathrm{Mod}(\mathcal{O}_X)$

e2) $f^*(\tilde{N}) \simeq (N \otimes_A B)^\sim$ for $N \in \mathrm{Mod}(\mathcal{O}_A)$:

- $N = A$: Trivial since $f^*(\tilde{A}) = f^*(\mathcal{O}_A) = \mathcal{O}_B = (A \otimes_A B)^\sim$
- $N = \bigoplus_{i \in I} A$: Follows by d) as f^* commutes w/ $\oplus$
- N arbitrary : Write $N = \mathrm{cok}\left(\bigoplus_{j \in J} A \longrightarrow \bigoplus_{i \in I} A\right)$ any presentation of N

& use exactness of $\sim$ in b)

& right exactness of $f^*(-)$. $\square$

Def Let X be a scheme. We say $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ is <u>quasicoherent</u> if $\exists$ open cover $X = \bigcup_{\alpha \in I} U_\alpha$ w/ $U_\alpha = \mathrm{Spec}\, A_\alpha$ affine sth $\forall \alpha \in I$:

$$\mathcal{M}|_{U_\alpha} \simeq \tilde{M}_\alpha \text{ for some } M_\alpha \in \mathrm{Mod}(A_\alpha).$$

Thm If $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ is quasicoherent, then for every open affine $U = \mathrm{Spec}\, A$ we have:

$$\mathcal{M}|_U \simeq \tilde{M} \text{ for } M := \mathcal{M}(U) \in \mathrm{Mod}(A).$$

Pf. ① If $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ is quasicoherent, then so is $\mathcal{M}|_U \in \mathrm{Mod}(\mathcal{O}_U)$ $\forall$ open $U \subseteq X$.

(pick a cover $X = \bigcup_{\alpha \in I} U_\alpha$ as in the definition

& for each α pick a cover $U \cap U_\alpha = \bigcup_{\beta \in J} U_{\alpha\beta}$

w/ $U_{\alpha\beta} = D(f_{\alpha\beta}) \subset U_\alpha = \mathrm{Spec}(A_\alpha)$

$\Rightarrow U = \bigcup_{\alpha,\beta} U_{\alpha\beta}$ & $\mathcal{M}|_{U_{\alpha\beta}} = \tilde{M}_\alpha|_{U_{\alpha\beta}} = \widetilde{M_{\alpha, f_{\alpha\beta}}}$)

② Hence wlog $X = U = \operatorname{Spec} A$ affine.

Let $X = \bigcup_{\alpha \in I} U_\alpha$ be as in the definition,

and write

$j_\alpha : U_\alpha = \operatorname{Spec} A_\alpha \hookrightarrow X = \operatorname{Spec} A$.

Since affine schemes X are quasicompact, may assume that I is a finite set. Since affine schemes X are separated we also know the $U_{\alpha\beta} := U_\alpha \cap U_\beta$ are affine, say $j_{\alpha\beta} : U_{\alpha\beta} = \operatorname{Spec} A_{\alpha\beta} \hookrightarrow X$.

By the sheaf axioms

$$\mathcal{M} \simeq \lim \Big(\underbrace{\prod_{\alpha \in I} j_{\alpha *}(\mathcal{M}|_{U_\alpha})}_{\substack{\| \\ \bigoplus_{\alpha \in I} \cdots}} \rightrightarrows \underbrace{\prod_{\alpha,\beta \in I} j_{\alpha\beta *}(\mathcal{M}|_{U_{\alpha\beta}})}_{\substack{\| \\ \bigoplus_{\alpha,\beta \in I} \cdots}} \Big)$$

(can replace $\prod$ by $\bigoplus$ because I is finite)

③ Now $\mathcal{M}|_{U_\alpha} = \tilde{M}_\alpha$ by assumption

$$\begin{aligned} \Rightarrow \mathcal{M}|_{U_{\alpha\beta}} &= (\mathcal{M}|_{U_\alpha})|_{U_{\alpha\beta}} \\ &= (\tilde{M}_\alpha)|_{U_{\alpha\beta}} \\ &= \tilde{M}_{\alpha\beta} \text{ w/ } M_{\alpha\beta} := A_{\alpha\beta} \otimes_{A_\alpha} M_\alpha \quad (\text{prop. part e)}) \end{aligned}$$

$$\overset{e)}{\Longrightarrow} j_{\alpha *}(\mathcal{M}|_{U_\alpha}) = \tilde{N}_\alpha \text{ w/ } N_\alpha := M_\alpha \text{ as } A\text{-module}$$

$$j_{\alpha\beta *}(\mathcal{M}|_{U_{\alpha\beta}}) = \tilde{N}_{\alpha\beta} \text{ w/ } N_{\alpha\beta} := M_{\alpha\beta} \text{ as } A\text{-module}$$

$$\overset{c),d)}{\Longrightarrow} \mathcal{M} \simeq \Big(\lim \big(\bigoplus_\alpha N_\alpha \rightrightarrows \bigoplus_{\alpha,\beta} N_{\alpha\beta} \big) \Big)^\sim$$

$\Rightarrow$ claim w/ $M := \operatorname{colim}(\ldots) \in \operatorname{Mod}(A)$ $\square$

<u>Def</u> We denote by

$\operatorname{QCoh}(X) \subseteq \operatorname{Mod}(\mathcal{O}_X)$

the full subcategory of quasicoherent sheaves.

Cor a) $\mathrm{QCoh}(X) \subseteq \mathrm{Mod}(\mathcal{O}_X)$ abelian subcategory

w/ $\bigoplus_{i\in I} \mathcal{M}_i \in \mathrm{QCoh}(X) \quad \forall\, \mathcal{M}_i \in \mathrm{QCoh}(X)$

$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathrm{QCoh}(X) \quad \forall\, \mathcal{M}, \mathcal{N} \in \mathrm{QCoh}(X)$.

b) For $X = \mathrm{Spec}\, A$ affine, get exact equivalence

$\mathrm{Mod}(A) \xrightarrow{\sim} \mathrm{QCoh}(X), \quad M \longmapsto \widetilde{M}$

w/ inverse $\mathcal{M} \mapsto \Gamma(\mathcal{M}) := \mathcal{M}(X)$.

In particular $\Gamma : \mathrm{QCoh}(X) \to \mathrm{Mod}(A)$ is exact.

c) For any $\varphi : Y \to X$ have:

- $\varphi^*(\mathcal{N}) \in \mathrm{QCoh}(Y) \quad \forall\, \mathcal{N} \in \mathrm{QCoh}(X)$
- If φ is qcqs, then also

  $\varphi_*(\mathcal{M}) \in \mathrm{QCoh}(X) \quad \forall\, \mathcal{M} \in \mathrm{QCoh}(Y)$.

Pf. a) Clear from the proposition (part d)

b) clear by previous thm.

c) follows from the proposition (part e) as in previous thm $\square$

Rem The above also shows:

$\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ quasicoherent

$\iff \forall\, U = \mathrm{Spec}\, A \subset X$ affine

$\mathcal{M}|_U \simeq \widetilde{M}$ for $M := \mathcal{M}(U) \in \mathrm{Mod}(A)$

$\iff \forall\, U = \mathrm{Spec}\, A \subset X$ affine

$\forall$ basic open $D(f) \subseteq U \quad (f \in A)$

the natural map

$$A_f \otimes_A \mathcal{M}(U) = \mathcal{M}(U)_f \longrightarrow \mathcal{M}(D(f))$$

is an isomorphism.

Ex $\varphi : Y = \coprod_{\alpha\in\mathbb{N}} \mathrm{Spec}\,\mathbb{Z} \longrightarrow X = \mathrm{Spec}\,\mathbb{Z}$

$\implies \mathcal{M} := \varphi_*(\mathcal{O}_Y)$ is NOT quasicoherent

Indeed: For $f \in \mathbb{Z} \setminus \{0, \pm 1\}$ & $U = X$,

$$\begin{array}{ccccc} \mathcal{M}(U)_f & = & \left(\prod_{\alpha\in\mathbb{Z}} \mathcal{O}_X(X)\right)_f & = & \left(\prod_{\alpha\in\mathbb{N}} \mathbb{Z}\right)[\tfrac{1}{f}] \\ \downarrow & & & & \downarrow \text{NOT onto!} \\ \mathcal{M}(D(f)) & = & \prod_{\alpha\in\mathbb{N}} \mathcal{O}_X(D(f)) & = & \prod_{\alpha\in\mathbb{N}} \mathbb{Z}[\tfrac{1}{f}] \end{array}$$

3. Coherent sheaves

Goal: Impose finiteness condition on modules

(e.g. finitely generated over Noetherian ring ...) !

Def Let $X = (|X|, \mathcal{O}_X)$ be a ringed space.

We say that $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ is

a) of finite type if $\forall p \in |X|$ $\exists$ open $U \ni x$

$\exists$ surjective hom. $\mathcal{O}_U^{\oplus n} \twoheadrightarrow \mathcal{F}|_U$ (some $n \in \mathbb{N}_0$)

b) of finite presentation if $\forall p \in |X|$ $\exists$ open $U \ni x$

$\exists$ exact sequence

$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$ (some $m, n \in \mathbb{N}_0$)

c) coherent if $\mathcal{F}$ is of finite type and for

all $\varphi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ (needn't be surjective)

w/ $n \in \mathbb{N}_0$ and $U \subseteq X$ open,

$\ker(\varphi)$ is again of finite type.

Prop Let X be a scheme & $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$.

Each of the following properties implies the next:

a) $\mathcal{F}$ is coherent.

b) $\mathcal{F}$ is of finite presentation

c) $\mathcal{F}$ is quasicoherent & of finite type.

d) $\mathcal{F}$ is quasicoherent & $\forall\, U = \mathrm{Spec}\, A \subseteq X$,

the A-module $\mathcal{F}(U)$ is fin. generated.

For X locally Noetherian, a) – d) are equivalent.

Pf. a) $\Rightarrow$ b) $\Rightarrow$ c) clear.

c) $\Rightarrow$ d) : Assume $\mathcal{F} \in \mathrm{QCoh}(X)$ of finite type.

Any $U = \mathrm{Spec}\, A \subseteq X$ has a finite cover $U = \bigcup_{\alpha \in I} U_\alpha$

w/ basic open $U_\alpha = D(f_\alpha)$ ($f_\alpha \in A$)

on which $\exists$ surjection $\mathcal{O}_{U_\alpha}^{\oplus n_\alpha} \twoheadrightarrow \mathcal{F}|_{U_\alpha}$ (some $n_\alpha \in \mathbb{N}_0$)

$\Longrightarrow$ $A_{f_\alpha}^{\oplus n_\alpha} \twoheadrightarrow \mathcal{F}(U_\alpha)$ since $U_\alpha = \mathrm{Spec}(A_{f_\alpha})$

& $\mathcal{F}$ quasicoherent

By quasicoherence & since $U = \operatorname{Spec} A$ is also affine,

$$\mathcal{F}(U_\alpha) = \mathcal{O}(U_\alpha) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) = A_{f_\alpha} \otimes_A \mathcal{F}(U).$$

$\Rightarrow \exists$ fin. gen. A-submodule $M \subseteq \mathcal{F}(U)$ sth

$$M_{f_\alpha} = A_{f_\alpha} \otimes_A M \overset{!}{=} A_{f_\alpha} \otimes_A \mathcal{F}(U) = \mathcal{F}(U_\alpha).$$

Wlog the same M works for all α (finitely many!).

$\Rightarrow \tilde{M} \twoheadrightarrow \mathcal{F}|_U$ surjective as it is so on each U_α

$\Rightarrow M \twoheadrightarrow \mathcal{F}(U)$ surjective

d) $\Rightarrow$ a): Suppose X locally Noetherian & d) holds.

Let $U \subseteq X$ open and $\varphi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}$.

Want to show $\ker(\varphi)$ is of finite type.

This is a local property, so wlog $U = \operatorname{Spec} A$ affine.

X locally Noetherian $\Rightarrow$ A Noetherian

$\Rightarrow \ker(\varphi(U) : A^{\oplus n} \to M)$ fin. gen. A-module

fin. gen. by assumption d)

$\Rightarrow \ker(\varphi) = (\ker(\varphi(U)))^\sim$ of finite type $\square$

<u>Cor</u> Let X be locally Noetherian.

a) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence in $\operatorname{QCoh}(X)$. Then:

$$\mathcal{F} \text{ coherent} \Leftrightarrow \mathcal{F}' \text{ and } \mathcal{F}'' \text{ coherent}$$

b) The coherent sheaves form a full abelian subcategory $\operatorname{Coh}(X) \subseteq \operatorname{QCoh}(X)$.

c) $\mathcal{F}, \mathcal{G} \in \operatorname{Coh}(X) \Rightarrow \mathcal{F} \oplus \mathcal{G} \in \operatorname{Coh}(X)$

$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \operatorname{Coh}(X)$

d) For $\varphi : Y \to X$ w/ X, Y locally Noetherian:

- $\varphi^*(\mathcal{N}) \in \operatorname{Coh}(Y) \ \forall \mathcal{N} \in \operatorname{Coh}(X)$
- If φ is a finite morphism, $\varphi_*(\mathcal{M}) \in \operatorname{Coh}(X) \ \forall \mathcal{M} \in \operatorname{Coh}(Y)$.

We'll later extend this to all proper φ

<u>Ex</u> $i : Y \hookrightarrow X$ closed immersion $\Rightarrow i_* : \operatorname{Coh}(Y) \to \operatorname{Coh}(X)$

<u>Ex</u> $j : Y = \mathbb{A}^1_k \setminus 0 \hookrightarrow X = \mathbb{A}^1_k \Rightarrow j_* \mathcal{O}_Y$ NOT coherent:

$\Gamma(j_* \mathcal{O}_Y) = k[z, z^{-1}]$ not fin. gen. over $\Gamma(\mathcal{O}_X) = k[z]$

Pf. a) Since X is locally Noetherian, the proposition says:

$$\mathcal{G} \text{ coherent} \Leftrightarrow \forall U = \operatorname{Spec} A \subset X,$$
$$\mathcal{G}(U) \text{ fin. gen. } A\text{-module}$$

We apply this to $\mathcal{G} \in \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$.

For $U = \operatorname{Spec} A$ affine, have exact sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0 \text{ of } A\text{-modules.}$$

Thus $\mathcal{F}(U)$ fin. gen. $/A$ $\Longleftrightarrow$ $\mathcal{F}'(U), \mathcal{F}''(U)$ fingen $/A$

A is Noetherian, hence submodules of fingen A-modules are again fingen over A

b) For $\mathcal{F}, \mathcal{G} \in \operatorname{Coh}(X)$ and $\varphi \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, part a) gives

$\ker(\varphi) \in \operatorname{Coh}(X)$ as a subsheaf of $\mathcal{F}$

$\operatorname{cok}(\varphi) \in \operatorname{Coh}(X)$ as a quotient of $\mathcal{G}$ etc.

c) As in a): $M, N \in \operatorname{Mod}(A)$ fin gen

$$\Rightarrow M \oplus N, \quad M \otimes_A N \in \operatorname{Mod}(A) \text{ fin gen.}$$

d) As in a):

- For any ring homom. $A \to B$:

  $$N \in \operatorname{Mod}(A) \text{ fin gen} \Rightarrow B \otimes_A N \text{ fin gen.}$$

  Therefore $\mathcal{N} \in \operatorname{Coh}(X)$ implies $\varphi^*(\mathcal{N}) \in \operatorname{Coh}(Y)$.

- If $A \to B$ is a finite ring extension, then

  $$M \in \operatorname{Mod}(B) \text{ fin gen} \Rightarrow M \text{ fin gen as } A\text{-module}$$

  So for $\varphi: Y \to X$ finite,

  $\mathcal{M} \in \operatorname{Coh}(Y)$ implies $\varphi_*(\mathcal{M}) \in \operatorname{Coh}(X)$. $\square$

<u>Upshot</u> For X locally Noetherian,

$$\operatorname{Coh}(X) = \{\mathcal{O}_X\text{-modules of finite type}\}$$

& these behave very nicely.

<u>Ex</u> For $\mathcal{F}, \mathcal{G} \in \operatorname{QCoh}(X)$ only quasicoherent,

it can happen that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \notin \operatorname{QCoh}(X)$:

e.g. for $\mathcal{M} = \mathcal{H}om_{\mathcal{O}_X}(\bigoplus_{n\in\mathbb{N}} \mathcal{O}_X, \mathcal{O}_X)$ on $X = \operatorname{Spec} A$:

$$\left.\begin{array}{l} \mathcal{M}(X) = \operatorname{Hom}_A(\bigoplus_{n\in\mathbb{N}} A, A) = \prod_{n\in\mathbb{N}} A \\ \mathcal{M}(D(f)) = \prod_{n\in\mathbb{N}} A_f \underset{\substack{\uparrow \\ \text{for } f \notin A^* \cup \{0\}}}{\not\simeq} (\prod_{n\in\mathbb{N}} A)_f \end{array}\right\} \Rightarrow \mathcal{M} \notin \mathrm{QCoh}(X)$$

For coherent sheaves this doesn't happen:

<u>Lemma</u> Let $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(X)$.

a) If $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{QCoh}(X)$.

b) If $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$ & X locally Noetherian, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{Coh}(X)$.

Pf. Wlog $X = \operatorname{Spec} A$ & $\mathcal{F} = \operatorname{cok}(\mathcal{O}_X^{\oplus m} \xrightarrow{\varphi} \mathcal{O}_X^{\oplus n})$.

$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$ left exact & commutes w/ finite $\oplus$

$\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \ker(\mathcal{F}^{\oplus n} \xrightarrow{(-)\circ\varphi} \mathcal{F}^{\oplus m})$

use $\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$

$\Rightarrow$ claim. $\square$

Similarly, finite presentations allow for a passage "local (stalkwise) $\rightsquigarrow$ local (on open sets)" as in statement b) below:

<u>Lemma</u> Let $X = (|X|, \mathcal{O}_X)$ be a ringed space.

a) $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$ w/ $\mathcal{F}$ of fin. presentation

$\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_p \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X,p}}(\mathcal{F}_p, \mathcal{G}_p) \;\forall p \in X$

b) $\mathcal{F}, \mathcal{G}$ both of finite presentation w/ $\mathcal{F}_p \simeq \mathcal{G}_p$ in $\mathrm{Mod}(\mathcal{O}_{X,p})$ for some $p \in X$

$\Rightarrow \exists U \ni p$ open w/ $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{G}|_U$

Pf. a) Wlog $\mathcal{F} \simeq \operatorname{cok}(\mathcal{O}_X^{\oplus m} \to \mathcal{O}_X^{\oplus n})$

$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$ & $\mathcal{H}om_{\mathcal{O}_{X,p}}(-, \mathcal{G}_p)$ right exact & commute with finite $\oplus$ $\Rightarrow$ wlog $\mathcal{F} = \mathcal{O}_X \Rightarrow$ done.

b) Apply part a) to $\mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p$ and its inverse. $\square$

4. Locally free sheaves

Def Let $X = (|X|, \mathcal{O}_X)$ be a ringed space.

An $\mathcal{O}_X$-module $\mathcal{E}$ is called

a) free if $\mathcal{E} \simeq \bigoplus_{i \in I} \mathcal{O}_X$ for some set I. (may be infinite)

Then $\mathrm{rk}(\mathcal{E}) := |I|$ is called the rank of $\mathcal{E}$

[$\mathrm{rk}(\mathcal{E})$ is well-defined if $\mathcal{O}_X \neq 0$: Let $p \in X$ w/ $\mathcal{O}_{X,p} \neq 0$ & pick a nontrivial $\varphi : \mathcal{O}_{X,p} \to k$ to a field k, then $\mathrm{rk}(\mathcal{E}) = \dim_k(\mathcal{E}_p \otimes_{\mathcal{O}_{X,p}} k)$]

b) locally free if $\exists$ open cover $X = \bigcup_\alpha U_\alpha$ sth $\mathcal{E}|_{U_\alpha}$ is free for each index α.

If X is connected & $\mathcal{O}_X|_{U_\alpha} \neq 0$ $\forall \alpha$, the rank $\mathrm{rk}(\mathcal{E}) := \mathrm{rk}(\mathcal{E}|_{U_\alpha})$ is well-defined & independent of the chosen U_α.

Ex Let $X = \mathbb{P}^1_k = \mathrm{Proj}\, R[x_0, x_1]$ and $0 := [0:1]$, $\infty := [1:0] \in X$.

Define a sheaf of modules $\mathcal{L} \in \mathrm{Mod}(\mathcal{O}_X)$ by

$$\mathcal{L}(U) := \{ f \in \mathcal{O}_X(U) \mid \text{if } 0 \in U, \text{ then } f(0) = 0 \}$$

$\Rightarrow$ For the charts $U_\infty = \mathrm{Spec}\, k[\tfrac{x_1}{x_0}] \ni \infty$
$U_0 = \mathrm{Spec}\, k[\tfrac{x_0}{x_1}] \ni 0$

we have $\mathcal{L}|_{U_\infty} = \mathcal{O}_X|_{U_\infty}$

$\mathcal{L}|_{U_0} = \mathcal{O}_X|_{U_0} \cdot \frac{x_0}{x_1} \simeq \mathcal{O}_X|_{U_0}$

$\Rightarrow$ $\mathcal{L} \in \mathrm{Mod}(\mathcal{O}_X)$ locally free of rank 1

But $\mathcal{L}$ is NOT free, since it has no global sections:

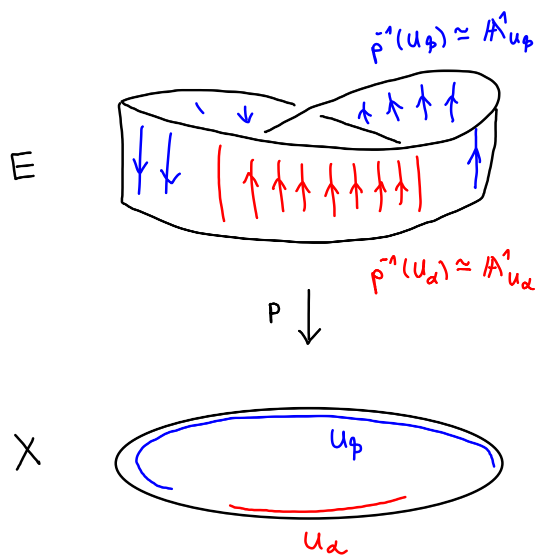
$$\mathcal{L}(X) = \{ f \in \underbrace{\mathcal{O}_X(X)}_{= k} \mid f(0) = 0 \} = \{0\}$$

Notation $\mathcal{O}_X(-1) := \mathcal{L}$ (reason will become clear later)

Ex Let X be a scheme & E a vector bdle on X, ie a scheme w/ a morphism $p : E \to X$ and w/ collection of iso's $\varphi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\sim} \mathbb{A}^n_{U_\alpha}$ over an open cover $X = \bigcup_{\alpha \in I} U_\alpha$ sth $\forall \alpha, \beta$ we have on $U_{\alpha\beta} := U_\alpha \cap U_\beta$:

$$\varphi_\beta \circ \varphi_\alpha^{-1} \big|_{U_{\alpha\beta}} : \mathbb{A}^n_{U_{\alpha\beta}} \longrightarrow \mathbb{A}^n_{U_{\alpha\beta}}$$

is given by a matrix $M \in GL_n(\mathcal{O}_X(U_{\alpha\beta}))$.



Define $\mathcal{E} := \mathcal{O}(E)$ "sheaf of sections of E" by

$$\mathcal{E}(U) := \mathrm{Hom}_U(U, p^{-1}(U))$$
$$= \{ s : U \to E \mid p \circ s = \mathrm{id} \}$$

$\Longrightarrow$ $\mathcal{E}$ is a locally free sheaf of $\mathrm{rk}(\mathcal{E}) = n$.

(for $U \subseteq U_\alpha$ the trivialization $\varphi_\alpha|_U : p^{-1}(U) \xrightarrow{\sim} \mathbb{A}^n_U$ induces an iso $\mathcal{E}(U) \simeq \mathrm{Hom}_U(U, \mathbb{A}^n_U) = \mathcal{O}_X^{\oplus n}(U)$ & these are $\mathcal{O}_X$-linear by def^n of vector bundles)

Thm Every locally free sheaf $\mathcal{E}$ of finite rank arises like this from a vector bundle E.

More precisely we get an equiv. of categories

$$VB(X) \xrightarrow{\sim} Loc(X), \quad E \mapsto \mathcal{O}(E)$$

where

$$Loc(X) := \{ \text{loc. free sheaves of finite rk} \} \subseteq Coh(X) \quad \text{as full subcat.}$$

$$VB(X) := \{ \text{vbdles of finite rk on } X \}$$

$\hookrightarrow$ where morphisms are defined as $f \in \mathrm{Hom}_X(E, F)$ that are "linear" in the sense that

$$\underset{\text{for } F \text{ on } V_\beta}{\psi_\beta} \circ f \circ \underset{\text{for } E \text{ on } U_\alpha}{\varphi_\alpha^{-1}} \big|_{U_{\alpha\beta}} \in \mathrm{Mat}(m \times n, \underset{U_{\alpha\beta} = U_\alpha \cap V_\beta}{\mathcal{O}(U_{\alpha\beta})}).$$

Pf. Given $\mathcal{E} \in Coh(X)$ locally free of rank n,

pick an open cover $X = \bigcup_{\alpha \in I} U_\alpha$ w/ $f_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}^{\oplus n}$

Since the f_α are $\mathcal{O}_X$-linear isomorphisms, we have

on $U_{\alpha\beta} := U_\alpha \cap U_\beta$:

$$\varphi_{\beta\alpha} := f_\beta \circ f_\alpha^{-1}|_{U_{\alpha\beta}} \in GL_n(\mathcal{O}_X(U_{\alpha\beta}))$$

$$\Longrightarrow \quad \begin{array}{ccc} E & := & \bigsqcup_\alpha \mathbb{A}^n_{U_\alpha} \Big/ \text{gluing via } \varphi_{\alpha\beta} \\ \downarrow & & \\ X & = & \bigcup_\alpha U_\alpha \end{array}$$

vector bdle with $\mathcal{O}(E) \simeq \mathcal{E}$.

The rest is left as an exercise. $\square$

Caution $Loc(X) \subset Coh(X)$ is NOT an abelian subcategory, e.g. for $X = \operatorname{Spec} k[z]$ we have $\mathcal{F} := \operatorname{cok}(\mathcal{O}_X \xrightarrow{z\cdot} \mathcal{O}_X) \notin Loc(X)$.

This is why we did NOT require morphisms in $VB(X)$ to have constant rank.

Local freeness can be read off on stalks:

Lemma Let X be a scheme. For $\mathcal{E} \in Coh(X)$, TFAE:

a) $\mathcal{E}$ is locally free

b) $\mathcal{E}_p$ is a free module over $\mathcal{O}_{X,p}$ $\forall p \in X$.

Pf. a) $\Rightarrow$ b) trivial

b) $\Rightarrow$ a): $\mathcal{E}_p \simeq (\mathcal{O}_X^{\oplus r})_p$

$\Rightarrow \exists U \ni p : \mathcal{E}|_U \simeq (\mathcal{O}_X^{\oplus r})|_U$

(coherent sheaves w/ isomorphic stalks at p are isomorphic in a nbhood of p, see the end of the section 3: coherent sheaves) $\square$

Caution $\mathcal{E}$ locally free does NOT imply that $\mathcal{E}|_U$ is free for ALL open affine $U \subseteq X$, only for some fine enough open cover!

For $X = \operatorname{Spec} R$ Noetherian & $M \in Mod(R)$ fin. gen, the above only says:

$$\mathcal{E} = \tilde{M} \text{ locally free} \Longleftrightarrow M_\mathfrak{p} \text{ free } R_\mathfrak{p}\text{-module } \forall \mathfrak{p} \in \operatorname{Spec} R$$

To rephrase this condition more "globally" we need some basic algebra:

<u>Recall</u> An R-module M is called <u>projective</u> if it satisfies the following equivalent properties:

i) $\exists N \in \mathrm{Mod}(R)$ with $M \oplus N \simeq \bigoplus_{i \in I} R$

ii) $\mathrm{Hom}_R(M,-) : \mathrm{Mod}(R) \to \mathrm{AbGps}$ is exact.

Pf. i) $\Rightarrow$ ii):
$$\begin{tikzcd} B \arrow[r, two heads] & C \\ M \arrow[u, dashed, "\exists ? \tilde{f}"] \arrow[ur, "f"'] \end{tikzcd}$$
Replace f by $f \oplus 0 : M \oplus N \to C$

$\Rightarrow$ wlog $M = \bigoplus_{i \in I} R$ free

& then existence of $\tilde{f}$ is clear

ii) $\Rightarrow$ i): Apply $\mathrm{Hom}_R(M,-)$ to an epi $\bigoplus_{i \in I} R \twoheadrightarrow M$. $\square$

<u>Prop</u> Let $M \in \mathrm{Mod}(R)$ be of finite presentation. Then TFAE:

a) M is a projective R-module.

b) $\forall \mathfrak{p} \in \mathrm{Spec}(R)$,

$M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module.

Pf. a) $\Rightarrow$ b):

$M \in \mathrm{Mod}(R)$ projective

$\Rightarrow \forall \mathfrak{p} : M_{\mathfrak{p}} \in \mathrm{Mod}(R_{\mathfrak{p}})$ projective (use characterization i))

$\Rightarrow$ wlog R local

Claim: Every fin. gen. module M over a local ring R is free.

To show this, let $k = R/\mathfrak{m}$.

$\Rightarrow M \otimes_R k \simeq M/\mathfrak{m}M$ fin. dim. vector space / k

Lifting a k-basis we get
$$\begin{tikzcd} R^n \arrow[r, "\varphi"] \arrow[d] & M \arrow[d] \\ k^n \arrow[r, "\sim"] & M \otimes_R k \end{tikzcd}$$

Nakayama $\Rightarrow \varphi : R^n \twoheadrightarrow M$ surjective

Since M is projective, applying $\mathrm{Hom}_R(M,-)$ gives (via ii)) a splitting
$$\begin{tikzcd} 0 \arrow[r] & \ker(\varphi) \arrow[r] & R^n \arrow[r, "\varphi"] & M \arrow[r] & 0 \\ & & M \arrow[u, "\exists"] \arrow[ur, "\mathrm{id}"'] \end{tikzcd}$$

$\Rightarrow R^n \simeq \ker(\varphi) \oplus M$

$\Rightarrow \ker(\varphi) \otimes_R k \simeq 0$ and so $\ker(\varphi) = 0$ by Nakayama

b) $\Rightarrow$ a):

If M is not projective, $\exists\, f : B \twoheadrightarrow C$ sth

$(-)\circ f :\ \mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C)$ not surjective.

these are again R-modules

$\Rightarrow\ N := \mathrm{cok}((-)\circ f) \neq 0$ in $\mathrm{Mod}(R)$

$\Rightarrow\ \exists\, \mathfrak{p} \in \mathrm{Spec}(R) :\ N_{\mathfrak{p}} \neq 0$

$$\begin{tikzcd} \mathrm{Hom}_R(M,B)_{\mathfrak{p}} \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_R(M,C)_{\mathfrak{p}} \arrow[d, "\cong"] \\ \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, B_{\mathfrak{p}}) \arrow[r, "(-)\circ f_{\mathfrak{p}}"] & \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) \end{tikzcd}$$

$\Rightarrow$ top map not surjective (since localization is an exact fctor)

where the vertical iso's hold since M is fin. presented

$\Rightarrow$ Bottom map $(-)\circ f_{\mathfrak{p}}$ not surjective even though still $B_{\mathfrak{p}} \twoheadrightarrow C_{\mathfrak{p}}$.

$\Rightarrow\ \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, -)$ not exact

$\Rightarrow\ M_{\mathfrak{p}} \in \mathrm{Mod}(R_{\mathfrak{p}})$ not projective (use ii)). $\square$

<u>Upshot</u> For $\mathcal{E} \in \mathrm{Coh}(X)$, TFAE:

a) $\mathcal{E}$ is locally free

b) For each $p \in X$, $\mathcal{E}_p$ is a free module over $\mathcal{O}_{X,p}$.

c) For each affine open $U = \mathrm{Spec}\, R \subseteq X$, $\mathcal{E}(U)$ is a projective module over $R = \mathcal{O}_X(U)$.

<u>Ex 1</u> $X = \mathrm{Spec}\, R$ w/ $R = R_1 \oplus R_2$

$= X_1 \sqcup X_2$ w/ $X_i = \mathrm{Spec}\, R_i$

$\Rightarrow$ Algebraically:

$M := R_1$ is not free ($\mathrm{Ann}(M) = 0 \oplus R_2 \neq 0$)

but projective over R ($M \oplus N \simeq R$ for $N := R_2$)

Geometrically:

$\mathcal{E} := \tilde{M}$ is locally free ($\mathcal{E}|_{X_1} \simeq \mathcal{O}_{X_1}$, $\mathcal{E}|_{X_2} \simeq 0$)

but not free (since $\mathrm{rk}(\mathcal{E})$ is not constant)

Ex 2 $X = \operatorname{Spec} R$ w/ $R = \mathcal{O}_K$ for a number field K

$= \{ a \in K \mid a \text{ integral over } \mathbb{Z} \}$

(e.g. $K = \mathbb{Q}(\sqrt{-5})$, $R = \mathbb{Z}[\sqrt{-5}]$)

$M = I \trianglelefteq R$ a non-zero ideal

$\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec} R$: $\underset{\substack{\uparrow \\ \text{hence principal}}}{M_{\mathfrak{p}} = I_{\mathfrak{p}}} \trianglelefteq \underset{\substack{\uparrow \\ \text{PID}}}{R_{\mathfrak{p}}}$

$\Rightarrow \tilde{M} \in \operatorname{Coh}(X)$ is locally free

BUT usually not free:

$\tilde{M}$ free $\iff$ $\tilde{M}$ free of rank 1 (take stalk at $\mathfrak{p}$)

$\iff I \trianglelefteq R$ principal ideal

e.g. $I := (2, 1+\sqrt{-5}) \trianglelefteq \mathbb{Z}[\sqrt{-5}]$ not principal,

though a direct summand in

$R^2 \simeq I \oplus \ker(p)$ for $p: R^2 \longrightarrow I$, $\exists s$

$(a,b) \mapsto 2a + (1+\sqrt{-5})b$

(p is split by $s: I \to R^2$, $x \mapsto (-x, \frac{1-\sqrt{-5}}{2} \cdot x) \ldots$)

Geometrically this $\tilde{M}$ is a "line bdle" (see below)

Functoriality?

Exercise a) $f: X \to Y$ morphism of schemes

& $\mathcal{E} \in \operatorname{Mod}(\mathcal{O}_Y)$ locally free

$\Rightarrow f^* \mathcal{E} \in \operatorname{Mod}(\mathcal{O}_X)$ locally free

(caution: analog for f_* usually NOT true)

b) $\mathcal{E} \in \operatorname{Mod}(\mathcal{O}_X)$ locally free of finite rank

$\Rightarrow \mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ locally free of same rank

and

$\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \quad \forall \mathcal{F} \in \operatorname{Mod}(\mathcal{O}_X)$.

In particular

$(\mathcal{E}^\vee)^\vee \simeq \mathcal{E}$ (replace $\mathcal{E}$ by $\mathcal{E}^\vee$ & take $\mathcal{F} = \mathcal{O}_X$).

c) Projection formula: $f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \xrightarrow{\sim} f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G})$

for $\mathcal{F}$ any $\mathcal{O}_X$-module

and $\mathcal{G}$ loc free $\mathcal{O}_Y$-module of finite rank

Line bundles

Prop For $\mathcal{L} \in \mathrm{Mod}(\mathcal{O}_X)$ TFAE :

a) $\mathcal{L}$ is a "line bundle"

ie. locally free of rank 1.

b) $\mathcal{L}$ is an "invertible sheaf"

ie the functor

$(-) \otimes_{\mathcal{O}_X} \mathcal{L} : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$

is an equivalence of categories.

If these conditions hold, then the inverse of the functor in b) is given by $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} (-)$, in particular $\exists$ natural iso $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$

Pf. a) $\Rightarrow$ b) :

$$\mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \underbrace{\longrightarrow}_{\text{evaluation hom}} \mathcal{O}_X$$ iso if $\mathcal{L}$ is locally free of rk 1 (check on stalks)

& then $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} (-)$ gives an inverse to $(-) \otimes_{\mathcal{O}_X} \mathcal{L}$.

b) $\Rightarrow$ a) :

If $(-) \otimes_{\mathcal{O}_X} \mathcal{L}$ is essentially surjective,

then $\exists \mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ w/ $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$

$\Rightarrow \forall p \in X : \mathcal{M}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{L}_p \simeq \mathcal{O}_{X,p}$

$\Rightarrow (-) \otimes_{\mathcal{O}_{X,p}} \mathcal{L}_p : \mathrm{Mod}(\mathcal{O}_{X,p}) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{O}_{X,p})$

equivalence of categories w/ inverse $\mathcal{M}_p \otimes_{\mathcal{O}_{X,p}} (-)$

$\Rightarrow \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{L}_p, -)$ ← left exact

$\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{M}_p \otimes \mathcal{L}_p, \mathcal{M}_p \otimes (-))$

right exact

$\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}, \mathcal{M}_p \otimes (-)) \simeq \mathcal{M}_p \otimes (-)$

$\Rightarrow \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{L}_p, -)$ exact,

ie $\mathcal{L}_p$ projective (= free) module over $\mathcal{O}_{X,p}$

$\Rightarrow \mathcal{L}$ locally free & then clearly of rk 1 $\square$

Cor / Def $\mathrm{Pic}(X) := \{\text{line bdles } \mathcal{L} \text{ on } X\} / \text{iso}$

forms an abelian group wrt the product $\otimes$.

We call it the Picard group of X.

Ex $X = \operatorname{Spec} R$ w/ R a PID

$\Rightarrow$ any f.g. R-module has the form $M \simeq R^n \oplus \text{Torsion}$ (some $n \in \mathbb{N}$)

$\Rightarrow$ any $\mathcal{M} \in \mathrm{Coh}(X)$ has the form

$\mathcal{M} = \mathcal{O}_X^n \oplus \mathcal{J}$ w/ $\mathrm{Supp}(\mathcal{J}) \overset{\neq}{\hookrightarrow} X$ proper closed subset

$\Rightarrow$ locally free = free

and hence $\mathrm{Pic}(X) = 0$

Special case: $\mathrm{Pic}(\mathbb{A}^1_k) = 0$ for any field k.

In fact $\mathrm{Pic}(\mathbb{A}^n_k) = 0 \ \forall n$ (easy, will be shown later)

Much harder (Quillen, Suslin 1976):

Locally free sheaves on $\mathbb{A}^n_k$ of any rank are free!

Rem $X = \operatorname{Spec} R$ w/ R a Noeth. domain

a) Every line bdle $\mathcal{L}$ on X is of the form $\mathcal{L} \simeq \tilde{I}$ w/ $0 \neq I \trianglelefteq R$ a "projective ideal" (ie an ideal which is projective as R-module)

↳ necessarily of rk 1, since a submodule of R

b) For such a projective ideal $I \trianglelefteq R$:

$$\tilde{I} \simeq \mathcal{O}_X \iff I = (a) \text{ principal ideal}$$

Pf. a) $\mathcal{L}$ invertible $\Rightarrow \exists \mathcal{M} \ \exists \varphi: \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$

X affine $\Rightarrow \mathcal{L} = \tilde{L}$ and $\mathcal{M} = \tilde{M}$ w/ $L, M \in \mathrm{Mod}(R)$

$\Rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L} = \widetilde{M \otimes_R L}$ (exercise, OK $\forall$ qcoh sheaves)

$\Rightarrow \exists$ iso $\varphi: M \otimes_R L \xrightarrow{\sim} R$

Pick any $m \in M \setminus \{0\}$, then we get an embedding

$$\varphi(m \otimes (-)): L \hookrightarrow R \quad \text{of } R\text{-modules}$$

$\Rightarrow L \simeq I$ for some ideal (= submodule) $I \trianglelefteq R$

b) $\mathcal{O}_X \xrightarrow[\alpha]{\sim} \tilde{I}$ gives $R \xrightarrow[\alpha]{\sim} I = (a)$ w/ $a = \alpha(1)$ ... $\square$

Ex $X = \operatorname{Spec} R$ w/ $R = \mathcal{O}_K$, K number field

$\Rightarrow$ every ideal $0 \neq I \trianglelefteq R$ is projective

(since $\forall \mathfrak{p} \in \operatorname{Spec} R$ the ring $R_\mathfrak{p}$ is DVR

so that $I_\mathfrak{p} \trianglelefteq R_\mathfrak{p}$ is a principal ideal)

$$\Rightarrow \operatorname{Pic}(X) \simeq \mathcal{C}\ell(K) := \frac{\{\text{fractional ideals}\}}{\{\text{p'pal ideals}\}}$$

the "ideal class group" from number theory.

e.g. $\operatorname{Pic}(X) \neq 0$ for $X = \operatorname{Spec} \mathbb{Z}[\sqrt{-5}]$ ...

Rem
- Same for any Dedekind domain.
- For R not Dedekind, not all ideals are projective.

  $\operatorname{Pic}(\operatorname{Spec} R)$ only sees the projective ones,

  eg. $R = k[x_1, .., x_n]$ for $n > 1$ NOT a PID

  but still $\operatorname{Pic}(\mathbb{A}^n_k) = 0$!

  ↳ will show this later, same for Spec(UFD) ...

A first non-affine example:

Ex $\operatorname{Pic}(\mathbb{P}^1_k) \simeq \mathbb{Z}$:

Write $X := \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$

$\cup$

$U_i = \operatorname{Spec} k[x_i]$ $(i = 0, 1)$

$\mathcal{L} \in \operatorname{Pic}(X)$

$k[x_i]$ PID

$\Rightarrow \mathcal{L}|_{U_i} \in \operatorname{Pic}(U_i) \simeq \operatorname{Pic}(\mathbb{A}^1_k) = 0$

$\Rightarrow \exists$ iso $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$

$\Rightarrow \varphi_1^{-1} \circ \varphi_0 : \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_U$ on $U = U_1 \cap U_2 \simeq \operatorname{Spec} k[t, t^{-1}]$

$\mathcal{O}_U$- linear, ie multiplication by some $f \in k[t, t^{-1}]$

but also an isomorphism

$\Rightarrow f \in \mathcal{O}_U^*(U)$, ie invertible on U

$\Rightarrow f(s, s^{-1}) = s^n$ (some $n \in \mathbb{Z}$)

But then gluing gives $\mathcal{L} \simeq \mathcal{O}_X(n)$

Thus $\mathbb{Z} \xrightarrow{\sim} \operatorname{Pic}(X)$ epi, hence iso.

$n \longmapsto \mathcal{O}_X(n)$

Extending sections from basic open sets

Def The *fiber* of $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ at $x \in |X|$ is

$$\mathcal{F}(x) := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \simeq \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$$

Get *evaluation homom* $\mathcal{F}_x \to \mathcal{F}(x)$

For $s \in \mathcal{F}(U)$, $x \in U$, we define the *value* $s(x) := \mathrm{image}(s_x) \in \mathcal{F}(x)$.

Rem For $\mathcal{L} \in \mathrm{Pic}(X)$ & $s \in \mathcal{L}(X) = \Gamma(X, \mathcal{L})$ the set

$$D(s) := \{ x \in |X| : s(x) \neq 0 \} \subset |X|$$

is open.

Pf. Check on an open cover $\Rightarrow$ wlog $\mathcal{L} \simeq \mathcal{O}_X$ & $X = \mathrm{Spec}\, A$

Then $D(s)$ basic open in Zariski top on $\mathrm{Spec}\, A$ $\square$

Prop Let X be qcqs, $\mathcal{F} \in \mathrm{QCoh}(X)$, $\mathcal{L} \in \mathrm{Pic}(X)$ and $s \in \Gamma(X, \mathcal{L})$.

a) For any $f \in \Gamma(X, \mathcal{F})$ w/ $f|_{D(s)} = 0$, $\exists n \in \mathbb{N}$ sth $f \otimes s^n = 0$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

b) For any $g \in \Gamma(D(s), \mathcal{F})$, $\exists n \in \mathbb{N}$ sth $g \otimes s^n \in \Gamma(D(s), \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ extends to a global section $h \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$

Pf. a) X qc $\Rightarrow$ $\exists$ finite open affine cover $X = \bigcup_{i=1}^r U_i$ w/ $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$

Say $U_i = \mathrm{Spec}\, A_i$, $s_i := s|_{U_i} \in A_i$

$$f|_{D(s)} = 0 \Rightarrow f|_{U_i} \in \Gamma(U_i, \mathcal{F})$$
$$\downarrow$$
$$0 \in \Gamma(U_i, \mathcal{F})_{s_i}$$

$\Rightarrow \exists n_i :\ s_i^{n_i} \cdot f|_{U_i} = 0$ in $\Gamma(U_i, \mathcal{F})$

$\Rightarrow n := \max_i \{n_i\}$ works

b) For all i, we have

$$\Gamma(\underbrace{U_i \cap D(s)}_{= D(s_i) \text{ basic open in } U_i = \operatorname{Spec} A_i}, \mathcal{F}) = \Gamma(U_i, \mathcal{F})_{s_i}$$

since $\mathcal{F}$ is quasicoherent & U_i is affine.

$\Rightarrow \exists N_i : \; s_i^{N_i} \cdot g|_{U_i \cap D(s)} \hat{=} g \otimes s_i^{\otimes N_i}$ extends

to a section $\in \Gamma(U_i, \mathcal{F})$ $\hookrightarrow \simeq \mathcal{F} \otimes \mathcal{L}^{\otimes N_i}$ on U_i since $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$

Take $N := \max_i \{N_i\}$

$\Rightarrow g \otimes s^{\otimes N} \in \Gamma(D(s), \mathcal{F} \otimes \mathcal{L}^{\otimes N})$

extends to sections $h_i \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^{\otimes N}) \; \forall i$

These h_i needn't glue. But $h_{ij} := h_i - h_j \in \Gamma(U_{ij}, \dots)$

on $U_{ij} = U_i \cap U_j$ has $h_{ij}|_{U_{ij} \cap D(s)} = 0$

$\Rightarrow$ by a), $\exists M : h_{ij} \otimes s^M = 0$ on all of U_{ij}

$\hookrightarrow$ U_{ij} qc since X qs!

$\Rightarrow$ the $h_i \otimes s^M$ glue, hence $n = MN$ works $\square$