Contents

V. Warm-Up: Algebraic Curves

0. Reminder: Dimension

Def The dimension of a scheme X is

$$\dim X := \sup\left\{ d\in\mathbb{N}_0 \;\middle|\; \begin{array}{l} \exists\, x_0,x_1,\dots,x_d\in|X| \\ \text{s.th } \forall i:\ x_i\in\overline{\{x_{i+1}\}}\end{array}\right\}$$

$$= \sup\left\{ d\in\mathbb{N}_0 \;\middle|\; \begin{array}{l}\exists\, X_0 \overset{\neq}{\hookrightarrow} X_1 \overset{\neq}{\hookrightarrow}\cdots\overset{\neq}{\hookrightarrow} X_d\hookrightarrow X\\ \text{w/ integral closed } X_i\subseteq X\end{array}\right\}$$

Rem a) $\dim X$ only depends on $|X|$

"Krull dim"

b) For $X = \operatorname{Spec} R$ affine, $\dim X = \dim R$:

e.g. $\dim \operatorname{Spec}\mathbb{Z} = 1$,

$\dim \mathbb{A}^n_k = n$ for any field k, etc

c) For any open cover $X = \bigcup_{i\in I} U_i$,

have $$\dim X = \sup_{i\in I} \dim U_i .$$

(any finite chain of specializations lies in some U_i)

Lemma For $f : X \to Y$ integral morphism, we have $\dim X \leq \dim Y$ & equality holds if f is dominant.

Pf. By c) above, wlog $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ affine.

$f : Y \to X$ integral

$\Rightarrow\ \varphi = f^\# : A \to B$ integral

$\Rightarrow\ \varphi : A/\ker\varphi \hookrightarrow B$ integral & injective

$\Rightarrow\ \dim(A/\ker\varphi) = \dim(B)$ by "going up"

$\wedge|$

$\dim(A)$ w/ equality if $\ker\varphi = 0$

$\Rightarrow\ \dim X \leq \dim Y$ w/ equality if f is dominant. $\square$

Cor X integral scheme of finite type over a field k

$\Rightarrow\ \dim X = \operatorname{trdeg}(k(X)/k)$

$= \dim U$ for any nonempty open $U \subseteq X$.

Pf. Wlog $X = \operatorname{Spec} R$ affine

R is a fin. gen. k-alg

$\Rightarrow$ Noether normalizat^n: For some $n \in \mathbb{N}_0$,

$\exists\, k[x_1, \dots, x_n] \hookrightarrow R$ finite extension.

$$\Rightarrow \dim X = n = \operatorname{trdeg}(k(x_1,\dots,x_n)/k)$$
$$= \operatorname{trdeg}(k(X)/k)$$

$k(x_1,\dots,x_n) \hookrightarrow \operatorname{Quot}(R) = k(X)$ finite

Moreover $k(X) = k(U)$ for any nonempty $U \subseteq X$. $\square$

Def A <u>function field in n variables over k</u> is a finite extension K of $k(x_1, \dots, x_n)$.

Ex $K = k(X)$ for an integral scheme X of finite type over k w/ $\dim X = n$.

... and that's the only example:

Prop For any fct^n field K in n variables $/k$, $\exists$ proper normal integral scheme X of finite type $/k$ w/ $\dim X = n$ sth $k(X) \simeq K$.

Note $k(X) = k(\overline{X})$ for $X \subset \overline{X}$ open dense

$k(X) = k(\tilde{X})$ for the normalizat^n $\tilde{X} \to X$

Pf. K/k fin.gen. $\Rightarrow \exists m \in \mathbb{N}_0\ \exists J = \sqrt{J} \trianglelefteq k[x_1,\dots,x_m]$

sth $K = \operatorname{Quot}(R)$

w/ $R = k[x_1,\dots,x_m]/J$

Let $X_0 := V(J) \subset \mathbb{A}^m_k$

$X_1 := \overline{X_0} \subset \mathbb{P}^m_k$ (proper over k)

$X := \tilde{X}_1$ normalization of X_1

$\Rightarrow X$ normal w/ $k(X) \simeq K$ & still proper $/k$

since $\tilde{X}_1 \to X_1$ is finite for X_1 of finite type $/k$ $\square$

1. Curves & function fields

Def A *curve* over a field k is a separated scheme C of finite type $/k$ which is integral of dimension $\dim C = 1$.

Rem If C is a curve $/k$ & $p \in |C|$ a closed pt, then $\mathcal{O}_{C,p}$ is a 1-dim Noetherian local domain.

Hence TFAE:

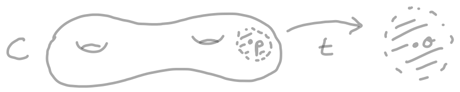
a) $\mathcal{O}_{C,p}$ is normal

b) $\mathfrak{m}_{C,p} \trianglelefteq \mathcal{O}_{C,p}$ is principal

c) $\mathcal{O}_{C,p}$ is a DVR

We call any element $t \in \mathcal{O}_{C,p}$ w/ $\mathfrak{m}_{C,p} = (t)$ a *local parameter* on C at the point p

(intuitively: a "local coordinate fct^n" on C)



Pf of a) $\Leftrightarrow$ b) $\Leftrightarrow$ c):

Let A be a 1-dim Noetherian local domain

Note: $\operatorname{Spec} A = \{0, \mathfrak{m}\}$.

1) A normal $\Rightarrow$ $\mathfrak{m}$ principal:

$a \in A \setminus (A^* \cup \{0\})$

$\Rightarrow V(a) = \{\mathfrak{m}\}$

$\Rightarrow \sqrt{(a)} = \mathfrak{m}$, ie $\exists\, n \in \mathbb{N}: \mathfrak{m}^n \subseteq (a)$ (use Noetherian)

Pick n minimal as above & let $b \in \mathfrak{m}^{n-1} \setminus (a)$

Then $x := \frac{b}{a} \in K = \operatorname{Quot}(A)$ but $x \notin A$

A normal $\Rightarrow$ x not integral over A,

ie $A[x]$ not fingen over A

$\Rightarrow x\mathfrak{m} \nsubseteq \mathfrak{m}$ (else $A[x] \hookrightarrow \operatorname{End}_A(\mathfrak{m})$ fin gen $/A$)

But $x\mathfrak{m} = \frac{b\mathfrak{m}}{a} \subseteq \frac{\mathfrak{m}^n}{a} \subseteq A$

$\Rightarrow x\mathfrak{m} = A$ (being a submodule of A & not $\subseteq \mathfrak{m}$)

$\Rightarrow \mathfrak{m} = tA$ for $t := x^{-1}$ (in particular $t \in A$)

2) $\mathfrak{m} \trianglelefteq A$ principal $\Rightarrow$ A DVR :

Say $\mathfrak{m} = (t)$. Note : $\bigcap_{n \geq 0} (t^n) = 0$:

$a \in \bigcap_{n\geq 0} (t^n) \Rightarrow a \in (at) \Rightarrow \exists b \in R : \; a = abt$

i.e. $a \cdot \underbrace{(1-bt)}_{\in A \setminus \mathfrak{m} = A^*} = 0 \Rightarrow a = 0$

For $a \in A \setminus \{0\}$, take $n \in \mathbb{N}_0$ maximal w/ $a \in (t^n)$

$\Rightarrow a = \varepsilon t^n$ w/ $\varepsilon = \frac{a}{t^n} \in A \setminus (t) = A^*$

Check that $A \setminus \{0\} \to \mathbb{N}_0$, $a \mapsto n$

gives a discrete valuation $K^* \to \mathbb{Z}$.

3) DVR's are PID's, hence normal. □

Cor A curve C over k is normal

iff $\mathcal{O}_{C,p}$ is a DVR for all closed pts $p \in |C|$.

$C = V(y^2 - x^3)$ not normal :

$\mathfrak{m}_p = (x,y) \trianglelefteq \mathcal{O}_{C,p}$ not principal (Exercise)

Prop ("extension of morphisms from curves")

Let C be a normal curve / k

& X a proper scheme / k.

$\Rightarrow$ any morphism $f_0 : C_0 \to X$

on a nonempty open $C_0 \subset C$

extends to a unique $f : C \to X$.

Pf. Uniqueness clear as C reduced & X separated.

Existence: Let $p \in |C| \setminus |C_0|$.

Consider

$$\begin{tikzcd}
\operatorname{Spec} k(C) \arrow[r, hook, "\text{generic pt}"] \arrow[d, hook] & C_0 \arrow[r, "f_0"] & X \arrow[d, "\text{proper}"] \\
\operatorname{Spec} \underbrace{\mathcal{O}_{C,p}}_{\text{DVR!}} \arrow[rru, dashed, "\exists f_p"] \arrow[rr] & & \operatorname{Spec} k
\end{tikzcd}$$

Valuative criterion : $\exists$ extension $f_p : \operatorname{Spec} \mathcal{O}_{C,p} \to X$

making the diagram commute

Now f_p extends to $f_p : U_p \to X$ w/ $p \in U_p \subset C$ open

(since X is of finite type over k)

By uniqueness these extensions glue to $f : C \to X$. □

Cor ("Unique compactification of curves")

For any affine normal curve C_0 over k,

$\exists!$ (up to iso) proper normal curve C over k

with an open embedding $C_0 \hookrightarrow C$.

Pf. **Existence:** Pick an embedding $C_0 \subset \mathbb{A}^n_k$ & let $C :=$ normalizatⁿ of the closure $\overline{C_0} \subset \mathbb{P}^n_k$ (see previous section)

Uniqueness: Let $C_0 \underset{\text{open}}{\hookrightarrow} C_i$ w/ C_i normal & proper $(i = 1, 2)$

By the previous proposition, can extend:

$$\begin{tikzcd} C_1 \arrow[dr, dashed, "\exists! \varphi"] & \\ C_0 \arrow[u, phantom, "\cup"] \arrow[r, hook] & C_2 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[dr, dashed, "\exists! \psi"] & \\ C_0 \arrow[u, phantom, "\cup"] \arrow[r, hook] & C_1 \end{tikzcd}$$

Uniqueness $\Rightarrow \varphi \circ \psi = \mathrm{id}$ & $\psi \circ \varphi = \mathrm{id} \Rightarrow C_1 \xrightarrow{\sim} C_2$ $\square$

Ex Let $f(x) \in k[x]$ monic & char $k \neq 2$.

By problem 9.2 (AG I),

$C_0 := V(y^2 - f(x)) \subset \mathbb{A}^2_k$

is normal iff f is square-free.

In this case,

a) describe the normal proper C w/ $C_0 \underset{\text{open}}{\subset} C$.

b) Show that $\mathrm{pr} : C_0 \to \mathbb{A}^1_k, (x,y) \mapsto x$ extends to a finite morphism $C \to \mathbb{P}^1_k$.

c) How many points are there in $C \setminus C_0$?

We call C an *elliptic curve* if $\deg f \in \{3, 4\}$, and a *hyperelliptic curve* for $\deg f \geq 5$.

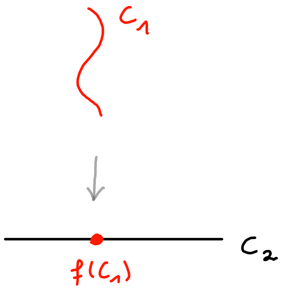
Lemma ("morphism between curves")

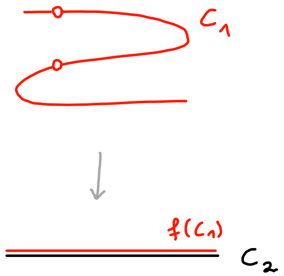
Let $f : C_1 \to C_2$ be a morphism of curves $/ k$.

Then

a) either $f(C_1)$ is a single point,

b) or f is dominant.





Pf. Assume $f(C_1)$ is not a single point.

Since C_1 is irreducible, so is the closure $\overline{f(C_1)} \subseteq C_2$

$\dim C_2 = 1 \Rightarrow \overline{f(C_1)} = C_2$ □

Rem If C_1 is proper, we'll see that in b) the morphism $f : C_1 \to C_2$ is finite.

Thm ("Curves = function fields")

The functor $C \mapsto k(C)$ gives an equivalence of categories between

a) the category of normal proper curves $/ k$, w/ dominant morphisms, and

b) the category of fct fiels in one variable $/ k$, w/ homomorphisms of k-algebras.

Pf. Essentially surjective: by previous section.

Full: Let $\varphi \in \mathrm{Hom}_{k\text{-alg}}(k(C_2), k(C_1))$.

$$\Rightarrow \varphi^a : \mathrm{Spec}\, \underbrace{k(C_1)}_{= \mathcal{O}_{C_1,\eta_1}} \longrightarrow \mathrm{Spec}\, \underbrace{k(C_2)}_{= \mathcal{O}_{C_2,\eta_2}} \longrightarrow C_2$$

Extend to a morphism $U \to C_2$ w/ $\emptyset \neq U \subseteq C_1$ open

(use that C_2 is of finite type $/ k$)

Prop. w/ C_1 normal & C_2 proper $\Rightarrow$ extends to $C_1 \xrightarrow{\text{dominant}} C_2$

Faithful: $f : C_1 \to C_2$ determined by $f|_U$ for any open dense $U \subset C_1$ (since C_1 reduced, C_2 separated), hence also by $f^* : k(C_2) \to k(C_1)$. □

Cor ("morphisms between proper curves")

Let $f : C' \to C$ be a non-constant morphism between curves over k.

If C' is proper, then so is C and then f is a finite surjective morphism.

Pf. $\begin{matrix} C' \text{ proper} / k \\ C \text{ separated} / k \end{matrix} \Rightarrow f$ proper (cancellation property)

f non-constant & proper & C curve $\Rightarrow f$ surjective
$\Rightarrow C$ also proper
(AGI, problem 11.2)

Pick $U = \operatorname{Spec} A \subset C$ any open affine

$\Rightarrow A \subset k(C) \subset k(C')$

Put $A' :=$ integral closure of A in $k(C')$

AG1 (II.7) $\Rightarrow A'$ is a finite A-module

Claim: $\operatorname{Spec} A' \simeq f^{-1}(U)$ over U ($\Rightarrow f$ finite / U)

Indeed:

$U' := \operatorname{Spec} A'$ affine normal curve

$\Rightarrow \exists!$ cpactification $U' \underset{\text{open}}{\subset} U''$

w/ U'' proper normal curve

But $k(U'') = k(U') = \operatorname{Quot}(A') = k(C')$

Uniqueness in previous thm $\Rightarrow U'' \simeq C'$

$\Rightarrow U' = \operatorname{Spec} A'$ is isomorphic to an open in C'

By construction $U' \subset f^{-1}(U)$

If $U' \neq f^{-1}(U)$, pick $p \in f^{-1}(U) \setminus U'$

Let $V = \operatorname{Spec} B \subset f^{-1}(U)$ open nbhd of p

Then

$$\begin{tikzcd} V' \arrow[d] \arrow[r, phantom, "\subset"] & C' \arrow[d] \\ U \arrow[r, phantom, "\subset"] & C \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} B \arrow[r, phantom, "\subset"] & k(C') \\ A \arrow[u, hook] \arrow[r, phantom, "\subset"] & k(C) \arrow[u, hook] \end{tikzcd}$$

$\Rightarrow B \supset A'$ since B is normal & contains A

$\Rightarrow V \subset U'$ but $p \notin U'$ ↯

□

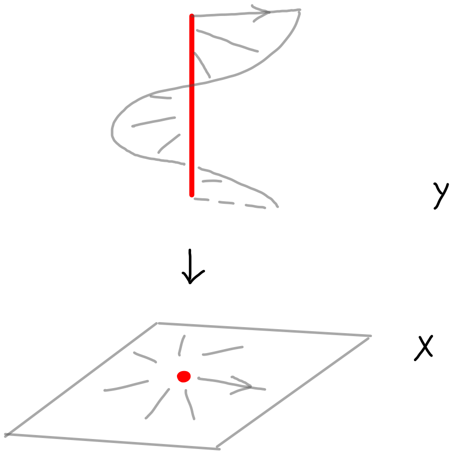
Upshot Proper normal curves are "the same thing" as fct^n fields in one variable !

Rem For $\dim X > 1$, $k(X)$ doesn't determine X :

e.g. $X = \mathbb{P}^2_k$ and $Y = Bl_p(X)$

are both normal (even smooth)

w/ $k(X) \simeq k(Y)$ but $X \not\simeq Y$ (exercise)



$\longrightarrow$ birational geometry ...

2. Divisors & the Picard group

Motivation Control zeroes & poles of rat$^{\ell}$ fct^{s} !

Def Let C be a normal curve over a field k.

A divisor on C is an element of the free abelian gp

$$\mathrm{Div}(C) := \bigoplus_{\substack{p \in |C| \\ \text{closed pt}}} \mathbb{Z} \cdot [p]$$

We write such divisors as $D = \sum_p n_p \cdot [p]$ with $n_p = n_p(D) \in \mathbb{Z}$ almost all zero, and put

$$\mathrm{Supp}(D) := \{ p \in |C| : n_p \neq 0 \}.$$

We call D an effective divisor if it lies in the semigroup

$$\mathrm{Div}^+(C) = \{ D \in \mathrm{Div}(C) \mid \forall p : n_p(D) \geq 0 \}.$$

We write $D \geq D'$ if $D - D' \in \mathrm{Div}^+(C)$.

Ex For any closed pt $p \in |C|$, we know $\mathcal{O}_{C,p}$ is a DVR.

Let $v_p : k(C)^* \to \mathbb{Z}$ be the unique valuation of $k(C) = \mathrm{Quot}(\mathcal{O}_{C,p})$ w/ $\mathcal{O}_{C,p} = \{ f \in k(C)^* \mid v_p(f) \geq 0 \} \cup \{0\}$.

For any rat$^{\ell}$ fct $f \in k(C)^*$ we define its divisor of zeroes & poles by

$$\mathrm{div}(f) := \sum_p v_p(f) \cdot [p]$$

Divisors of the form $\mathrm{div}(f)$ w/ $f \in k(C)^*$ are called principal divisors.

We get a group homomorphism

$$\begin{aligned} \mathrm{div} : k(C)^* &\longrightarrow \mathrm{Div}(C) \\ f &\longmapsto \mathrm{div}(f) \end{aligned}$$

Q What's its kernel & cokernel ?

Lemma $\ker(\mathrm{div}: k(C)^* \to \mathrm{Div}(C)) = \Gamma(C, \mathcal{O}_C)^*$

Pf. Any $f \in k(C)^*$ corresponds to a rational map $f: C \dashrightarrow \mathbb{A}^1_k$ and for $p \in |C|$ closed we have:

$$v_p(f) \geq 0 \iff f \in \mathcal{O}_{C,p}$$
$$\iff f : C \dashrightarrow \mathbb{A}^1_k \text{ is defined at } p$$

Thus $\Gamma(C, \mathcal{O}_C) = \mathrm{Hom}_k(C, \mathbb{A}^1_k)$

$$= \{ f \in k(C)^* \mid \mathrm{div}(f) \geq 0 \}$$

$$\Rightarrow \Gamma(C, \mathcal{O}_C)^* = \{ f \in k(C)^* \mid f, f^{-1} \in \Gamma(C, \mathcal{O}_C) \}$$

$$= \{ f \in k(C)^* \mid \mathrm{div}(f) \geq 0 \ \& \ \underbrace{\mathrm{div}(f^{-1})}_{= -\mathrm{div}(f)} \geq 0 \}$$

$$= \{ f \in k(C)^* \mid \mathrm{div}(f) = 0 \}$$

$$= \ker(\mathrm{div}: k(C)^* \to \mathrm{Div}(C)). \quad \square$$

Rem If C is proper & k alg closed, then $\Gamma(C, \mathcal{O}_C) = k$ (exercise).

For coker(div) we need more work:

Def Let $D = \sum_p n_p \cdot [p] \in \mathrm{Div}(C)$.

Define $\mathcal{O}_C(D) \in \mathrm{Mod}(\mathcal{O}_C)$ by

$$\Gamma(U, \mathcal{O}_C(D)) := \{ f \in k(C) \mid \forall p \in |U| : v_p(f) \geq -n_p \},$$

formally put $v_p(f) = \infty$ for the zero function $f = 0$

ie for $f \in \Gamma(U, \mathcal{O}_C(D))$:

- $n_p \geq 0 \Rightarrow f$ has a pole of order $\leq n_p$ at p
- $n_p < 0 \Rightarrow f$ has a zero of order $\geq -n_p$ at p

Ex $C = \mathbb{P}^1_k$ & $p = [1:0] \in |C|$

$\Rightarrow$ For $D = n \cdot [p]$ w/ $n \in \mathbb{Z}$, we recover $\mathcal{O}_C(D) = \mathcal{O}(n)$:

$$\cdots \subset \mathcal{O}_C(-[p]) \subset \mathcal{O}_C \subset \mathcal{O}_C([p]) \subset \cdots$$

For any normal curve C / k we have:

Lemma For any $D \in \mathrm{Div}(C)$,

$\mathcal{O}_C(D)$ is a line bundle.

Pf. Let $D = \sum_p n_p \cdot [p]$.

For any closed point p, pick $f \in \mathcal{O}_{C,p}$ w/ $v_p(f) = |n_p|$

Pick $U \underset{\text{open}}{\hookrightarrow} C$ w/ $p \in U$,

$f \in \mathcal{O}_C(U)$,

$\mathrm{Supp}(D) \cap U = \mathrm{Supp}(\mathrm{div}(f)) \cap U = \{p\}$.

$$\begin{tikzcd} \text{Then} \quad \mathcal{O}_C|_U \arrow[r, "\sim"] & \mathcal{O}_C(D)|_U : \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ s \arrow[r, mapsto] & f^e \cdot s \quad \text{w/ } e := \mathrm{sgn}(n_p) \in \{\pm 1\} \end{tikzcd}$$

For any $V \subset U$ & $s \in k(C)$,

$$t \in (\mathcal{O}_C(D))(V) \iff \forall q \in V : v_q(t) \geq -n_q = e \cdot v_q(f)$$

$$\iff \forall q \in V : v_q(t/f^e) \geq 0$$

$$\iff s := t/f^e \in \mathcal{O}_C(V)$$

□

Lemma For any $D, D' \in \mathrm{Div}(C)$,

$\exists$ iso

$$\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D') \xrightarrow{\sim} \mathcal{O}_C(D+D').$$

Pf. For $U \subset C$ open, have a homom.

$$\begin{tikzcd} \Gamma(U, \mathcal{O}_C(D)) \otimes \Gamma(U, \mathcal{O}_C(D')) \arrow[r] & \Gamma(U, \mathcal{O}_C(D+D')) \\ f \otimes g \arrow[r, mapsto] & f \cdot g \end{tikzcd}$$

$\uparrow$
product in $k(C)$

since

$v_p(fg) = v_p(f) + v_p(g)$ for all closed pt^s $p \in |C|$.

These homom. are compatible w/ restriction, hence pass to sheafification. Being an iso can be checked on stalks:

$$\underbrace{\mathcal{O}_C(D)_p}_{t^{-n_p} \cdot \mathcal{O}_{C,p}} \otimes_{\mathcal{O}_{C,p}} \underbrace{\mathcal{O}_C(D')_p}_{t^{-n'_p} \cdot \mathcal{O}_{C,p}} \xrightarrow{\sim} \underbrace{\mathcal{O}_C(D+D')_p}_{t^{-n_p - n'_p} \cdot \mathcal{O}_{C,p}}$$

□

Cor $\mathrm{Div}(C) \to \mathrm{Pic}(C)$, $D \mapsto \mathcal{O}_C(D)$

is a group homomorphism.

Prop $\mathrm{Div}(C) \to \mathrm{Pic}(C)$ is surjective.

Pf. Let $\eta \in C$ generic pt

$\mathcal{L} \in \mathrm{Pic}(C) \Rightarrow \mathcal{L}_\eta$ is a 1-dim vspace over $K := \mathcal{O}_{C,\eta} = k(C)$

Fix $s \in \mathcal{L}_\eta$ w/ $\mathcal{L}_\eta = K \cdot s$.

For any closed $p \in |C|$,

$\mathcal{L}_p \hookrightarrow \mathcal{L}_\eta = K \cdot s$ free $\mathcal{O}_{C,p}$-submodule of rk 1,

say $\mathcal{L}_p = \mathcal{O}_{C,p} \cdot fs$ for some $f \in K$

$= \{ gs \mid g \in K, v_p(g) \geq -n_p \}$ w/ $n_p = -v_p(f)$

Claim: $n_p = 0$ for almost all p:

Let $\emptyset \neq U \subset C$ open w/ $s \in \mathcal{L}(U) \subset \mathcal{L}_\eta$.

Shrinking U may assume $\forall$ closed pt^s $p \in |U|$:

$$s(p) \neq 0 \in \mathcal{L}_p \otimes_{\mathcal{O}_{C,p}} \kappa(p)$$

$\Rightarrow \forall p \in |U| : n_p = 0$

$\Rightarrow D := \sum_p n_p \cdot [p]$ is a divisor

and by constructⁿ $\mathcal{O}_C(D) \xrightarrow{\sim} \mathcal{L}$

$f \longmapsto f \cdot s$ $\square$

Thm The following sequence is exact:

$$0 \to \Gamma(C, \mathcal{O}_C)^* \to k(C)^* \to \mathrm{Div}(C) \to \mathrm{Pic}(C) \to 0$$

$f \longmapsto \mathrm{div}(f)$

$D \longmapsto \mathcal{O}_C(D)$

Pf. Only remains to show:

$$\mathcal{O}_C(D) \simeq \mathcal{O}_C \iff \exists f \in k(C)^* : D = \mathrm{div}(f)$$

"$\Leftarrow$": For $D = \mathrm{div}(f)$ get $\mathcal{O}_C(D) \xrightarrow{\sim} \mathcal{O}_C$

$s \longmapsto f \cdot s$

"$\Rightarrow$": For $D \in \mathrm{Div}(C)$ w/ $\mathcal{O}_C(D) \xrightarrow{\sim} \mathcal{O}_C$,

consider the induced iso on generic stalks

$$\begin{tikzcd} (\mathcal{O}_C(D))_\eta \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{O}_{C,\eta} \\ \mathcal{O}_{C,\eta} \arrow[ur] & \end{tikzcd}$$

canonically since $\eta \in C \setminus \mathrm{Supp}(D)$

multiplicatⁿ by some $f \in K^* = \mathcal{O}_{C,\eta}^*$ (automorphism of a 1-dim vspace)

Then $\mathrm{div}(f) = D$ by construction. $\square$

Upshot $\mathrm{Pic}(C) \simeq \mathrm{Div}(C)/_{\sim}$

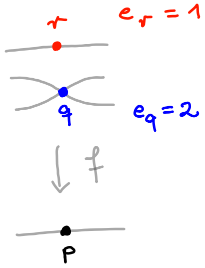
where $\sim$ is *linear equivalence* of divisors, defined by

$$D \sim D' \;:\Longleftrightarrow\; \exists f \in k(C)^* \,:\, D - D' = \mathrm{div}(f).$$

Functoriality?

Def Let $f: C' \to C$ be a dominant morphism of normal curves / k. For closed pt^s $p \in |C|$, pick a local parameter $t \in \mathcal{O}_{C,p}$. The *ramification index* of f at a point $q \in f^{-1}(p)$ is

$$e_q(f) := \nu_q(f^\#(t)) \in \mathbb{N}_0.$$



We define a gp homom.

$$f^*: \mathrm{Div}(C) \to \mathrm{Div}(C')$$

by $f^*([p]) := \sum_{q \mapsto p} e_q(f) \cdot [q]$.

Exercise Have a commutative diagram

$$\begin{tikzcd}
k(C)^* \arrow[r] \arrow[d, "f^*"'] & \mathrm{Div}(C) \arrow[r] \arrow[d, "f^*"'] & \mathrm{Pic}(C) \arrow[d, "f^*"'] \\
k(C')^* \arrow[r] & \mathrm{Div}(C') \arrow[r] & \mathrm{Pic}(C')
\end{tikzcd}$$

The group $\mathrm{Div}(C)$ has a "discrete" part:

Def For $D = \sum_p n_p \cdot [p] \in \mathrm{Div}(C)$ consider the *degree*

$$\deg(D) := \sum_p n_p \cdot \underbrace{[\kappa(p):k]}_{\text{degree of residue field of the pt } p \text{ (see next proposit}^n\text{)}}.$$

We get a homomorphism $\deg: \mathrm{Div}(C) \to \mathbb{Z}$.

Prop Let $f: C' \to C$ be a finite morphism of normal curves / k. Then for any divisor $D \in \mathrm{Div}(C)$, we have

$$\deg(f^* D) = \deg(f) \cdot \deg(D).$$

Pf. Wlog $D = [p]$ & $C = \operatorname{Spec} A$

$C' = \operatorname{Spec} A'$

Localize $A \hookrightarrow A'$ at $S := A \setminus m_p$

$\Rightarrow \mathcal{O}_{C,p} \hookrightarrow S^{-1} A' =: B \subset k(C')$

B is a torsion free module over the PID $\mathcal{O}_{C,p}$

$\Rightarrow$ free of rank $r = \deg(f)$

$\Rightarrow$ for a local parameter $t \in \mathcal{O}_{C,p}$,

B/tB is a vector space of dim r over $\kappa(p)$

But $\operatorname{Spec} B/tB = C' \times_C \operatorname{Spec} \kappa(p) = f^{-1}(p)$

$$= \bigsqcup_{q \mapsto p} \operatorname{Spec} \mathcal{O}_{C',q} / (f^\#(t))$$

$$\Rightarrow \underbrace{\dim_{\kappa(p)} B/tB}_{= \deg(f)} = \sum_{q \mapsto p} \underbrace{\dim_{\kappa(p)} \mathcal{O}_{C',q} / (f^\#(t))}_{= e_q(f) \cdot [\kappa(q) : \kappa(p)]}$$

$$\begin{aligned} \Rightarrow \deg(f) \cdot \deg(D) &= \deg(f) \cdot [\kappa(p) : k] \\ &= \sum_{q \mapsto p} e_q(f) \cdot [\kappa(q) : k] \\ &= \deg(f^* D). \end{aligned}$$

□

Let $\operatorname{Div}^\circ(C) := \ker(\deg : \operatorname{Div}(C) \to \mathbb{Z})$

then we get:

$$\begin{tikzcd} \operatorname{Div}^\circ(C) \arrow[r, hook] \arrow[d, dashed, "\exists ! f^*"'] & \operatorname{Div}(C) \arrow[d, "f^*"] \\ \operatorname{Div}^\circ(C') \arrow[r, hook] & \operatorname{Div}(C') \end{tikzcd}$$

<u>Cor</u> Let C be a proper normal curve $/ k$.

Then principal divisors on C have degree zero:

For all $f \in k(C)^*$ we have

$$\operatorname{div}(f) \in \operatorname{Div}^\circ(C).$$

Pf. If $f \in k(C)^*$ is not constant,

it gives a dominant morphism $f : C \to \mathbb{P}^1_k$.

By constuction $\operatorname{div}(f) = f^*([0] - [\infty])$

$\Rightarrow \deg(\operatorname{div}(f)) = \deg(f) \cdot \deg([0] - [\infty]) = 0$ □

<u>Cor</u> Get induced homom

$$\deg : \operatorname{Pic}(C) \longrightarrow \mathbb{Z}.$$

Ex

$$\begin{tikzcd} \mathrm{Pic}(\mathbb{P}^1_k) \arrow[r, "\sim"] & \mathbb{Z} & \text{iso} \\ \mathcal{O}(n) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & n \arrow[u, phantom, sloped, "\in"] & \end{tikzcd}$$

Thus $\forall\, p, q \in |\mathbb{P}^1_k|$ closed points,

we have $[p] \sim [q]$.

Rem For any proper normal curve C over k

w/ $C \not\cong \mathbb{P}^1_k$ we have

$$\mathrm{Pic}^\circ(C) := \ker(\deg : \mathrm{Pic}(C) \to \mathbb{Z}) \neq 0.$$

For $C(k) \neq \emptyset$ we get

$$0 \to \underset{\substack{\wr \\ \text{"continuous part"}}}{\mathrm{Pic}^\circ(C)} \to \mathrm{Pic}(C) \longrightarrow \underset{\substack{\wr \\ \text{"discrete part"}}}{\mathbb{Z}} \longrightarrow 0$$

(→ underlies an abelian variety ...)

3. Riemann-Roch

C normal proper curve / k = alg closed field

$\mathcal{L} \simeq \mathcal{O}_C(D) \in \mathrm{Pic}(C)$ w/ $D \in \mathrm{Div}(C)$

since C is proper, see later...

Def $h^0(C, D) := h^0(C, \mathcal{L}) := \dim_k \Gamma(C, \mathcal{L}) < \infty$

Rem a) For $\deg D < 0$ always $h^0(C, D) = 0$

b) For $\deg D = 0$ we have:

$$h^0(C, D) > 0 \iff \mathcal{O}_C(D) \simeq \mathcal{O}_C$$

$$\iff D \sim 0$$

ie $\exists f \in k(C)^* : D = \mathrm{div}(f)$

Pf. $\Gamma(C, \mathcal{L}) = \Gamma(C, \mathcal{O}_C(D))$

$$= \{ f \in k(C)^* \mid \mathrm{div}(f) \geq -D \} \cup \{0\}$$

for $\deg D < 0$: impossible

for $\deg D = 0$: equivalent to $\mathrm{div}(f) = -D$

(using that $\deg \mathrm{div}(f) = 0$) ☐

What about divisors of degree $\deg D > 0$?

Ex Let $p \in C$ be a closed point. Then

$$h^0(C, [p]) > 1 \iff C \simeq \mathbb{P}^1_k .$$

Pf. $\Gamma(C, \mathcal{O}_C([p])) = \{ f \in k(C)^* \mid \mathrm{div}(f) \geq -[p] \} \cup \{0\}$

$$\supseteq \Gamma(C, \mathcal{O}_C)$$

proper inclusion $\iff f \in k(C)^*$ w/ $\mathrm{div}(f) = [q] - [p]$ for some point $q \neq p$

$\iff f : C \to \mathbb{P}^1_k$ w/ $f^*([\infty]) = [p]$

ie $\deg f = 1$

ie f isomorphism ☐

Ex For $C = \mathbb{P}^1_k$, any divisor $D \in \mathrm{Div}(C)$ satisfies $D \sim \deg(D) \cdot [p]$ for any pt p

$$\Rightarrow h^0(C, D) = \begin{cases} \deg(D) + 1 & \text{if } \deg(D) \geq 0, \\ 0 & \text{if } \deg(D) < 0. \end{cases}$$

For other curves we need to introduce one more notion:

Outlook: Kähler differentials

Def Let A be a k-algebra (for any ring k)

We define the module of Kähler differentials as the A-module

formal basis vectors

$$\Omega^1_{A/k} := \bigoplus_{x\in A} A\cdot dx \Big/ \langle a), b), c) \rangle$$

w/ the relations a) $dc = 0 \quad \forall c \in k$

b) $d(x+y) = dx + dy \quad \forall x,y \in A$

c) $d(xy) = x\,dy + y\,dx \quad \forall x,y\in A$

Ex $\Omega^1_{k[t]/k} = k[t]\cdot dt$ for any field k,

here

$$d\Big(\sum_n c_n t^n\Big) = \Big(\sum_n n c_n t^{n-1}\Big)\cdot dt.$$

Caution: $d(t^p) = 0$ if char $k = p$.

Exercise a) For any multiplicative set $S \subset A$,

$\exists$ natural iso $\Omega^1_{S^{-1}A/k} \xrightarrow{\sim} S^{-1}\Omega^1_{A/k}$.

b) For any scheme X over k,

$\exists$ qcoh sheaf $\Omega^1_{X/k} \in QCoh(X)$

sth $\forall$ open affine $U = \operatorname{Spec} A \subset X$:

$\Gamma(U, \Omega^1_{X/k}) \simeq \Omega^1_{A/k}$.

Ex $X = \mathbb{P}^1_k = U_0 \cup U_\infty$ w/ $U_0 = \operatorname{Spec} k[z_0]$

$U_\infty = \operatorname{Spec} k[z_\infty]$

glued via $z_\infty = 1/z_0$

$$\Rightarrow \Gamma(U_0, \Omega^1_{X/k}) = k\cdot dz_0$$

$$\Gamma(U_\infty, \Omega^1_{X/k}) = k\cdot dz_\infty$$

$dz_\infty = d(1/z_0)$

$= -\frac{1}{z_0^2}\cdot dz_0$

$$\Rightarrow \Omega^1_{X/k} \simeq \mathcal{O}_X(-2)$$

Lemma For any normal curve C/k, the sheaf $\Omega_{C/k}$ is a line bundle. Its stalk at a closed pt $p \in C$ is

$$\Omega^1_{C/k,p} = \mathcal{O}_{C,p}\cdot dt$$

for any local parameter $t \in \mathcal{O}_{C,p}$ at the point p.

Pf. Consider $\mathcal{O}_{C,p} \cdot dt \hookrightarrow \Omega^1_{C/k,p}$

& the quotient $M := \Omega^1_{C/k,p} / \mathcal{O}_{C,p} \cdot dt$

fin.gen. over $\mathcal{O}_{C,p}$

Claim: $M \simeq 0$

Nakayama: Enough to show $\mathfrak{m} M = M$ w/ $\mathfrak{m} = (t) \trianglelefteq \mathcal{O}_{C,p}$.

For this pick any $df \in \Omega^1_{C/k,p}$ (here $f \in \mathcal{O}_{C,p}$).

Put

$$c_0 := f(p) \in \kappa(p) = \mathcal{O}_{C,p} / \mathfrak{m} \underset{\text{since } k \text{ is alg closed}}{=} k$$

$\Rightarrow$ $f - c_0 \in \mathfrak{m} = (t)$,

say $f = c_0 + gt$ w/ $g \in \mathcal{O}_{C,p}$

$$\Rightarrow df = \underbrace{dc_0}_{\substack{=0 \\ \text{since} \\ c_0 \in k}} + \underbrace{g\,dt}_{\substack{=0 \\ \text{in } M}} + \underbrace{t\,dg}_{\in \mathfrak{m} M} \in \mathfrak{m} M$$

Hence $M \simeq 0$ & so $\Omega^1_{C/k,p} \simeq \mathcal{O}_{C,p} \cdot dt$ free of rk 1

True for all p $\Rightarrow$ $\Omega^1_{C/k}$ locally free of rk 1 $\square$

Def For any proper normal curve C over k, its genus is $g := h^0(C, \Omega^1_{C/k})$.

Ex $C = \mathbb{P}^1_k$ has genus $g = h^0(C, \mathcal{O}_C(-2)) = 0$.

Thm (Riemann-Roch) Let C be a proper normal curve of genus g over k. Then $\forall \mathcal{L} \in \mathrm{Pic}(C)$, we have

$$h^0(C, \mathcal{L}) - h^0(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g.$$

Rem a) Taking $\mathcal{L} = \Omega^1_{C/k}$ & using $h^0(C, \mathcal{O}_C) = 1$, we get $\deg \Omega^1_{C/k} = 2g - 2$.

b) Hence $\deg(\Omega^1_{C/k} \otimes \mathcal{L}^\vee) = 2g - 2 - \deg \mathcal{L}$

$\Rightarrow$ For $\deg \mathcal{L} > 2g - 2$ the "error term" $h^0(C, \Omega^1_{C/k} \otimes \mathcal{L}^\vee)$ vanishes & we get $h^0(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$.

Pf of RR (outlook)

① True for $\mathcal{L} = \mathcal{O}_C$ by our definition of g.

② Now add / subtract points:

For $p \in C$ closed pt, put $\mathcal{L}(-p) := \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{O}_C(-p)$

& consider the exact sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \underbrace{i_* \mathcal{O}_{\{p\}}}_{\substack{\text{skyscraper sheaf arising} \\ \text{via } i:\, \operatorname{Spec} \kappa(p) \hookrightarrow X}} \to 0$$

$$\Rightarrow 0 \to \Gamma(C,\mathcal{L}(-p)) \to \Gamma(C,\mathcal{L}) \to \Gamma(\{p\}, \mathcal{O}_{\{p\}}) \to H^1(C,\mathcal{L}(-p)) \to H^1(C,\mathcal{L}) \to 0$$

(see later: Sheaf cohomology!)

$\Rightarrow$ For $\chi(C,-) := \dim_k \Gamma(C,-) - \dim_k H^1(C,-)$

we get:

$$\chi(C,\mathcal{L}) = \chi(C,\mathcal{L}(-p)) + 1.$$

③ Serre duality:

$$H^1(C,\mathcal{E}) \simeq \left(\Gamma(C,\Omega^1_{C/k} \otimes \mathcal{E}^\vee)\right)^\vee \quad \forall \text{ loc. free } \mathcal{E} \in \mathrm{Coh}(C)$$

$$\Rightarrow \chi(C,\mathcal{E}) = \Gamma(C,\mathcal{E}) - \Gamma(C,\Omega^1_{C/k} \otimes \mathcal{E}^\vee)$$

By step ② then

RR for $\mathcal{L}$ $\Leftrightarrow$ RR for $\mathcal{L}(-p)$

& therefore we reduce inductively to ① $\square$

Ex Let C be a normal proper curve of genus g over k.

a) $g = 0$:

$\Rightarrow h^0(C, \mathcal{O}_C([p])) \geq 2$

$\Rightarrow \exists f \in \Gamma(C, \mathcal{O}_C([p])) \setminus \Gamma(C, \mathcal{O}_C)$

$\Rightarrow \exists$ iso $f : C \xrightarrow{\sim} \mathbb{P}^1_k$

↳ since $\deg f = \deg [p] = 1$

$\Rightarrow$ "Genus 0 = rational curves"!

b) $g = 1$:

RR $\Rightarrow$ $h^0(C, \mathcal{L}) = \deg \mathcal{L}$ for $\deg \mathcal{L} > 0$

$\Rightarrow$ Fix $p \in C$ closed pt, then $\exists\, x, y$ sth

$$H^0(C, \mathcal{O}_C(p)) = \langle 1 \rangle$$
$$H^0(C, \mathcal{O}_C(2p)) = \langle 1, x \rangle$$
$$H^0(C, \mathcal{O}_C(3p)) = \langle 1, x, y \rangle$$

Now

- $\dim_k H^0(C, \mathcal{O}_C(6p)) = 6$
- $\underbrace{1, x, x^2, x^3, xy, y, y^2}_{\text{7 sections} \Rightarrow \text{linearly dependent!}} \in H^0(C, \mathcal{O}_C(6p))$

$\Rightarrow$ $\exists$ relation $y^2 + axy + by = f(x)$

w/ $\deg(f) = 3$ (since $1, x, x^2, xy, y$ lin. indep)

$\Rightarrow$ morphism

$\varphi : C \longrightarrow V_+(F) \subset \mathbb{P}^2_k$

w/ a homogenous cubic $F \in k[x,y,z]$

Fact: φ is an embedding: "Genus 1 = elliptic curves"!

What's next?

- embeddings $\varphi : X \hookrightarrow \mathbb{P}^n_k$ via sections of line bundles
  → ample bundles, linear series & divisors
- sheaf cohomology & Serre duality

global

- sheaves of differentials, smoothness ...

local

VI. More about line bdles & divisors

1. Coherent sheaves on Proj(A)

Recall $R = \bigoplus_{n \geq 0} R_n$ graded ring

$\rightsquigarrow$ scheme $X := \mathrm{Proj}(R)$ w/ basis of open charts

given by $D_+(f) := \mathrm{Spec}(\underset{\substack{\uparrow \\ \text{degree zero part} \\ \text{of localiz}^{\text{n}}\ R_f}}{R_{f,0}})$ for homogenous $f \in R$

Def For a graded R-module $M = \bigoplus_{n \geq 0} M_n$,

(ie $R_k \cdot M_\ell \subset M_{k+\ell}$ $\forall\, k, \ell \geq 0$)

define $\tilde{M} \in \mathrm{Mod}(\mathcal{O}_X)$ on the above basis

by

$$\tilde{M}(D_+(f)) := M_{f,0} \quad (:= \text{degree zero part of } M_f)$$

Rem a) Get exact fctor $\mathrm{GrMod}(R) \to \mathrm{Mod}(\mathcal{O}_X)$, $M \mapsto \tilde{M}$

b) The stalk of $\tilde{M}$ at a point (= homog. prime ideal) $\mathfrak{p} \in X$

is $(\tilde{M})_{\mathfrak{p}} \simeq (M_{\mathfrak{p}})_0$

c) R Noetherian & M fin. gen. / R $\Rightarrow$ $\tilde{M} \in \mathrm{Coh}(X)$.

Caution: a) $M \mapsto \tilde{M}$ is NOT faithful:

For a converse see [Ha, exer. II.5.9]

For $M = \bigoplus_{d \geq 0} M_d \in \mathrm{GrMod}(R)$ & any $d_0 \geq 0$,

put $N := \bigoplus_{d \geq d_0} N_d \in \mathrm{GrMod}(R)$. Then $\tilde{M} \simeq \tilde{N}$.

$$\left[\ \tilde{M}(D_+(f)) = \left\{ \frac{m}{f^k} \;\middle|\; f \in R_d,\ m \in M_{kd} \right\} \ \right]$$

$\frac{m}{f^k} = \frac{m f^\ell}{f^{k+\ell}}$: so it suffices to consider M_d for d as large as we want

b) In particular, in general $\Gamma(X, \tilde{M}) \neq M_0$.

We can generalize the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ as follows:

Def For $M \in \mathrm{GrMod}(R)$ & $n \in \mathbb{Z}$,

define $M(n) \in \mathrm{GrMod}(R)$ by $(M(n))_d := M_{n+d}$.

Put $\mathcal{O}_X(n) := \widetilde{R(n)} \in \mathrm{QCoh}(X)$.

Ex $X = \mathbb{P}^1_k = \mathrm{Proj}\, k[s,t]$

$$\Rightarrow \Gamma(D_+(s), \mathcal{O}_X(1)) = k[s,t]_{s,1} \overset{\text{degree 1 piece}}{=} k[\tfrac{t}{s}] \cdot s \ni s$$

$$\downarrow$$

$$\Gamma(D_+(t), \mathcal{O}_X(1)) = \cdots = k[\tfrac{s}{t}] \cdot t \ni \underbrace{\tfrac{s}{t}}_{\substack{\text{transition fct} \\ \text{for } \mathcal{O}_{\mathbb{P}^1}(1)}} \cdot t$$

Caution: Let $R = k[x_1, x_2, x_3]$

w/ $R_0 := k$ & $\deg x_i := i$ for $i = 1,2,3$.

Exercise: On $X = \operatorname{Spec} R$,

a) $\mathcal{O}_X(1)$ is NOT a line bdle,

b) $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \not\cong \mathcal{O}_X(2)$.

Hint: Compute on the open $\mathcal{U} = D_+(x_3) \subset X$

& look at fibers in a suitable point of $\mathcal{U}$

But:

Lemma Let $X = \operatorname{Proj}(R)$. Then for $f \in R_d$ homogeneous of degree $d > 0$, we have

$$\mathcal{O}_X(nd)|_{D_+(f)} \simeq \mathcal{O}_{D_+(f)} \quad \text{for all } n \in \mathbb{Z}.$$

Pf. $\mathcal{O}_X(m)|_{D_+(f)} =$ qcoh sheaf on $D_+(f) = \operatorname{Spec}(R_{f,0})$

w/ $M := (R(m))_{f,0} \in \operatorname{Mod}(R_{f,0})$.

For $m = nd$ we have

$$\begin{array}{ccc} R_{f,0} & \xrightarrow{\sim} & R(m)_{f,0} \\ x & \longmapsto & f^n \cdot x \end{array}$$

$\square$

Def We say R is generated in degree 1

if $R \simeq R_0[x_0, \dots, x_n]/J$ w/ $\deg x_i = 1 \ \forall i$.

Cor Let $X = \operatorname{Proj} R$ w/ R generated in degree 1.

a) $\mathcal{O}_X(n) \in \operatorname{Pic}(X)$ for all $n \in \mathbb{Z}$.

b) For $m, n \in \mathbb{Z}$ $\exists$ natural iso

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \xrightarrow{\sim} \mathcal{O}_X(m+n).$$

c) For all $M \in \operatorname{GrMod}(R)$, $n \in \mathbb{Z}$,

$\exists$ iso $\quad \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \xrightarrow{\sim} \widetilde{M(n)}$.

Pf. a) $R \simeq R_0[x_0, \dots, x_n]/J \Rightarrow X = \bigcup_i D_+(x_i)$

$\Rightarrow \mathcal{O}_X(n)$ line bdle by the lemma w/ $f = x_i$, $\deg f = 1$

b) & c):

$\exists$ always natural morphisms $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(m+n)$

$\tilde{M} \otimes \mathcal{O}_X(n) \to \widetilde{M(n)}$

& on charts $D_+(x_i)$ w/ $\deg x_i = 1$ they are an iso (exercise). $\square$

2. Morphisms to $\mathbb{P}^n_A$

Motivation Any proper normal curve C over an alg closed field k of genus $g = 1$ is isomorphic to a cubic in $\mathbb{P}^2_k$:

$\exists\, x, y \in \Gamma(C, \mathcal{O}_C(3p))$ w/

$f := [1 : x : y] : C \xrightarrow{\sim} \text{cubic} \subset \mathbb{P}^2_k.$

Here $1, x, y$ "generate" $\mathcal{O}_C(3p) \simeq f^*\mathcal{O}_{\mathbb{P}^2}(1)$.

To explain this construction more generally, we need:

Def Let X be a scheme & $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$.

We say that $\mathcal{F}$ is generated by global sections if the following equivalent conditions hold:

a) $\exists$ family $(s_i)_{i \in I}$ of sections $s_i \in \mathcal{F}(X)$ sth

$$\begin{array}{ccc} \bigoplus_{i \in I} \mathcal{O}_X & \twoheadrightarrow & \mathcal{F} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (f_i) & \longmapsto & \sum_i f_i s_i \end{array}$$

is an epimorphism.

($\Leftrightarrow \forall p \in |X|$, the $s_{i,p}$ generate $\mathcal{F}_p$ as $\mathcal{O}_{X,p}$-module)

b) the morphism $\mathcal{F}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \twoheadrightarrow \mathcal{F}$ is an epimorphism.

c) $\mathcal{F}$ is a quotient of a free $\mathcal{O}_X$-module.

Ex a) X affine $\Rightarrow$ every $\mathcal{F} \in \mathrm{QCoh}(X)$ is generated by global sections

b) $\mathcal{O}_{\mathbb{P}^1_k}(n)$ is generated by global sections iff $n > 0$

c) X proper normal curve over a field k & $D \in \mathrm{Div}_+(C)$,

$\uparrow$ effective!

then TFAE:

i) $\mathcal{O}_C(D)$ gen. by global sections

ii) $\forall p \in \mathrm{Supp}(D)$: $h^0(C, D-p) < h^0(C, D)$.

Prop Let X be a scheme over a ring R.

a) For any $f : X \to \mathbb{P}^n_R = \mathrm{Proj}\, R[x_0, \dots, x_n]$ over R, the line bundle $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by the global sections $s_i := f^*(x_i)$, $i = 0, 1, \dots, n$.

b) Conversely, given $\mathcal{L} \in \mathrm{Pic}(X)$ which is generated by global sections $s_0, \dots, s_n \in \mathcal{L}(X)$,

$\exists!\, f : X \to \mathbb{P}^n_R$ w/ $f^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq \mathcal{L}$ & $f^*(x_i) = s_i$

We also write $f = [s_0 : \dots : s_n]$.

Pf. a) Consider

$$\begin{array}{ccc} f^*: \Gamma(\mathbb{P}^n, \mathcal{O}(1)) & \longrightarrow & \Gamma(X, \underbrace{f^*\mathcal{O}(1)}_{=:\mathcal{L}}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x_i & \longmapsto & s_i \end{array}$$

Let $x \in X$ and $p = f(x) \in \mathbb{P}^n$.

$\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

$$\Longrightarrow (\mathcal{O}_{\mathbb{P}^n}(1))_p = \mathcal{O}_{\mathbb{P}^n,p} \cdot x_0 + \dots + \mathcal{O}_{\mathbb{P}^n,p} \cdot x_n$$

$$\Longrightarrow \mathcal{L}_x = (f^*\mathcal{O}_{\mathbb{P}^n}(1))_x = (\mathcal{O}_{\mathbb{P}^n}(1))_p \otimes_{\mathcal{O}_{\mathbb{P}^n,p}} \mathcal{O}_{X,x}$$

$$= \mathcal{O}_{X,x} \cdot s_0 + \dots + \mathcal{O}_{X,x} \cdot s_n$$

b) $\mathcal{L}$ generated by $s_0, \dots, s_n$

$\Longrightarrow X = \bigcup_{i=0}^{n} D_\mathcal{L}(s_i)$ for the open subsets

$$D_\mathcal{L}(s_i) := \{ p \in X \mid s_i(p) \neq 0 \text{ in } \mathcal{L}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p) \}$$

(else $\exists\, p \in X$ w/ $s_{i,p} \in \mathfrak{m}_p \cdot \mathcal{L}_p$ for all i,

but then $s_{0,p}, \dots, s_{n,p}$ cannot generate $\mathcal{L}_p$ $\lightning$)

Now define

$$\begin{array}{ccc} D_\mathcal{L}(s_i) & \longrightarrow & D_+(x_i) = \operatorname{Spec} R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \\ \cap & & \cap \\ X & & \mathbb{P}^n_R \end{array}$$

by the ring homom.

$$\begin{array}{ccc} R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] & \longrightarrow & \mathcal{O}_X(D(s_i)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \frac{x_j}{x_i} & \longmapsto & f_{ij} \text{ w/ } s_j = f_{ij} \cdot s_i \end{array}$$

$\Longrightarrow$ morphisms $D_\mathcal{L}(s_i) \to \mathbb{P}^n_R$ & these glue to $f: X \to \mathbb{P}^n_R$.

Uniqueness: Exercise. $\square$

Rem Let X be a scheme $/R$ & $\mathcal{L} \in \operatorname{Pic}(X)$ generated by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$, then TFAE:

a) $f := [s_0 : \dots : s_n]: X \to \mathbb{P}^n_R$ is a closed immersion

b) For all i, the open subset $D_\mathcal{L}(s_i) \subset X$ is affine and the R-algebra $\mathcal{O}_X(D_\mathcal{L}(s_i))$ is generated by the f_{ij} defined by $s_j = f_{ij} \cdot s_i$ on $D_\mathcal{L}(s_i)$.

Pf. Clear since $D_\mathcal{L}(s_i) = f^{-1}(D_+(x_i))$ & $f_{ij} = f^\#(\frac{x_j}{x_i})$. $\square$

Let's take a look at the case when $R = k$ is a field.

First a coordinate-free reformulation:

Def A linear series on a scheme X over k is a pair $(\mathcal{L}, V)$ where $\mathcal{L} \in \mathrm{Pic}(X)$ and $0 \neq V \subset \Gamma(X, \mathcal{L})$ a vector subspace of finite dim $/ k$.

We also call $|V| := \mathbb{P}V \subset \mathbb{P}\Gamma(X, \mathcal{L})$ a linear series. For $V = \Gamma(X, \mathcal{L})$ we put $|\mathcal{L}| := |V|$ and call it a complete linear series.

The base locus of a linear series $(\mathcal{L}, V)$ is the set

$$\mathrm{Bs}(V) := X \setminus \bigcup_{s \in V} D_{\mathcal{L}}(s)$$

$$= \{ p \in X \mid \forall s \in V : s_p \in m_p \cdot \mathcal{L}_p \}$$

We call V base-point free if $\mathrm{Bs}(V) = \emptyset$. In that case the proposition gives a morphism

$$\varphi_{\mathcal{L},V} : X \longrightarrow \mathbb{P}V^* \qquad \text{w/ } V^* = \mathrm{Hom}_k(V, k).$$

Prop Assume k is algebraically closed. Let X be a proper scheme $/ k$ & $(\mathcal{L}, V)$ a baseptfree linear series.

Then $\varphi_{\mathcal{L},V} : X \longrightarrow \mathbb{P}V^*$ is a closed immersion iff the following two conditions hold:

a) V separates points:

$\forall$ closed pts $p \neq q \in X$,

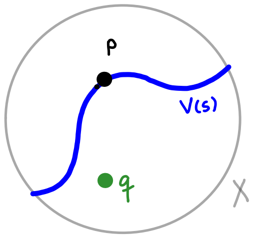
$\exists\, s \in V$ w/ $s(p) = 0$, $s(q) \neq 0$ $\left(\Longleftrightarrow \begin{array}{l} s_p \in m_p \mathcal{L}_p \\ s_q \notin m_q \mathcal{L}_q \end{array} \right)$

b) V separates tangent vectors:

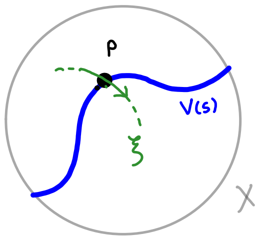
$\forall$ morphism $\xi :$

$$\begin{tikzcd} \operatorname{Spec} k[\varepsilon] \arrow[r] & X \\ \text{closed pt} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & p \arrow[u, phantom, sloped, "\in"] \end{tikzcd} \qquad \left(\begin{array}{l} k[\varepsilon] := k[t]/(t^2) \\ \varepsilon := t \bmod t^2 \end{array} \right)$$

$\exists\, s \in V$ w/ $s(p) = 0$ and $\xi^*(s) \neq 0$.



s doesn't vanish at q



s doesn't vanish along ξ

Pf. $\Rightarrow$: Exercise

$\Leftarrow$: Let $\varphi := \varphi_{X,V}$.

X proper / k $\Rightarrow$ φ proper

$\Rightarrow$ φ has closed image

Point separation a) $\Rightarrow$ φ injective on closed pts

$\Rightarrow$ diag: $X \to X \times_{\mathbb{P}V} X$ surjective

on k-points $=$ closed points,

$\uparrow$ k alg closed

hence surjective (by properness)

$\Rightarrow$ φ injective (also on nonclosed pts)

Thus $|\varphi| : |X| \to |\mathbb{P}V^*|$ is an injective closed map, hence $|X|$ has the subspace topology from $|\mathbb{P}V|$.

Remains to show:

$$\varphi^\# : \mathcal{O}_{\mathbb{P}V^*} \to \varphi_* \mathcal{O}_X \text{ is surjective.}$$

Enough to check this on stalks at closed points.

So let $p \in X$ be a closed pt & $w := \varphi(p) \in \mathbb{P}V$

Want:

$$\varphi^\#_p : \mathcal{O}_{\mathbb{P}V^*, w} \longrightarrow \mathcal{O}_{X,p} \quad \text{epi.}$$

Note that φ is proper w/ finite fibers $\Rightarrow$ finite

So we can use:

Let $f : R \to S$ be a local hom. of local rings sth S is finite / R. Suppose f induces an

i) iso $R/m_R \xrightarrow{\sim} S/m_S$

ii) epi $m_R/m_R^2 \twoheadrightarrow m_S/m_S^2$

Then $f : R \twoheadrightarrow S$ is surjective.

[Pf. By Nakayama / R we only need $R/m_R \twoheadrightarrow S/m_R S$.
By i) this becomes $S/m_S \twoheadrightarrow S/m_R S$, ie $m_R S = m_S$.
By Nakayama / S this becomes $m_R S/m_R m_S \twoheadrightarrow m_S/m_S^2$
which holds by ii).]

Check the conditions in our case:

i) $\underset{\kappa(\omega)}{\underset{\parallel}{\mathcal{O}_{\mathbb{P}V^*,\omega}/m_\omega}} \xrightarrow{\sim} \underset{\kappa(p)}{\underset{\parallel}{\mathcal{O}_{X,p}/m_p}}$ since $\kappa(\omega) = \kappa(p) = k$ for k algebraically closed

ii) $m_\omega/m_\omega^2 \twoheadrightarrow m_p/m_p^2$ surjective:

If not, then

$$\begin{tikzcd}
f_1|_{m_p} \in & \mathrm{Hom}_k(m_p/m_p^2, k) \arrow[r] & \mathrm{Hom}_k(m_\omega/m_\omega^2, k) \text{ not injective} \\
f_0+\varepsilon f_1 \in \arrow[u, mapsto] & \mathrm{Hom}_k(\mathcal{O}_{X,p}, k[\varepsilon]) \arrow[u, equal] & \mathrm{Hom}_k(\mathcal{O}_{\mathbb{P}V,\omega}, k[\varepsilon]) \arrow[u, equal] \\
 & \mathrm{Hom}_{\text{at } p}(\operatorname{Spec} k[\varepsilon], X) \arrow[u, equal] & \mathrm{Hom}_{\text{at } \omega}(\operatorname{Spec} k[\varepsilon], \mathbb{P}V) \arrow[u, equal] \\
 & \exists\, 0 \neq \xi \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \varphi \circ \xi = 0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

Tangent separation b) $\Rightarrow \exists\, s \in V: \ \xi^*(s) \neq 0$

But $s = \varphi^*(\text{a section of } \mathcal{O}_{\mathbb{P}V^*}(1)) \Rightarrow \xi^*(s) = 0$

since $\varphi \circ \xi = 0$ ↯

$\square$

Def Let X be a scheme over a ring R.

A line bdle $\mathcal{L} \in \mathrm{Pic}(X)$ is *very ample* if $\exists\, n \in \mathbb{N}\ \exists\, s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ giving a locally closed immersion

$$\varphi = [s_0 : \cdots : s_n] : \ X \hookrightarrow \mathbb{P}^n_R .$$

Cor Let k be an alg closed field & C a proper normal curve $/ k$.

a) Let $D \in \mathrm{Div}_+(C)$ be an effective divisor sth $\forall$ closed $p, q \in C$ (not necessarily distinct):

$$h^0(C, D-p-q) < h^0(C, D-p) < h^0(C, D)$$

Then $\mathcal{O}_C(D)$ is very ample.

b) In particular: Any $\mathcal{L} \in \mathrm{Pic}(C)$ w/ $\deg \mathcal{L} > 2g$ is very ample.

↳ genus of C

Pf. a) The definitions show that:

$$H^0(C, \mathcal{O}_C(D-p)) = \{ s \in H^0(C, \mathcal{O}(D)) \mid s_p \in m_p \mathcal{L}_p \}$$

For $p \neq q$:

$$H^0(C, \mathcal{O}_C(D-p-q)) = \left\{ s \in H^0(C, \mathcal{O}(D)) \;\middle|\; \begin{matrix} s_p \in m_p \mathcal{L}_p \\ s_q \in m_q \mathcal{L}_p \end{matrix} \right\}$$

For $p = q$:

$$H^0(C, \mathcal{O}_C(D-2p)) = \{ s \in H^0(C, \mathcal{O}_C(D)) \mid s_p \in m_p^2 \mathcal{L}_p \}$$

$$\Updownarrow$$

$\xi^*(s) = 0$

$\forall\, \xi : \operatorname{Spec} k[\varepsilon] \to C$

w/ $\xi(\text{closed pt}) = P$

Thus

$h^0(D-p) < h^0(D) \Rightarrow \mathcal{O}_C(D)$ basepoint-free

$h^0(D-p-q) < h^0(D-p) < h^0(D)$:

$p \neq q \Rightarrow$ separation of pt^s

$p = q \Rightarrow$ separation of tangent vectors

$\Rightarrow$ claim follows from proposition.

b) follows from a) via Riemann-Roch.

Note: $h^0(C, \mathcal{L}) > 0$

$\Rightarrow$ writing $\mathcal{L} \simeq \mathcal{O}_C(D')$, $\exists f \in k(C)^*$: $\operatorname{div}(f) \geq -D'$

$\Rightarrow$ $\operatorname{div}(f) = -D' + D$ for some $D \in \operatorname{Div}_+(C)$

$\Rightarrow \exists$ effective $D \in \operatorname{Div}_+(C)$ w/ $\mathcal{L} \simeq \mathcal{O}_C(D)$ $\square$

<u>Ex</u> C of genus $g = 1$

$\Rightarrow \mathcal{O}_C(3p)$ very ample

(as $\deg \mathcal{O}_C(3p) = 3 > 2g = 2$)

$$\Rightarrow \underset{\substack{\| \\ [1:x:y]}}{|\mathcal{O}_C(3p)|} : C \hookrightarrow \mathbb{P}^2 \text{ immersion}$$

as claimed at the beginning!

3. Ample line bdles

Recall For $X \in Sch_R$, we say $\mathcal{L} \in Pic(X)$ is very ample if $\exists\, s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ sth the morphism

$$[s_0 : \dots : s_n] : X \to \mathbb{P}^n_R$$ is a locally closed immersion.

Rem • Notion depends on $X \to Spec\ R$,

eg take $X = \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{L} = \mathcal{O}(1) \in Pic(X)$:

$$[x_0 : \dots : x_n] : X \to \mathbb{P}^n_R \quad \begin{cases} \text{iso for } R = \mathbb{C}, \\ \text{NOT an immersion for } R = \mathbb{R}. \end{cases}$$

• $\forall m \geq 1$: $\mathcal{L}$ very ample $\Rightarrow$ $\mathcal{L}^{\otimes m}$ very ample

$$\not\Leftarrow$$

eg $\mathcal{L} = \mathcal{O}_C(p)$ on a curve of genus $g > 0$ is NOT very ample but $\mathcal{L}^{\otimes m}$ is for $m > g$.

Q: Weaker, more flexible notion which is "absolute" & for which "$\Leftrightarrow$" holds?

Idea: Tensor powers have more sections!

Def Let X be a scheme. We say $\mathcal{L} \in Pic(X)$ is ample if $\exists\, n \in \mathbb{N}$ $\exists\, m_0, \dots, m_n \geq 1$ $\exists\, s_i \in \Gamma(X, \mathcal{L}^{\otimes m_i})$ sth for $D(s_i) := \{p \in X \mid s_{i,p} \notin \mathfrak{m}_p \cdot \mathcal{L}_p^{\otimes m_i}\}$ we have

$$X = \bigcup_{i=0}^{n} D(s_i) \quad \& \text{ each } D(s_i) \text{ is affine.}$$

Equivalently:

- X is quasicompact, and
- $\forall p \in X$ $\exists\, m \geq 1$ $\exists\, s \in \Gamma(X, \mathcal{L}^{\otimes m})$ sth $D(s)$ is affine & contains p.

Rem • definition doesn't involve a base ring R

• $\mathcal{L}$ ample $\Leftrightarrow$ $\mathcal{L}^{\otimes m}$ ample for any $m \geq 1$

Main Thm. For X of fin. type over a Noeth. ring R

$\mathcal{L} \in Pic(X)$ ample $\Leftrightarrow$ $\exists\, m \geq 1$: $\mathcal{L}^{\otimes m}$ very ample / R

Slogan: X is quasiprojective iff $\exists$ ample $\mathcal{L} \in Pic(X)$!

Some preliminary remarks:

Lemma Let X be a scheme.

a) For any $\mathcal{L} \in \mathrm{Pic}(X)$ & any $s \in \Gamma(X, \mathcal{L}^{\otimes m})$, $m \geq 1$, the inclusion $D(s) \hookrightarrow X$ is an affine morphism.

b) If X is affine, then every $\mathcal{L} \in \mathrm{Pic}(X)$ is ample

c) If $\exists$ ample $\mathcal{L} \in \mathrm{Pic}(X)$, then X is qcqs.

d) Let X be qcqs & $s \in \Gamma(X, \mathcal{L})$, $\mathcal{L} \in \mathrm{Pic}(X)$.

$\Rightarrow \forall \mathcal{M} \in \mathrm{QCoh}(X)$:

$$\Gamma(D(s), \mathcal{M}) \simeq$$

$$\mathrm{colim}\left(\Gamma(X,\mathcal{M}) \xrightarrow{\cdot s} \Gamma(X, \mathcal{M}\otimes\mathcal{L}) \xrightarrow{\cdot s} \Gamma(X, \mathcal{M}\otimes\mathcal{L}^{\otimes 2}) \xrightarrow{\cdot s} \cdots\right)$$

"localization of $\Gamma(X, \mathcal{M})$ at $s \in \Gamma(X, \mathcal{L})$"

OR: "$\forall f \in \Gamma(D(s), \mathcal{M})\ \exists m \geq 1$: $s^m f \in \Gamma(D(s), \mathcal{M}\otimes\mathcal{L}^{\otimes m})$ extends to X"

e) Let X be qc & $\mathcal{L} \in \mathrm{Pic}(X)$. Then TFAE:

- $\mathcal{L}$ is ample
- the $D(s)$ w/ varying $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ & varying $m \in \mathbb{N}$ form a nbhood basis of X.

Pf. a) Claim is local on X

$\Rightarrow$ wlog $X \simeq \mathrm{Spec}\, A$, $\mathcal{L} \simeq \mathcal{O}_X$

$\Rightarrow$ $s \in A^{\otimes m} \simeq A \underset{A}{\otimes} \cdots \underset{A}{\otimes} A \simeq A$

$\Rightarrow$ $D(s) = \mathrm{Spec}\, A_s \hookrightarrow X = \mathrm{Spec}\, A$ affine morphism

b) $X = \mathrm{Spec}\, A$ affine & $\mathcal{L} = \tilde{L} \in \mathrm{Pic}(X)$

$\Rightarrow$ $\forall p \in X\ \exists s \in L = \Gamma(X, \mathcal{L})$: $p \in D(s)$ ← affine by a)

(lift a generator of $\mathcal{L}_p \otimes \kappa(p) \simeq L \underset{A}{\otimes} \kappa(p)$)

$\Rightarrow$ $\exists$ finite cover of X by such opens as X is qc

c) $\mathcal{L}$ ample $\Rightarrow$ X qc by def^n

Check quasiseparatedness:

Let $X = \bigcup_{i=0}^{n} D(s_i)$ w/ $s_i \in \Gamma(X, \mathcal{L}^{\otimes m_i})$ sth $D(s_i)$ is affine

$$D(s_i) \cap D(s_j) = \underset{\text{for } \mathcal{L}^{\otimes(m_i+m_j)}}{D(s_i s_j)} \underset{\text{affine morphism by a)}}{\hookrightarrow} \underbrace{D(s_i)}_{\text{affine}}$$

$\Rightarrow$ $D(s_i) \cap D(s_j)$ again affine, hence qc

$\Rightarrow$ X qs

d) X qcqs $\Rightarrow$ $\exists$ finite open cover $X = \bigcup_i U_i$ w/ $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$

sth

- $U_i \simeq \operatorname{Spec} A_i$ affine, and
- $U_i \cap U_j = \bigcup_{\text{finite}} U_{ijk}$ w/ $U_{ijk} = \operatorname{Spec} A_{ijk}$ affine

$\Rightarrow \Gamma(D(s), \mathcal{M})$

$$= \lim \Big(\bigoplus_i \underbrace{\Gamma(D(s)\cap U_i, \mathcal{M})}_{\operatorname{colim}_m \Gamma(U_i, \mathcal{M}\otimes\mathcal{L}^{\otimes m})} \rightrightarrows \bigoplus_{i,j,k} \underbrace{\Gamma(D(s)\cap U_{ijk}, \mathcal{M})}_{\operatorname{colim}_m \Gamma(U_{ijk}, \mathcal{M}\otimes\mathcal{L}^{\otimes m})} \Big)$$

because U_i is affine (localization wrt the basic open $D(s) \subset \operatorname{Spec} A_i$)

because U_{ijk} is affine (localization wrt the basic open $D(s) \subset \operatorname{Spec} A_{ijk}$)

General fact: Filtered colim commute w/ finite lim $\Rightarrow$ claim

e) "$\Rightarrow$": Let $p \in U \subset X$ open.

$\mathcal{L}$ ample $\Rightarrow$ $\exists\, s \in \Gamma(X, \mathcal{L}^{\otimes m})$: $p \in D(s) \simeq \operatorname{Spec} A$

Pick $f \in A$ sth the basic open $V = \operatorname{Spec} A_f \subset \operatorname{Spec} A = D(s)$

satisfies $p \in V \subset U \cap D(s)$. Now $f \in \Gamma(D(s), \mathcal{O}_X)$.

By c), d) $\exists\, g \in \Gamma(X, \mathcal{L}^{\otimes r})$: $f = \frac{g}{s^r}$

$\Rightarrow V = D(g) \cap D(s) = D(\underbrace{gs}_{=:\, \tilde{s} \in \Gamma(X, \mathcal{L}^{\otimes(m+r)})})$ open affine $\ni p$

"$\Leftarrow$": Exercise. $\square$

Cor $\varphi : X \hookrightarrow Y$ locally closed immersion, then:

$$\mathcal{L} \in \operatorname{Pic}(Y) \text{ ample} \Rightarrow \varphi^*(\mathcal{L}) \in \operatorname{Pic}(X) \text{ ample.}$$

Pf. $\varphi = j \circ i$ w/ j open immersion, i closed immersion

$\Rightarrow$ left with two cases:

a) $\varphi = j$ open immersion:

Then $\varphi^*\mathcal{L}$ is ample by part d) of the lemma.

b) $\varphi = i$ closed immersion:

Let $s_0, \dots, s_n \in \Gamma(Y, \mathcal{L}^{\otimes m})$ w/ $Y = \bigcup_i D(s_i)$ & all $D(s_i)$ affine

For $t_i := \varphi^* s_i \in \Gamma(X, \varphi^*(\mathcal{L})^{\otimes m})$ we get:

$$\begin{tikzcd} D(t_i) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, hook, "\varphi"] \\ \text{affine } D(s_i) \arrow[r, hook] & Y \end{tikzcd}$$

closed imm, hence affine

$\Rightarrow D(t_i)$ affine ($\&\ X = \bigcup_i D(t_i)$)

$\Rightarrow \varphi^*\mathcal{L}$ ample

$\square$

Pf of main thm.

" $\Leftarrow$ " : $\mathcal{L}^{\otimes m}$ very ample (over R)

$\Rightarrow \exists \varphi : X \hookrightarrow \mathbb{P}^n_R$ loc closed

w/ $\mathcal{L}^{\otimes m} \simeq \varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$

$\Rightarrow \mathcal{L}^{\otimes m}$ ample by previous corollary

$\Rightarrow \mathcal{L}$ ample

" $\Rightarrow$ " : Let $\mathcal{L} \in \mathrm{Pic}(X)$ be ample,

say $X = \bigcup_{i=0}^{r} D(s_i)$ w/ $s_i \in \Gamma(X, \mathcal{L}^{\otimes m_i})$

& $D(s_i) \simeq \mathrm{Spec}\, A_i$ affine. Wlog all $m_i = 1$

(replace $\mathcal{L}$ & s_i by powers)

By part d) of the lemma,

$$A_i = \Gamma(D(s_i), \mathcal{O}_X) \simeq \mathrm{colim}\left(\Gamma(X,\mathcal{O}_X) \to \Gamma(X,\mathcal{L}) \to \Gamma(X,\mathcal{L}^{\otimes 2}) \to \cdots\right)$$

$\uparrow$

finite type / R (since X is so),

say generated by x_{ij} $(j = 1, \ldots, n_i)$ as R-algebra

$$\Rightarrow \exists\, m\ \exists\, \tilde{x}_{ij} \in \Gamma(X, \mathcal{L}^{\otimes m}) \text{ w/ } x_{ij} = \frac{\tilde{x}_{ij}|_{D(s_i)}}{s_i^m}.$$

Consider

$$\varphi := [s_0^m : \cdots : s_r^m : \tilde{x}_{1,1} : \cdots : \tilde{x}_{r,n_r}] : X \longrightarrow \mathbb{P}^N_R$$

w/ $N = r + n_1 + \cdots + n_r$

1) φ is everywhere defined:

$s_0^m, \ldots, s_r^m$ already generate $\mathcal{L}^{\otimes m}$ since $X = \bigcup_{i=1}^{r} D(s_i^m)$.

2) φ is a locally closed immersion:

Let $\mathbb{P}^N_R = \mathrm{Proj}\, R[z_0, \ldots, z_N]$ w/

$\varphi^*(z_0) = s_0^m$
$\vdots$
$\varphi^*(z_r) = s_r^m$
$\varphi^*(z_{r+1}) = \tilde{x}_{1,1}$
$\vdots$
$\varphi^*(z_N) = \tilde{x}_{r,n_r}$

Since $X = \bigcup_{i=0}^{r} D(s_i^m)$

$= \bigcup_{i=0}^{r} \varphi^{-1}(D_+(z_i))$,

we have a factorization

$$\varphi : X \longrightarrow U := \bigcup_{i=0}^{r} D_+(z_i) \xhookrightarrow{\text{open}} \mathbb{P}^N_R$$

Claim: $\varphi : X \to U$ is a closed immersion.

Check on open charts $D_+(z_i) \subset U$ w/ $i = 0, 1, \dots, r$:

(enough charts for U !)

For $i \le r$ we have:

$$\begin{tikzcd}
D_+(z_i) \arrow[r, phantom, "="] & \operatorname{Spec} R\left[\frac{z_0}{z_i}, \dots, \frac{z_N}{z_i}\right] & \frac{z_\alpha}{z_i} \text{ (some } \alpha) \arrow[d, mapsto] \\
\varphi^{-1}(D_+(z_i)) \arrow[u, hook] \arrow[r, phantom, "\underset{\substack{\uparrow \\ i \le r}}{=}"] & D(s_i^m) \simeq \operatorname{Spec} A_i & \frac{\tilde{x}_{ij}|_{D(s_i)}}{s_i^m} = x_{ij}
\end{tikzcd}$$

$$\Longrightarrow \quad k\left[\frac{z_0}{z_i}, \dots, \frac{z_N}{z_i}\right] \twoheadrightarrow A_i \text{ surjective}$$

$$\Longrightarrow \quad \varphi^{-1}(D_+(z_i)) \hookrightarrow D_+(z_i) \text{ closed immersion.} \qquad \square$$

$\exists$ another useful characterization of ampleness:

Thm Let X be a Noetherian scheme. For $\mathcal{L} \in \operatorname{Pic}(X)$, TFAE:

a) $\mathcal{L}$ is ample

b) $\forall \mathcal{M} \in \operatorname{Coh}(X)\ \exists m > 0 : \mathcal{M} \otimes \mathcal{L}^{\otimes m}$ globally generated

c) $\forall$ coherent ideal sheaves $\mathcal{J} \trianglelefteq \mathcal{O}_X\ \exists m > 0 : \mathcal{J} \otimes \mathcal{L}^{\otimes m}$ g.g.

b') $\forall \mathcal{M} \in \operatorname{Coh}(X)\ \exists m_0 > 0\ \forall m \geq m_0 : \mathcal{M} \otimes \mathcal{L}^{\otimes m}$ g.g.

c') $\forall$ coherent $\mathcal{J} \trianglelefteq \mathcal{O}_X\ \exists m_0 > 0\ \forall m \geq m_0 : \mathcal{J} \otimes \mathcal{L}^{\otimes m}$ g.g.

Pf. We show

$$\begin{tikzcd}
a) \arrow[r, Rightarrow] & b) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & b') \arrow[d, Rightarrow] \\
 & c) \arrow[ul, Rightarrow] & c') \arrow[l, Rightarrow]
\end{tikzcd}$$

a) $\Rightarrow$ b):

$\mathcal{L}$ ample $\Longrightarrow \exists m \geq 1\ \exists s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes m})$

w/ $D(s_i)$ affine & $X = \bigcup_{i=0}^{r} D(s_i)$

Wlog $m = 1$ (replace $\mathcal{L}$ by a power)

Write $U_i := D(s_i) \simeq \operatorname{Spec} A_i$.

$\mathcal{M} \in Coh(X) \Rightarrow M_i := \Gamma(U_i, \mathcal{M})$ fin. gen. A_i-module

say $= \sum_{j=1}^{n_i} A_i \cdot m_{ij}$

Lemma d) shows $M_i = \operatorname{colim}_{m, \cdot s_i} \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m})$

$\Rightarrow \exists m \ \forall i,j \ \exists \tilde{m}_{ij} \in \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes m}) : m_{ij} = \frac{\tilde{x}_{ij}|_{U_{ij}}}{s_i^m}$

$\Rightarrow \tilde{m}_{i1}, \dots, \tilde{m}_{i,n_i}$ generate $(\mathcal{M} \otimes \mathcal{L}^{\otimes m})|_{U_i}$

$\Rightarrow \tilde{m}_{1,1}, \dots, \tilde{m}_{r,n_r}$ generate $\mathcal{M} \otimes \mathcal{L}^{\otimes m}$

b) $\Rightarrow$ c) : trivial

c) $\Rightarrow$ a) :

X Noetherian $\Rightarrow$ X qc

Let $p \in X$ & pick an affine nborhood $p \in U = \operatorname{Spec} A \subset X$.

Write : $Z := (X \setminus U)^{red} = V(\mathcal{J}) \underset{\text{closed}}{\hookrightarrow} X$ w/ $\mathcal{J} \trianglelefteq \mathcal{O}_X$.

By c) $\exists m \geq 1$: $\mathcal{J} \otimes \mathcal{L}^{\otimes m}$ globally generated.

Thus $\exists s \in \Gamma(X, \mathcal{J} \otimes \mathcal{L}^{\otimes m})$: $s(p) \neq 0$ in $(\dots)_p \otimes \kappa(p)$

Let $t := \text{image}(s) \in \Gamma(X, \mathcal{L}^{\otimes m}) = \Gamma(X, \mathcal{O}_X \otimes \mathcal{L}^{\otimes m})$

via $\mathcal{J} \underset{\text{incl}}{\hookrightarrow} \mathcal{O}_X$

$\Rightarrow Z \subset V(t) := X \setminus D(t)$ by construction

$\Rightarrow D(t) \subset U$

& clearly $t(p) \neq 0$ (as $s(p) \neq 0$ & $\mathcal{J} \hookrightarrow \mathcal{O}_X$ iso over $\underset{p}{\overset{\Psi}{U}}$)

Moreover $D(t)$ is affine (basic open in $U = \operatorname{Spec} A$)

Thus : $\forall p \ \exists t \in \Gamma(X, \mathcal{L}^{\otimes m})$: $\underset{p}{\overset{\Psi}{D(t)}}$ affine

$\Rightarrow \mathcal{L}$ ample

b) $\Rightarrow$ b') :

$\mathcal{L}$ ample (using b) $\Rightarrow$ a)), say $X = \bigcup_i \overset{\swarrow \text{affine}}{D(s_i)}$

w/ $s_i \in \Gamma(X, \mathcal{L}^{\otimes r})$

$\mathcal{M} \in Coh(X) \Rightarrow \forall j \in \{0, 1, \dots, r-1\} \ \exists m_j$:

$\underbrace{(\mathcal{M} \otimes \mathcal{L}^{\otimes j})}_{=: \mathcal{M}_j} \otimes \underbrace{(\mathcal{L}^{\otimes r})}_{=: \mathcal{L}_j \text{ (ample, hence satisfies b))}}{}^{\otimes m_j}$ globally gen.

Then $\forall\, m \geq m_0 := \max\{ j + r m_j \}$,

$\mathcal{M} \otimes \mathcal{L}^{\otimes m}$ is globally generated:

$\exists\, j \in \{0, 1, \ldots, r-1\}$ w/ $m - j = r \cdot (m_j + s)$ w/ $s \in \mathbb{N}_0$,

& then

$$\mathcal{M} \otimes \mathcal{L}^{\otimes m} = \underbrace{\mathcal{M} \otimes \mathcal{L}^{\otimes j} \otimes \mathcal{L}^{\otimes r m_j}}_{\substack{\text{globally generated} \\ \text{by previous page}}} \otimes \underbrace{\mathcal{L}^{\otimes r} \otimes \cdots \otimes \mathcal{L}^{\otimes r}}_{\substack{s \text{ factors } \mathcal{L}^{\otimes r} \\ \text{each globally gen.} \\ \text{by } s_0, \ldots, s_r}}$$

$\Rightarrow \mathcal{M} \otimes \mathcal{L}^{\otimes m}$ g.g. (as a tensor product of g.g. bdles)

b') $\Rightarrow$ c') $\Rightarrow$ c): trivial. $\square$

<u>Cor</u> $X \hookrightarrow \mathbb{P}^n_R$ locally closed subscheme, $\mathcal{M} \in \mathrm{Coh}(X)$

$\Rightarrow$ epi $\mathcal{O}_X(-d)^{\oplus N} \twoheadrightarrow \mathcal{M}$ for $N, d \gg 0$.

Pf. Pick d s.th. $\mathcal{M} \otimes \mathcal{O}_X(d)$ is globally generated, and tensor $\mathcal{O}_X^{\oplus N} \twoheadrightarrow \mathcal{M} \otimes \mathcal{O}_X(d)$ by $\mathcal{O}_X(-d)$. $\square$

4. Divisors & line bdles

Let X be a loc. Noetherian integral scheme

Def a) A *prime divisor* on X is an integral closed subscheme $Z \hookrightarrow X$ w/

$$\operatorname{codim}(Z,X) := \underset{\substack{\uparrow \\ \text{Krull dim}}}{\dim}\,(\mathcal{O}_{X,\underset{\substack{\uparrow \\ \eta_Z := \text{generic pt of } Z}}{\eta_Z}}) = 1$$

$$\mathcal{P}(X) := \{\text{prime divisors } Z \subset X\}$$

b) A *Weil divisor* on X is a formal sum

$$D = \sum_{Z \in \mathcal{P}(X)} n_Z \cdot [Z]$$

which is *locally finite*, ie. $\forall p \in X \ \exists$ open $\overset{\overset{p}{\cap}}{U} \subset X$:

$n_Z = 0$ for almost all $Z \in \mathcal{P}(X)$ w/ $Z \cap U \neq \emptyset$

c) Notation:

$$\begin{array}{ll} \operatorname{Div}(X) := \{\text{Weil divisors on } X\} & \leftarrow \text{ab. gp} \\ \cup & \\ \operatorname{Div}^+(X) := \{\sum_Z n_Z [Z] \mid \text{all } n_Z \geq 0\} & \leftarrow \text{effective Weil divisors} \end{array}$$

$$D \geq D' \ :\Longleftrightarrow \ D - D' \in \operatorname{Div}^+(X)$$

Ex Let $Z \in \mathcal{P}(X)$.

For $0 \neq f \in \mathcal{O}_{X,\eta_Z}$ put

$$v_Z(f) := \operatorname{length}_{\mathcal{O}_{X,\eta_Z}}(\mathcal{O}_{X,\eta_Z}/(f))$$

One easily sees $v_Z(ab) = v_Z(a) + v_Z(b)$.

$\Longrightarrow$ well-defined group homom.

$$\begin{array}{rcl} v_Z : k(X)^* & \longrightarrow & \mathbb{Z} \\ \Psi & & \Psi \\ f = \frac{g}{h} & \longmapsto & v_Z(f) := v_Z(g) - v_Z(h) \end{array}$$

For $f \in k(X)^*$ its *Weil divisor of zeroes & poles* is defined by

$$\operatorname{div}(f) := \sum_{Z \in \mathcal{P}(X)} v_Z(f) \cdot [Z] \in \operatorname{Div}(X).$$

Such divisors are called *principal divisors*.

We get a gp homom. $\operatorname{div} : k(X)^* \to \operatorname{Div}(X)$

& put $\operatorname{Cl}(X) := \operatorname{Div}(X)/\sim$ "Weil class gp"

where $D \sim D' \ :\Longleftrightarrow \ D - D'$ is principal.

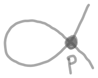
Caution In general, for $f \in k(X)^*$ we have:

$$f \in \mathcal{O}_{X,\eta_Z} \underset{\not\Longleftarrow}{\Longrightarrow} v_Z(f) \geq 0$$

Converse NOT true!

e.g. for the nodal cubic

$X = V(y^2 - x^2(x+1)) \subset \operatorname{Spec} \mathbb{C}[x,y]$



the rat^e fct^n $f := \frac{y}{x} \in k(X)^*$

is not defined at $p = (0,0)$,

ie $f \notin \mathcal{O}_{X,p}$. Nevertheless

$$v_p(f) = \underbrace{v_p(y)}_{=2} - \underbrace{v_p(x)}_{=2} = 0$$

$$\left[\begin{aligned} \text{eg } v_p(y) &= \text{length } \mathbb{C}[x,y]_{(x,y)} / (y, y^2 - x^2(x+1)) \\ &= \text{length } \mathbb{C}[x]_{(x)} / (x^2) = 2 \end{aligned} \right]$$

But: If X is normal, then $\mathcal{O}_{X,\eta_Z}$ is a DVR

and $v_Z : k(X)^* = \operatorname{Quot}(\mathcal{O}_{X,\eta_Z})^* \to \mathbb{Z}$

is the discrete valuation w/ valuation ring $\mathcal{O}_{X,\eta_Z}$.

Then $v_Z(f) \geq 0 \iff f \in \mathcal{O}_{Z,\eta_Z}$.

Cor If X is normal, then

$$\ker(\operatorname{div} : k(X)^* \longrightarrow \operatorname{Div}(X)) = \Gamma(X, \mathcal{O}_X)^*$$

Pf. "$\supseteq$" always true.

"$\subseteq$": Let $f \in k(X)^*$ w/ $v_Z(f) = 0$ for all $Z \in \mathcal{P}(X)$

$\Rightarrow f, f^{-1} \in \mathcal{O}_{X,\eta_Z} \subseteq k(X)$ for all $Z \in \mathcal{P}(X)$

(since X is normal)

$\Rightarrow f, f^{-1} \in \bigcap_{Z \in \mathcal{P}(X)} \mathcal{O}_{X,\eta_Z} \subseteq k(X)$

$\Rightarrow$ for any open affine $U = \operatorname{Spec} A \subset X$,

$$f|_U, f^{-1}|_U \in \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p}) = 1}} A_{\mathfrak{p}} = A \subseteq \operatorname{Quot}(A)$$

for A a normal Noetherian domain (see comm. algebra)

$\Rightarrow f|_U, f^{-1}|_U \in \Gamma(U, \mathcal{O}_X)$ for all open affine $U \subseteq X$

$\Rightarrow f, f^{-1} \in \Gamma(X, \mathcal{O}_X)$ by glueing. $\square$

More generally, can replace $\mathcal{O}_X$ by line bdles:

Def A *rational section* of $\mathcal{L} \in \mathrm{Pic}(X)$ is a germ $s \in \mathcal{L}_\eta$ of the stalk at the generic pt $\eta \in X$.

For $Z \in \mathcal{P}(X)$ we then define the *order of vanishing* $v_{Z,\mathcal{L}}(s) \in \mathbb{Z}$ for $s \neq 0$ as follows:

Let $\xi \in Z$ be the generic pt of Z

$\mathcal{L}_\xi$ is a free module of rk 1 over $\mathcal{O}_{X,\xi}$

$\Longrightarrow \exists\, t_\xi \in \mathcal{L}_\xi$ w/ $\mathcal{L}_\xi = \mathcal{O}_{X,\xi} \cdot t_\xi$

Represent t_ξ by $t \in \mathcal{L}(U)$ on some open $U \ni \xi$

& consider the stalk $t_\eta \in \mathcal{L}_\eta = \mathcal{O}_{X,\eta} \cdot s_\eta$.

$\Longrightarrow \exists!\, f \in \mathcal{O}^*_{X,\eta} = k(X)^*: \quad s = f \cdot t_\eta$.

Put
$$v_{Z,\mathcal{L}}(s) := v_Z(f) \in \mathbb{Z}.$$

Lemma $v_{Z,\mathcal{L}}(s)$ is well-defined.

Pf. Let $\tilde{t}_\xi \in \mathcal{L}_\xi$ be another generator: $\mathcal{L}_\xi = \mathcal{O}_{X,\xi} \cdot t_\xi$.

Let $\tilde{t} \in \mathcal{L}(\tilde{U})$ represent the germ $\tilde{t}_\xi$.

$\Longrightarrow t = u \cdot \tilde{t}$ with $u \in \mathcal{O}^*_{X,\xi} \subset k(X)^*$

$$\Longrightarrow \quad s = f \cdot t_\eta = \underbrace{uf}_{=:\tilde{f}} \cdot \tilde{t}_\eta$$

$$\Longrightarrow \quad v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/(f))$$
$$= \mathrm{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi}/(uf)) = v_Z(\tilde{f}). \quad \square$$

Def For $\mathcal{L} \in \mathrm{Pic}(X)$ & a rational section $0 \neq s \in \mathcal{L}_\eta$ we put

$$\mathrm{div}(s) := \sum_{Z \in \mathcal{P}(X)} v_{Z,\mathcal{L}}(s) \cdot [Z] \in \mathrm{Div}(X).$$

Rem For any two rational sections $s, s' \in \mathcal{L}_\eta \setminus \{0\}$,

$\exists f \in k(X)^* : \quad s = f \cdot s'$

$\Longrightarrow \quad \mathrm{div}(s) \sim \mathrm{div}(s')$

Thus we get a homom.

$$\begin{tikzcd}
\mathrm{cl}: \mathrm{Pic}(X) \arrow[r] & \mathrm{Cl}(X) \\
\mathcal{L} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathrm{cl}(\mathcal{L}) := (\mathrm{div}(s) \bmod \sim) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

any rational section $s \in \mathcal{L}_\eta \setminus \{0\}$

Prop If X is normal, then

$\mathrm{cl}: \mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ is injective.

Pf. Let $\mathcal{L} \in \mathrm{Pic}(X)$ w/ $\mathrm{cl}(\mathcal{L}) = 0$.

Pick any rat^l section $0 \neq s \in \mathcal{L}_\eta \Rightarrow \mathrm{div}(s) \sim 0$

$\Longrightarrow \exists f \in k(X)^* : \quad \mathrm{div}(s) = \mathrm{div}(f)$

Replace s by $\frac{1}{f} \cdot s \quad \Longrightarrow$ wlog $\mathrm{div}(s) = 0$.

$\Longrightarrow$ by normality: $s \in \Gamma(X, \mathcal{L})$ & $\mathcal{O}_X \xrightarrow{\sim} \mathcal{L}$ via $f \mapsto fs$

(claim local, so reduces to the case $\mathcal{L} = \mathcal{O}_X$

where we have seen this in the above corollary) $\square$

Rem Even if X is normal, $\mathrm{cl}: \mathrm{Pic}(X) \to \mathrm{Cl}(X)$

need NOT be surjective:

If $D = \mathrm{div}(s)$ for some rat sect $s \in \mathcal{L}_\eta$,

then D is <u>locally principal</u>: Let $X = \bigcup_i U_i$

sth $\exists$ iso $\mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$, $1 \mapsto s_i$. Then

$\exists! f_i \in k(X)^*$ w/ $s|_{U_i} = f \cdot s_i$ and

$\mathrm{div}(s)|_{U_i} = \mathrm{div}(f_i) \sim 0$ in $\mathrm{Div}(U_i)$.

Ex Let $X = V(z^2 - x^2 - y^2) \subset \mathrm{Spec}\, \mathbb{C}[x, y, z]$

$\Longrightarrow X$ is normal

& $Z := V(z - x, y) \in \mathcal{P}(X)$

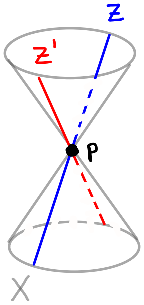
but $[Z]$ is NOT locally principal

(hint: for any linear form $\ell(x, y, z)$,

$\mathrm{div}(\ell) = [Z] + [Z']$

& Z' determines ℓ up to a scalar.

Only $2 \cdot [Z]$ becomes principal...)



So we need a better notion of divisors.

This doesn't even need X to be integral or loc. Noeth:

<u>**Def**</u> Let X be any scheme.

For $U = \operatorname{Spec} A \subset X$ open affine,

let $S := \{a \in A \mid \forall b \in A \setminus \{0\} : ab \neq 0\}$

and put $\mathcal{K}_X^{\mathrm{pre}}(U) := S^{-1}A$.

Let $\mathcal{K}_X :=$ sheafification of $(U \mapsto \mathcal{K}_X^{\mathrm{pre}}(U))$,

the <u>sheaf of total quotient rings</u> of X.

We have

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, hook] & \mathcal{K}_X & \leftarrow \text{sheaves of rings} \\ \mathcal{O}_X^* \arrow[u, phantom, "\cup"] \arrow[r, hook] & \mathcal{K}_X^* \arrow[u, phantom, "\cup"] & \leftarrow \text{sheaves of unit gps} \end{tikzcd}$$

A <u>Cartier divisor</u> on X is an element of the group $\operatorname{CaDiv}(X) := \Gamma(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$.

We call it <u>principal</u> if it is in the image of the homom. $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

We write $D \sim D' :\Longleftrightarrow D - D'$ is principal

and put $\operatorname{CaCl}(X) := \operatorname{CaDiv}(X)/\sim$.

<u>**Rem**</u> a) Explicitly, a Cartier divisor $D \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ can be described by giving

- an open cover $X = \bigcup_{i \in I} U_i$
- rational fct^s $f_i \in \Gamma(U_i, \mathcal{K}_X^*) = k(U_i)^*$

for U_i integral & small enough

sth $\forall i, j \in I : \frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$

b) If X is loc. Noetherian & integral so that we can talk about Weil divisors, we have by construction a commutative diagram

$$\begin{tikzcd} \{(f_i, U_i)\}_{i \in I} \arrow[r, mapsto] & D \text{ with } D|_{U_i} = \operatorname{div}(f_i) \\ \operatorname{CaDiv}(X) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r] \arrow[d] & \operatorname{Div}(X) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d] \\ \operatorname{CaCl}(X) \arrow[r] & \operatorname{Cl}(X) \end{tikzcd}$$

where the image of the horizontal arrows are precisely the "locally principal Weil divisors".

Def Let X be any scheme.

For $D \in \mathrm{CaDiv}(X)$ given by $\{(f_i, U_i)\}_{i \in I}$ as above, define a subsheaf

$$\mathcal{L} = \mathcal{O}_X(D) \subset \mathcal{K}_X$$

by $\mathcal{L}|_{U_i} := \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_X|_{U_i}$

(well-defined as $f_i/f_j \in \mathcal{O}^*_{U_i \cap U_j}$ on overlaps)

Prop a) $\mathcal{L}(D) \in \mathrm{Pic}(X)$

b) We get a bijection

$$\begin{array}{ccc} \mathrm{CaDiv}(X) & \xrightarrow{\sim} & \{\text{invertible subsheaves of } \mathcal{K}_X\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ D & \longmapsto & \mathcal{O}_X(D) \end{array}$$

which induces an injective grp homom

$$\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X).$$

c) If X is integral, this is an iso.

"Cartier divisors compute the Picard group!"

Pf. a) Clear since $\mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$ via $1 \mapsto f_i^{-1}$.

b) From the subsheaf $\mathcal{O}_X(D) \hookrightarrow \mathcal{K}_X$ we can recover D by taking a cover $X = \bigcup_{i \in I} U_i$ sth $\exists$ iso $\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$ & taking $f_i := \varphi_i(1)^{-1}$, as then $\{(f_i, U_i)\}_{i \in I} = D$ in $\Gamma(X, \mathcal{K}^*_X / \mathcal{O}^*_X)$.

For $D_1 = \{(f_i, U_i)\}$ & $D_2 = \{(g_i, U_i)\}$, wlog with the same open U_i (else refine covers), we have $D_1 - D_2 = \{(f_i g_i^{-1}, U_i)\}$ and so

$$\mathcal{O}_c(D_1 - D_2) = \underbrace{\mathcal{O}_c(D_1) \cdot \mathcal{O}_c(D_2)^{-1}}_{\simeq\, \mathcal{O}_c(D_1) \otimes \mathcal{O}_c(D_2)^{-1}} \subset \mathcal{K}_X \qquad (*)$$

It remains to show:

$$D_1 \sim D_2 \iff \mathcal{O}_c(D_1) \simeq \mathcal{O}_c(D_2)$$

Passing to $D := D_1 - D_2$ and using $(*)$, we are reduced to:

$$D \text{ principal} \iff \mathcal{O}_c(D) \simeq \mathcal{O}_c .$$

If D is principal, defined by $f \in \Gamma(X, \mathcal{K}_X^*)$ then $\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X(D)$ via $1 \mapsto f^{-1}$

Conversely, if $\exists$ iso $\varphi : \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X(D)$, then D is principal, defined by $f := (\varphi(1))^{-1}$.

c) Want: $\forall\, \mathcal{L} \in \operatorname{Pic}(X)$ $\exists$ embedding $\mathcal{L} \hookrightarrow \mathcal{K}_X$.

X integral $\implies$ $\mathcal{K}_X$ constant sheaf w/ fiber $K = k(X)$.

Given $\mathcal{L} \in \operatorname{Pic}(X)$, consider $\mathcal{M} := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$.

$\forall$ open $U \subset X$ with $\mathcal{L}|_U \simeq \mathcal{O}_X$, have $\mathcal{M}|_U \simeq \mathcal{K}_X|_U$ constant sheaf.

X irreducible $\implies$ a sheaf on X is constant if it is so on each open of some cover

$\implies$ $\mathcal{M}$ constant, hence $\mathcal{M} \simeq \mathcal{K}_X$

$\implies$ $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}_X = \mathcal{M} \simeq \mathcal{K}_X$ as wanted. $\square$

Back to Weil divisors:

Let X be integral & locally Noetherian.

We have shown:

$$\operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X)$$

injective if X normal

Recall normality of X does NOT imply $\operatorname{Pic}(X) \twoheadrightarrow \operatorname{Cl}(X)$, as shown by $X = \operatorname{Spec} \mathbb{C}[x,y,z]/(z^2 - x^2 - y^2)$. But we can now say precisely what is needed:

Thm For X integral & locally Noetherian, TFAE:

a) X is normal and $\operatorname{Pic}(X) \overset{\sim}{\twoheadrightarrow} \operatorname{Cl}(X)$

b) X is locally factorial, ie all local rings of X are UFD's.

↳ not only codim 1...

For the proof we reduce to the local case:

Def Let X be an integral locally Noetherian scheme. For $U \subset X$ open & $p \in U$ consider the homom.

$$\begin{tikzcd}
\mathrm{Div}(X) \arrow[r] \arrow[d] & \mathrm{Div}(U) \arrow[r] \arrow[d] & \mathrm{Div}(\mathrm{Spec}\, \mathcal{O}_{X,p}) \arrow[d] \\
\mathrm{Cl}(X) \arrow[r] & \mathrm{Cl}(U) \arrow[r] & \mathrm{Cl}(\mathrm{Spec}\, \mathcal{O}_{X,p})
\end{tikzcd}$$

defined on prime divisors Z by

$$[Z] \longmapsto \begin{cases} [Z \cap U] & \text{if } Z \cap U \neq \emptyset \\ 0 & \text{else} \end{cases}$$

resp

$$[Z] \longmapsto \begin{cases} [Z \times_U \mathrm{Spec}\, \mathcal{O}_{X,p}] & \text{if } p \in Z \\ 0 & \text{else} \end{cases}$$

Lemma For $Z \in \mathrm{Div}(X)$, TFAE:

a) $D \in \mathrm{im}(\mathrm{Pic}(X) \to \mathrm{Cl}(X))$.

b) $\forall p \in X \ \exists$ open $U \subset X$ w/ $p \in U$ sth $D \cap U = 0$ in $\mathrm{Cl}(U)$.

c) $\forall p \in X$ we have $D \cap \mathrm{Spec}\, \mathcal{O}_{X,p} = 0$.

(ie "Cartier divisors $\hat{=}$ locally principal Weil divisors")

Pf. a) $\Rightarrow$ b) $\Rightarrow$ c) clear from

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(U) \arrow[d] & \\
\mathrm{Cl}(X) \arrow[r] & \mathrm{Cl}(U) \arrow[r] & \mathrm{Cl}(\mathrm{Spec}\, \mathcal{O}_{X,p})
\end{tikzcd}$$

since line bdles are locally trivial:

$$\forall \mathcal{L} \in \mathrm{Pic}(X) \ \forall p \in X \ \exists \underset{\substack{\cup \\ p}}{U} \subset X \text{ open} : \mathcal{L}|_U \simeq \mathcal{O}_U.$$

c) $\Rightarrow$ a): Let $p \in X$.

Suppose $D \cap \mathrm{Spec}\, \mathcal{O}_{X,p} = 0$ in $\mathrm{Cl}(\mathrm{Spec}\, \mathcal{O}_{X,p})$

$\Rightarrow \exists f_p \in \mathrm{Quot}(\mathcal{O}_{X,p})^* = k(X)^*$:

$$D \cap \mathrm{Spec}\, \mathcal{O}_{X,p} = \mathrm{div}(f_p) \text{ in } \mathrm{Div}(\mathrm{Spec}\, \mathcal{O}_{X,p})$$

$$\Rightarrow \exists \underset{\substack{\cup \\ p}}{U_p} \subset X \text{ open} : \underbrace{\mathrm{div}(f_p)}_{\substack{\text{seen in } \mathrm{Div}(X) \\ \text{via } f_p \in k(X)^*}} \cap U_p = D \cap U_p$$

If this holds for all $p \in X$:

Pick $(p_i)_{i \in I}$ sth $X = \bigcup_{i \in I} U_{p_i}$ & let $f_i := f_{p_i}$

$\Rightarrow \{(f_i, U_i)\}_{i \in I} \in \mathrm{CaDiv}(X)$ induces $D \in \mathrm{Div}(X)$. $\square$

We are left with the affine case:

Prop Let $X = \operatorname{Spec} A$ w/ a normal Noetherian domain A.

Then for prime divisors $Z \subset X$, TFAE:

a) $[Z] \sim 0$ in $\mathrm{Cl}(X)$.

b) $\exists f \in A : [Z] = \operatorname{div}(f)$.

Pf. b) $\Rightarrow$ a) trivial

a) $\Rightarrow$ b): Let $f \in k(X)^* = \operatorname{Quot}(A)^*$: $[Z] = \operatorname{div}(f)$

$$\Rightarrow \forall W \in \mathcal{P}(X): \quad v_W(f) = \begin{cases} 1 & \text{if } W = Z \\ 0 & \text{else} \end{cases}$$

Each W is $= V(\mathfrak{p})$ w/ $\mathfrak{p} \in \operatorname{Spec} A$ of ht $\mathfrak{p} = 1$,

& X normal implies $\mathcal{O}_{X,W} = A_\mathfrak{p}$ DVR w/ valuation v_W

$\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec} A$ of ht $\mathfrak{p} = 1$: $f \in A_\mathfrak{p}$

$$\Rightarrow f \in \bigcap_{\text{ht}\,\mathfrak{p}=1} A_\mathfrak{p} = A$$

since A is a normal Noetherian domain

Let $\mathfrak{q} \in \operatorname{Spec} A$ w/ $V(\mathfrak{q}) = Z$

$\Rightarrow$ for any $g \in \mathfrak{q}$, one gets $\frac{g}{f} \in A$ as above

$\Rightarrow \mathfrak{q} = (f)$ & hence $Z = \operatorname{div}(f)$ $\square$

Cor 1 Let $X = \operatorname{Spec} A$ w/ a Noetherian domain A.

Then TFAE:

a) X is normal & $\mathrm{Cl}(X) = 0$

b) A is a UFD

Pf. A UFD

$\Leftrightarrow$ all prime ideals of ht 1 in A are p'pal (see appendix)

$\Leftrightarrow$ X normal, and $\forall Z \in \mathcal{P}(X)$: $[Z] \sim 0$ in $\mathrm{Cl}(X)$

by the proposition $\square$

Ex For any UFD R, $\mathrm{Pic}(\mathbb{A}^n_R) = \mathrm{Cl}(\mathbb{A}^n_R) = 0$.

Cor 2 For X integral & loc Noetherian, TFAE:

a) X is normal & $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X)$

b) X is locally factorial.

Pf. X loc. factorial $:\Leftrightarrow \forall p \in X$: $\mathcal{O}_{X,p}$ UFD

cor 1 $\Leftrightarrow \forall p \in X$: $\operatorname{Spec} \mathcal{O}_{X,p}$ normal w/ $\mathrm{Cl}(\dots) = 0$

$\Leftrightarrow$ X normal & $\forall D \in \mathcal{P}(X)\ \forall p \in X$: $D \cap \operatorname{Spec} \mathcal{O}_{X,p} \sim 0$

lemma $\Leftrightarrow$ X normal & $\forall D \in \mathcal{P}(X)$: $D \in \operatorname{im}(\mathrm{Pic}\, X \to \mathrm{Cl}\, X)$

$\Leftrightarrow$ X normal & $\mathrm{Pic}\, X \twoheadrightarrow \mathrm{Cl}\, X$. $\square$

Ex $X = \operatorname{Spec} \mathcal{O}_K$ w/ a number field K

$\Rightarrow \operatorname{Pic} X = Cl\, X =$ ideal class grp of K

Same for any Dedekind domain.

Warning In general, X locally factorial

$\not\Rightarrow$ $\exists$ open cover $X = \bigcup_{i \in I} U_i$ w/ $U_i = \operatorname{Spec}$ UFD !

Ex X normal proper curve of genus $g > 0$ over $\mathbb{C}$

$\Rightarrow$ open sets: $U = X \setminus \Sigma$ w/ $\Sigma \subset X$ finite set of closed points

$\Rightarrow$ exact sequence (exercise):

$$\underset{\substack{\wr \\ \text{countable}}}{\bigoplus_{p \in \Sigma} \mathbb{Z} \cdot [p]} \longrightarrow \underset{\substack{\wr \\ \text{uncountable since} \\ \{\text{closed pts of } X\} \hookrightarrow Cl(X) \\ \text{for curves of genus } g>0}}{Cl(X)} \longrightarrow Cl(U) \longrightarrow 0$$

$\Rightarrow Cl(U) \neq 0$ even though U is affine for $\Sigma \neq \emptyset$

Here $U = \operatorname{Spec}$ (Dedekind domain),

but $Cl(U)$ is NOT finitely generated!

Some more examples:

Recall: For X integral loc Noetherian,

$$\operatorname{Pic}(X) \xrightarrow[\substack{\wr \\ \text{mono if } X \text{ normal} \\ \text{iso if } X \text{ loc. factorial}}]{} Cl(X).$$

Ex For any UFD R,

$$\operatorname{Pic}(\mathbb{P}^n_R) \simeq Cl(\mathbb{P}^n_R) = \mathbb{Z} \cdot [H]$$

for any hyperplane $H = V_+(x_i) \subset \mathbb{P}^n_R$.

Pf. $X = \mathbb{P}^n_R$ is locally factorial, so $\operatorname{Pic} X \simeq Cl\, X$.

For $\mathcal{L} \in \operatorname{Pic} X$ $\exists$ iso $\mathcal{L}|_U \simeq \mathcal{O}_U$ on $U = X \setminus H \simeq \mathbb{A}^n_R$

because $\operatorname{Pic} U \simeq \operatorname{Pic} \mathbb{A}^n_R = 0$. Then $1 \in \mathcal{O}_U(U)$

gives a rational section $s \in \mathcal{L}_\eta$ w/ $\operatorname{div}(s) \in \mathbb{Z} \cdot [H]$

$$\Rightarrow \quad \begin{array}{ccccc} \mathbb{Z} & \twoheadrightarrow & \operatorname{Pic} X & \simeq & Cl\, X \\ m & \longmapsto & \mathcal{O}_{\mathbb{P}^n_R}(m) & \mapsto & m \cdot [H] \end{array}$$

Mono via pullback to any $\mathbb{P}^1_R \hookrightarrow \mathbb{P}^n_R$ (apply deg on $\mathbb{P}^1_R$) $\square$

More generally, Weil class gps satisfy "excision":

Lemma Let X be an integral loc Noeth scheme & $U = X \setminus Z$ for a proper closed $Z \subset X$.

Then

a) $\exists$ epi $Cl\, X \twoheadrightarrow Cl\, U$, $D \mapsto D \cap U$.

b) For $\operatorname{codim}(Z, X) \geq 2$ this is an iso.

c) For Z irreducible of $\operatorname{codim}(Z, X) = 1$, $\exists$ exact sequence

$$\begin{array}{ccccccc} \mathbb{Z} & \to & Cl(X) & \to & Cl(U) & \to & 0 \\ m & \longmapsto & m \cdot [Z] & & & & \\ & & D & \longmapsto & D \cap U & & \end{array}$$

(needn't be injective, eg take $\operatorname{Pic} X = 0$)

(Z needn't be Cartier)

Pf. a) $W \subset X$ prime divisor $\Rightarrow$ $W \cap U$ prime divisor or $= \emptyset$

So we get $\operatorname{Div}(X) \to \operatorname{Div}(U)$ & $Cl(X) \to Cl(U)$.

Epi: $W = W \cap U$ for any $W \in P(U)$ w/ closure $\overline{W} \in P(X)$.

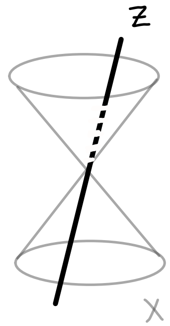
b) If $\operatorname{codim}(Z, X) > 1$, then $P(X) = P(U)$.

c) If $Z \in P(X)$, then $\ker(Cl(X) \to Cl(U)) = \mathbb{Z} \cdot [Z]$. $\square$

Ex For $f \in k[x_0, \ldots, x_n]$ irred & homog of degree $d = \deg f$ & $U = \mathbb{P}^n_k \setminus V_+(f)$, we get:

$$\operatorname{Pic}(U) \simeq Cl(U) \simeq \mathbb{Z}/d\mathbb{Z}.$$

Ex For $X = \operatorname{Spec} k[x,y,z]/(z^2 - xy)$ and $Z = V(y,z) \subset X$ we get



$$\begin{array}{ccccccc} \mathbb{Z} & \to & Cl(X) & \to & Cl(U) & \to & 0 \\ m & \longmapsto & m \cdot [Z] & & & & \end{array}$$

where $U = X \setminus Z$
$= X \setminus V(y)$
$= \operatorname{Spec} \underbrace{k[x,y,z,y^{-1}]/(z^2 - xy)}$

$\simeq k[y, y^{-1}, z]$ UFD

$\Rightarrow Cl(U) = 0$

$\Rightarrow \mathbb{Z} \twoheadrightarrow Cl(X)$, $m \mapsto m[Z]$ epi

Moreover $2[Z] \sim 0$ but $[Z] \not\sim 0$ (since not Cartier)

$\Rightarrow Cl(X) \simeq \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \operatorname{Pic}(X) \simeq 0$ (since $\xhookrightarrow{\neq} Cl(X)$)

Weil class groups are also "homotopy invariant":

Prop Let X be an integral locally Noetherian scheme.

$$\Rightarrow \quad Cl(X) \simeq Cl(X \times \mathbb{A}^1)$$

Pf. Let $\pi : X \times \mathbb{A}^1 \to X$ be the projection, and

define $\pi^* : \operatorname{Div}(X) \longrightarrow \operatorname{Div}(X \times \mathbb{A}^1)$

$$D = \sum_Z n_Z \cdot [Z] \longmapsto \sum_Z n_Z \cdot [\pi^{-1} Z]$$

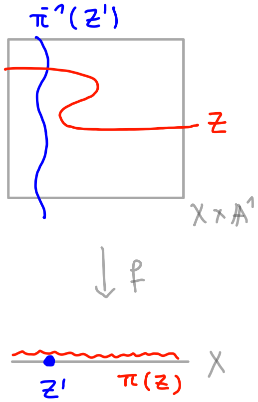
$\Rightarrow$ descends to $\pi^* : Cl(X) \to Cl(X \times \mathbb{A}^1)$

Now $\exists$ only two types of prime divisors $Z \in \mathcal{P}(X \times \mathbb{A}^1)$:

- "vertical": $Z = \pi^{-1}(Z')$

  for some $Z' \in \mathcal{P}(X)$

- "horizontal": $\pi : Z \to X$ dominant



(for $Z \in \mathcal{P}(X \times \mathbb{A}^1)$ the generic fiber of $\pi : Z \to X$ is either $\mathbb{A}^1$ or finite)

Therefore $\pi^* : Cl(X) \to Cl(X \times \mathbb{A}^1)$ is

- injective: $\pi^*(D) = \operatorname{div}(f)$, $f \in k(X \times \mathbb{A}^1)^* = K(t)^*$

  with $K := k(X)$

  $\Rightarrow \operatorname{div}(f)$ has no horizontal cpt

  $\Rightarrow f \in K^* \subset K(t)^*$ & $D = \operatorname{div}(f)$ principal

- surjective: Say $Z \in \mathcal{P}(X \times \mathbb{A}^1)$ is horizontal.

  $\Rightarrow$ Localize at generic pt $\eta \in X$

  to get $Z_\eta \in \mathcal{P}(\mathbb{A}^1_K)$ $\quad (K = k(X) = \mathcal{O}_{X,\eta})$

  corresponding to a prime ideal $\mathfrak{p} \in \operatorname{Spm} K[t]$

$K[t]$ PID $\Rightarrow \exists f \in K[t] \setminus \{0\} : \mathfrak{p} = (f)$

$\Rightarrow \operatorname{div}(f) = [Z_\eta] -$ vertical divisors

$\Rightarrow [Z_\eta] \sim$ vertical divisor $\in \operatorname{image}(\pi^*)$ $\square$

Ex For $X := \mathbb{P}^1_R \times_R \mathbb{P}^1_R$ (any UFD R),

one has $\mathrm{Pic}\, X \simeq \mathrm{Cl}\, X \simeq \mathbb{Z} \times \mathbb{Z}$

(use excision & homotopy invariance w/ $X = \mathbb{P}^1_R$)

Rem a) Similarly one can show

$\mathrm{Pic}\,(X) \xrightarrow{\sim} \mathrm{Pic}\,(X \times \mathbb{A}^1)$ for X normal.

Normality is needed:

E.g. $X = \mathrm{Spec}\, R$ w/ $R = (k[s,t]/(s^2-t^3))_{(s,t)}$

has $\mathrm{Pic}\, X = 0$ (since R is a local ring)

but $\mathrm{Pic}\, X \times \mathbb{A}^1 \not\simeq 0$ (exercise ...) !

b) For any proper normal curve of genus $g > 0$

over a field one can show

$\mathrm{Pic}(X \times X) \not\simeq \mathrm{Pic}(X) \times \mathrm{Pic}(X)$

$\Rightarrow$ $\mathrm{Pic}(\cdot\cdot)$ (and $\mathrm{Cl}(\cdot\cdot)$) usually

NOT compatible w/ products !

Appendix

For any Noetherian domain A, we have:

i) A is a UFD

ii) every ht 1 prime ideal $\mathfrak{p} \trianglelefteq A$ is principal.

Pf. i) $\Rightarrow$ ii): Let $\mathfrak{p} \in \mathrm{Spec}\, A$ w/ $\mathrm{ht}\, \mathfrak{p} = 1$.

Pick $f \in \mathfrak{p} \setminus (0)$ & write $f = f_1 \cdots f_r$ w/ irred $f_i \in A$.

$\mathfrak{p}$ prime $\Rightarrow \exists i: \; f_i \in \mathfrak{p}$

$\Rightarrow \mathfrak{p} \supseteq (f_i) \longleftarrow$ prime since irred in UFD

$\uparrow$

$\mathrm{ht} = 1 \Rightarrow \mathfrak{p} = (f_i)$

ii) $\Rightarrow$ i): Assume every ht 1 prime $\mathfrak{p} \trianglelefteq A$ is principal.

A Noetherian $\Rightarrow$ every $f \in A \setminus (A^* \cup \{0\})$ is a finite prod of irred.

Want: Every irreducible $f \in A$ is prime.

Let $\mathfrak{p} \in \mathrm{Spec}\, A$ be a minimal prime containing f

$\Rightarrow \mathrm{ht}(\mathfrak{p}) \leq 1$ by Krull's principal ideal thm

$\Rightarrow$ By assumption $\exists g \in A: \; (g) = \mathfrak{p} \supseteq (f)$

$\Rightarrow$ f irreducible forces $\mathfrak{p} = (f)$

□

VII. Smoothness & differentials

0. Motivation

For $f \in \mathbb{C}[x,y]$ consider the plane curve

$$X := V(f) \subset \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y].$$

Implicit function thm: Let $p \in X(\mathbb{C})$.

If $\vec{\nabla} f(p) := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(p) \neq (0,0)$,

then $\exists$ analytic nborhood $p \in U \subset \mathbb{C}^2$

$\exists$ biholomorphic map

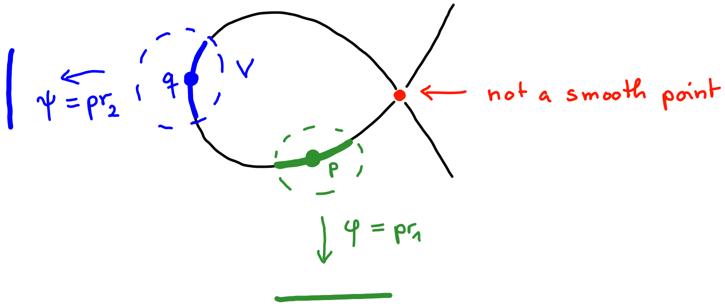
$$\varphi : U \cap X(\mathbb{C}) \xrightarrow{\sim} V \underset{\text{open}}{\subset} \mathbb{C}$$

Thus $X_0 := \{ p \in X(\mathbb{C}) \mid \vec{\nabla} f(p) \neq 0 \}$ is a cplex submanifold of $\mathbb{C}^2$ w/ tangent spaces

$$T_p X_0 = \ker \vec{\nabla} f(p)$$

$$= \left\{ \binom{u}{v} \in T_p \mathbb{C}^2 \;\middle|\; u \frac{\partial f}{\partial x}(p) + v \frac{\partial f}{\partial y}(p) = 0 \right\}$$

Ex $f(x,y) = y^2 - x^2(x+2)$:



$$\varphi^{-1}(x) = (x, x \cdot \underbrace{\sqrt{x+2}}_{}\ )$$

$$= 1 + \frac{x+1}{2} - \frac{(x+1)^2}{8} + \dots$$

holomorphic near $x = -1$

Q Algebraic analogue?

NO "local inverse":

- analytic open $\neq$ Zariski open
- power series $\neq$ polynomial

BUT: Over any field k, can still talk about $df \in \Gamma(X, \Omega^1_{X/k})$!

→ Kähler differentials

1. Kähler differentials

Let R be a commutative ring.

Def A derivation of an R-algebra A w/ values in an R-module M is an R-linear map $d : A \to M$ satisfying the "Leibniz rule"

$$d(fg) = f\,dg + g\,df \qquad \forall f, g \in A$$

We denote by $\mathrm{Der}_R(A, M) \in \mathrm{Mod}(R)$ the R-module of such derivations.

Rem The elements of $R \cdot 1 \subset A$ are "constant" ie $d(r \cdot 1) = 0$ for all $r \in R$

($d(r \cdot 1) = r \cdot d(1)$ by R-linearity and $d(1) = d(1 \cdot 1) = d(1) + d(1)$ by Leibniz so that $d(1) = 0$)

Ex $A = \mathcal{C}^\infty(U) \longrightarrow M = \mathcal{C}^\infty(U) \times \cdots \times \mathcal{C}^\infty(U)$ $\quad\Big|\quad U \subset \mathbb{R}^n$

$f \longmapsto \vec{\nabla} f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ $\quad\Big|\quad R = \mathbb{R}$

There is a "universal derivation":

up to canonical iso

Prop For any R-algebra A, $\exists!$ pair $(\Omega^1_{A/R}, d_{A/R})$ w/ $\Omega^1_{A/R} \in \mathrm{Mod}(A)$ & $d_{A/R} \in \mathrm{Der}_R(A, \Omega^1_{A/R})$ sth $\forall M \in \mathrm{Mod}(A)$:

"algebra" $\quad \mathrm{Hom}_A(\Omega^1_{A/R}, M) \xrightarrow{\sim} \mathrm{Der}_R(A, M) \quad$ "calculus"

$$\varphi \longmapsto \varphi \circ d_{A/R}$$

formal basis vector

Pf. Take $\Omega^1_{A/R} := \bigoplus_{f \in A} A \cdot df \,\Big/\, N$

where N is the submodule generated by the relations

$$\begin{cases} d(af + g) = a\,df + dg \\ d(fg) = f\,dg + g\,df \end{cases} \quad \text{for } a \in R,\ f, g \in A.$$

$\Rightarrow$ $\exists$ natural map $d : A \to \Omega^1_{A/R}$, $f \mapsto df$ which is an R-linear derivation & has the universal property ($\Rightarrow$ uniqueness). $\square$

Def We call $\Omega^1_{A/R}$ the module of *Kähler differentials*,

$d = d_{A/R} : A \longrightarrow \Omega^1_{A/R}$ the *differential* of A/R.

Ex $A = R[t_1, \dots, t_n]$ polynomial ring

$\Rightarrow \Omega^1_{A/R} \simeq \bigoplus_{i=1}^{n} A \cdot dt_i$ free module

w/ $d : A \longrightarrow \Omega^1_{A/R} \simeq \bigoplus_{i=1}^{n} A \cdot dt_i$

$$f \longmapsto df = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} \cdot dt_i$$

In general, formal symbols df & relations between them can be avoided:

Prop Let A be an R-algebra, $\mu : A \otimes_R A \longrightarrow A$ its multiplication and $J := \ker(\mu) \trianglelefteq A \otimes_R A$.

Then

$$(\Omega^1_{A/R}, d) \simeq (J/J^2, \delta)$$

w/ $\delta : A \longrightarrow J/J^2$, $a \mapsto a \otimes 1 - 1 \otimes a$.

Pf. 0) The A-module structure on J/J^2:

J is an $A \otimes_R A$-module

$\Rightarrow$ comes with TWO A-module structures,

either via $A \longrightarrow A \otimes_R A$, $a \longmapsto a \otimes 1$

or via $A \longrightarrow A \otimes_R A$, $a \longmapsto 1 \otimes a$.

But for any $x \in J$ we have:

$$(a \otimes 1) \cdot x - (1 \otimes a) \cdot x = (a \otimes 1 - 1 \otimes a) \cdot x \in J^2$$

$\Rightarrow$ both A-module structures agree on J/J^2

1) $\delta \in \mathrm{Der}_R(A, J/J^2)$:

For $a \in A$ we have $a \otimes 1 - 1 \otimes a \in J = \ker_\mu$

$\Rightarrow \delta(a) \in J/J^2$ well-defined

The map $\delta : A \to J/J^2$ is R-linear:

For $r \in R$, $a, b \in A$ we have

$$\begin{aligned} \delta(ra+b) &= (ra+b) \otimes 1 - 1 \otimes (ra+b) \\ &= ra \otimes 1 - 1 \otimes ra + b \otimes 1 - 1 \otimes b \\ &= r \cdot \delta(a) + \delta(b) \quad (\text{since } \otimes \text{ is over } R) \end{aligned}$$

Leibniz rule for δ:

$$\delta(fg) = fg\otimes 1 - 1\otimes fg$$

$$= (f\otimes 1)(g\otimes 1 - 1\otimes g) + (1\otimes g)(f\otimes 1 - 1\otimes f)$$

$$= f\cdot\delta(g) + g\cdot\delta(f)$$

module structure of J/J^2

2) Claim: J is generated as an $(A\underset{R}{\otimes} 1)$-module by elements $(a\otimes 1 - 1\otimes a)$ w/ $a\in A$.

Indeed, let $a_i, b_i \in A$ w/ $\sum_i a_i\otimes b_i \in J \trianglelefteq A\underset{R}{\otimes}A$

$\Rightarrow \sum_i a_i b_i = 0$ in A

$\Rightarrow \sum_i a_i b_i \otimes 1 = 0$ in $A\otimes A$

$\Rightarrow \sum_i a_i\otimes b_i = \sum_i (a_i\otimes 1)\cdot(1\otimes b_i - b_i\otimes 1)$

3) $(J/J^2, \delta)$ has the universal property:

Let $d: A\to M$ be any R-linear derivation.

Have an $A\otimes 1$-linear map

$$\varphi: A\underset{R}{\otimes}A \to M,\quad x\otimes y \mapsto x\cdot d(y).$$

We have $\varphi(J^2) = 0$: By 2) this follows from

$$\varphi((a\otimes 1 - 1\otimes a)(b\otimes 1 - 1\otimes b))$$

$$= \varphi(ab\otimes 1) - \varphi(a\otimes b) - \varphi(b\otimes a) + \varphi(1\otimes ab)$$

$$= ab\cdot d(1) - a\cdot d(b) - b\cdot d(a) + d(ab)$$

$$= 0$$

since d is a derivation

$\Rightarrow$ get induced map $\bar\varphi := \varphi|_J : J/J^2 \to M$

sth

$$d = \bar\varphi\circ\delta$$

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "d"'] & J/J^2 \arrow[d, dashed, "\exists!\bar\varphi"] \\ & M \end{tikzcd}$$

By 2) the map δ is surjective, so $\bar\varphi$ is unique & by the above it is A-linear (as φ is $A\otimes 1$-linear) $\square$

<u>Cor</u> If $R \to A$ is a categorical epimorphism

(e.g. $A = S^{-1}R$ for some multiplicative set S),

then
$$\Omega^1_{A/R} \simeq 0$$

Pf. By step 2) above,

J is generated as $A \otimes 1$-module by

elements $a \otimes 1 - 1 \otimes a$ with $a \in A$.

But the ring homom. $i_\nu : A \longrightarrow A \otimes_R A$ w/ $\begin{array}{l} i_1(a) = a \otimes 1 \\ i_2(a) = 1 \otimes a \end{array}$

satisfy
$$i_1 \circ \varphi = i_2 \circ \varphi \quad \text{for the map } \varphi : R \to A.$$

If φ is epi, we get $i_1 = i_2$

$\Longrightarrow$ $a \otimes 1 = 1 \otimes a$

$\Longrightarrow$ $J = 0$ & so $\Omega^1_{A/R} \simeq J/J^2 = 0$. $\square$

The description via J/J^2 sheafifies nicely:

<u>Def</u> Let X be a scheme over a scheme S.

The diagonal $\Delta : X \longrightarrow X \times_S X$ is a locally

closed immersion, say $X \underset{\text{closed}}{\overset{\Delta}{\hookrightarrow}} W \underset{\text{open}}{\hookrightarrow} X \times_S X$.

Let $\mathcal{J} = (\text{vanishing ideal of } \Delta(X)) \trianglelefteq \mathcal{O}_W$.

Define the <u>sheaf of Kähler differentials</u> on X/S

by
$$\Omega^1_{X/S} := \Delta^*(\mathcal{J}/\mathcal{J}^2) \in \mathrm{QCoh}(\mathcal{O}_X)$$

(by def^n of sheaf-theoretic inverse image this

doesn't depend on the chosen open W)

and let
$$d_{X/S} : \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$$
$$f \longmapsto (pr_2^* f - pr_1^* f) \in J/J^2$$

for the projections $pr_i : X \times_S X \rightrightarrows X$.

The universal property also has a sheaf version:

Def Let $\varphi : X \to S$ be a morphism of schemes. An S-derivation of $\mathcal{O}_X$ w/ values in an $\mathcal{O}_X$-module $\mathcal{M}$ is a $\varphi^{-1}\mathcal{O}_S$-linear morphism $d : \mathcal{O}_X \longrightarrow \mathcal{M}$ satisfying the "Leibniz rule"

$$d(fg) = f\,dg + g\,df \qquad \forall f, g \in \mathcal{O}_X$$

We put

$$\mathrm{Der}_S(\mathcal{O}_X, \mathcal{M}) := \{ S\text{-derivations } \mathcal{O}_X \to \mathcal{M} \}.$$

Cor $\exists$ natural S-derivation $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ which is universal, i.e. $\forall\, \mathcal{M} \in \mathrm{QCoh}(\mathcal{O}_X)$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{M}) \xrightarrow{\sim} \mathrm{Der}_S(\mathcal{O}_X, \mathcal{M})$$
$$\varphi \longmapsto \varphi \circ d_{X/S}$$

Pf. Glue previous results on affine pieces. $\square$

Concretely: $\forall$ affine open

$$\begin{tikzcd} U = \mathrm{Spec}\, A \arrow[d] & \subset & X \arrow[d] \\ V = \mathrm{Spec}\, R & \subset & S \end{tikzcd}$$

$$\begin{tikzcd} \exists!\ \text{iso} \quad \Omega^1_{X/S}|_U \arrow[r, "\sim"] & \widetilde{\Omega^1_{A/R}} \\ \mathcal{O}_U \arrow[u, "d_{X/S}"] \arrow[r, equal] & \widetilde{A} \arrow[u, "\tilde{d}_{A/R}"'] \end{tikzcd}$$

Ex For $X = \mathbb{A}^n_S = S \times \mathrm{Spec}\, \mathbb{Z}[t_1, \dots, t_n]$ we have

$$\Omega^1_{X/S} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_X \cdot dt_i \quad \text{trivial vbdle of rk } n.$$

We can then compute $\Omega^1_{Y/S}$ for any affine scheme $Y \xhookrightarrow[\text{closed}]{} X = \mathbb{A}^1_S$ using functoriality of Kähler differentials (see next section)...

Ex For $X = \mathbb{P}^1_R$ one has $\Omega^1_{X/R} \simeq \mathcal{O}_{\mathbb{P}^1_R}(-2)$

since $\left.\begin{aligned} \Omega^1_{X/R}|_{U_0} &= \mathcal{O}_{U_0} \cdot dt \\ \Omega^1_{X/R}|_{U_\infty} &= \mathcal{O}_{U_\infty} \cdot ds \end{aligned}\right\}$ glued via $ds = -\frac{1}{t^2}\,dt$

For higher-dimensional projective space we have:

Thm (Euler sequence) ∃ exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_R/R} \to \mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_R} \to 0$$

Pf. Write $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$

& consider the surjective morphism

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)} \arrow[r, "h"] & \mathcal{O}_{\mathbb{P}^n_R} \\ (s_0, \dots, s_n) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \sum_{i=0}^{n} s_i x_i \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Claim: ∃ natural morphism $\Omega^1_{\mathbb{P}^n_R/R} \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)}$

We construct g first on affine charts,

say on $U_0 = \operatorname{Spec} R[y_1, \dots, y_n]$ w/ $y_i = \frac{x_i}{x_0}$:

Idea: On $\mathbb{A}^{n+1}_R \setminus \{0\}$ we have

$$d\left(\frac{x_i}{x_0}\right) = -\frac{x_i}{x_0^2}\, dx_0 + \frac{1}{x_0}\, dx_i = \frac{1}{x_0} \cdot \left(-\frac{x_i}{x_0}\, dx_0 + dx_i\right)$$

Thus we define an $\mathcal{O}_{U_0}$-linear map

$$\begin{tikzcd} \Omega^1_{\mathbb{P}^n_R/R}\big|_{U_0} \arrow[r, "g_0"] \arrow[d, phantom, "\parallel"] & \left(\mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)}\right)\big|_{U_0} \arrow[d, phantom, "\parallel"] \\ \bigoplus_{i=1}^{n} \mathcal{O}_{U_0} \cdot dy_i & \frac{1}{x_0} \cdot \mathcal{O}_{U_i}^{\oplus(n+1)} \end{tikzcd}$$

by $dy_i \longmapsto \frac{1}{x_0} \cdot (\underset{\substack{\uparrow \\ 0}}{-y_i}, 0, \dots, \underset{\substack{\uparrow \\ i}}{1}, \dots, 0)$

$(i = 1, \dots, n)$

A direct computation shows $\operatorname{im}(g_0) = \ker(h|_{U_0})$.

Similarly one defines for $j = 1, \dots, n$

$g_j : \Omega^1_{\mathbb{P}^n_R/R}\big|_{U_j} \to \left(\mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)}\right)\big|_{U_j}$

w/ $\operatorname{im}(g_j) = \ker(h|_{U_j})$ on the chart $U_j \subset \mathbb{P}^n_R$.

Exercise: These maps agree on overlaps,

hence glue to $\Omega^1_{\mathbb{P}^n_R/R} \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n_R}(-1)^{\oplus(n+1)}$ □

2. Functoriality

To control $\Omega^1_{X/S}$ for more general $X \in Sch_S$, we must understand behaviour under various functors.

Prop ("Pullback of differentials")

For any commutative diagram
$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$
$\exists$ natural morphism

$$df : f^*\Omega^1_{X/S} \longrightarrow \Omega^1_{X'/S'}.$$

For Cartesian diagrams this is an iso.

Pf. Consider
$$\begin{tikzcd} X' = V(\mathcal{J}') \arrow[r, hook] \arrow[d, "p"'] & W' \arrow[r, hook, "\text{open}"'] \arrow[d] & X' \times_{S'} X' \arrow[d] \\ X = V(\mathcal{J}) \arrow[r, hook] & W \arrow[r, hook, "\text{open}"'] & X \times_S X \end{tikzcd}$$

$\Rightarrow$ $\exists$ natural morphism $p^*\mathcal{J} \longrightarrow \mathcal{J}'$

Claim: The induced morphism $p^*(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}'/\mathcal{J}'^2$ is an iso if $X' \simeq X \times_S S'$.

Check on affine charts:

Wlog
$$\begin{tikzcd} X = \operatorname{Spec} A \arrow[d] & X' = \operatorname{Spec} A' \arrow[l] \arrow[d] \\ S = \operatorname{Spec} R & S' = \operatorname{Spec} R' \arrow[l] \end{tikzcd}$$
w/ $A' = A \otimes_R R'$

Consider
$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & A \otimes_R A \arrow[r, "\mu"] & A \arrow[r] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$$
split via $a \otimes 1 \mapsfrom a$

$(-) \otimes_R R'$ gives:

$$\begin{tikzcd} 0 \arrow[r] & J \otimes_R R' \arrow[r] \arrow[d] & A \otimes_R A \otimes_R R' \arrow[r] \arrow[d, "\wr"] & A \otimes_R R' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J' \arrow[r] & A' \otimes_{R'} A' \arrow[r] & A' \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ $J \otimes_R R' \xrightarrow{\sim} J'$ iso

$\Rightarrow$ $J^2 \otimes_R R' \twoheadrightarrow (J')^2$ surjective

We get:

$$\begin{tikzcd} & J^2 \otimes_R R' \arrow[r] \arrow[d] & J \otimes_R R' \arrow[r] \arrow[d, "\wr"] & J/J^2 \otimes_R R' \arrow[r] \arrow[d, "\wr\ \text{hence iso}"] & 0 \\ 0 \arrow[r] & (J')^2 \arrow[r] & J' \arrow[r] & J'/(J')^2 \arrow[r] & 0 \end{tikzcd}$$

$\square$

Prop ("Relative cotangent sequence")

Any morphism $f : Y \to X$ of schemes over S induces a short exact sequence

$$f^*\Omega^1_{X/S} \xrightarrow{df} \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/X} \longrightarrow 0 .$$

Pf. The morphisms in the sequence arise from the previous proposition applied to

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d] & X \arrow[d] & \text{resp} & Y \arrow[r, "\mathrm{id}"] \arrow[d] & Y \arrow[d] \\
S \arrow[r, equal] & S & & X \arrow[r] & S
\end{tikzcd}$$

Exactness of the sequence can then be checked locally,

so wlog
$Y = \operatorname{Spec} B$
$X = \operatorname{Spec} A$
$S = \operatorname{Spec} R$.

Claim: The sequence of B-modules

$$\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0 \quad \text{is exact.}$$

Equivalently (using left exactness of Hom),

$\forall\, M \in \mathrm{Mod}(B)$ the following sequence is exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_B(\Omega^1_{B/A}, M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_B(\Omega^1_{B/R}, M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_B(\Omega^1_{A/R} \otimes_A B, M) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Der}_A(B, M) \arrow[r, "\mathrm{incl}"] & \mathrm{Der}_R(B, M) \arrow[dr, "(-)|_A"'] & \mathrm{Hom}_A(\Omega^1_{A/R}, M) \arrow[d, equal] \\
 & & & \mathrm{Der}_R(A, M)
\end{tikzcd}$$

Clearly incl is injective & $(\mathrm{incl}(-))|_A = 0$.

Conversely, let $D \in \mathrm{Der}_R(B, M)$ w/ $D|_A = 0$, then D is A-linear:

$\forall\, a \in A,\ b, c \in B$:

$$D(ab+c) = aD(b) + D(c) + \underbrace{bD(a)}_{=0 \text{ for } D|_A = 0}$$

□

Ex a) $f = pr_1 : Y = \mathbb{A}^2_k \longrightarrow X = \mathbb{A}^1_k, \quad (x,y) \mapsto x$

$$\begin{tikzcd} \Rightarrow & f^*\Omega^1_{X/k} \arrow[r, "df"] \arrow[d, equal] & \Omega^1_{Y/k} \arrow[r] \arrow[d, equal] & \Omega^1_{Y/X} \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}_Y \cdot dx & \mathcal{O}_Y \cdot dx \oplus \mathcal{O}_Y \cdot dy & \mathcal{O}_Y \cdot dy & \end{tikzcd}$$

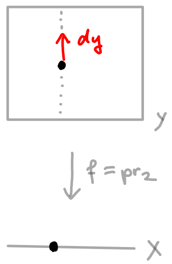
$\Rightarrow \Omega^1_{Y/X} = \text{coker}(df)$ "relative differential forms"

b) $f = \text{incl} : Y = \mathbb{A}^1_k \hookrightarrow X = \mathbb{A}^2_k, \quad y \mapsto (0,y)$

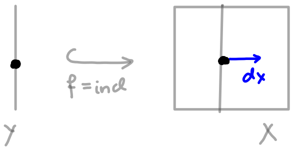
$$\begin{tikzcd} \Rightarrow & f^*\Omega^1_{X/k} \arrow[r] \arrow[d, equal] & \Omega^1_{Y/k} \arrow[r] \arrow[d, equal] & \Omega^1_{Y/X} \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}_Y \cdot dx \oplus \mathcal{O}_Y \cdot dx & \mathcal{O}_Y \cdot dy & 0 & \end{tikzcd}$$

$\Rightarrow df : f^*\Omega^1_{X/S} \to \Omega^1_{Y/S}$ NOT injective,

$\ker(df)$ = "conormal vectors to $Y \hookrightarrow X$"



$\text{coker}(df) = \mathcal{O}_Y \cdot dy$



$\ker(df) = \mathcal{O}_Y \cdot dx$

Generalizing b) above, we can for any closed $Y \overset{i}{\hookrightarrow} X$ compute $\Omega^1_{Y/S}$ as a quotient of $i^*\Omega^1_{X/S}$:

Prop ("Conormal sequence")

Let $i : Y = V(\mathcal{J}) \hookrightarrow X$ be a closed immersion defined by an ideal sheaf $\mathcal{J} \trianglelefteq \mathcal{O}_X$.

$\Rightarrow \Omega^1_{Y/X} = 0$ and $\exists$ exact sequence

$$\mathcal{J}/\mathcal{J}^2 \xrightarrow{\exists \varphi} i^*\Omega^1_{X/S} \xrightarrow{di} \Omega^1_{Y/S} \to 0$$

Pf. Wlog $X = \text{Spec } A$, $S = \text{Spec } R$ and $Y = \text{Spec } B$ with $B = A/J$

Have $d_{A/R} : A \longrightarrow \Omega^1_{A/R}$

Leibniz rule $\Rightarrow d_{A/R}(J^2) \subset J \cdot \Omega^1_{A/R}$

$\Rightarrow$ get induced homom of B-modules

$$J/J^2 \xrightarrow{\varphi} \Omega^1_{A/R} / J \cdot \Omega^1_{A/R} = \Omega^1_{A/R} \underset{A}{\otimes} B$$

Consider the sequence

$$J/J^2 \xrightarrow{\varphi} \Omega^1_{A/R} \otimes_A B \xrightarrow{di} \Omega^1_{B/R} \longrightarrow 0$$

- di is surjective since $\Omega^1_{B/A} = 0$

  ($A \twoheadrightarrow B$ is surjective, hence an epimorphism)

- $\ker(di) \subset \operatorname{im}(\varphi)$:

  For $a \in J$ w/ class $\bar{a} := (a \bmod J^2) \in J/J^2$ we have

  $$\varphi(\bar{a}) = da \otimes 1 \in \Omega^1_{A/R} \otimes_A B$$

  $$\begin{aligned} \Rightarrow (di)(\varphi(\bar{a})) &= (di)(a \otimes 1) \\ &= d(i^\# a) \otimes 1 \\ &= 0 \quad \text{since } i^\# a = 0 \in B \\ &\qquad \text{for } a \in J = \ker(A \to B) \end{aligned}$$

- Thus we get $\Omega^1_{B/R} \simeq \dfrac{\Omega^1_{A/R} \otimes_A B}{\ker(di)} \twoheadrightarrow \operatorname{coker}(\varphi)$

Claim: This is an iso

Indeed:

$$\operatorname{cok}(\varphi) = \bigoplus_{a \in A} B \cdot da \otimes 1 \Big/ \ldots$$

B-submodule generated by the relations

$(d(ra+a') - rda + da') \otimes 1$

$(d(aa') - ada' + a'da) \otimes 1$

w/ $a, a' \in A$, $r \in R$

AND $dc \otimes 1$ w/ $c \in J$

$\Longrightarrow$ $\exists$ R-linear derivation

$$\begin{aligned} \delta : B \simeq A/J &\longrightarrow \operatorname{cok}(\varphi) \\ \bar{a} = (a \bmod J) &\longmapsto da \end{aligned}$$

& one easily checks that $(\operatorname{cok}(\varphi), \delta)$ has the universal property of $(\Omega^1_{B/R}, d)$. $\square$

<u>Cor</u> For $Y = V(f_1, \ldots, f_m) \hookrightarrow \mathbb{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n]$:

$\Omega^1_{Y/R} = \operatorname{coker}(\operatorname{Jac}(f_1, \ldots, f_m))$ for the *Jacobian matrix* $\operatorname{Jac}(f_1, \ldots, f_m) := \left(\frac{\partial f_j}{\partial x_i} \Big|_Y \right) \in \operatorname{Mat}_{n \times m}(\mathcal{O}_Y)$.

Ex a) $Y = V(y-x^2) \subset \mathbb{A}^2_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x,y]$

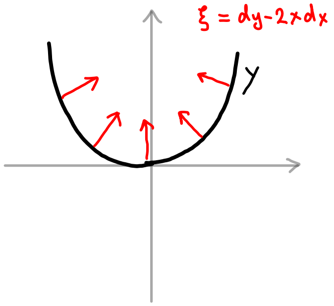
$$\Rightarrow \Omega^1_{Y/\mathbb{R}} = (\mathcal{O}_Y \cdot dx \oplus \mathcal{O}_Y \cdot dy) / \mathcal{O}_Y \cdot \underbrace{d(y-x^2)}_{= dy - 2x dx =: \xi}$$

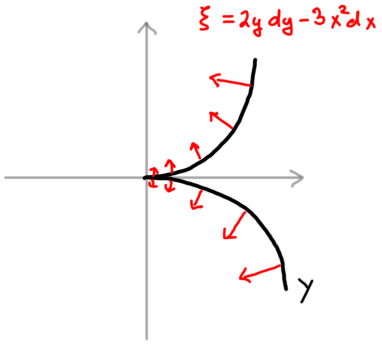
locally free of rank 1,
spanned by dx mod $\mathcal{O}_Y \cdot \xi$

b) $Y = V(y^2 - x^3) \subset \mathbb{A}^2_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x,y]$

$$\Rightarrow \Omega^1_{Y/\mathbb{R}} = (\mathcal{O}_Y \cdot dx \oplus \mathcal{O}_Y \cdot dy) / \mathcal{O}_Y \cdot \underbrace{d(y^2-x^3)}_{= 2y\,dy - 3x^2 dx =: \xi}$$

locally free of rk 1 on $Y \setminus \{(0,0)\}$
but NOT locally free at $p = (0,0)$!





Def For a closed subscheme $Y = V(\mathcal{J}) \overset{i}{\hookrightarrow} X$, we call $\mathcal{C}_{Y/X} := \mathcal{J}/\mathcal{J}^2 \in \mathrm{QCoh}(\mathcal{O}_Y)$ the conormal sheaf to Y in X. Thus for $X \in \mathrm{Sch}_S$ $\exists$ exact sequence

$$\mathcal{C}_{Y/X} \to i^* \Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow 0 .$$

3. Smooth, unramified & étale morphisms

Recall A morphism $f: X \to S$ of smooth manifolds is called an immersion / submersion / local diffeomorphism if $\forall p \in X$ the map on tangent spaces $T_p f : T_p X \to T_{f(p)} S$ is a mono- / epi- / isomorphism

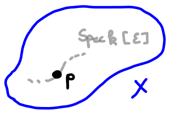
ie $\forall v \in T_{f(p)} S$,

$\exists$ at most / at least / precisely one lift $w \in T_p X$.

Def For schemes X over a field k, we define the tangent space to X at $p \in X(k)$ by

$$T_p X := \{ \varphi \in \mathrm{Hom}_k(\mathrm{Spec}\, k[\varepsilon], X) \mid \varphi(\overset{\text{closed pt}}{0}) = p \}$$

$$= \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[\varepsilon])$$



w/ $k[\varepsilon] := k[t]/(t^2)$, $\varepsilon := (t \bmod t^2)$.

Here $Z = \mathrm{Spec}\, k[\varepsilon]$ is a "first order thickening" of $Z_0 = \mathrm{Spec}\, k$.

More generally:

Def Let Z_0 be any scheme & $n \in \mathbb{N}$. An n-th order thickening of Z_0 is a closed immersion $Z_0 = V(\mathcal{J}) \overset{i}{\hookrightarrow} Z$ given by an ideal sheaf $\mathcal{J} \trianglelefteq \mathcal{O}_Z$ with $\mathcal{J}^n = (0)$.

Ex We call it split if $\exists\, s : Z \to Z_0$ w/ $s \circ i = \mathrm{id}_{Z_0}$.

$\exists$ bijection of sets

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{split 1st order} \\ \text{thickenings of } Z_0 \end{array} \right\} & \xrightarrow{\sim} & \mathrm{QCoh}(Z_0) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (i : Z_0 = V(\mathcal{J}) \hookrightarrow Z) & \longmapsto & \mathcal{M} := \mathcal{J} \\ Z := \mathrm{Spec}_{\mathcal{O}_{Z_0}}(\mathcal{O}_{Z_0}[\mathcal{M}]) & \longmapsfrom & \mathcal{M} \end{array}$$

relative spectrum of the $\mathcal{O}_{Z_0}$-algebra

$\mathcal{O}_{Z_0}[\mathcal{M}] := \mathcal{O}_{Z_0} \oplus \mathcal{M} \in \mathrm{Mod}(\mathcal{O}_{Z_0})$

w/ multiplicatⁿ $(a \oplus m)(b \oplus n) := (ab \oplus (an + bn))$

Notation: $a + m\varepsilon := a \oplus m \in \mathcal{O}_{Z_0}[\mathcal{M}]$

e.g. $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\varepsilon]$ split

$\mathrm{Spec}\, \mathbb{Z}/p \hookrightarrow \mathrm{Spec}\, \mathbb{Z}/p^2$ not split

Def A morphism $f : X \to S$ of schemes is called

a) formally unramified / smooth / étale at $p \in X$

if $\exists$ open $S' \subset S$, $X' \subset f^{-1}(S')$ w/ $p \in X'$

sth $\forall$ first order thickening $i : Z_0 \hookrightarrow Z = \operatorname{Spec} R$

$\uparrow$ affine!

$\forall$ commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r, "u_0"] \arrow[d, hook, "i"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "v"] \arrow[ur, dashed, "\exists u?"] & S' \end{tikzcd}$$

$\exists$ at most / at least / precisely one u w/ $u \circ i = u_0$, $f \circ u = v$.

b) formally unramified / smooth / étale if it is so at all $p \in X$.

c) unramified / smooth / étale if it is formally so and moreover lft / lfp / lfp. Here

lft / lfp := locally of finite type / presentation

$\downarrow$

i.e. $\forall p \in X$ $\exists$ open $U \subset X$, $U \ni p$
sth

$$\begin{tikzcd} U = V(J) \arrow[r, hook, "\text{closed}"'] \arrow[dr, "f|_U"'] & \mathbb{A}^n_S \arrow[d, "pr"] \\ & S \end{tikzcd}$$

w/ $J \trianglelefteq \mathcal{O}_{\mathbb{A}^n_S}$ finitely generated

Ex a) If $X, S \in \mathrm{Sch}_k$ for a field k, then

$f : X \to S$ unramified / smooth / étale

$\Rightarrow$ $\forall p \in X$: $T_p f : T_p X \to T_{f(p)} S$ mono / epi / iso.

b) Open immersions $j : U \hookrightarrow S$ are étale:

Here a morphism $v : Z \to S$ factors over $j : U \hookrightarrow S$ iff $v(|Z|) \subset |U|$, hence the claim follows since $|Z| = |Z_0|$ $\forall$ first order thickening $Z_0 \hookrightarrow Z$.

c) Closed immersions $i : X \hookrightarrow S$ are

- unramified since they satisfy $X(Z) \hookrightarrow S(Z)$ for any scheme Z. $\leftarrow$ this is why we only imposed lft, not lfp...
- in general NOT formally smooth: think of $Z_0 = X \hookrightarrow Z = S$ any non-split first order thickening...

Thus unramified $\not\Rightarrow$ smooth!

d) $X = \mathbb{A}^n_R \longrightarrow S = \operatorname{Spec} R$

- smooth (by universal property of $R[x_1, \dots, x_n]$)
- NOT unramified (generators x_i can map anywhere)

Thus smooth $\not\Rightarrow$ unramified !

e) More geometric example for c):

$X = V(y^2 - x^3) \hookrightarrow S = \operatorname{Spec} k[x,y]$

Here existence of lifts fails for the morphisms u_0, v given by $x \mapsto 0$ & $y \mapsto t$ in the diagram below:

$$\begin{tikzcd}
Z_0 = \operatorname{Spec} k[t]/(t^2) \arrow[r, "u_0"] \arrow[d] & X = \operatorname{Spec} k[x,y]/(y^2 - x^3) \arrow[d] \\
Z = \operatorname{Spec} k[t]/(t^3) \arrow[r, "v"] \arrow[ur, dashed, "\nexists u"] & S = \operatorname{Spec} k[x,y]
\end{tikzcd}$$

Note: This cannot be seen by taking only $Z_0 = \operatorname{Spec} k \hookrightarrow Z = \operatorname{Spec} k[\varepsilon]$ as test scheme !

(Tangent spaces don't see enough if source is singular)

f) $X = \operatorname{Spec} k[t] \xrightarrow{f} S = \operatorname{Spec} k[x]$ w/ $f^\#(x) = t^n$

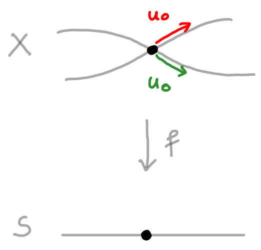
is

- étale for $n = \pm 1$,
- unramified but not smooth for $n = 0$,
- neither smooth nor unramified for $n \neq 0, \pm 1$:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "u_0"] \arrow[d] & \operatorname{Spec} k[t] \arrow[d] & t^n \\
\operatorname{Spec} k[\varepsilon] \arrow[r, "v"] \arrow[ur, dashed, "\exists ? u"] & \operatorname{Spec} k[x] & x \arrow[u, mapsto]
\end{tikzcd}$$

take $v^\#(x) := \varepsilon \Rightarrow$ no u exists

take $v^\#(x) := 0 \Rightarrow$ u not unique



$f(u_0) = 0$

for all $u_0 \in T_p X$

<u>Rem</u> a) Our definition only uses "local" 1st order lifting properties near each $p \in X$, hence smoothness can be checked on an open cover. In fact "global" 1st order lifting then follows, but that's non-trivial.

b) Could as well take n-th order thickenings (induction).

Lemma a) The class of (formally) unramified / smooth / étale morphisms is stable under

- base change
- products
- composition.

b) For $Y \xrightarrow{g} X \xrightarrow{f} S$ w/ f (formally) unramified

we have the "cancellation property":

$f \circ g$ (formally) unramified / smooth / étale

$\Longrightarrow$ g ——〃——

Pf. a) Exercise. b) Say $f \circ g$ is formally smooth.

Consider a diagram

$$\begin{tikzcd} Z_0 \arrow[r, "u_0"] \arrow[d, hook, "i"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "v"] \arrow[ur, dashed, "\exists u ?"] \arrow[d, equal] & X \arrow[d, "f"] \\ Z \arrow[r, "w"] & S \end{tikzcd}$$

w/ $Z_0 \hookrightarrow Z$ first order thickening.

$f \circ g$ formally smooth $\Longrightarrow$ $\exists u$ with $\begin{cases} i \circ u = u_0 \\ f \circ g \circ u = w \end{cases}$

f formally unramified

$\Longrightarrow$ $g \circ u = v$ by looking at

$$\begin{tikzcd} Z_0 \arrow[r, "u_0"] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \arrow[r, "w"] \arrow[ur, bend left, "v"] \arrow[ur, bend right, "g \circ u"'] & S \end{tikzcd}$$

If we drop "formally" it remains to check:

" f lft & $f \circ g$ lfp $\Longrightarrow$ g lfp "

Wlog $S = \operatorname{Spec} A$

$X = \operatorname{Spec} B$

$Y = \operatorname{Spec} C$

Let $A \xrightarrow[\alpha := f^\#]{} B \xrightarrow[\beta := g^\#]{} C$ w/ α fin. type, $\beta \circ \alpha$ fin. pres.

Write

$$\beta : B \xrightarrow{id \otimes \beta\alpha} B \otimes_A C \longrightarrow C$$

base change of the fin pres morphism $\beta\alpha$ $\Rightarrow$ fin. pres.

base change of $B \otimes_A B \xrightarrow{\mu} B$ which is fin. pres for $A \to B$ fin type, since $\ker(\mu)$ is gen. by $b \otimes 1 - 1 \otimes b$ for $b \in B$

$\Longrightarrow$ β fin. presented $\square$

Prop For any smooth morphism $f : X \to S$,

$\Omega^1_{X/S}$ is locally free of finite rank.

Pf. Properties local

$\Longrightarrow$ wlog $f : X = \mathrm{Spec}\, B \to S = \mathrm{Spec}\, A$

& 1st order lifting property holds on $X' = X$ and $S' = S$

Note that $\Omega^1_{X/S}$ is a locally finitely presented $\mathcal{O}_X$-module since $f : X \to S$ is lfp.

Thus:

$\Omega^1_{X/S}$ locally free

$\Longleftrightarrow$ $\Omega^1_{B/A}$ projective B-module

$\Longleftrightarrow$ $\forall$ surjection $M \twoheadrightarrow N$ in $\mathrm{Mod}(B)$:

$$\begin{tikzcd}
\mathrm{Hom}_B(\Omega^1_{B/A}, M) \arrow[r, two heads] \arrow[d, equal] & \mathrm{Hom}_B(\Omega^1_{B/A}, N) \arrow[d, equal] \\
\mathrm{Der}_A(B, M) \arrow[r] & \mathrm{Der}_A(B, N)
\end{tikzcd}$$

surjectivity follows from 1st order lifting property of $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ applied to the diagram:

(use remark a) below)

$$\begin{tikzcd}
\mathrm{Spec}\, B[N] \arrow[r] \arrow[d, hook] & \mathrm{Spec}\, B \arrow[d] \\
\mathrm{Spec}\, B[M] \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\, A
\end{tikzcd}$$

$\square$

Rem a) We have

$pr : B[M] \twoheadrightarrow B$

$$\begin{tikzcd}
\mathrm{Der}_A(B, M) \arrow[r, "\sim"] & \{\varphi \in \mathrm{Hom}_A(B, B[M]) \mid pr \circ \varphi = id\} \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\delta \arrow[r, mapsto] & \varphi := id + \delta
\end{tikzcd}$$

Indeed: $\delta(ab) = a\delta(b) + b\delta(a)$

$\Longleftrightarrow 1 + \delta(ab) = (1+\delta(a))(1+\delta(b))$ (as $M^2 = 0$)

$\Longleftrightarrow \varphi(ab) = \varphi(a)\varphi(b)$

b) Converse of the proposition is not true:

$\Omega^1_{X/S}$ locally free of finite rk $\not\Longrightarrow$ $X \to S$ smooth !

e.g. closed immersions $X \overset{\neq}{\hookrightarrow} S$ are NOT smooth

not flat

but have $\Omega^1_{X/S} \simeq 0$ locally free.

e.g. $X = \mathrm{Spec}(\mathbb{F}_p(t)) \to S = \mathrm{Spec}(\mathbb{F}_p(t^p))$

is NOT smooth but has $\Omega^1_{X/S} = \mathcal{O}_X \cdot dt$ free.

wrong rank (insep)

[BUT for irreducible schemes X over a field k, we will see later:

$X \to \mathrm{Spec}\, k$ smooth $\Longleftrightarrow$ $\Omega^1_{X/k}$ loc free of rk $= \dim X$]

(automatic if char $k = 0$)

First order lifts are controlled by derivations (generalizing remark a)):

Lemma Consider for an ideal $J \trianglelefteq R$ w/ $J^2 = 0$ a diagram

$$\begin{tikzcd}
\operatorname{Spec} R/J \arrow[r, "u_0"] \arrow[d, hook] & \operatorname{Spec} B \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists ? u"] & \operatorname{Spec} A
\end{tikzcd} \qquad (*)$$

Then the set $\text{Lifts}(u_0) := \{ u \mid (*) \text{ commutes} \}$ is a torsor under the group $\text{Der}_A(B, J)$.

More precisely: If $\exists\, u \in \text{Lifts}(u_0)$, then the B-module structure on J via $u^\#: B \to R$ only depends on u_0 (not on u) and for fixed u we get a bijection

$$\begin{tikzcd}
\text{Der}_A(B, J) \arrow[r, "\sim"] & \text{Lifts}(u_0) \\
\delta \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & u + \delta \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

Pf. We look at ring homom. $\varphi = u^\#: B \to A$ sth the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & R \arrow[d] \\
B \arrow[r] \arrow[ur, dashed, "\varphi"] & R/J
\end{tikzcd} \qquad (**)$$

Given $\varphi_1, \varphi_2 : B \to R$ making $(**)$ commute, we have $\varphi_1(b) - \varphi_2(b) \in J$ for all $b \in B$

$\Rightarrow (\varphi_1(b) - \varphi_2(b)) \cdot c = 0$ for all $c \in J$ since $J^2 = (0)$

$\Rightarrow \varphi_1(b) \cdot c = \varphi_2(b) \cdot c$ //

Thus if $\exists$ any $\varphi \in \text{Hom}_A(B, R)$ making $(**)$ commute, we get a B-module structure (independent of φ) $J \in \text{Mod}(B)$ via $b \cdot c := \varphi(b) \cdot c$ for $b \in B$, $c \in J$.

Given any such φ & $\delta \in \mathrm{Der}_A(B, J) \subset \mathrm{Der}_A(B, R)$, the map $\psi := \varphi + \delta : B \to R$ also makes (**) commute & it is a ring homom: For $b, c \in B$,

$$\begin{aligned} \psi(bc) &= \varphi(bc) + \delta(bc) \\ &= \varphi(b)\varphi(c) + b\,\delta(c) + c\,\delta(b) \\ &= (\varphi(b)+\delta(b)) \cdot (\varphi(c)+\delta(c)) = \psi(b)\psi(c) \end{aligned}$$

$\uparrow$

since $\delta(b) \cdot \delta(c) \in J^2 = (0)$

Conversely: Given $\varphi, \psi \in \mathrm{Hom}_A(B, R)$ which both make (**) commute, then $\delta := \varphi - \psi$ takes values in $J = \ker(R \to R/J)$ and is an A-linear derivation:

- A-linear by construction
- derivation since for $b, c \in B$,

$$\begin{aligned} \delta(bc) &= \varphi(bc) - \psi(bc) \\ &= \varphi(b)\varphi(c) - \psi(b)\psi(c) \\ &= \varphi(b) \cdot (\varphi(c) - \psi(c)) + \psi(c) \cdot (\varphi(b) - \psi(b)) \\ &= b \cdot \delta(c) + c \cdot \delta(b) \end{aligned}$$

□

<u>Cor</u> $X \to S$ is formally unramified iff $\Omega^1_{X/S} \simeq 0$.

Pf. Consider

$$\begin{tikzcd} Z_0 \arrow[r, "u_0"] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \arrow[r, "v"'] \arrow[ur, dashed, "\exists ? u"] & Y \end{tikzcd}$$

w/ $Z_0 \hookrightarrow Z$ first order thickening

f formally unramified $:\Longleftrightarrow$ uniqueness of u for all such diagrams, all 1st order $Z_0 \hookrightarrow Z$ w/ image in suitable open covering subsets $S' \subset S$ and $X' \subset f^{-1}(S')$

Thus wlog $X = \mathrm{Spec}\, B \xrightarrow{f} S = \mathrm{Spec}\, A$. Then:

$$\Omega^1_{X/S} \simeq 0 \iff \mathrm{Der}_S(B, M) = 0 \ \forall M$$

$$\underset{\text{previous lemma}}{\iff} f \text{ formally unramified}$$

(any M arises as $M = J$ for the 1st order $R = A[M]$)

□

4. Splitting of the cotangent sequence & local coordinates

Recall for $X \xrightarrow{f} Y \xrightarrow{g} S$ we had the cotangent sequence

$$0 \underset{?}{\longrightarrow} f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \qquad (*)$$

Prop ("splitting of the cotangent sequence")

a) If f is formally smooth, then $(*)$ is exact (ie $f^*\Omega^1_{Y/S} \hookrightarrow \Omega^1_{X/S}$ injective) and locally split.

b) Conversely, if $g \circ f$ is formally smooth & $(*)$ is exact and locally split, then f is formally smooth.

Pf. Conditions are local

$\Rightarrow$ wlog $X = \operatorname{Spec} C$
$Y = \operatorname{Spec} B$
$S = \operatorname{Spec} A$.

a) Assume $f: \operatorname{Spec} C \to \operatorname{Spec} B$ formally smooth.

Goal: The cotangent sequence is split exact:

$$\begin{tikzcd} 0 \arrow[r] & C \otimes_B \Omega^1_{B/A} \arrow[r, "df^\#"] & \Omega^1_{C/A} \arrow[r] \arrow[l, dashed, bend left] & \Omega^1_{C/B} \arrow[r] & 0 \end{tikzcd}$$

$\exists$ splitting s

w/ $s(\underbrace{c \cdot d(f^\#(b))}_{df^\#(c \otimes db)}) = c \otimes db$

To find s, use formal smoothness of f. After a localization:

$$\begin{tikzcd} C & C \arrow[l, "id"'] \arrow[ld, dashed, "\exists \eta"'] \\ C[M] \arrow[u, "pr"] & B \arrow[l, "f^\# \oplus (1 \otimes d_{B/A})"] \arrow[u, "f^\#"'] \end{tikzcd}$$

w/ $M := C \otimes_B \Omega^1_{B/A} \in \operatorname{Mod}(C)$

this is A-linear (not B-linear!)

$\Rightarrow \exists \eta = id \oplus \delta \in \operatorname{Hom}_A(C, C[M])$

w/ $\delta \in \operatorname{Der}_A(C, M)$

$\downarrow \qquad \|$

Define: $s \in \operatorname{Hom}_C(\Omega^1_{C/A}, M) \Rightarrow s(dc) = \delta(c) \; \forall c \in C$

$\Rightarrow s(c \cdot d(f^\#(b))) = c \cdot s(d(f^\#(b)))$
$= c \cdot \delta(f^\#(b)) = c \otimes b$ as wanted.

b) Conversely, assume $g \circ f$ is formally smooth and

$0 \to C \underset{B}{\otimes} \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$ is split exact.

Goal: f is formally smooth.

For this consider any diagram

$$\begin{tikzcd}
R/J & C \arrow[l, "\gamma_0"'] \arrow[dl, dashed, "\exists ? \gamma"'] \\
R \arrow[u, "p"] & B \arrow[l, "\beta"] \arrow[u, "f^\#"'] \\
R \arrow[u, equal] & A \arrow[l, "\alpha"] \arrow[u, "g^\#"']
\end{tikzcd}$$

w/ $J \trianglelefteq R$

$J^2 = (0)$

& put

$\alpha := \beta \circ g^\#$

$g \circ f$ formally smooth $\Rightarrow \exists\, \gamma' : C \to R$ w/ $p \circ \gamma' = \gamma_0$

$\gamma' \circ f^\# \circ g^\# = \alpha$

We look for $\delta \in \mathrm{Der}_A(C, J)$ sth $\gamma := \gamma + \delta$

satisfies $\gamma \circ f^\# = \beta$,

ie. $\beta - \gamma' \circ f^\# = \delta \circ f^\#$.

Existence of such δ follows from the fact that

$$\begin{tikzcd}
\mathrm{Der}_A(C, J) \arrow[r, "\delta \mapsto \delta \circ f^\#"'] \arrow[d, equal] & \mathrm{Der}_A(B, J) \arrow[d, equal] \\
\mathrm{Hom}_C(\Omega^1_{C/A}, J) \arrow[r, two heads] & \mathrm{Hom}_C(C \underset{B}{\otimes} \Omega^1_{B/A}, J)
\end{tikzcd}$$

is surjective, using the splitting of the cotangent sequence. ☐

Cor 1 ("étale morphisms & differentials")

If $f \in \mathrm{Hom}_S(X, Y)$ is formally étale,

then $f^* \Omega^1_{Y/S} \xrightarrow{\sim} \Omega_{X/S}$ is an iso.

Pf. Epi because $\Omega^1_{X/Y} = 0$ for f formally unramified.

Mono by the proposition a) for f formally smooth. ☐

Cor 2 ("uniformizing parameters / étale coordinates")

A morphism $h : X \to S$ of schemes is smooth at a point $p \in X$ iff $\exists$ open $p \in U \subset X$ $\exists$ $f_1, \dots, f_n \in \mathcal{O}_X(U)$ giving an étale morphism $f := (f_1, \dots, f_n) : U \to \mathbb{A}^n_S$:

$$\begin{tikzcd} U \arrow[r, "\overset{\text{étale}}{f}"] \arrow[dr, "h|_U"'] & \mathbb{A}^n_S \arrow[d, "g = pr"] \\ & S \end{tikzcd}$$

Pf. "$\Leftarrow$" : $h|_U = pr \circ f$ = smooth $\circ$ étale implies that $h|_U$ is smooth

"$\Rightarrow$" : For h smooth we know $\Omega^1_{X/S}$ locally free, say locally $\Omega^1_{X/S}|_U = \bigoplus_{i=1}^n \mathcal{O}_U \cdot df_i$ w/ $f_1, \dots, f_n \in \mathcal{O}_X(U)$.

Then for $f = (f_1, \dots, f_n) : U \to Y = \mathbb{A}^n_S$ we get :

$$f^* \Omega^1_{Y/S} = \bigoplus_{i=1}^n \mathcal{O}_U \cdot dx_i \xrightarrow{\sim} \Omega^1_{X/S}|_U = \bigoplus_{i=1}^n \mathcal{O}_U \cdot df_i$$

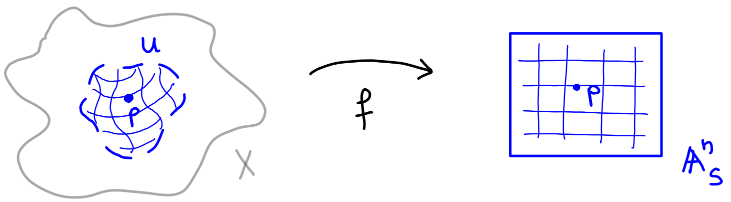
$$dx_i \longmapsto df_i$$

So the cotangent sequence

$$0 \to f^* \Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{U/S} \to \Omega^1_{U/Y} \to 0$$

is trivially split exact. Since $g \circ f|_U = h|_U$ is smooth, prop. b) implies $f|_U$ is smooth. $\square$

Def We call $(f_1, \dots, f_n)$ local uniformizing parameters or étale-local coordinates on $X \in Sch_S$.



Caution In general U is NOT a local isomorphism!

e.g. $f : \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, $z \mapsto z^2$ is étale but has no local inverse using polynomials: We need power series, eg $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots \in \mathbb{C}[[x]]$.

Def For a scheme X and $p \in X$, consider the complete local ring

$$\hat{\mathcal{O}}_{X,p} := \varprojlim \mathcal{O}_{X,p}/\mathfrak{m}_p^n .$$

Lemma Let $f : X \to Y$ be formally étale and $x \in X$, $y = f(x) \in Y$ w/ $\kappa(x) \simeq \kappa(y)$.

$\Rightarrow$ $f^\#$ induces an isomorphism

$$\hat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}$$

Pf. Put $k := \kappa(x)$. Let $\mathcal{C}_n$ be the category w/

- objects: (R, ι) where R is a local ring w/ $\mathfrak{m}_R^n = (0)$ & $\iota : k \xrightarrow{\sim} R/\mathfrak{m}_R$ is an iso.
- morphisms: local ring homom. $\varphi : R \to R'$ inducing a comm. diagram

$$\begin{tikzcd} k \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & R/\mathfrak{m}_R \arrow[d, "\bar{\varphi}"] \\ & R'/\mathfrak{m}_{R'} \end{tikzcd}$$

Have representable functors

$$\mathcal{F}_n = \mathrm{Hom}_{\mathcal{C}_n}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n, -) : \mathcal{C}_n \longrightarrow \mathrm{Sets}$$

$$\mathcal{G}_n = \mathrm{Hom}_{\mathcal{C}_n}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n, -) : \mathcal{C}_n \longrightarrow \mathrm{Sets}$$

and a natural trafo $\mathcal{F}_n \Rightarrow \mathcal{G}_n$, $\varphi \mapsto \varphi \circ f^\#$.

Goal: This is an iso for all n

(so by Yoneda $\mathcal{O}_{Y,y}/\mathfrak{m}_y^n \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ for all n).

Use induction:

- $n = 1$ by assumption $\kappa(y) \simeq \kappa(x)$.
- Assume claim for $n-1$. Then for $A \in \mathcal{C}_n$ we have:

$$A_0 := A/\mathfrak{m}_A^{n-1} \in \mathcal{C}_{n-1} \Rightarrow \mathcal{F}_n(A_0) = \mathcal{F}_{n-1}(A_0)$$
$$\underset{\substack{\uparrow \\ \text{induction}}}{=} \mathcal{G}_{n-1}(A_0) = \mathcal{G}_n(A_0)$$

$\Rightarrow$ $\mathcal{F}_n(A) = \mathcal{G}_n(A)$ since for f formally étale we have unique lifting in the diagram

$$\begin{tikzcd} A_0 & \mathcal{O}_{X,x} \arrow[l, "u_0"'] \arrow[dl, dashed, "\exists ! \tilde{u}"'] \\ A \arrow[u] & \mathcal{O}_{Y,y} \arrow[l, "v"] \arrow[u] \end{tikzcd}$$

$\square$

Here is an example how to use this:

<u>Cor</u> Let X be a scheme over a field k and $p \in X$ a closed point at which $X \to \operatorname{Spec} k$ is smooth.

$$\Longrightarrow \dim_{\kappa(p)} \Omega^1_{X/k} \otimes_{\mathcal{O}_{X,p}} \kappa(p) = \dim_p X$$

(where $\dim_p X := \dim \mathcal{O}_{X,p}$ is the "Krull dim of X at p")

Pf. Base change from k to $\kappa(p) \Rightarrow$ wlog $\kappa(p) = k$

(exerise: This doesn't change $\dim_p X$ ... e.g. use Noether normalization)

- Since $X \to \operatorname{Spec} k$ is smooth at p,

  $\exists$ open $\underset{\overset{\cup\!\shortmid}{p}}{U} \subset X$ $\exists$ étale coordinate system $f : U \longrightarrow \mathbb{A}^n_k$.

  $$\Rightarrow \Omega^1_{X/S}|_U \simeq (f^* \Omega^1_{\mathbb{A}^n/k})|_U \quad \text{loc. free of rank } n$$

- Moreover $\kappa(p) = k = \kappa(f(p))$ implies $\hat{\mathcal{O}}_{X,p} \simeq \hat{\mathcal{O}}_{\mathbb{A}^n, f(p)}$

  $$\Rightarrow \dim_p X = \dim \mathcal{O}_{X,p} = \dim \hat{\mathcal{O}}_{X,p} = \dim \hat{\mathcal{O}}_{\mathbb{A}^n, f(p)} = n.$$

  $\dim A = \dim \hat{A}$ for any Noetherian local ring A
  (see e.g. Atiyah-MacDonald, cor. 11.19)

$\square$

5. Splitting of the conormal sequence & the Jacobian criterion

Now consider a closed immersion in Schs

$$\begin{tikzcd}
X = V(J) \arrow[r, hook, "i"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\
 & S
\end{tikzcd}$$

w/ conormal sequence

$$0 \xrightarrow{?} J/J^2 \longrightarrow i^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \to 0 \qquad (**)$$

Prop ("splitting of the conormal sequence")

a) If h is formally smooth, then $(**)$ is exact (ie $J/J^2 \hookrightarrow i^*\Omega^1_{X/S}$ injective) and locally split.

b) Conversely, if g is formally smooth & $(**)$ is exact and locally split, then h is formally smooth.

Pf. Again wlog $S = \mathrm{Spec}\, A$
$Y = \mathrm{Spec}\, B$
$X = \mathrm{Spec}\, C$ w/ $C = B/J$.

a) Want a section s of $d: J/J^2 \longrightarrow C \otimes_B \Omega^1_{B/A}$.

Consider

$$\begin{tikzcd}
B/J & B/J \arrow[l, "id"'] \arrow[dl, dashed, "\exists ? \varphi"'] & B \arrow[l, two heads, "p"'] \\
B/J^2 \arrow[u] & A \arrow[l, "g^\# \bmod J^2"] \arrow[u, "h^\#"'] &
\end{tikzcd}$$

h smooth $\Rightarrow$ $\exists \varphi$ making the diagram commute

But then $\varphi \circ p$ & the quotient map $q: B \twoheadrightarrow B/J^2$ both lift $p: B \twoheadrightarrow B/J$. Hence we can put

$$\begin{tikzcd}
\delta := q - \varphi \circ p \arrow[dd, mapsto] & \in & \mathrm{Der}_A(B, J/J^2) \arrow[d, equal] \\
 & & \mathrm{Hom}_B(\Omega^1_{B/A}, J/J^2) \arrow[d, equal] \\
s := \mathrm{image}(\delta) & \in & \mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, J/J^2)
\end{tikzcd}$$

By construction, for $b \in J$ w/ image $\bar b \in J/J^2$ we have

$$s(d\bar b) = s(1 \otimes db) = \delta(b)$$
$$= q(b) - \varphi(\underbrace{p(b)}_{=0}) = \bar b.$$

b) To show h is formally smooth, we need to solve a lifting problem

$$\begin{tikzcd} R/\mathcal{J} & C = B/J \arrow[l, "u_0"'] \arrow[dl, dashed, "\exists ? u"'] \\ R \arrow[u] & A \arrow[l] \arrow[u, "h^\#"'] \end{tikzcd}$$

w/ $\mathcal{J} \trianglelefteq R$
$\mathcal{J}^2 = (0)$

By assumption g is formally smooth, hence $\exists v$ as shown below:

$$\begin{tikzcd} R/\mathcal{J} & C = B/J \arrow[l, "u_0"'] \arrow[dl, dashed, "\exists ? u"'] & B \arrow[l, "i^\#"'] \arrow[dll, dashed, "\exists v"'] \\ R \arrow[u] & A \arrow[l] \arrow[ur, "g^\#"'] & \end{tikzcd}$$

Idea: Modify v sth $J \subseteq \ker(v)$!

Any other lift has the form $v + \delta$ w/ $\delta \in \operatorname{Der}_A(B, \mathcal{J})$.

We want δ with $\delta|_J = -v|_J$. This exists since

$$\underset{\underset{\operatorname{Hom}_C(C \otimes_B \Omega^1_{B/A}, \mathcal{J})}{\shortparallel}}{\operatorname{Der}_A(B, \mathcal{J})} \twoheadrightarrow \operatorname{Hom}_B(J/J^2, \mathcal{J})$$

is surjective, because by assumption the sequence $(**)$ splits.

$\square$

<u>Cor</u> ("Jacobian criterion") Let $i : Z \hookrightarrow \mathbb{A}^n_R = \operatorname{Spec} R[z_1, .., z_n]$ a closed immersion which is locally of finite presentation.

Then for any $p \in Z$, TFAE:

a) $Z \to \operatorname{Spec} R$ is smooth at p

b) $\exists \underset{\underset{p}{\ni}}{U} \subset \mathbb{A}^n_R \ \exists f_1, \dots, f_r \in \mathcal{O}_{\mathbb{A}^n_R}(U)$ sth

- $Z \cap U = V(f_1, \dots, f_r)$, and
- $\operatorname{rk}\left(\frac{\partial f_j}{\partial z_i}(p)\right) = r$

"Jacobian matrix has maximal rank"

Pf. b) $\Rightarrow$ a): Put $\operatorname{Jac} := \left(\frac{\partial f_j}{\partial z_i}\right) \in \operatorname{Mat}_{n \times r}(\mathcal{O}_U)$.

Consider

$$\begin{tikzcd} \bigoplus_{j=1}^r \mathcal{O}_{Z \cap U} \arrow[r, "\operatorname{Jac}"] \arrow[d, "(f_1, \dots, f_r)"] & \bigoplus_{i=1}^n \mathcal{O}_{Z \cap U} \cdot dz_i \arrow[d, "\wr"] & & \\ J/J^2|_{Z \cap U} \arrow[r, "d"] & i^*(\Omega^1_{\mathbb{A}^n_R/R})|_{Z \cap U} \arrow[r] & \Omega^1_{Z/R} \arrow[r] & 0 \end{tikzcd}$$

By assumption:

- $Z \cap U = V(f_1, \dots, f_r)$

  $$\Rightarrow (f_1, \dots, f_r) : \mathcal{O}_{Z \cap U}^{\oplus n} \twoheadrightarrow \mathcal{J}/\mathcal{J}^2|_U \quad \text{epi}$$

- $\mathrm{rk}(\mathrm{Jac})(p) = r$

  $$\Rightarrow \mathrm{Jac} : \bigoplus_{j=1}^{n} \mathcal{O}_{Z,p} \hookrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{Z,p} \cdot dz_i \quad \text{mono}$$

The diagram then gives:

$$(f_1, \dots, f_r) : \mathcal{O}_{Z,p}^{\oplus n} \xrightarrow{\sim} (\mathcal{J}/\mathcal{J}^2)_p \quad \text{iso}$$

$$\& \quad d : (\mathcal{J}/\mathcal{J}^2)_p \hookrightarrow i^*(\Omega^1_{\mathbb{A}^n_R/R})_p \quad \text{split mono}$$

(exercise: For a local ring A w/ residue field $k = A/\mathfrak{m}$ and any $M \in \mathrm{Mat}_{n \times r}(A)$, we have:

$A^r \xrightarrow{M} A^n$ split injective $\Leftrightarrow$ $k^r \xrightarrow{M} k^n$ injective)

$$\Rightarrow 0 \to \mathcal{J}/\mathcal{J}^2 \to i^*\Omega^1_{\mathbb{A}^n_R/R} \to \Omega^1_{Z/R} \to 0$$

is locally split exact at p, so proposition (part b)) implies that $h : Z \to \operatorname{Spec} R$ is smooth at p

a) $\Rightarrow$ b): Assume $h : Z \to \operatorname{Spec} R$ is smooth at p.

By the proposition (part a)) then

$$0 \to \mathcal{J}/\mathcal{J}^2 \to i^*\Omega^1_{\mathbb{A}^n_R/R} \to \Omega^1_{Z/R} \to 0$$

is split exact at p. Hence $\exists\, U \ni p$ $\exists\, f_1, \dots, f_r \in \mathcal{J}(U)$:

$$(\mathcal{J}/\mathcal{J}^2)|_{Z \cap U} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{Z \cap U} \cdot \bar{f}_i \quad \text{w/ } \bar{f}_i := f_i \bmod \mathcal{J}^2.$$

The same diagram as above then shows

$$\bigoplus_{j=1}^{r} \mathcal{O}_{Z,p} \xhookrightarrow{\left(\frac{\partial f_j}{\partial z_i}\right)} \bigoplus_{i=1}^{n} \mathcal{O}_{Z,p} \cdot dz_i \quad \text{mono}$$

Hence $\mathrm{rk}\left(\frac{\partial f_j}{\partial z_i}(p)\right) = r$.

Moreover, shrinking U we may assume $Z \cap U = V(f_1, \dots, f_r)$ by Nakayama:

$M := \left(\mathcal{J}/(f_1, \dots, f_r)\right)_p$ fin gen. module over $\mathcal{O}_{Z,p}$

$$M/\mathcal{J}M = \left((\mathcal{J}/\mathcal{J}^2)/(\bar{f}_1, \dots, \bar{f}_r)\right)_p = 0 \underset{\substack{\wr \\ \mathcal{J} \trianglelefteq \mathfrak{m}_p}}{\Rightarrow} M = 0 \qquad \square$$

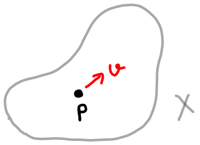
6. Smoothness and Regularity

Recall For a scheme X over a field k and $p \in X$ we define the tangent space to X at p by

$$T_pX := \{ v \in \mathrm{Hom}_k(\mathrm{Spec}(\kappa(p)[\varepsilon]), X) \mid v(0) = p \}$$

ie the set of lifts making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Spec}\,\kappa(p) \arrow[r, "\text{point } p"] \arrow[d, hook] & X \arrow[d] \\
\mathrm{Spec}\,\kappa(p)[\varepsilon] \arrow[r] \arrow[ur, dashed, "v"] & \mathrm{Spec}\,k
\end{tikzcd}$$



Lemma We have a natural identification

$$T_pX \simeq \mathrm{Hom}_{\mathcal{O}_{X,p}}(\Omega^1_{X/k,p}, \kappa(p))$$

Pf. Wlog $X = \mathrm{Spec}\,\mathcal{O}_{X,p}$ by localizing. Then the set of lifts in the diagram is in bijection with

$$\mathrm{Der}_k(\mathcal{O}_{X,p}, (\varepsilon)) = \mathrm{Der}_k(\mathcal{O}_{X,p}, \kappa(p))$$
$$= \mathrm{Hom}_{\mathcal{O}_{X,p}}(\Omega^1_{X/k,p}, \kappa(p)).$$

□

Cor a) T_pX is a vector space over $\kappa(p)$

b) If $\kappa(p)$ is a finite separable extension of k, we have an isomorphism of vector spaces

$$T_pX \simeq (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee := \mathrm{Hom}_{\kappa(p)}(\mathfrak{m}_p/\mathfrak{m}_p^2, \kappa(p)).$$

Pf. a) clear from the lemma.

b) The lemma says

$$T_pX \simeq \mathrm{Hom}_{\kappa(p)}(\Omega^1_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p), \kappa(p))$$

The claim follows since

$$d_{X/k}: \mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{\sim} \Omega^1_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p) \quad \text{is an iso,}$$

by the conormal sequence for

$$\begin{tikzcd}
\mathrm{Spec}\,\kappa(p) = V(\mathfrak{m}_p) \arrow[r, hook] \arrow[dr, "h"'] & X \arrow[d] \\
& \mathrm{Spec}\,k
\end{tikzcd}$$

if $\kappa(p) \supset k$ is finite separable, then h is étale (see appendix)

$\Rightarrow$ conormal sequence split exact & $\Omega^1_{\kappa(p)/k} = 0$

□

Def Let X be a scheme and $p \in X$. We call

$$T_p^\vee X := \mathfrak{m}_p / \mathfrak{m}_p^2$$ the cotangent space and

$$T_p X := \mathrm{Hom}_{\kappa(p)}(T_p^\vee X, \kappa(p))$$ the tangent space to X at p.

Ex Let k be a field, $f_1, \dots, f_m \in k[x_1, \dots, x_n]$

and $X = V(f_1, \dots, f_m) \hookrightarrow \mathbb{A}^n_k$.

For $p \in X(k)$ then

$$T_p X = \bigcap_{j=1}^{m} (\vec{\nabla} f_i(p))^\perp$$

$$= \{ (\xi_1, \dots, \xi_n) \in \mathbb{A}^n_k \mid \forall j : \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \cdot \xi_i = 0 \}$$

"dual coordinates"

Pf. We know $\Omega^1_{X/k} \simeq \mathrm{coker}\left(\bigoplus_{j=1}^{m} \mathcal{O}_X \xrightarrow{\left(\frac{\partial f_j}{\partial x_i}\right)} \bigoplus_{i=1}^{n} \mathcal{O}_X \right)$.

Since $\kappa(p) = k$ is separable over k, the previous corollary gives

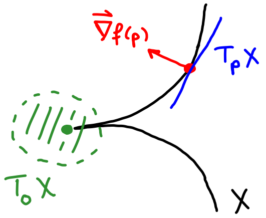
$$T_p X = \mathrm{Hom}_{\mathcal{O}_{X,p}}(\Omega^1_{X/k,p}, \kappa(p))$$

$$\simeq \ker\left(\bigoplus_{i=1}^{n} \mathcal{O}_X \xrightarrow{\left(\frac{\partial f_j}{\partial x_i}\right)} \bigoplus_{j=1}^{m} \mathcal{O}_X \right)$$ $\square$

Ex a) For the cusp $X = V(y^2 - x^3) \subset \mathbb{A}^2_k$

and $p = (x, y)$,

$$T_p X = \{ (\xi, \eta) \mid 2y\eta = 3x^2\xi \} \text{ has dim} = \begin{cases} 1 & \text{if } p \neq 0 \\ 2 & \text{if } p = 0 \end{cases}$$

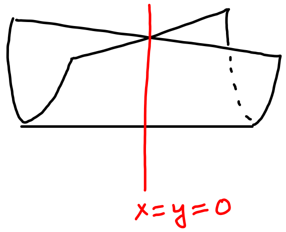


b) For the Whitney umbrella $X = V(x^2 - y^2 z) \subset \mathbb{A}^2_k$

and $p = (x, y, z)$,

$$T_p X = \{ (\xi, \eta, \zeta) \mid 2x\xi = y \cdot (2z\eta + y\zeta) \} \text{ has}$$

$$\dim = \begin{cases} 2 & \text{if } (x,y) \neq (0,0) \\ 3 & \text{if } (x,y) = (0,0) \end{cases}$$



Lemma For any locally Noetherian scheme X, we have

$$\dim_{\kappa(p)} T_p X \geq \dim_p X := \dim \mathcal{O}_{X,p} \quad \forall p \in X.$$

Pf. Put $A = \mathcal{O}_{X,p} \trianglerighteq m = m_p$.

Nakayama: The ideal $m \trianglelefteq A$ can be generated by $n := \dim_{A/m} (m/m^2)$ elements.

Krull's principal ideal thm (inductively) then implies $\dim A = ht(m) \leq n$. $\square$

Def We call $p \in X$ a regular (or nonsingular) point if $\dim_{\kappa(p)} T_p X = \dim \mathcal{O}_{X,p}$, i.e. if $\mathcal{O}_{X,p}$ is a regular local ring.

↳ a Noetherian local ring A satisfying the following equivalent conditions:

a) $\dim A = \dim_{A/m} (m/m^2)$,

b) m can be generated by $\dim A$ elements.

A scheme X is called regular if it is locally Noetherian and regular at all points $p \in X$.

Lemma For a locally Noetherian scheme X, TFAE:

a) X is regular

b) $\mathcal{O}_{X,p}$ is a regular local ring for all closed points $p \in X$.

Pf. a) $\Rightarrow$ b) by definition.

For b) $\Rightarrow$ a) use:

- If a local Noetherian ring A is regular, then $A_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} A$.
- Any locally Noetherian scheme has closed points. $\square$

Rem a) Any regular scheme X is reduced & normal.

(since by commutative algebra, Noetherian regular local rings are UFD & UFD are reduced & integrally closed).

b) A Noetherian scheme X with $\dim X = 1$ is regular iff it is normal

(for Noetherian local rings of $\dim = 1$ we have UFD $\Leftrightarrow$ DVR $\Leftrightarrow$ normal).

Regularity vs smoothness?

- Smoothness is a relative property (of morphisms).

  Regularity is an absolute property (of schemes).

- Regularity is not stable under base change:

  e.g. $X = V(x^p - y^p - a) \subset \mathbb{A}^2_k$ w/ char $k = p$, $\sqrt[p]{a} \notin k$

  $\Longrightarrow$ X regular but $X_{\bar{k}}$ not regular (exercise)

  (and $X \to \operatorname{Spec} k$ not smooth)

- But regularity is useful in mixed characteristic,

  e.g. $X = \mathbb{Z}[x,y]/(xy - p)$ w/ p prime

  $\Longrightarrow$ X regular (but not smooth over any base) (exercise)

Prop Let X be a scheme of finite type over a field k, and let $p \in X$ be a closed point. TFAE:

a) $X \to \operatorname{Spec} k$ is smooth at p.

b) $\dim_{\kappa(p)} \Omega^1_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p) \leq \dim_p X$.

c) $\dim_{\kappa(p)} \Omega^1_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p) = \dim_p X$.

If $\kappa(p) \supset k$ is separable, these are equivalent to:

d) p is a regular point of X.

Slogan: Over perfect fields, smooth $\Leftrightarrow$ regular

Pf. ① Assume first $\kappa(s) \supset k$ is separable. Wlog $X = \operatorname{Spec} A$ affine. Have

$$\begin{tikzcd} Y = \operatorname{Spec} \kappa(p) = V(m_p) \arrow[r, hook] \arrow[dr, "h"'] & X \arrow[d] \\ & S = \operatorname{Spec} k \end{tikzcd}$$

$\kappa(p)/k$ finite (since p closed & X finite type $/k$) & separable (by assumption) $\Longrightarrow$ h étale (see appendix)

$\Rightarrow$ conormal sequence for $Y \hookrightarrow X$ gives

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & A/\mathfrak{m} \otimes_A \Omega^1_{A/k} \arrow[r] & \Omega^1_{\kappa(p)/k} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & & & 0 & \end{tikzcd}$$

injective since h is smooth

since h is unramified

$$\Rightarrow \quad \mathfrak{m}/\mathfrak{m}^2 \simeq A/\mathfrak{m} \otimes_A \Omega^1_{A/k}$$

$\wr$

$\dim_{\kappa(p)}(\mathfrak{m}/\mathfrak{m}^2) \geq \dim_p X$,

with equality iff X is regular at p

Thus: a) $\Rightarrow$ b) $\Leftrightarrow$ c) $\Leftrightarrow$ d)

via étale coord^s — for $\kappa(p) \supset k$ separable

② Next we show c), d) $\Rightarrow$ a) for $k = \bar{k}$ alg closed:

In that case $\kappa(p) = k$

$\Rightarrow$ Wlog $\exists$ closed immersion

$$X = V(J) \hookrightarrow Z = \mathbb{A}^n_k$$

with $\overset{\in}{p} \longmapsto \overset{\in}{0}$

We now reduce to $X \hookrightarrow Z$ being a "complete intersection":

We have an exact sequence

$$J/\mathfrak{m}_{Z,p} \cdot J \xrightarrow{\alpha} \mathfrak{m}_{Z,p}/\mathfrak{m}^2_{Z,p} \xrightarrow{\beta} \underbrace{\mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p}}_{\substack{\dim_k = \dim_p X \\ \text{since } \mathcal{O}_{X,p} \text{ is regular by d)}}} \to 0$$

$\Rightarrow$ For $d := \dim_p X$

and $c := n - d$,

$\exists\, f_1, \dots, f_c \in J$ whose images in $\mathfrak{m}_{Z,p}/\mathfrak{m}^2_{Z,p}$

generate the kernel $\ker(\beta)$ as $\mathcal{O}_{Z,p}$-module

Put $J' := (f_1, \dots, f_c) \subset J$ & $X' := V(J') \subset Z$.

Still get exact sequence

$$J'/\mathfrak{m}_{Z,p} \cdot J' \xrightarrow{\alpha'} \mathfrak{m}_{Z,p}/\mathfrak{m}^2_{Z,p} \xrightarrow{\beta'} \mathfrak{m}_{X',p}/\mathfrak{m}^2_{X',p} \to 0$$

where $\operatorname{im}(\alpha') = \operatorname{im}(\alpha)$ by choice of J'

$$\Rightarrow \quad \dim_k \mathfrak{m}_{X',p}/\mathfrak{m}^2_{X',p} = \dim_k \mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p} \overset{\text{again by d)}}{=} d$$

But also $\dim \mathcal{O}_{X',p} \geq \dim \mathcal{O}_{X,p} = d$ since $\mathcal{O}_{X',p} \twoheadrightarrow \mathcal{O}_{X,p}$

$(X = V(J) \hookrightarrow X' = V(J'))$

$\Rightarrow \dim \mathcal{O}_{X',p} \geq \dim_k \mathfrak{m}_{X',p}/\mathfrak{m}^2_{X',p}$

$\Rightarrow$ equality holds & $\mathcal{O}_{X',p}$ is regular of dim d

Thus $X \underset{\text{closed}}{\hookrightarrow} X'$ w/ $\dim_p X = \dim_p X'$ & $\mathcal{O}_{X',p}$ reduced

$\Rightarrow$ Shrinking X & X' we may assume $X = X'$

Upshot: Wlog $X = V(f_1, \dots, f_c) \subset \mathbb{A}^n_k$

where $c = n - d$ ("complete intersection")

Now by assumption c),

$$\Omega^1_{X/k,p} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{X,p} \cdot dx_i \Big/ \mathrm{span}(df_1, \dots, df_c)$$

has $\dim_k \Omega^1_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p) = d = n - c$

$\Rightarrow \mathrm{rank}(df_1, \dots, df_c)(p) = c$

$\Rightarrow$ X smooth / k at p by Jacobian criterion

③ Finally we deduce a) $\Leftrightarrow$ b) $\Leftrightarrow$ c) for ANY k & $\kappa(p)$:

Pick $K \supset k$ sth $\exists\, \bar{p} \in \bar{X}(K)$ above p, where $\bar{X} := X_K$.

Then:

- $\dim_{\kappa(p)} \Omega^1_{X/k} \otimes_{\mathcal{O}_{X,p}} \kappa(p) = \dim_K \Omega^1_{\bar{X}/K,\bar{p}} \otimes_{\mathcal{O}_{\bar{X},\bar{p}}} K$
- $\dim_p X = \dim_{\bar{p}} \bar{X}$ (exercise)
- X smooth / k at p $\Leftrightarrow$ $\bar{X}$ smooth / K at $\bar{p}$

see lemma below

$\square$

Lemma ("descent of smoothness").

Let X be a scheme of finite type / k.

Then for any field extension $K \supset k$ we have:

X smooth / k $\Leftrightarrow$ X_K smooth / K.

Pf.

Wlog $\exists$ closed immersion $i : X = V(J) \hookrightarrow Y = \mathbb{A}^n_k$.

Then:

X smooth $/k$ $\iff$ $0 \to J/J^2 \to i^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to 0$

is exact & locally split

$\iff$ $d: J/J^2 \to i^*\Omega^1_{Y/k}$ injective

and $\Omega^1_{X/k}$ locally free.

Likewise:

X_K smooth $/K$ $\iff$ $d_K : J_K/J_K^2 \to i_K^* \Omega^1_{Y_K/K}$ injective

and $\Omega^1_{X_K/K}$ locally free

Write $Y = \operatorname{Spec} R$ w/ $R = k[x_1, .., x_n]$,

$X = \operatorname{Spec} S$ w/ $S = R/J$.

Then

$$d : J/J^2 \longrightarrow \Omega^1_{R/k} \otimes_R S,$$

$$d_K = d \otimes id : J/J^2 \otimes_k K \longrightarrow \Omega^1_{R/k} \otimes_R S \otimes_k K.$$

Hence d injective $\iff$ d_K injective

(since $k \hookrightarrow K$ faithfully flat)

Moreover:

$\Omega^1_{X/k}$ locally free $\mathcal{O}_X$-module

$\iff$ $\Omega^1_{S/k}$ flat S-module (for finitely presented modules, loc. free $\iff$ flat (see next sect^n))

$\iff$ $\Omega^1_{S/k} \otimes_k K$ flat $S \otimes_k K$-module (since $k \hookrightarrow K$ is faithfully flat)

$\iff$ $\Omega^1_{X_K/K}$ locally free $\mathcal{O}_{X_K}$-module $\square$

Cor Let X be a scheme over a field k. Then TFAE:

a) $f : X \to \operatorname{Spec} k$ is smooth.

b) $\forall$ alg. closed $K \supset k$: X_K is regular.

c) $\exists$ alg. closed $K \supset k$: X_K is regular.

Slogan: Over any field, smooth $\iff$ geometrically regular

Pf. Clear by the prop & the lemma. $\square$

Appendix: Separable field extensions

We have used several times:

Thm Let K/k be a fin. gen. field extension.

Then TFAE:

a) K/k is finite & separable.

b) $\operatorname{Spec} K \to \operatorname{Spec} k$ is étale

c) ———— // ———— is unramified.

Pf. a) $\Rightarrow$ b): Write $K \simeq k[x]/(p(x))$

w/ $p(x) \in k[x]$ separable

(possible by thm of primitive element)

Then $X = \operatorname{Spec} K \to Y = \operatorname{Spec} k$ is

- unramified as $\Omega^1_{X/Y} = K\cdot dx / K \cdot dP(x)$

$$\underset{\substack{\uparrow \\ dP = P'(x)\,dx}}{\simeq} k[x]/(P(x), P'(x)) \underset{\substack{\uparrow \\ (P,P')=1}}{=} 0$$

- Smooth by Jacobian criterion for $X = V(P) \hookrightarrow \mathbb{A}^1_k$.

b) $\Rightarrow$ c): trivial

c) $\Rightarrow$ a): Since K/k is a fingen field extension,

$\exists\, t_1, \dots, t_n \in K$ algebraically independent over k

sth $K \supset K' = k(t_1, \dots, t_n)$ is a finite extension.

Consider

$$\begin{tikzcd} X = \operatorname{Spec} K \arrow[r, "f"] \arrow[rr, bend right, "h := g \circ f"'] & Y = \operatorname{Spec} K' \arrow[r, "g"] & S = \operatorname{Spec} k \end{tikzcd}$$

By assumption, h is unramified, so $\Omega^1_{X/S} = 0$

$\Rightarrow$ $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$

implies $\Omega^1_{X/Y} = 0$, so f is unramified

Claim: Then $K \supset K'$ is separable:

If not, $\exists\, K \supset K'' \supset K'$ w/ $K \supset K''$ purely inseparable

Wlog $K = K''[x]/(P(x))$ w/ $P'(x) = 0$ in $K''(x)$

$\Rightarrow$ $\Omega^1_{X/\operatorname{Spec} K''} \simeq K\cdot dx / K \cdot dP \simeq K \cdot dx \neq 0$

$\Rightarrow$ $X \to \operatorname{Spec} K''$ not unramified ↯ f unramified!

Thus $K \supset K'$ is (finite and) separable

By the already known implication a) $\Rightarrow$ b) (for $K \supset K'$)

then $f : X = \operatorname{Spec} K \to Y = \operatorname{Spec} K'$ is étale

$$\Rightarrow \quad f^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S} \text{ iso}$$

But we had $\Omega^1_{X/S} = 0$ by unramifiedness of h

$$\Rightarrow \quad f^*\Omega^1_{Y/S} = 0$$

$$\Rightarrow \quad \Omega^1_{Y/S} = 0 \quad \text{since } f : X \to Y \text{ surjective}$$

On the other hand:

$Y = \operatorname{Spec} K'$ w/ $K' = k(t_1, \dots, t_n)$ & $S = \operatorname{Spec} k$

$$\Rightarrow \quad \Omega^1_{Y/S} \simeq \bigoplus_{i=1}^{n} K' \cdot dt_i$$

This vanishes only if $n = 0$,

ie if $K' = k$, and then $K \supset k$ is finite separable.

$\square$

7. Generic smoothness

Def A scheme X over a field k is said to be geometrically reduced if $X_{\bar k} = X \times_k \bar k$ reduced.

Thm ("Generic smoothness over fields")

Let X be a geom. reduced scheme of fin. type $/ k$

$\Longrightarrow \exists\, U \subset X$ open dense w/ $U \to \operatorname{Spec} k$ smooth.

Pf. Wlog X integral. Then:

(exercise in field theory)

X geom. reduced $\Rightarrow k(X) \otimes_k k$ reduced $\Longrightarrow k(X) \supset k$ separable

$$\Longrightarrow \dim_{k(X)} \Omega^1_{k(X)/k} = \operatorname{trdeg}(k(X)/k)$$

(similar computation as in appendix to §6)

$$\Longrightarrow \dim_{\kappa(\eta)} \Omega^1_{X/k,\eta} \otimes_{\mathcal{O}_{X,\eta}} \kappa(\eta) = \dim X$$

$\Longrightarrow \exists\, U \subset X$ open dense $\forall p \in U$:

$$\dim_{\kappa(p)} \Omega^1_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p) = \dim X$$

(look at a local presentatⁿ of $\Omega^1_{X/k}$)

$\Longrightarrow U \to \operatorname{Spec} k$ smooth by proposition in §6. $\square$

Rem The thm applies to any integral scheme X of finite type over a perfect field k, since over perfect fields $k(X) \supset k$ is always separable. Separabity is needed:

e.g. $X = \operatorname{Spec} k[s]/(s^p - t)$ over $k = \mathbb{F}_p(t)$

has $X \simeq \operatorname{Spec} \mathbb{F}_p(t^{1/p})$ reduced,

but $X_{\bar k} \simeq \operatorname{Spec} \overline{\mathbb{F}_p}[s]/(s - t^{1/p})^p$ non-reduced

$\Longrightarrow X$ has NO point where it is smooth $/ k$

Thm ("Generic smoothness on the source")

Let $f : X \to Y$ be a dominant morphism between integral schemes of finite type over a field k.

Then TFAE:

a) $\exists$ open dense $U \subset X$ w/ $f|_U : U \to Y$ smooth.

b) the extension $k(X) \supseteq k(Y)$ is separable.

Pf. General case can be done by a "spreading out" argument (see exercises, or Poonen, Rational pts on varieties, §3.2).

Alternative argument for X, Y geom. reduced (eg k perfect, or X, Y smooth):

Then by previous thm wlog X smooth $/ k$ (shrink X).

Consider
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "g"] \\ & S = \operatorname{Spec} k \end{tikzcd}$$

We have:

$$f \text{ smooth} \overset{g \circ f \text{ smooth}}{\iff} 0 \to f^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

exact and locally split

$$\underset{g \text{ smooth}}{\iff} \Omega^1_{X/Y} \text{ loc free of rank } \dim X - \dim Y$$
$$= \operatorname{trdeg}(k(X)/k(Y))$$

This condition holds on an open dense subset of X iff $\dim_{k(X)} \Omega^1_{k(X)/k(Y)} = \operatorname{trdeg}(k(X)/k(Y))$,

ie. iff $k(X) \supset k(Y)$ is separable. $\square$

Ex Why separability is needed:

$f: X = \mathbb{A}^1_{\mathbb{F}_p} \to Y = \mathbb{A}^1_{\mathbb{F}_p}$, $t \mapsto t^p$

is a dominant morphism of integral schemes $/ \mathbb{F}_p$

but nowhere smooth:

$d: f^* \Omega^1_{Y/\mathbb{F}_p} \to \Omega^1_{X/\mathbb{F}_p}$ is the zero map!

Thm ("Generic smoothness on the target")

Let $f: X \to Y$ be a morphism between smooth schemes of finite type over a perfect field k.

$\Rightarrow \exists$ open dense $U \subset Y$ w/ $f^{-1}(U) \to U$ smooth.

Pf. Wlog X, Y integral.

Since X, Y are smooth $/ k$, we have:

$f: X \to Y$ smooth at $p \in X \iff \Omega^1_{X/Y, p}$ free of rank $\dim X - \dim Y$ over $\mathcal{O}_{X,p}$

(as in previous proof)

For closed pt^s $p \in X(k)$ (wlog, else pass to an extension field)

this is equivalent to surjectivity of $T_p f : T_p X \to T_p Y$

In general, for $r \in \mathbb{N}_0$ consider the loci

$$X_r := \{ \text{closed pts } p \in X \mid \operatorname{rk}(T_p X \to T_p Y) \le r \}.$$

Claim: $\dim \overline{f(X_r)} \le r$.

↑ closure of $f(X_r)$ inside Y

(⇒ for $d = \dim Y$ we get $\overline{f(X_{d-1})} \neq Y$

hence $U := Y \setminus \overline{f(X_{d-1})}$ will work)

To check the claim, pick an irred cpt $Y_r \subset \overline{f(X_r)}$

Let $X_r' \subset \overline{X}_r$ be an irred cpt dominating Y_r.

k perfect ⇒ $k(X_r') \supset k(Y_r)$ separable

⇒ By previous theorem

∃ open dense $U_r' \subset X_r'$ w/ $U_r' \to Y_r$ smooth.

For closed points $p \in U' \cap X_r$ we have:

$$\begin{tikzcd}
T_p U_r' \arrow[r] \arrow[d, "\text{surjective as } U_r' \to Y_r \text{ is smooth}"'] & T_p X \arrow[d, "\text{image has dim} \le r \text{ since } p \in X_r"] \\
T_{f(p)} Y_r \arrow[r, hook, "\text{injective since induced by an immersion}"'] & T_p Y
\end{tikzcd}$$

$\Rightarrow \dim T_{f(p)} Y_r \le r$

$\Rightarrow \dim Y_r \le r$

This works for all cpts $Y_r \subseteq \overline{f(X_r)}$

$\Rightarrow \dim \overline{f(X_r)} \le r$ □

Another dimension counting argument shows:

not really needed, only for simplicity.

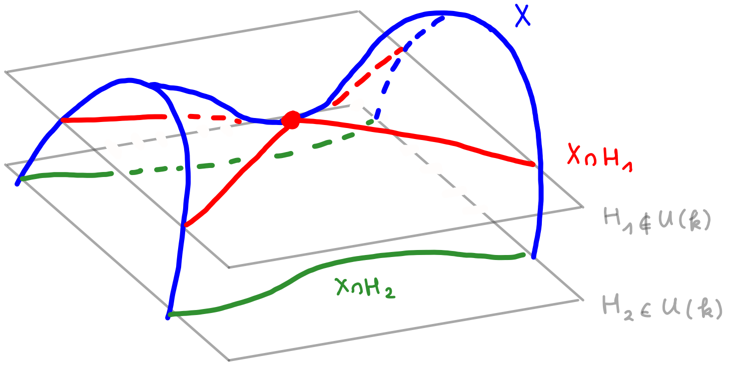
Thm (Bertini) Let k be an alg closed field,

$X \subset \mathbb{P}^n_k = \mathbb{P}V$ smooth irred. closed subscheme

Then $\exists$ open dense

$U \subset \mathbb{P}V^\vee = \{\text{hyperplanes in } \mathbb{P}V\}$

sth $\forall\, H \in U(k)$: $X \cap H$ is smooth.



Pf. For $p \in X(k)$, consider the set of "bad" hyperplanes

$$B_p := \{ H \in \mathbb{P}V^\vee(k) \mid X \cap H \text{ is singular at } p, \text{ or } X \subseteq H \}$$

Pick $f_0 \in V^\vee = \Gamma(\mathbb{P}V, \mathcal{O}(1))$ sth $H_0 := V_+(f_0) \not\ni p$.

$$\begin{tikzcd} \Longrightarrow \varphi : V^\vee \arrow[r] & \mathcal{O}_{\mathbb{P}V}(\underbrace{\mathbb{P}V \setminus H_0}_{D_+(f_0)}) \\ f \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \frac{f}{f_0} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Put $\varphi_p(f) := [\varphi(f)|_{X \setminus H_0}] \in \mathcal{O}_{X,p}$

For $H = V_+(f)$ we have:

- $p \in X \cap H \iff \varphi_p(f) \in m_p$
- If $\varphi_p(f) \neq 0$, then

  $p \in \mathrm{Sing}(X \cap H) \iff \mathcal{O}_{X\cap H,p} = \mathcal{O}_{X,p}/\varphi_p(f)$ not regular

  $\iff \varphi_p(f) \in m_p^2$
- If $\varphi_p(f) = 0$, then $\exists\, \underset{\substack{\rotatebox{90}{$\in$} \\ p}}{W} \subset \mathbb{P}V$ open w/ $X \cap W \subset H$,

  hence $X \subset H$ by irreducibility

Thus writing $\bar{\varphi}_p = (\varphi_p \bmod m_p^2) : V^\vee \to \mathcal{O}_{X,p}/m_p^2$, we have:

$$B_p = \{ H = V_+(f) \mid f \in \ker \bar{\varphi}_p \}.$$

Count dimensions:

$$\begin{tikzcd} \overline{\varphi}_p : V^\vee \arrow[r, two heads] \arrow[d, phantom, "\wr"] & \mathcal{O}_{X,p}/\mathfrak{m}_p^2 \arrow[d, phantom, "\wr"] \\ \dim_k(\ldots) = n+1 & \dim_k(\ldots) = d+1 \end{tikzcd}$$

epi since $\mathfrak{m}_p/\mathfrak{m}_p^2$ is spanned by linear forms in the coordinates & $k = \overline{k}$

for $d := \dim_p X$

since $\mathcal{O}_{X,p}$ is regular & $\mathcal{O}_{X,p}/\mathfrak{m}_p \simeq k$

$\Rightarrow \dim_k \ker \overline{\varphi}_p = n - d$

$\Rightarrow B_p \subset \mathbb{P}V^\vee = |\mathcal{O}_{\mathbb{P}V}(1)|$ linear series of dim $n-d-1$

Now vary the point p:

$\mathcal{B} := \{ (p, H) \mid H \in B_p \} \subset X \times \mathbb{P}V^\vee$

closed subset, which we endow w/ reduced subscheme structure.

Have seen: $pr_1 : \mathcal{B} \twoheadrightarrow X$ surjective w/ fibers $\simeq \mathbb{P}^{n-d-1}$

$\Rightarrow \mathcal{B}$ irreducible w/ $\dim \mathcal{B} = d + (n-d-1) = n-1$

$\Rightarrow pr_2 : \mathcal{B} \to \mathbb{P}V^\vee$ not surjective ($\dim \mathbb{P}V^\vee = n$)

But $pr_2(\mathcal{B}) \subset \mathbb{P}V^\vee$ is closed by properness of pr_2

$\Rightarrow \mathcal{U} := \mathbb{P}V^\vee \setminus pr_2(\mathcal{B})$ works. $\square$

8. Flatness

Motivation a) $f : X \to S$ w/ smooth fibers:

$X_p \to \operatorname{Spec} \kappa(p)$ smooth $\forall p \in S$

$\not\Rightarrow$ f smooth

e.g. $X = V(xy) \subset \mathbb{A}^2 \xrightarrow{pr} S = \mathbb{A}^1$

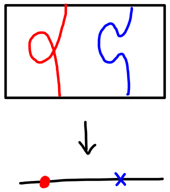
or $X = Bl_p S \to S = \mathbb{A}^2$ etc

BUT we'll see:

f smooth $\Leftrightarrow$ f flat w/ smooth fibers

b) Flatness is a very general notion, e.g.

- any dominant morphism $f : X \to S$ w/ X, S integral & $\dim S = 1$ is flat.
- flat families may have singular fibers



$\to$ important for moduli!

8.1. Algebraic definition of flatness

Def Let A be a ring.

a) An A-module M is flat if the functor

$(-) \otimes_A M : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ is exact.

b) A ring homom. $\varphi : A \to B$ is flat

if B is a flat A-module via φ.

Ex
- free modules $M \simeq \bigoplus_{i \in I} A$ are flat
- over fields $A = k$, every module is flat
- localizations $A \to B = S^{-1}A$ are flat

Ex Quotients $A \to B = A/J$ by an ideal $J \trianglelefteq A$

w/ $J/J^2 \neq 0$ are NOT flat:

$J \hookrightarrow A$ is injective but

$$\begin{tikzcd} J \otimes_A A/J \arrow[r] \arrow[d, equal] & A \otimes_A A/J \arrow[d, equal] \\ J/J^2 \arrow[r, "0"] & A/J \end{tikzcd}$$

is NOT injective

Lemma An A-module M is flat iff $\forall$ ideal $J \trianglelefteq A$

the map $J \otimes_A M \to M$ is injective.

$a \otimes m \longmapsto a \cdot m$

Pf. If M is flat over A, consider $\text{incl} : J \hookrightarrow A$

$\Rightarrow$ $\text{incl} \otimes_A M : J \otimes_A M \to A \otimes_A M = M$ injective by flatness

Conversely, assume $J \otimes_A M \to M$ injective $\forall J \trianglelefteq A$.

$(-) \otimes_A M$ is always right exact (exercise),

so we only need to show it preserves injections.

Let $N' \hookrightarrow N$ be an injection of A-modules.

① Reduction to free modules N:

Pick a surjection $p : \tilde{N} := \bigoplus_{i \in I} A \twoheadrightarrow N$ (I may be infinite)

& let $\tilde{N}' := p^{-1}(N')$. We get diagrams w/ exact rows:

$$\begin{tikzcd}
\ker(p) \arrow[r] \arrow[d, equal] & \tilde{N}' \arrow[r] \arrow[d, hook] & N' \arrow[r] \arrow[d, hook] & 0 \\
\ker(p) \arrow[r] & \tilde{N} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

right exactness of $\otimes M$ $\rightsquigarrow$

$$\begin{tikzcd}
\ker(p) \otimes_A M \arrow[r] \arrow[d, equal] & \tilde{N}' \otimes_A M \arrow[r] \arrow[d, "\tilde{\varphi}"] & N' \otimes_A M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\ker(p) \otimes_A M \arrow[r] & \tilde{N} \otimes_A M \arrow[r] & N \otimes_A M \arrow[r] & 0
\end{tikzcd}$$

Snake lemma: If $\tilde{\varphi}$ is injective, then so is φ.

$\Rightarrow$ wlog $N = \tilde{N}$ free (of any rank)

② Reduction to finite rank free modules N:

Now, $\forall x \in N' \otimes_A M$ $\exists$ direct summand $N_0 \subset N$ of finite rk

sth $x \in \text{im}\left((N' \cap N_0) \otimes_A M \to N' \otimes_A M\right)$.

$$\begin{tikzcd}
(N' \cap N_0) \otimes_A M \arrow[r] \arrow[d] & N' \otimes_A M \arrow[d] \\
N_0 \otimes_A M \arrow[r, hook, "\wr"' , ""] & N \otimes_A M
\end{tikzcd}$$

injective because $N_0 \hookrightarrow N$ is split

$\Rightarrow$ enough to show $(N' \cap N_0) \otimes_A M \hookrightarrow N_0 \otimes_A M$

$\Rightarrow$ wlog $N = N_0$ free of finite rank

③ Induction on $r = \text{rk}(N)$:

$r = 1$: by our assumption that $J \otimes_A M \hookrightarrow M$

$r > 1$: Write $N = N_1 \oplus N_2$ w/ $\text{rk}\, N_1, \text{rk}\, N_2 < r$

Put $N_1' := N' \cap N_1$ & $N_2' := \text{im}(N' \to N_2 = N/N_1)$.

We have a diagram w/ exact rows:

$$\begin{tikzcd}
N_1' \otimes_A M \arrow[r] \arrow[d, "\text{injective by ind}^n \ \wr"'] & N' \otimes_A M \arrow[r] \arrow[d] & N_2' \otimes_A M \arrow[d, "\wr\ \text{injective by ind}^n"] \\
N_1 \otimes_A M \arrow[r, "\wr"'] & N \otimes_A M \arrow[r] & N_2 \otimes_A M
\end{tikzcd}$$

injective since $N_1 \hookrightarrow N$ split

$\Rightarrow$ $N' \otimes_A M \hookrightarrow N \otimes_A M$ injective $\square$

Cor For any domain A & $M \in \mathrm{Mod}(A)$:

a) M flat / A $\Rightarrow$ M torsion-free
(ie $am \neq 0 \;\forall\, a \in A \setminus \{0\},\, m \in M \setminus \{0\}$)

b) If A is a PID, then the converse also holds.

Pf. For principal ideals $0 \neq J = (a) \trianglelefteq R$, the map $J \otimes_A M \to M$ is injective iff $M = A \otimes_A M \xrightarrow{\sim} J \otimes_A M \longrightarrow M$ is injective,

$$m = 1 \otimes m \longmapsto a \otimes m \longmapsto am$$

ie iff $am \neq 0$ for all $m \neq 0$. Now apply the lemma. □

To "globalize" the notion of flatness to schemes we need:

Lemma For $M \in \mathrm{Mod}(A)$ (resp a ring hom $\varphi: A \to B$), TFAE:

a) M is flat / A (resp B is flat / A)

b) $M_{\mathfrak{p}}$ is flat / $A_{\mathfrak{p}}$ $\forall\, \mathfrak{p} \in \mathrm{Spec}\, A$.
(resp $B_{\mathfrak{q}}$ is flat / $A_{\mathfrak{p}}$ $\forall\, \mathfrak{q} \in \mathrm{Spec}\, B$, $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$)

c) $M_{\mathfrak{m}}$ is flat / $A_{\mathfrak{m}}$ $\forall\, \mathfrak{m} \in \mathrm{Spm}\, A$.
(resp $B_{\mathfrak{m}}$ is flat / $A_{\mathfrak{n}}$ $\forall\, \mathfrak{m} \in \mathrm{Spm}\, B$, $\mathfrak{n} = \varphi^{-1}(\mathfrak{m})$)

Pf. a) $\Rightarrow$ b): localization is exact. b) $\Rightarrow$ c): trivial.

c) $\Rightarrow$ a): Let $N' \overset{i}{\hookrightarrow} N$ be an injection in $\mathrm{Mod}(A)$.

$\Rightarrow$ $0 \to \ker(i \otimes \mathrm{id}_M) \to N' \otimes_A M \to N \otimes_A M$ exact

$$\begin{tikzcd}
0 \arrow[r] & \ker(i \otimes \mathrm{id}_M)_{\mathfrak{m}} \arrow[r] & (N' \otimes_A M)_{\mathfrak{m}} \arrow[r] \arrow[d, phantom, "\parallel"] & (N \otimes_A M)_{\mathfrak{m}} \arrow[d, phantom, "\parallel"] & \text{exact} \\
 & & N'_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \arrow[r, hook] & N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} &
\end{tikzcd}$$

$\Rightarrow$

injective by assumption since $N'_{\mathfrak{m}} \hookrightarrow N_{\mathfrak{m}}$ and since $M_{\mathfrak{m}}$ is flat / $A_{\mathfrak{m}}$

$\Rightarrow$ $\ker(i \otimes \mathrm{id}_M)_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \mathrm{Spm}\, A$

$\Rightarrow$ $\ker(i \otimes \mathrm{id}_M) = 0$

Proof for ring hom. $\varphi: A \to B$ works the same. □

Cor If A is a Dedekind domain, then

$M \in \mathrm{Mod}(A)$ is flat $\Leftrightarrow$ M is torsion-free.

So any injective hom $A \hookrightarrow B$ to a domain B is flat.

Pf. For A Dedekind, all $A_{\mathfrak{m}}$ are DVR's. Thus

M flat / A $\Leftrightarrow$ $\forall\, \mathfrak{m}$: $M_{\mathfrak{m}}$ flat / $A_{\mathfrak{m}}$ (by the lemma)

$\Leftrightarrow$ $\forall\, \mathfrak{m}$: $M_{\mathfrak{m}}$ torsionfree / $A_{\mathfrak{m}}$ (by cor. b)

$\Leftrightarrow$ M torsionfree □

Prop Let A be a ring & $M \in \mathrm{Mod}(A)$.

a) M free $\Rightarrow$ M projective $\Rightarrow$ M flat.

b) Conversely, if A is a local ring, then any fin. generated flat A-module M is free.

Pf. a) $M \simeq \bigoplus_{i\in I} A$ free

$\Rightarrow$ $\mathrm{Hom}_A(M,-) \simeq \prod_{i\in I} \mathrm{Hom}_A(A,-)$

exact, ie. $M \in \mathrm{Mod}(A)$ projective

M projective $\Rightarrow$ M direct summand in a free module

(pick an epi $\bigoplus_{i\in I} A \twoheadrightarrow M$, for M projective it splits)

$\Rightarrow$ $M \otimes_A (-)$ exact, ie. M flat.

b) Let $\mathfrak{m} \trianglelefteq A$ be the max. ideal & $k := A/\mathfrak{m}$.

Pick $v_1, \dots, v_n \in M$ sth $\bar{v}_1, \dots, \bar{v}_n \in M/\mathfrak{m}M = M \otimes_A k$ are linearly independent over k.

Claim: $v_1, \dots, v_n \in M$ are linearly independent over A.

Let $a \cdot v^t := \sum_{i=1}^{n} a_i v_i = 0$ w/ $a = (a_1, .., a_n) \in A^n$
$v = (v_1, .., v_n) \in M^n$

Put $f : A^n \to A,\ b \mapsto b \cdot a^t$.

By flatness of M we get exactness of

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \otimes_A M \arrow[r] & M^n \arrow[r, "f \otimes \mathrm{id}_M"] & M \arrow[r] & 0 \\ & & v \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & 0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \end{tikzcd}$$

$\Rightarrow$ $v \in \ker(f) \otimes_A M$,

ie $\exists\ b_\alpha \in \ker(f),\ w_\alpha \in M :\quad v = \sum_{\alpha=1}^{N} b_\alpha \otimes w_\alpha$.

But $v_i \notin \mathfrak{m}M \Rightarrow \exists \alpha : b_\alpha = (b_{\alpha 1}, .., b_{\alpha n}) \notin \mathfrak{m} \cdot A^n$

Say $b_{\alpha 1} \notin M$. Then $b_\alpha \in \ker(f)$ implies

$$a_1 + a_2 \cdot c_2 + \dots + a_n c_n = 0 \quad \text{w/} \quad c_i := \frac{b_{\alpha i}}{b_{\alpha 1}} \in A.$$

If $n = 1$ then $a_1 = 0$.

If $n > 1$ then multiplying by v_1 & subtracting $a \cdot v^t = 0$

we get $a_2(c_2 v_1 - v_2) + \dots + a_n(c_n v_1 - v_2) = 0$

By induction on n then $a_2 = .. = a_n = 0$, hence $a_1 = 0$.

This proves the claim.

Now let $\bar{v}_1, .., \bar{v}_n$ be a basis of $M/\mathfrak{m}M$ over κ.

$$\Rightarrow \quad N := \underbrace{Av_1 + \cdots + Av_n}_{\simeq A^n \text{ by the claim}} \subset M$$

It only remains to show $N = M$.

Indeed $Q := M/N \in \mathrm{Mod}(A)$ is fin. gen.

& $Q \otimes_A \kappa = M \otimes_A \kappa / N \otimes_A \kappa = 0$ by construction

$\Rightarrow Q = 0$ by Nakayama. $\square$

Exercise a) Is $M = \mathbb{Q} \in \mathrm{Mod}(\mathbb{Z})$ free / projective / flat ?

b) Let $A = \prod_{i \in \mathbb{N}} \mathbb{F}_2$.

Show that $M = A / \{ (a_i)_{i \in I} \mid a_i = 0 \text{ for almost all } i \}$

is fin. generated & flat but NOT projective over A.

8.2. Geometric notion of flatness

Def A morphism $f : X \to S$ is *flat at $p \in X$* if the ring extension $f^\# : \mathcal{O}_{S, f(p)} \to \mathcal{O}_{X,p}$ is flat.

We call f *flat* if it is so at all points $p \in X$.

Ex a) Open immersions are flat.

b) flat morphisms are stable under composition, base change and fiber products.

c) A morphism of affine schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat iff the ring homom. $f^\# : A \to B$ is flat.

Ex Closed immersions are not flat, unless they are also open (= iso on a connected cpt)

Lemma Let $f : Y \to X$ be a flat morphism to an irreducible scheme X. Then every open subset $\emptyset \neq U \subset Y$ dominates X, i.e. has dense image $f(U) \subset X$.

Pf. Wlog $U = \operatorname{Spec} B \to X = \operatorname{Spec} A$.

Y flat over X & $U \subset Y$ open $\Rightarrow$ B flat over A

$$B / \underbrace{\mathrm{Rad}(A)}_{\text{nilradical of } A} \cdot B = B \otimes_A A^{\mathrm{red}} \underset{\text{by flatness}}{\hookrightarrow} B \otimes_A \mathrm{Quot}(A^{\mathrm{red}}) = B \otimes_A \kappa(\eta)$$

where $\eta :=$ generic pt of X

If $\eta \notin f(U)$, then $U_\eta = \emptyset$, hence $B \otimes_A \kappa(\eta) = 0$

and hence $B = \mathrm{Rad}(A) \cdot B$ would be nilpotent, so $B = 0$ ↯

Thus $\eta \in f(U)$ & $f|_U : U \to X$ is dominant. $\square$

Converse holds over Dedekind schemes (eg smooth curves):

Def A scheme C is a Dedekind scheme if $\exists$ open cover $C = \bigcup_i U_i$ w/ $U_i \simeq \operatorname{Spec}$ (Dedekind domain).

Cor Let $f : Y \to C$ be a morphism from a reduced Y to a Dedekind scheme C. TFAE:

a) f is flat

b) every irred cpt of Y dominates C.

Pf. a) $\Rightarrow$ b) always true by lemma.

b) $\Rightarrow$ a): Let $p \in Y$ and $q = f(p) \in C$.

- $q = \eta \in C$ generic pt $\Rightarrow$ $\mathcal{O}_{C,q} = \Bbbk(C)$ a field
  $\Rightarrow$ $\mathcal{O}_{Y,p}$ flat / $\mathcal{O}_{C,q}$
- $q \in C$ a closed pt & $\mathfrak{m}_q = (\pi) \trianglelefteq \mathcal{O}_{C,q}$:

  Since every irred cpt of Y dominates C, we know $\forall$ minimal prime ideals $\mathfrak{p} \trianglelefteq \mathcal{O}_{Y,p}$: $f^\#(\pi) \notin \mathfrak{p}$.

  $\Longrightarrow$ $f^\#(\pi)$ not a zero divisor in $\mathcal{O}_{Y,p}$

  (exercise, use that Y is reduced)

  $\Longrightarrow$ $M = \mathcal{O}_{Y,p}$ torsion-free over $\mathcal{O}_{C,q}$

  So $\mathcal{O}_{C,q}$ Dedekind implies M is flat / $\mathcal{O}_{C,q}$. $\square$

Ex Any non-constant morphism from an integral scheme to a Dedekind scheme is flat.

Ex Dominant but non-flat morphisms:

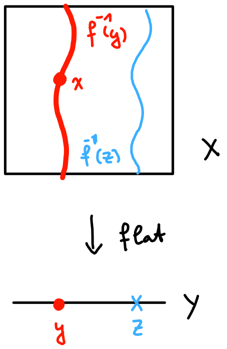
- $X = \mathrm{Bl}_p Y \to Y = \mathbb{A}^2_{\Bbbk}$ (note: Y not Dedekind)
- $X = \operatorname{Spec} \Bbbk[x,y]/(xy, y^2) \to Y = \operatorname{Spec} \Bbbk[x]$
  (note: X not reduced)

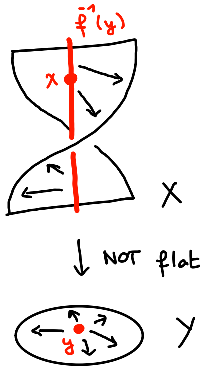
Flat morphisms of irreducible Noeth. schemes have constant fiber dimension, more precisely:

Thm (Flatness & fiber dim) Let $f : X \to Y$ be a morph. of loc. Noeth schemes. Then for all $x \in X$ & $y = f(x) \in Y$ we have:

a) $\dim_x f^{-1}(y) + \dim_y Y \geq \dim_x X$.

b) If f is flat, then equality holds.





Pf. Recall $\dim_y Y = \dim \mathcal{O}_{Y,y}$ etc.

Base change under $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ preserves the local rings

$\Rightarrow$ wlog $Y = \operatorname{Spec} A$ w/ A Noetherian local, y closed pt

Put $X_y = f^{-1}(y)$. We use induction on $\dim Y$:

- $\dim Y = 0$: Y is a "fat point", so $(X_y)^{red} = X_{red}$
  $\Rightarrow \dim X_y = \dim X$

- $\dim Y \geq 1$:
  Replace Y by Y^{red} & X by $X \times_Y Y^{red}$
  (doesn't change local dimension, and it preserves flatness)

  $\Rightarrow$ wlog $Y = \operatorname{Spec} A$ reduced

  Let $t \in A$ not a zero divisor & not a unit

  $\Rightarrow \dim A/_{tA} = \dim A - 1$ (by Krull's p'pal ideal thm)

  $\dim B/_{tB} \geq \dim B - 1$ for $B := \mathcal{O}_{X,x}$.

  ↳ ("$=$" if f is flat, since then t is not a zero divisor in B)

  Let $Y' := \operatorname{Spec} A/_{tA}$ & $X' := X \times_Y Y'$

  By induction $\dim_x X'_y + \dim_y Y' \geq \dim_x X'$ w/ equality if f is flat

  Here $X'_y = X_y$, $\dim_y Y' = \dim_y Y - 1$ and $\dim_x X' \geq \dim_x X - 1$ w/ equality if f is flat. □

Thm (Generic flatness) Let $f : X \to S$ of finite type with S integral & Noetherian.

$\Rightarrow \forall\, \mathcal{E} \in \mathrm{Coh}(X)\ \exists$ open dense $U \subset S$:

$f : f^{-1}(U) \to U$ is flat and

$\mathcal{E}|_{f^{-1}(U)}$ is flat over $\mathcal{O}_U$

(ie $\mathcal{E}_x$ flat over $\mathcal{O}_{U,f(x)}$ $\forall\, x \in f^{-1}(U)$)

Pf. Wlog $X = \mathrm{Spec}\, B \longrightarrow S = \mathrm{Spec}\, A$.

Goal: $\forall$ fin gen $M \in \mathrm{Mod}(B)\ \exists\, f \in A \setminus \{0\}$: M_f free / A_f.

$B_K := B \otimes_A K$ fin. type algebra over $K := \mathrm{Quot}(A)$.

Use induction on $d = \dim(B_K)$ (Krull dim, which is $< \infty$)

M fin gen / A $\Rightarrow \exists$ filtration $0 \subset M_1 \subset \dots \subset M_n = M$

w/ $M_i / M_{i-1} \simeq B/\mathfrak{p}_i$ ($\mathfrak{p}_i \in \mathrm{Spec}\, B$)

$\dim((B/\mathfrak{p}_i)_K) \leq \dim(B_K)$
& extensions of free A_f-modules are free] $\Rightarrow$ wlog $M = B$ & B a domain (replace by $B/\mathfrak{p}_i$)

If $\exists\, 0 \neq f \in \ker(A \to B)$, then $B_f \simeq 0$ & we're done.

Hence wlog $A \hookrightarrow B$ injective.

Noether normalization: $\exists\, f \in A$ w/ $A_f \hookrightarrow A_f[x_1, .., x_n] \overset{\text{finite}}{\hookrightarrow} B_f$

Wlog $f = 1$ (replace A, B by A_f, B_f)

Pick a basis $b_1, .., b_m \in B$ of $\mathrm{Quot}(B)$ over $K(x_1, \dots, x_n)$.

$$\Rightarrow\ 0 \to A[x_1, .., x_n]^{\oplus m} \xrightarrow{\varphi} B \to Q \to 0 \quad (*)$$

$$(p_1, .., p_m) \longmapsto \sum_i p_i b_i$$

w/ $Q := \mathrm{cok}(\varphi) \in \mathrm{Mod}(A[x_1, \dots, x_n])$ fin. gen.

and $\mathrm{Supp}(\tilde{Q}) \underset{\text{closed}}{\overset{\neq}{\hookrightarrow}} \mathrm{Spec}(A[x_1, .., x_n])$

$\Rightarrow Q \in \mathrm{Mod}(S')$ w/ $S' = A[x_1, .., x_n]/(g)$, some $g \neq 0$

But $\dim(S'_K) < d$, so by induction $\exists\, f \in A$: Q_f free / A_f

$\Rightarrow$ by $(*)$ then B_f is free over A_f & we're done. $\square$

Rem Over non-reduced bases generic flatness may fail:

e.g. $f : X = \mathrm{Spec}\, k \to S = \mathrm{Spec}\, k[t]/(t^2)$ ...

Thm (fiberwise smoothness criterion) Let $f : X \to S$ be a fin. type morphism between Noetherian schemes. Then TFAE:

a) f is smooth

b) f is flat w/ smooth fibers

(ie $\forall s \in S$: $X_s \to \operatorname{Spec} \kappa(s)$ smooth)

c) f is flat w/ smooth geometric fibers

(ie $\forall \operatorname{Spec} K \to S$ w/ K alg closed field, the morphism $X_K \to \operatorname{Spec} K$ is smooth)

Pf. b) $\Leftrightarrow$ c) by descent of smoothness over fields.

a) $\Rightarrow$ b):

f smooth $\Rightarrow$ fibers are smooth by base change

Goal: f is also flat.

Claim local on X, so wlog $X = V(f_1, \dots, f_c) \subset \mathbb{A}^n_S$

w/ $\operatorname{rk}(df_1, \dots, df_c)(p) = c$ at a given $p \in X$

(Jacobian criterion for smoothness).

Put $X_i := V(f_1, \dots, f_i) \subset \mathbb{A}^n_S$ $(i = 0, 1, \dots, c)$

Jacobian criterion $\Rightarrow$ X_i smooth $/S$ for all i

Now show flatness by descending induction on i:

- $X_0 = \mathbb{A}^n_S$ flat $/S$
- Assume now $W := X_{i+1}$ flat $/S$ for some i.

  Goal: Then $Z := X_i$ is also flat $/S$.

Note: $Z = V(f) \subset W$ w/ $f := f_i|_W$.

By construction $f(p) = 0$ & $df \neq 0$ in $\Omega^1_{W/S} \otimes_{\mathcal{O}_{W,p}} \kappa(p)$.

We'll use the local flatness criterion:

A local Noeth. ring & $M \in \operatorname{Mod}(A)$,

then M flat $/A$ $\Leftrightarrow$ $\mathfrak{m}_A \otimes_A M \to M$ injective.

So we want to show:

$$\mathfrak{m}_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Z,p} \hookrightarrow \mathcal{O}_{Z,p} \text{ injective} \quad (s := f(p))$$

Consider $\mathcal{K} := \ker(\mathcal{O}_{W,p} \twoheadrightarrow \mathcal{O}_{Z,p})$.

We have a diagram w/ exact rows:

$$\begin{tikzcd}
 & \mathfrak{m}_{S,s} \otimes \mathcal{K} \arrow[r] \arrow[d, "\varphi'"] & \mathfrak{m}_{S,s} \otimes \mathcal{O}_{W,p} \arrow[r] \arrow[d, "\varphi"] & \mathfrak{m}_{S,s} \otimes \mathcal{O}_{Z,p} \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{O}_{W,p} \arrow[r] & \mathcal{O}_{Z,p} \arrow[r] & 0
\end{tikzcd}$$

Here $\ker\varphi = 0$ since W is flat / S (by induction).

Snake lemma:

$\Rightarrow$ only need to show $\operatorname{cok}\varphi' \to \operatorname{cok}\varphi$ is injective!

$$\begin{tikzcd}
\operatorname{cok}\varphi' \arrow[r] \arrow[d, equal] & \operatorname{cok}\varphi \arrow[d, equal] \\
\mathcal{K}/\mathfrak{m}_{S,s}\cdot\mathcal{K} & \mathcal{O}_{W_s,p}
\end{tikzcd}$$

$(W_s = f^{-1}(s) \cap W)$

For this use

$$\begin{tikzcd}
\mathcal{O}_{W,p} \arrow[rr, "\cdot f"] \arrow[dr, two heads] & & \mathcal{O}_{W,p} \arrow[r] & \mathcal{O}_{Z,p} \arrow[r] & 0 \\
 & \mathcal{K} \arrow[ur] & & &
\end{tikzcd}$$

Modulo $\mathfrak{m}_{S,s}$:

$$\begin{tikzcd}
\mathcal{O}_{W_s,p} \arrow[rr, "\cdot \bar{f}"] \arrow[dr, two heads] & & \mathcal{O}_{W_s,p} = \operatorname{cok}(\varphi) \\
 & \operatorname{cok}(\varphi') \arrow[ur] &
\end{tikzcd}$$

By b), W_s is smooth / $k(s)$ $\Rightarrow$ regular, hence $\mathcal{O}_{W_s,p}$ is a domain and thus $0 \neq \bar{f} \in \mathcal{O}_{W_s,p}$ is not a zero divisor!

$\Rightarrow$ $\operatorname{cok}(\varphi') \hookrightarrow \operatorname{cok}(\varphi)$ injective

b) $\Rightarrow$ a):

Locally may assume $X = \operatorname{Spec} A \to S = \operatorname{Spec} R$,

w/ $A = B/J$ for $B = R[x_1, \ldots, x_n]$.

Let $p \in X$ & $s = f(p) \in S$. Put $A_s := A \otimes_R k(s)$
$B_s := B \otimes_R k(s)$

X flat / S $\Rightarrow$ $J_s := \ker(A_s \to B_s) \overset{!}{=} J \otimes_R k(s)$

By right exactness of $\otimes$ then $(J/J^2) \underset{R}{\otimes} k(s) = J_s/J_s^2$.

By assumption $X_s \to \operatorname{Spec} k(s)$ is smooth

$\Rightarrow$ locally split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & J_s/J_s^2 \arrow[r] \arrow[d, equal, "\text{flatness}"'] & A_s \underset{B_s}{\otimes} \Omega^1_{B_s/k(s)} \arrow[r] \arrow[d, equal] & \Omega^1_{A_s/k(s)} \arrow[r] \arrow[d, equal] & 0 \\
 & (J/J^2) \underset{R}{\otimes} k(s) & (A \underset{B}{\otimes} \Omega^1_{B/R}) \underset{R}{\otimes} k(s) & \Omega^1_{A/R} \underset{R}{\otimes} k(s) &
\end{tikzcd}$$

$\Rightarrow$ $0 \to J/J^2 \to A \underset{B}{\otimes} \Omega^1_{B/R} \to \Omega^1_{A/R} \to 0$

is also locally split exact near p (Nakayama) ☐

VIII. Sheaf cohomology

Motivation: For top spaces S, the functor

$$H^0(S,-) : Sh(S) \longrightarrow Ab$$
$$\mathcal{F} \longmapsto \mathcal{F}(S)$$

is only left exact: For a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in $Sh(S)$, we only get exactness of the black sequence below:

$$\begin{tikzcd}
0 \arrow[r] & H^0(S,\mathcal{F}') \arrow[r] & H^0(S,\mathcal{F}) \arrow[r] & H^0(S,\mathcal{F}'') \arrow[dll, rounded corners] \\
 & H^1(S,\mathcal{F}') \arrow[r] & H^1(S,\mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

Goal: Extend to a long exact sequence in a "minimal / universal" way, using just homological algebra!

8.1. Derived functors

Recall A category $\mathcal{A}$ is said to be *preadditive* if

- $\exists$ object $0 \in \mathcal{A}$ that is both initial & final,
- $\forall\, X, Y \in \mathcal{A}$ $\exists$ coproduct $X \sqcup Y$ & product $X \times Y$ and the morphism

  $$(id \times 0) \sqcup (0 \times id) : X \sqcup Y \longrightarrow X \times Y \text{ is an iso.}$$

  We then write $X \oplus Y := X \times Y = X \sqcup Y$.

Rem Any preadditive category $\mathcal{A}$ is "enriched in monoids":

- $\forall\, X, Y \in \mathcal{A}$, define $0 \in Hom_{\mathcal{A}}(X,Y)$ as the composite $0 := (X \to 0 \to Y)$.
- For $f, g \in Hom_{\mathcal{A}}(X,Y)$ define

  $$f+g : X \xrightarrow{diag} X \oplus X \simeq X \sqcup X \xrightarrow{f \sqcup g} X$$

- This makes $Hom_{\mathcal{A}}(X,Y)$ an abelian monoid, and

  $$\circ : Hom_{\mathcal{A}}(X,Y) \times Hom_{\mathcal{A}}(Y,Z) \longrightarrow Hom_{\mathcal{A}}(X,Z)$$

  is a monoid homomorphism (exercise).

Def An <u>additive category</u> is a preadditive category $\mathcal{A}$ where all the monoids $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ are groups.

An <u>abelian category</u> is an additive category $\mathcal{A}$ sth $\forall\, X, Y \in \mathcal{A}$ $\forall\, f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$:

- $$\exists \ker(f) := \lim \left(\begin{tikzcd} 0 \arrow[dr] & \\ & Y \\ X \arrow[ur, "f"'] & \end{tikzcd} \right) \in \mathcal{A}$$
- $$\exists \operatorname{cok}(f) := \operatorname{colim} \left(\begin{tikzcd} & 0 \\ X \arrow[ur] \arrow[dr, "f"'] & \\ & Y \end{tikzcd} \right) \in \mathcal{A}$$
- the natural morphism $$\operatorname{coim}(f) \longrightarrow \operatorname{im}(f) \text{ is an iso.}$$

Here $\operatorname{coim}(f) := \operatorname{cok}(\ker(f) \to X)$,

$\operatorname{im}(f) := \ker(Y \to \operatorname{cok}(f))$.

Ex $\mathcal{A}$ = abelian gps, R-modules, sheaves of ab gps on a space,
quasicoherent sheaves on a scheme,
coherent sheaves on a Noetherian scheme, etc.

Rem $\mathcal{A}$ being abelian is a property, not extra structure!
We have: $\mathcal{A}$ abelian $\Longleftrightarrow$ $\mathcal{A}^{op}$ abelian.

Def Let $\mathcal{A}, \mathcal{B}$ be abelian categories.
A functor $F : \mathcal{A} \to \mathcal{B}$ is <u>additive</u> if

- $F(0) \simeq 0$
- $F(X \oplus Y) \simeq F(X) \oplus F(Y)$ for all $X, Y \in \mathcal{A}$.

It then preserves split exact sequences:

Def We say a sequence $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ <u>splits</u> if the following equivalent conditions hold:

a) $\exists$ section $s \in \mathrm{Hom}_{\mathcal{A}}(Z,Y)$ w/ $g \circ s = \mathrm{id}$

b) $\exists$ retraction $p \in \mathrm{Hom}_{\mathcal{A}}(Y,X)$ w/ $p \circ f = \mathrm{id}$

c) $\exists$ iso
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, "s"'] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "\mathrm{incl}"] & X \oplus Z \arrow[r, "\mathrm{pr}"] & Z \arrow[r] & 0 \end{tikzcd}$$

Def An object $X \in \mathcal{A}$ is <u>injective</u> if every mono $i : X \to Y$ splits, or equivalently if $\mathrm{Hom}_{\mathcal{A}}(-, X)$ is exact.

An object $Z \in \mathcal{A}$ is <u>projective</u> if every epi $p : Y \to Z$ splits, or equivalently if $\mathrm{Hom}_{\mathcal{A}}(Z, -)$ is exact.

Ex a) An abelian gp M is injective iff it is divisible.

b) Any abelian gp M embeds in a divisible ab gp $\tilde{M}$.

Pf. a) M injective

$\Rightarrow$ the injection $\mathbb{Z} \xhookrightarrow{n\cdot} \mathbb{Z}$ induces a surjection

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}, M) \arrow[r, two heads] \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}, M) \arrow[d, equal] \\
M \arrow[r, "n\cdot"] & M
\end{tikzcd}$$

$\Rightarrow$ M divisible

Conversely, assume M divisible. Let $N \hookrightarrow \tilde{N}$ be any injection of abelian gps. We want to show

$$\mathrm{Hom}(\tilde{N}, M) \twoheadrightarrow \mathrm{Hom}(N, M) \text{ is surjective.}$$

Given any $f \in \mathrm{Hom}(N, M)$, consider the set of pairs (N', f') w/ $N \subset N' \subset \tilde{N}$ and $f' \in \mathrm{Hom}(N', M)$, $f'|_N = f$.

These are partially ordered & Zorn's lemma gives a maximal pair (N', f'). If $N' \neq \tilde{N}$, pick $a \in \tilde{N} \setminus N$.

Consider $\varphi : \mathbb{Z} \to \tilde{N}$, $n \mapsto n \cdot a$.

and write $\varphi^{-1}(N') = m\mathbb{Z} \subset \mathbb{Z}$ (some $m \in \mathbb{Z}$).

Now

$$\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & ma \arrow[d, phantom, sloped, "\in"] & \\
m\mathbb{Z} \arrow[r, "\varphi"] \arrow[d, phantom, sloped, "\subset"] & N' \arrow[r, "f'"] & M \\
\mathbb{Z} \arrow[rru, dashed, "\exists g"'] & &
\end{tikzcd}$$

(for $m = 0$ trivially, for $m \neq 0$ we only need to pick $v \in M$ w/ $mv = f'(ma)$ using M is divisible)

$\Rightarrow$ extension of $f' : N' \to M$ to $f'' : N'' := N' + \mathbb{Z} \cdot a \to M$ $\lightning$

b) Let M be any abelian gp.

Let $0 \neq a \in M$ $\Rightarrow$ $\mathbb{Z} \cdot a \subset M$ is $\simeq \mathbb{Z}/n\mathbb{Z}$ $(n \in \mathbb{N}_0)$

$\Rightarrow$ $\mathbb{Z} \cdot a \hookrightarrow \mathbb{Q}/n\mathbb{Z} =: \tilde{M}(a)$

$\uparrow$ divisible, hence injective!

$\tilde{M}(a)$ injective

$$\begin{tikzcd}
\mathbb{Z} \cdot a \arrow[r, hook] \arrow[d, phantom, sloped, "\subset"] & \tilde{M}(a) \\
M \arrow[ru, dashed, "\exists f_a"'] &
\end{tikzcd}$$

$\Rightarrow$ (maybe not injective)

$\Rightarrow$ $\prod f_a : M \hookrightarrow \tilde{M} := \prod_{a \in M \setminus \{0\}} \tilde{M}(a)$ works. $\square$

Def We say an abelian cat $\mathcal{A}$ has <u>enough injectives</u> if $\forall\, X \in \mathcal{A}$ $\exists$ mono $X \hookrightarrow \tilde{X}$ w/ $\tilde{X}$ injective.

What about non-split sequences?

Def An additive functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories is *left exact* if for all short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ the sequence $0 \to F(X) \to F(Y) \to F(Z)$ is exact.

Ex
- $F = \mathrm{Hom}_R(R, -)$ on $\mathcal{A} = \mathrm{Mod}(R)$
- $F = \Gamma(S, -)$ on $\mathcal{A} = \mathrm{Sh}(S)$, etc.

We want to remedy the failure of right exactness in a minimal way:

Def Let $F : \mathcal{A} \to \mathcal{B}$ be left exact.

A *cohomological δ-functor extending F* is a sequence of additive functors $F^i : \mathcal{A} \to \mathcal{B}$ w/ $F^0 = F$ together with natural boundary maps $\delta : F^i(Z) \to F^{i+1}(X)$ for each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, ...

... s.th. $\forall$ such sequences we get a long exact sequence

$$0 \to F^0 X \to F^0 Y \to F^0 Z \xrightarrow{\delta} F^1 X \to F^1 Y \to F^1 Z \xrightarrow{\delta} \cdots$$

The cohomological δ-functors extending F form a category.

By a *universal cohomological δ-functor extending F* we mean an initial object of this category.

Q Do they exist? How to construct them?

The key is the following notion:

Def An additive functor $G : \mathcal{A} \to \mathcal{B}$ is *effaceable* if $\forall X \in \mathcal{A}$ $\exists$ mono $X \hookrightarrow \tilde{X}$ w/ $G(\tilde{X}) = 0$.

Rem a) G effaceable $\Longrightarrow$ $G(X) = 0$ $\forall$ injective $X \in \mathcal{A}$

b) If $\mathcal{A}$ has enough injectives, then $\Longleftrightarrow$ holds.

Pf. a) Let $X \hookrightarrow \tilde{X}$ w/ $G(\tilde{X}) = 0$.

If X is injective, then $\tilde{X} \simeq X \oplus (\dots)$, hence $G(X) = 0$.

b) Follows from a). $\square$

Def A cohomological δ-functor $(F^i)_{i\geq 0}$ is *effaceable* if $G := F^i$ is so $\forall\, i > 0$ (we don't include $i=0$).

Thm a) Any effaceable cohomological δ-functor is universal.

b) If $\mathcal{A}$ has enough injectives, then any left exact functor $F : \mathcal{A} \to \mathcal{B}$ extends to an effaceable, hence universal cohomological δ-functor $(R^iF)_{i\geq 0}$.

We call $R^iF : \mathcal{A} \to \mathcal{B}$ the *ith right derived fctor* of F.

Pf. a) Let $(F^i)_{i\geq 0}$ be effaceable, $(G^i)_{i\geq 0}$ any coh. δ-functor $\Big\}$ extending F.

Construct natural trafos $\varphi_i : F^i \to G^i$ inductively:

For $i=0$ have $\varphi_0 : F^0 = F \xrightarrow{id} G^0 = F$ by assumption.

For $i > 0$: Suppose we already have $\varphi_0, \dots, \varphi_{i-1}$.

Given $X \in \mathcal{A}$, $\exists\, \tilde{X} \in \mathcal{A}$ w/ $F^i(\tilde{X}) = 0$.

We get

$$\begin{tikzcd}
F^{i-1}(\tilde{X}) \arrow[r] \arrow[d, "\varphi_{i-1}"] & F^{i-1}(\tilde{X}/X) \arrow[r, "\delta"] \arrow[d, "\varphi_{i-1}"] & F^i(X) \arrow[r] \arrow[d, dashed, "\exists!\ \varphi_i"] & F^i(\tilde{X}) = 0 \\
G^{i-1}(\tilde{X}) \arrow[r] & G^{i-1}(\tilde{X}/X) \arrow[r, "\delta"] & G^i(X) &
\end{tikzcd}$$

$F^i(\tilde{X}) = 0 \Rightarrow F^i(X) = \operatorname{cok}(F^{i-1}(\tilde{X}) \to F^{i-1}(\tilde{X}/X))$

$\Rightarrow \exists!\ \varphi_i$ as shown

Check that φ_i is independent of the chosen $\tilde{X}$:

Given another mono $X \hookrightarrow \tilde{X}'$ w/ $F^i(\tilde{X}') = 0$,

also have $X \xhookrightarrow{diag} \tilde{X} \oplus \tilde{X}'$ w/ $F^i(\tilde{X} \oplus \tilde{X}') = 0$

$\Rightarrow$ wlog $\tilde{X} \hookrightarrow \tilde{X}'$

Then

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, hook] \arrow[d, equal] & \tilde{X} \arrow[r] \arrow[d, hook] & \tilde{X}/X \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & X \arrow[r, hook] & \tilde{X}' \arrow[r] & \tilde{X}'/X \arrow[r] & 0
\end{tikzcd}$$

gives

$$\begin{tikzcd}
F^{i-1}(\tilde{X}) \arrow[r] \arrow[d] & F^{i-1}(\tilde{X}/X) \arrow[r] \arrow[d] & F^i(X) \arrow[r] \arrow[d, equal] & 0 \\
F^{i-1}(\tilde{X}') \arrow[r] & F^{i-1}(\tilde{X}'/X) \arrow[r] & F^i(X) \arrow[r] & 0
\end{tikzcd}$$

& similarly for G^i, hence $\tilde{X}$ and $\tilde{X}'$ induce the same φ_i.

Exercise: These $(\varphi_i)_{i\geq 0}$ give a morphism of δ-functors.

b) First let's find a candidate for an effaceable $(R^iF)_{i\geq 0}$,

in fact there is no choice:

Let $X \in \mathcal{A}$. Pick $X \hookrightarrow J^0$ w/ J^0 injective,

and let $X^1 := J^0/X^0$.

$$\Rightarrow \quad F(J^0) \to F(X^1) \to R^1F(X) \to 0 = R^1F(J^0)$$

since R^1F effaceable
and J^0 injective

$\Rightarrow$ necessarily

$$F^1(X) \simeq \mathrm{cok}(F(J^0) \longrightarrow F(X^1)).$$

Proceed inductively:

Pick $X^1 \hookrightarrow J^1$ w/ J^1 injective, $X^2 := J^1/X^1 \hookrightarrow J^2$ etc

$$\Rightarrow \quad 0 \to X \to J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots \quad \text{exact}$$

w/ all $J^i \in \mathcal{A}$ injective (an "injective resolution $X \to J^\bullet$")

Put $X^i := \mathrm{cok}(J^{i-2} \to J^{i-1}) = J^{i-1}/X^{i-1}$

$$\Rightarrow \quad R^iF(X) \simeq R^{i-1}F(X^1) \simeq \cdots \simeq R^1F(X^{i-1})$$
$$\simeq \mathrm{cok}(F(J^{i-1}) \to F(X^i))$$
$$= H^i(F(J^\bullet))$$

where

$$H^i(F(J^\bullet)) := \frac{\ker(F(J^i) \to F(J^{i+1}))}{\mathrm{im}(F(J^{i-1}) \longrightarrow F(J^i))}.$$

Upshot: To compute $R^iF(X)$,

pick an injective resolution $X \to J^\bullet$

& take ith cohomology of the cplex $F(J^\bullet)$.

We need to check:

- this is independent of the chosen injective resolut^n $X \to J^\bullet$
- the sequence $(R^iF)_{i\geq 0}$ forms a cohom. δ-functor.

See the two propositions below.

Effaceability is then automatic:

For X injective, take $X \xrightarrow{id} J^\bullet$ w/ $J^i = \begin{cases} X, & i=0 \\ 0, & \text{else} \end{cases}$

then

$$R^iF(X) = 0 \quad \text{for all } i > 0.$$

□

2. Injective resolutions & homotopy categories

Q Let $\mathcal{A}$ be an ab cat w/ enough injectives. For every $X \in \mathcal{A}$ $\exists$ injective resolution $X \to J^\bullet$ but there is no natural choice for it. How to make the assignment $X \mapsto J^\bullet$ functorial?

Def Let $\mathcal{A}$ be an additive cat.

a) $C(\mathcal{A}) :=$ category of cochain cplexes in $\mathcal{A}$, ie

- objects:

$$X^\bullet = (\cdots \to X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots)$$

w/ $d^{i+1} \circ d^i = 0 \ \forall i \in \mathbb{Z}$,

- morphisms:

$$f = (f^i : X^i \to Y^i)_{i \in \mathbb{Z}} \text{ w/ } d^i \circ f^i = f^i \circ d^i \ \forall i \in \mathbb{Z}.$$

Subcategories of bounded cplexes:

$$C^{\geq 0}(\mathcal{A}) := \{ X^\bullet \mid \forall i < 0 : X^i = 0 \} \subset C(\mathcal{A}),$$

$$C^{+}(\mathcal{A}) := \{ X^\bullet \mid \exists i_0 \ \forall i < i_0 : X^i = 0 \} \subset C(\mathcal{A}).$$

b) Two morphisms $f, g : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ are called chain homotopic if $\exists \, (h^i : X^i \to Y^{i-1})_{i \in \mathbb{Z}}$ sth

$$f^i - g^i = h^{i+1} \circ d^i - d^{i-1} \circ h^i \quad \text{for all } i \in \mathbb{Z}:$$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl] & X^i \arrow[r, "d^i"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h^i"'] & X^{i+1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h^{i+1}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "d^{i-1}"'] & Y^i \arrow[r, "d^i"'] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

We write $f \sim g$ and call $(h^i)_{i \in \mathbb{Z}}$ a chain homotopy.

c) The homotopy category of $\mathcal{A}$ is the category $K(\mathcal{A})$ w/ the same objects as $C(\mathcal{A})$, but morphisms up to chain homotopy:

$$\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet) := \mathrm{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet) / \sim$$

This is still an additive category. We define $K^{\geq 0}(\mathcal{A}), K^{+}(\mathcal{A}) \subset K(\mathcal{A})$ the subcategories of objects which are in $C^{\geq 0}(\mathcal{A}), C^{+}(\mathcal{A})$.

Note: These are not stable under iso in $K(\mathcal{A})$!

Lemma ("homotopy invariance of cohomology")

For $\mathcal{A}$ abelian, the cohomology functor

$$H^i : C(\mathcal{A}) \to \mathcal{A},$$

$$X^\bullet \mapsto H^i(X^\bullet) := \frac{\ker(X^i \to X^{i+1})}{\operatorname{im}(X^{i-1} \to X^i)}$$

factors over the homotopy category:

$$\begin{tikzcd} C(\mathcal{A}) \arrow[rr, "H^i"] \arrow[dr] & & \mathcal{A} \\ & K(\mathcal{A}) \arrow[ur, dashed, "\exists!\, H^i"'] & \end{tikzcd}$$

Pf. Let $f, g \in \operatorname{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$ w/ $f \sim g$.

Pick a chain homotopy $(h^i)_{i\in\mathbb{Z}}$ between f and g.

Let $Z^i(X^\bullet) := \ker(d^i : X^i \to X^{i+1})$,

$\cup$

$B^i(X^\bullet) := \operatorname{im}(d^{i-1} : X^{i-1} \to X^i)$, ditto for $Y^\bullet$

$\Rightarrow$ $f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$

maps $Z^i(X^\bullet)$ into $B^i(Y^\bullet)$ since $d^i(Z^i(X^\bullet)) = 0$

$\Rightarrow$ $H^i(f-g) = 0 : H^i(X^\bullet) \to H^i(Y^\bullet)$

$\Rightarrow$ $H^i(f) = H^i(g)$ $\square$

Prop ("Functorial injective resolutions")

Let $\mathcal{A}$ be an abelian cat w/ enough injectives.

Let $\operatorname{Inj}(\mathcal{A})$ be the category of injectives in $\mathcal{A}$,

and consider the full subcategory

$$\mathcal{C} := \{ X^\bullet \mid \forall i \neq 0 : H^i(X^\bullet) = 0 \} \subset K^{\geq 0}(\operatorname{Inj}(\mathcal{A}))$$

Then $H^0 : \mathcal{C} \to \mathcal{A}$ is an equivalence of categories.

($\Rightarrow$ $\exists$ quasi-inverse $\mathcal{A} \to \mathcal{C}$, $X \mapsto J^\bullet$)

Pf. $H^0 : \mathcal{C} \to \mathcal{A}$ essentially surjective

since every $X \in \mathcal{A}$ has an injective resolutⁿ $X \to J^\bullet$

($\mathcal{A}$ has enough injectives) & then $J^\bullet \in \mathcal{C}$, $H^0(J^\bullet) \simeq X$.

For full faithfulness we must check:

Given $f : X \to Y$ in $\mathcal{A}$ & inj. res $X \to J^\bullet$, $Y \to \mathcal{J}^\bullet$,

$\exists$ $f^\bullet : J^\bullet \to \mathcal{J}^\bullet$ in $C(\mathcal{A})$, unique up to chain homotopy,

sth

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & J^0 \arrow[r] \arrow[d, dashed, "f^0"] & J^1 \arrow[r] \arrow[d, dashed, "f^1"] & J^2 \arrow[r] \arrow[d, dashed, "f^2"] & \cdots \\ 0 \arrow[r] & Y \arrow[r] & \mathcal{J}^0 \arrow[r] & \mathcal{J}^1 \arrow[r] & \mathcal{J}^2 \arrow[r] & \cdots \end{tikzcd}$$

commutes.

Existence:

J^0 injective $\Rightarrow$ $\mathrm{Hom}(-, J^0)$ exact

$$\begin{tikzcd} \Rightarrow \mathrm{Hom}(I^0, J^0) \arrow[r, two heads] & \mathrm{Hom}(X, J^0) \text{ surjective} \\ \exists f^0 \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (X \xrightarrow[f]{} Y \to J^0) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Proceed inductively:

$$\begin{tikzcd} \cdots \arrow[r, dashed] & I^n \arrow[r] \arrow[d, "f^n"'] & I^{n+1} \arrow[d, dashed, "\exists f^{n+1}"] \\ \cdots \arrow[r, dashed] & J^n \arrow[r] & J^{n+1} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(I^{n+1}, J^{n+1}) \arrow[r, two heads] & \mathrm{Hom}(I^n, J^{n+1}) \\ \exists f^{n+1} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (I^n \xrightarrow[f^n]{} J^n \to J^{n+1}) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Uniqueness up to homotopy:

Given two solutions $(f^n)_{n\geq 0}$ and $(g^n)_{n\geq 0}$, put $\varphi^n := f^n - g^n$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "d^{-1}"] \arrow[d, "0"'] & I^0 \arrow[r, "d^0"] \arrow[d, "\varphi^0"'] \arrow[dl, "h^0"'] & I^1 \arrow[r, "d^1"] \arrow[d, "\varphi^1"'] \arrow[dl, "h^1"'] & I^2 \arrow[r] \arrow[d, "\varphi^2"'] \arrow[dl, "h^2"'] & \cdots \\ 0 \arrow[r] & Y \arrow[r, "d^{-1}"'] & J^0 \arrow[r, "d^0"'] & J^1 \arrow[r, "d^1"'] & J^2 \arrow[r] & \cdots \end{tikzcd}$$

Goal: $\exists \, (h^n)_{n\geq 0}$ w/ $\varphi^i = h^{i+1} \circ d^i - d^{i-1} \circ h^i$.

Put $h^0 := 0$.

By induction, suppose we have $\varphi^{n-1} = h^n \circ d^{n-1} - d^{n-2} \circ h^{n-1}$ in the diagram below (for the induction start we put $I^{-1} = X$, $J^{-1} = Y$ and $I^m = J^m = 0$ for $m < -1$):

$$\begin{tikzcd} I^{n-2} \arrow[r, "d^{n-2}"] \arrow[d, "\varphi^{n-2}"'] & I^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\varphi^{n-1}"'] \arrow[dl, "h^{n-1}"'] & I^n \arrow[r, "d^n"] \arrow[d, "\varphi^n"'] \arrow[dl, "h^n"'] & I^{n+1} \arrow[dl, dashed, "\exists h^{n+1} ?"] \\ J^{n-2} \arrow[r, "d^{n-2}"'] & J^{n-1} \arrow[r, "d^{n-1}"'] & J^n & \end{tikzcd}$$

Goal: $\exists \, h^{n+1}$ with $\varphi^n + d^{n-1} \circ h^n = h^{n+1} \circ d^n$

By injectivity of J^n it suffices to construct h^{n+1} on $\mathrm{im}(d^n) \subset I^{n+1}$ (extension to $h^n : I^{n+1} \to J^n$ is then automatic).

But $\mathrm{im}(d^n) \simeq I^n / \mathrm{im}(d^{n-1})$

$\Rightarrow$ enough to show $(\varphi^n + d^{n-1} \circ h^n) \circ d^{n-1} = 0$

This is trivial:

- $\varphi^n \circ d^{n-1} = d^{n-1} \circ \varphi^{n-1}$
- $d^{n-1} \circ h^n \circ d^{n-1} = d^{n-1} \circ (\varphi^{n-1} + d^{n-2} \circ h^{n-1})$ by induction
  $= d^{n-1} \circ \varphi^{n-1}$ as $d^{n-1} \circ d^{n-2} = 0$ $\square$

Upshot If $\mathcal{A}$ has enough injectives,

$\exists$ functor $\mathcal{A} \to K^{\geq 0}(\mathrm{Inj}\,\mathcal{A})$, $X \mapsto J^\bullet = J^\bullet_X$

sending X to an injective resolution of X.

$\Rightarrow$ For a left exact functor $F : \mathcal{A} \to \mathcal{B}$, we get well-defined functors

using $F : K^{\geq 0}(\mathrm{Inj}\mathcal{A}) \to K^{\geq 0}(\mathcal{B})$

$$R^iF_* : \mathcal{A} \to \mathcal{B}$$
$$X \longmapsto H^i(F(J^\bullet_X))$$

We also know $R^iF_*(X) = 0$ for X injective & $i > 0$.

To check $(R^iF_*)_{i\geq 0}$ form a universal cohomological δ-functor, we only need the long exact sequence.

Lemma $\mathcal{A}$ ab. cat w/ enough injectives

$0 \to X \to Y \to Z \to 0$ exact in $\mathcal{A}$

$\Rightarrow$ $\exists$ injective resolutions $X \to J^\bullet$
$Y \to \mathcal{J}^\bullet$
$Z \to K^\bullet$

sth the following diagram commutes and has exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^0 \arrow[r] \arrow[d] & \mathcal{J}^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^1 \arrow[r] \arrow[d] & \mathcal{J}^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Pf (outline).

- Pick an injective resolution $X \to J^\bullet$
- Construct an injective resolution $Y \to \mathcal{J}^\bullet$ w/ $J^\bullet \to \mathcal{J}^\bullet$ sth moreover $J^\bullet \to \mathcal{J}^\bullet$ is a monomorphism in each degree
- Put $K^i := \mathcal{J}^i / J^i$

Note: $0 \to J^i \to \mathcal{J}^i \to K^i \to 0$ split since J^i injective

$\Rightarrow$ K^i direct summand of $\mathcal{J}^i$, hence injective $\square$

Rem All rows except the first one split, since the J^n are injective.

$\Rightarrow$ For any additive functor $F : \mathcal{A} \to \mathcal{B}$, still get exact sequence of complexes

$$0 \to F(J^\bullet) \to F(\mathcal{J}^\bullet) \to F(K^\bullet) \to 0$$

hence a long exact cohomology sequence

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H^i(F(J^\bullet)) \arrow[r] & H^i(F(\mathcal{J}^\bullet)) \arrow[r] & H^i(F(K^\bullet)) \arrow[dll, rounded corners] \\
& H^{i+1}(F(J^\bullet)) \arrow[r] & \cdots &
\end{tikzcd}$$

(diagram chase using snake lemma)

$\Rightarrow$ For F left exact this reads:

$$\begin{tikzcd}
\cdots \arrow[r] & R^iF(X) \arrow[r] & R^iF(Y) \arrow[r] & R^iF(Z) \arrow[dll, rounded corners] \\
& R^{i+1}(X) \arrow[r] & \cdots &
\end{tikzcd}$$

so $(R^iF)_{i \geq 0}$ is a univ. coh. δ-functor extending F.

3. Sheaf cohomology

Recall $\mathcal{A}$ ab cat w/ enough injectives

$F : \mathcal{A} \to \mathcal{B}$ left exact fctor

to another abelian cat $\mathcal{B}$,

then for $i \geq 0$ we get functors

$$\begin{aligned} R^i F : \mathcal{A} &\longrightarrow \mathcal{B} \\ K &\longmapsto H^i(F(J^\bullet)) \end{aligned}$$

w/ $K \to J^\bullet$ inj. res.

sth for any short exact sequence

$$0 \to K \to L \to M \to 0$$

we get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & FK \arrow[r] & FL \arrow[r] & FM \arrow[dll, "\delta"] & \\ & R^1FK \arrow[r] & R^1FL \arrow[r] & \cdots \end{tikzcd}$$

Def ("sheaf cohomology")

X top space

$\mathcal{A} = Sh(X) :=$ sheaves of ab gps on X

$\mathcal{B} = AbGps :=$ abelian groups

$F = \Gamma_{Sh(X)}(X, -) : Sh(X) \to AbGps$

$\rightsquigarrow H^i_{Sh(X)}(X, -) := R^i\Gamma_{Sh(X)}(X, -)$

Def ("cohomology of $\mathcal{O}_X$-modules")

X ringed space

$\mathcal{A} = Mod(\mathcal{O}_X)$

$\mathcal{B} = AbGps$

$F = \Gamma_{Mod\,\mathcal{O}_X}(X, -) : Mod(\mathcal{O}_X) \to AbGps$

$\rightsquigarrow H^i_{Mod\,\mathcal{O}_X}(X, -) := R^i\Gamma_{Mod\,\mathcal{O}_X}(X, -)$

Goal For $\mathcal{F} \in Mod(\mathcal{O}_X)$,

$\exists$ natural iso

$H^i_{Sh(X)}(X, \mathcal{F}) \simeq H^i_{Mod(\mathcal{O}_X)}(X, \mathcal{F})$.

Recall:

a) $H^i_{Mod(\mathcal{O}_X)}(X, \mathcal{F}) = H^i(\mathcal{I}^\bullet(X))$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is a resolution

by injective objects $\mathcal{I}^k$ in $Mod(\mathcal{O}_X)$

b) $H^i_{Sh(X)}(X, \mathcal{F}) = H^i(\mathcal{J}^\bullet(X))$

where $\mathcal{F} \to \mathcal{J}^\bullet$ is a resolution

by injective objects $\mathcal{J}^k$ in $Sh(X)$

Obvious attempt:

Pick $\mathcal{I}^\bullet$ as in a)

Take $\mathcal{J}^\bullet := forget(\mathcal{I}^\bullet)$ in b)

w/ $forget: Mod(\mathcal{O}_X) \to Sh(X)$.

Clearly $H^i(forget(\mathcal{I}^\bullet)(X)) \simeq H^i(\mathcal{I}^\bullet(X))$.

BUT: In general forget does NOT send injectives to injectives !!!

Ex $X = pt$

$\mathcal{O}_X(X) = k$ a field of char $p > 0$

$\Rightarrow$ $Hom_k(-, k)$ exact,

$Hom_{AbGps}(-, k)$ not exact ($(k,+)$ not divisible)

$\Rightarrow$ k injective in $Mod(\mathcal{O}_X)$,

but NOT in $Sh(X)$.

$\rightsquigarrow$ replace injectives by a larger class of objects:

Def Let $\mathcal{A}$ be an ab cat w/ enough injectives and $F: \mathcal{A} \to \mathcal{B}$ a left exact functor.

An object $J \in \mathcal{A}$ is *F-acyclic* if

$$R^iF(J) \simeq 0 \text{ for all } i > 0.$$

Lemma Let $K \in \mathcal{A}$ and $K \to J^\bullet$ a resolution by F-acyclic objects $J^k \in \mathcal{A}$. Then $R^iF(K) \simeq H^i(F(J^\bullet))$ for all i.

Pf. As in the uniqueness proof for effaceable δ-fctors:

Put $J^{-1} := K$, $J^k := 0$ for $k < -1$,

and

$$C^k := \mathrm{im}\,(J^{k-1} \to J^k) = \ker\,(J^k \to J^{k+1}).$$

From $0 \to C^k \to J^k \to C^{k+1} \to 0$

& F-acyclicity of J^k we get for all $i > 0$:

$$\begin{tikzcd} R^iF(J^{k-1}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & R^iF(C^k) \arrow[r, "\simeq"] & R^{i+1}F(C^{k-1}) \arrow[r] & R^{i+1}F(J^{k-1}) \arrow[d, phantom, "\shortparallel"] \\ 0 & & & 0 \end{tikzcd}$$

$$\begin{aligned} \Longrightarrow R^iF(K) &= R^iF(C^0) \\ &\simeq R^{i-1}F(C^1) \\ &\vdots \\ &\simeq R^1F(C^{i-1}) \\ &\simeq \mathrm{cok}(F(J^i) \to F(C^i)) \\ &\simeq H^i(F(J^\bullet)) \qquad \square \end{aligned}$$

$F(C^i)$
$= \ker(F(J^i) \to F(J^{i+1}))$
as F is left exact

Back to sheaves:

Def Let X be a top space.

A sheaf $\mathcal{F} \in Sh(X)$ is *flabby* (or *flasque*, german: *welk*) if $\forall$ open $U \subset V \subset X$ the restriction map $\mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$ is surjective.

Prop a) On any ringed space $(X, \mathcal{O}_X)$, injective $\mathcal{O}_X$-modules are flabby sheaves.

b) On any top space X, flabby sheaves on X are acyclic for $\Gamma_{Sh(X)}(X,-) : Sh(X) \to AbGps$.

Pf. a) Consider open subsets

$$\begin{tikzcd} U \arrow[r, hook, "j_U"] \arrow[d, phantom, "\cup"] & X \\ V \arrow[ur, hook, "j_V"'] & \end{tikzcd}$$

For $\mathcal{F} \in Mod(\mathcal{O}_X)$ we have

$$\mathcal{F}(U) \simeq Hom_{\mathcal{O}_U}(\mathcal{O}_U, j_U^* \mathcal{F}) \simeq Hom_{\mathcal{O}_X}(j_{U!}\mathcal{O}_U, \mathcal{F})$$

$$\mathcal{F}(V) \simeq Hom_{\mathcal{O}_V}(\mathcal{O}_V, j_V^* \mathcal{F}) \simeq Hom_{\mathcal{O}_X}(j_{V!}\mathcal{O}_V, \mathcal{F})$$

If $\mathcal{F}$ is injective in $\mathrm{Mod}(\mathcal{O}_X)$, then applying $\mathrm{Hom}_{\mathcal{O}_X}(-,\mathcal{F})$ to $j_{V!}\mathcal{O}_V \hookrightarrow j_{U!}\mathcal{O}_U$ we get

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{O}_X}(j_{U!}\mathcal{O}_U, \mathcal{F}) \arrow[r, two heads] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{O}_X}(j_{V!}\mathcal{O}_V, \mathcal{F}) \arrow[d, equal] \\ \mathcal{F}(U) & \mathcal{F}(V) \end{tikzcd}$$

$\Rightarrow$ $\mathcal{F} \in \mathrm{Sh}(X)$ is flabby.

b) Assume $\mathcal{F} \in \mathrm{Sh}(X)$ is flabby.

Claim: $\forall$ exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ the map $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective.

Indeed: Let $h \in \mathcal{H}(X)$.

Pick a maximal pair (U, g) w/ $U \subset X$ open & $g_U \in \mathcal{G}(U)$ lifting $h|_U \in \mathcal{H}(U)$ (such pairs exist since $\mathcal{G} \twoheadrightarrow \mathcal{H}$; now use Zorn's lemma).

If $U \neq X$, pick $p \in X \setminus U$.

Pick $V \ni p$ sth $\exists\, g_V \in \mathcal{G}(V)$ lifting $h|_V$.

$\Rightarrow$ On $W := U \cap V$,

$$s := g_U|_W - g_V|_W \in \mathcal{F}(W)$$
$$= \ker(\mathcal{G}(W) \to \mathcal{H}(W))$$

$\mathcal{F}$ flabby $\Rightarrow$ $\exists\, f \in \mathcal{F}(X)$: $f|_W = s$

$\Rightarrow$ g_U and $\tilde{g}_V := g_V - s|_V$ agree on $W = U \cap V$, hence glue to a section $g_{U \cup V} \in \mathcal{G}(U \cup V)$ lifting $h|_{U \cup V}$ $\lightning$ maximality $\lightning$

This proves the claim:

$$\mathcal{G}(X) \twoheadrightarrow \mathcal{H}(X).$$

Now embed $\mathcal{F} \hookrightarrow \mathcal{G}$ in an injective $\mathcal{G} \in Sh(X)$, and put $\mathcal{H} := \mathcal{G} / \mathcal{F}$. Consider

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}(X) \arrow[r, two heads, "\text{surjective by the above}"'] & \mathcal{H}(X) \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & H^1(X, \mathcal{G}) \arrow[r] \arrow[d, equal] & \cdots \\
 & & & & 0 \text{ since injective obj. are acyclic} &
\end{tikzcd}$$

$\Rightarrow H^1(X, \mathcal{F}) = 0$

and $H^i(X, \mathcal{H}) \simeq H^{i+1}(X, \mathcal{F}) \quad \forall i \geq 1$

$\Rightarrow H^i(X, \mathcal{F}) = 0 \quad \forall i \geq 1$ by induction,

since $\mathcal{H}$ is again flabby:

- $\mathcal{G}$ injective in $Sh(X) = Mod(\mathbb{Z}_X)$, hence flabby by part a)
- for $V \hookrightarrow U \hookrightarrow X$ open then

$$\begin{tikzcd}
\mathcal{G}(U) \arrow[r] \arrow[d, "\text{epi since } \mathcal{G} \text{ flabby}"'] & \mathcal{H}(U) \arrow[d, "\text{hence epi}"] \\
\mathcal{G}(V) \arrow[r, two heads, "\text{epi by claim}"'] & \mathcal{H}(V)
\end{tikzcd}$$

$\square$

<u>Cor</u> For any $\mathcal{F} \in Mod(\mathcal{O}_X)$, $\exists$ natural iso

$$H^i_{Sh(X)}(X, \mathcal{F}) \simeq H^i_{Mod(\mathcal{O}_X)}(X, \mathcal{F}).$$

Pf. Pick $\mathcal{F} \to \mathcal{J}^\bullet$ injective resolut^n in $Mod(\mathcal{O}_X)$.

Put $\mathscr{J}^k := \text{forget}(\mathcal{J}^k) \in Sh(X)$.

Prop b) $\Rightarrow$ $\mathscr{J}^k$ flabby

Prop a) $\Rightarrow$ $\mathscr{J}^k$ acyclic wrt $\Gamma(X, -)$

So $\mathcal{F} \to \mathscr{J}^\bullet$ acyclic resolution in $Sh(X)$

$\Rightarrow H^i_{Sh(X)}(X, \mathcal{F}) \simeq H^i(\mathscr{J}^\bullet(X))$ by above lemma

$\simeq H^i(\mathcal{J}^\bullet(X))$ as forget is exact

$\simeq H^i_{Mod(\mathcal{O}_X)}(X, \mathcal{F})$ $\square$

<u>Notation</u>: $H^i(X, \mathcal{F}) := H^i_{Sh(X)}(X, \mathcal{F})$

Rem a) For a ringed space $(X, \mathcal{O}_X)$ & $R := \mathcal{O}_X(X)$, can also derive $\Gamma(X,-): \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(R)$.

Since $\mathrm{forget}: \mathrm{Mod}(R) \to \mathrm{AbGps}$ is exact, we get a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{O}_X) \arrow[d, "\exists"'] \arrow[dr, "H^i(X,-)"] & \\
\mathrm{Mod}(R) \arrow[r, "\mathrm{forget}"'] & \mathrm{AbGps}
\end{tikzcd}$$

$\Rightarrow$ $H^i(X, \mathcal{F})$ are naturally R-modules

b) If X is a scheme, the abelian category $\mathrm{QCoh}(X)$ of quasicoh. sheaves has enough injectives by a result of Gabber (see e.g. Stacks 077K)

$\Rightarrow$ can also derive

$\Gamma_{\mathrm{QCoh}(X)}(X,-): \mathrm{QCoh}(X) \to \mathrm{AbGps}$.

If X is Noetherian, then any $\mathcal{F} \in \mathrm{QCoh}(X)$ has a resolution $\mathcal{F} \to \mathcal{J}^\bullet$ by quasicoherent flabby sheaves $\mathcal{J}^k \in \mathrm{QCoh}(X)$

(take a finite cover $X = \bigcup_\alpha U_\alpha$ w/ $U_\alpha = \mathrm{Spec}\, R_\alpha$, let $M_\alpha := \mathcal{F}(U_\alpha) \in \mathrm{Mod}(R_\alpha)$, pick an embedding $M_\alpha \hookrightarrow J_\alpha$ w/ J_α injective R_α-module then $\mathcal{F} \hookrightarrow \mathcal{J}^0 := \bigoplus_\alpha j_{\alpha *} \tilde{J}_\alpha$

& $\mathcal{J}^0$ is flabby and qcoh [Hartshorne, cor. III.3.6])

$\Rightarrow$ For X Noetherian we get a comm. diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[d] \arrow[dr, "R^i\Gamma_{\mathrm{QCoh}}(X,-)"] & \\
\mathrm{Mod}(\mathcal{O}_X) \arrow[r, "H^i(X,-)"'] & \mathrm{AbGps}
\end{tikzcd}$$

Caution:
- This can fail for X not Noetherian.
- Even for X Noetherian, usually $\mathrm{Coh}(X)$ does NOT have enough injectives.

That's why we defined sheaf cohomology on all of $\mathrm{Mod}(\mathcal{O}_X)$...

4. Serre vanishing on affine schemes

Goal: QCoh sheaves on affine schemes

have no higher cohomology!

We follow Kempf (1980): No Noetherian assumption,

yet neither spectral sequences nor Čech cohomology needed ...

Prop Let X be a top space and $\mathcal{F} \in Sh(X)$.

Assume $\exists$ nbhood basis $\mathcal{B}$ for X $\exists\, n \geq 1$

sth $\forall\, U \in \mathcal{B}$:

$$H^i(U, \mathcal{F}) = 0 \text{ for all } 0 < i < n.$$

Then for any $\alpha \in H^n(X, \mathcal{F})$,

$\exists$ cover $X = \bigcup_\nu U_\nu$ by subsets $U_\nu \in \mathcal{B}$

sth $\forall \nu$:

$$\alpha \in \ker\left(H^n(X,\mathcal{F}) \to H^n(X, j_{\nu *}\mathcal{F}|_{U_\nu})\right)$$

induced by $\mathcal{F} \to j_{\nu *} j_\nu^* \mathcal{F}$ w/ $j_\nu : U_\nu \hookrightarrow X$

Pf. Let $\alpha \in H^n(X, \mathcal{F})$.

Goal: $\forall\, p \in X$ $\exists$ nbhood $V \in \mathcal{B}$ of p

sth $\alpha \mapsto 0 \in H^n(X, j_* \mathcal{F}|_V)$, $\quad j : V \hookrightarrow X$.

We use induction on n:

Pick an embedding $\mathcal{F} \hookrightarrow \mathcal{G}$ w/ $\mathcal{G}$ flabby,

put $\mathcal{F}' := \mathcal{G}/\mathcal{F}$ and consider

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_*\mathcal{F}|_V \arrow[r] & j_*\mathcal{G}|_V \arrow[r] & \underset{\mathcal{K}}{\cancel{j_*\mathcal{F}'}|_V} \arrow[r] & 0
\end{tikzcd}$$

w/ $\mathcal{K} := \text{image}\,(\mathcal{F}' \to j_* \mathcal{F}'|_V)$

$= \text{image}\,(j_*\mathcal{G}|_V \to j_*\mathcal{F}'|_V)$.

If $n = 1$:

$$\begin{tikzcd}
 & \exists \alpha' \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & \alpha \arrow[d, phantom, sloped, "\in"] \\
 & H^0(X, \mathcal{F}') \arrow[r, two heads, "\text{epi since } \mathcal{G} \text{ flabby}"'] \arrow[d, "\varphi'"'] & H^1(X, \mathcal{F}) \arrow[d, "\varphi"] \\
H^0(X, j_*\mathcal{G}|_V) \arrow[r] \arrow[d, phantom, sloped, "\ni"] & H^0(X, \mathcal{K}) \arrow[r] \arrow[d, phantom, sloped, "\ni"] & H^1(X, j_*\mathcal{F}|_V) \\
\exists \gamma \arrow[r, mapsto] & \beta := \varphi'(\alpha') & \\
\text{(shrinking } V \text{)} & &
\end{tikzcd}$$

$j_*\mathcal{G}|_V \to \mathcal{K}$ sheaf surjective

$\Longrightarrow$ shrinking V we can assume $\exists \gamma \in H^0(X, j_*\mathcal{G}|_V)$
w/ $\gamma \mapsto \beta$

$\Longrightarrow$ $\beta \mapsto 0 \in H^1(X, j_*\mathcal{F}|_V)$

$\Longrightarrow$ $\alpha \in \ker(\varphi)$ as required

Induction step for $n > 1$:

By assumptⁿ $H^1(U, \mathcal{F}) = 0$ for all $U \in \mathcal{B}$.

Fixing $V \overset{j}{\hookrightarrow} X$, we get for any $U \in \mathcal{B}$
w/ $U \subset V$:

$$\begin{tikzcd}
\mathcal{G}(U) \arrow[r, two heads, "\text{epi}"] \arrow[d, equal] & \mathcal{F}'(U) \arrow[d, equal] \\
(j_*\mathcal{G}|_V)(U) & (j_*\mathcal{F}'|_V)(U)
\end{tikzcd}$$

$\Longrightarrow$ $\mathcal{K} = j_*\mathcal{F}'|_V$ by varying $U \subset V$ $\quad (*)$

Now consider the diagram:

$$\begin{tikzcd}
\exists \alpha' \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & \alpha \arrow[d, phantom, sloped, "\in"] \\
H^{n-1}(X, \mathcal{F}') \arrow[r, "\sim", "\text{iso since } \mathcal{G} \text{ flabby}"'] \arrow[d, "\varphi'"'] & H^n(X, \mathcal{F}) \arrow[d, "\varphi"] \\
H^{n-1}(X, \mathcal{K}) \arrow[r, "\sim", "\text{iso since } j_*\mathcal{G}|_V \text{ flabby}"'] \arrow[d, equal, "\text{by } (*)"'] & H^n(X, j_*\mathcal{F}|_V) \\
H^{n-1}(X, j_*\mathcal{F}'|_V) &
\end{tikzcd}$$

We have:

$\alpha \in \ker(\varphi) \iff \alpha' \in \ker(\varphi')$

So we're done by induction, replacing $\mathcal{F}$ by $\mathcal{F}'$ and α by α'

Assumptions on $\mathcal{F}'$ hold for $n' := n-1$, since

$H^i(U, \mathcal{F}') \xrightarrow{\sim} H^{i+1}(U, \mathcal{F}) = 0$

for all $0 < i < n-1$ & all $U \in \mathcal{B}$. □

Thm (Serre vanishing on affine schemes)

For X affine, any $\mathcal{F} \in \mathrm{QCoh}(X)$ satisfies

$H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Pf. Let $\mathcal{B}$ be the nborhood basis of the open subsets $D(f) \subset X = \operatorname{Spec} A$ $(f \in A)$.

$\Rightarrow$ For $j : D(f) \hookrightarrow X$ we have

$$j_* \mathcal{F}|_{D(f)} \simeq \tilde{M} \quad \text{w/} \quad M := \Gamma(X, \mathcal{F})_f \in \mathrm{Mod}(A).$$

By induction, assume $H^i(Y, \mathcal{G}) = 0$ for $i = 1, \dots, n-1$, all affine schemes Y and all $\mathcal{G} \in \mathrm{QCoh}(Y)$.

Let $\alpha \in H^n(X, \mathcal{F})$.

Proposit^n $\Rightarrow$ $\exists$ cover $X = V_1 \cup \dots \cup V_p$ by basic open subsets $V_k \overset{j_k}{\hookrightarrow} X$ in $\mathcal{B}$ sth $\alpha \mapsto 0 \in H^n(X, \underbrace{\bigoplus_k j_{k*} \mathcal{F}|_{V_k}}_{=:\mathcal{G}})$.

The long exact sequence of

$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ w/ $\mathcal{H} := \mathcal{G}/\mathcal{F}$

gives

$$\alpha \in \text{image}\left(H^{n-1}(X,\mathcal{H}) \xrightarrow{\delta} H^n(X,\mathcal{F})\right).$$

But for $n>1$ we have $H^{n-1}(X,\mathcal{H})=0$ by induction, since $\mathcal{H}$ is still quasicoherent;

and for $n=1$ we have $\delta=0$ since the sequence $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$ is exact ($\Gamma(X,-)$ being exact on quasicoh. sheaves because X is affine).

$\Longrightarrow$ $\alpha = 0$ in all cases

Since $\alpha \in H^n(X,\mathcal{F})$ was arbitrary, we get $H^n(X,\mathcal{F}) = 0$. $\square$

<u>Cor</u> Let $f : X \to Y$ be an affine morphism of schemes and $\mathcal{F} \in QCoh(X)$. Then

$$H^i(X,\mathcal{F}) \simeq H^i(Y, f_*\mathcal{F}) \text{ for all } i \geq 0.$$

Pf. Take a resolution $\mathcal{F} \to \mathcal{G}^\bullet$ w/ all $\mathcal{G}^k$ flabby. For $V \subset Y$ open affine, $U := f^{-1}(V) \subset X$ is affine.

Serre vanishing for $\mathcal{F}|_U \in QCoh(U)$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, equal] & \mathcal{G}^0(U) \arrow[r, "d^0"] \arrow[d, equal] & \mathcal{G}^1(U) \arrow[r, "d^1"] \arrow[d, equal] & \mathcal{G}^2(U) \arrow[r, "d^2"] \arrow[d, equal] & \cdots \\ & (f_*\mathcal{F})(V) & (f_*\mathcal{G}^0)(V) & (f_*\mathcal{G}^1)(V) & (f_*\mathcal{G}^2)(V) & \end{tikzcd}$$

is exact, since $\ker(d^i)/_{\mathrm{im}(d^{i-1})} = H^i(U,\mathcal{F}) = 0 \;\forall\, i>0$.

$\Longrightarrow$ Since V can be arbitrarily small, we get that $f_*\mathcal{F} \longrightarrow f_*\mathcal{G}^\bullet$ is exact, i.e. a flabby resolution of $f_*\mathcal{F}$

$$\Rightarrow \; H^i(Y, f_*\mathcal{F})$$

$$\simeq H^i\big((f_*\mathcal{G}^0)(Y) \to (f_*\mathcal{G}^1)(Y) \to \cdots\big)$$

$$\simeq H^i\big(\mathcal{G}^0(X) \to \mathcal{G}^1(X) \to \cdots\big)$$

$$\simeq H^i(X, \mathcal{F}). \qquad \square$$

Serre vanishing in fact characterizes affine schemes:

Thm (Serre criterion for affinity) For any qcqs scheme X, the following are equivalent:

a) X is affine.

b) $\forall\, \mathcal{F} \in \mathrm{QCoh}(X)\ \forall\, i > 0$: $H^i(X, \mathcal{F}) = 0$.

c) $\forall$ ideal sheaves $\mathcal{J} \trianglelefteq \mathcal{O}_X$: $H^1(X, \mathcal{J}) = 0$.

Pf. a) $\Rightarrow$ b) is Serre vanishing, b) $\Rightarrow$ c) trivial.

We show c) $\Rightarrow$ a):

Let $A = \Gamma(X, \mathcal{O}_X)$.

Have a natural morphism $g: X \to \operatorname{Spec} A$.

Claim: $\exists\, f_1, \dots, f_n \in A$ sth

- $(f_1, \dots, f_n) = (1)$ (unit ideal in A)
- $D_X(f_i) \subset X$ is affine for all i

(since $D_X(f_i) = g^{-1}(D_{\operatorname{Spec} A}(f_i))$ it then follows that the morphism g is affine, hence X is affine)

To prove the claim, let $p \in X$ be a closed pt

Pick an affine open nborhood $p \in U \subset X$.

For $Z := X \setminus U$ we have an exact sequence

$$0 \to \mathcal{J}_{Z \cup \{p\}} \to \mathcal{J}_Z \to i_* \kappa(p) \to 0$$

where $i: \operatorname{Spec} \kappa(p) \hookrightarrow X$ is the inclusion.

We got

$$\begin{tikzcd} H^0(X,\mathcal{J}_Z) \arrow[r, "\text{hence epi}"'] & H^0(X, i_*\kappa(p)) \arrow[r] & H^1(X, \mathcal{J}_{Z\cup\{p\}}) \arrow[d, phantom, "\parallel"] \\ & & 0 \text{ by assumpt}^{n} \text{ c)} \end{tikzcd}$$

$\Longrightarrow \exists f \in H^0(X,\mathcal{J}_Z) : \quad f(p) = 1 \in \kappa(p)$

$\Longrightarrow p \in D_X(f) \underset{\substack{\uparrow \\ \text{since } Z \subset V(f)}}{=} D_U(f)$

Now $D_U(f)$ is affine (principal open in the affine U)

$\Longrightarrow D_X(f)$ affine nbhood of p

X qc $\Longrightarrow$ can take finitely many such $f_1,\dots,f_n \in A$

(for finitely many closed pts $p_1,\dots,p_n \in X$, recalling any qcqs scheme has closed pts)

sth

$D_X(f_i)$ affine & $X = \bigcup_{i=1}^{n} D_X(f_i)$.

Still need to show $(f_1,\dots,f_n) = (1)$ unit ideal in A.

Since $X = \bigcup_i D_X(f_i)$, we have epi

$$\varphi : \mathcal{O}_X^n \twoheadrightarrow \mathcal{O}_X, \quad (a_1,\dots,a_n) \mapsto \sum_i a_i f_i .$$

Put $\mathcal{F} := \ker \varphi$. We get:

$$\begin{tikzcd} H^0(X,\mathcal{O}_X^n) \arrow[r] \arrow[d, phantom, "\parallel"] & H^0(\mathcal{O}_X) \arrow[r] \arrow[d, phantom, "\parallel"] & H^1(X,\mathcal{F}) \\ A^n \arrow[r] & A & \\ (a_1,\dots,a_n) \arrow[r, mapsto] & \sum_i a_i f_i & \end{tikzcd}$$

$\Longrightarrow$ enough to show that $H^1(X,\mathcal{F}) = 0$.

For this consider $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$

where $\mathcal{F}_i := \mathcal{F} \cap \mathcal{O}_X^i$ via $\mathcal{O}_X^i \hookrightarrow \mathcal{O}_X^n$

$(a_1,\dots,a_i) \mapsto (a_1,\dots,a_i,0,\dots,0)$

Then $\mathcal{J}_i := \mathcal{F}_i / \mathcal{F}_{i-1} \subset \mathcal{O}_X^i / \mathcal{O}_X^{i-1} \simeq \mathcal{O}_X$

is a sheaf of ideals, hence $H^1(X,\mathcal{J}_i) = 0 \ \forall i$

$\Longrightarrow H^1(X,\mathcal{F}) = 0$ $\square$

Example Recall an infinitesimal thickening of a scheme X_0 is a scheme X sth $X_0 = V(J) \hookrightarrow X$ for some ideal sheaf $J \trianglelefteq \mathcal{O}_X$ w/ $J^n = 0$ for $n \gg 0$.

We can now show:

If X_0 is affine, then so is X.

Pf. Consider $X_0 = V(J) \hookrightarrow V(J^2) \hookrightarrow \cdots \hookrightarrow V(J^n) = X$

$\Rightarrow$ Wlog $i: X_0 \hookrightarrow X$ is a 1st order thickening,

ie $X_0 = V(J) \hookrightarrow X$ with $J^2 = (0)$.

But then we have $J = i_*(J_0)$ for the ideal sheaf $J_0 = J/\cancel{J^2} \trianglelefteq \mathcal{O}_{X_0}$ on X_0.

$$X_0 \text{ affine} \Rightarrow \underset{\underset{H^1(X, J)}{\|}}{H^1(X_0, J_0)} = 0 \text{ by Serre}$$

$i: X_0 \hookrightarrow X$ closed immersion, hence an affine morphism

More generally, for any $\mathcal{J} \trianglelefteq \mathcal{O}_X$ we have an exact sequence

$$0 \to i_*(\mathcal{J}_0) \to \mathcal{J} \to i_*(\mathcal{J}/\mathcal{J}_0) \to 0$$

with $\mathcal{J}_0 := \dfrac{\mathcal{J} \cap J}{\cancel{J \cdot (\mathcal{J} \cap J)}} \trianglelefteq \mathcal{O}_{X_0}$.

Thus

$$\underset{\underset{0}{\|}}{H^1(X_0, \mathcal{J}_0)} \to H^1(X, \mathcal{J}) \to \underset{\underset{0}{\|}}{H^1(X_0, \mathcal{J}/\mathcal{J}_0)}$$

$\Rightarrow H^1(X, \mathcal{J}) = 0$

This holds for all $\mathcal{J} \trianglelefteq \mathcal{O}_X$, so X is affine by Serre's criterion. □

Upshot Higher cohomology is useful even when it all vanishes!

5. Cech cohomology

Goal Compute sheaf cohomology in terms of open covers!

Motivation Let $X = \bigcup_{i \in I} U_i$

$$\Rightarrow H^0(X, \mathcal{F}) = \{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \ \forall i,j \}$$

$$= \ker \Big(\prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \Big)$$

extend this to a cplex computing sheaf cohomology in all degrees!

Def Let X be a top space w/ an open cover $X = \bigcup_{i \in I} U_i$. Fix a well-ordering $<$ on I.

For $i_0, \dots, i_p \in I$ put $U_{i_0, \dots, i_p} := U_{i_0} \cap \dots \cap U_{i_p}$.

The <u>Čech complex</u> of a sheaf $\mathcal{F} \in Sh(X)$ wrt the cover $\mathcal{U} = (U_i)_{i \in I}$ is

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) := [C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \dots]$$

where

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0, \dots, i_p})$$

and

$$(d^p \alpha)_{i_0, \dots, i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \cdot \alpha_{i_0, \dots, \underset{\underset{\text{omitted}}{\uparrow}}{\hat{i}_k}, \dots, i_{p+1}} \Big|_{U_{i_0, \dots, i_{p+1}}}$$

We have $d^{p+1} \circ d^p = 0$

$\Rightarrow$ $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex

We define the *Čech cohom* of $\mathcal{F}$ wrt $\mathcal{U}$ by

$$\check{H}^i(\mathcal{U}, \mathcal{F}) := H^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F})).$$

Ex a) $\check{H}^0(\mathcal{U}, \mathcal{F}) = H^0(X, \mathcal{F})$.

b) For the trivial cover $\mathcal{U} = (X)$,

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) = [\mathcal{F}(X) \to 0 \to 0 \to \cdots]$$

Here

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = \begin{cases} H^0(X, \mathcal{F}), & n = 0 \\ 0, & n > 0 \end{cases}$$

$\Rightarrow$ short exact sequences of sheaves in general need NOT induce a long exact sequence in Čech coh!

Q When does Čech coh agree w/ sheaf cohom?

Def For $i_0, \dots, i_p \in I$, consider the inclusion $j_{i_0,\dots,i_p} : U_{i_0,\dots,i_p} \hookrightarrow X$. We define the "sheaf version" $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of the Čech cplex by

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} j_{i_0,\dots,i_p *}(\mathcal{F}|_{U_{i_0,\dots,i_p}})$$

w/ differentials

$$d^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

given by same formula as above.

Rem By construction

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \simeq \Gamma(X, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$$

$$:= [\Gamma(X, \mathcal{C}^0(\mathcal{U}, \mathcal{F})) \to \Gamma(X, \mathcal{C}^1(\mathcal{U}, \mathcal{F})) \to \cdots]$$

Prop $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is a resolution of $\mathcal{F}$,

ie $\exists$ sheaf hom $\varepsilon : \mathcal{F} \to \check{\mathcal{C}}^0(\mathcal{U},\mathcal{F})$

giving an exact sequence

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \check{\mathcal{C}}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \check{\mathcal{C}}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \cdots$$

Pf. Let ε be the product of the adjunction morphisms $\mathcal{F} \to j_{i*}(\mathcal{F}|_{U_i})$ $(i \in I)$.

$\Longrightarrow$ $\ker(d^0) = \operatorname{im}(\varepsilon)$ by sheaf axioms.

Exactness in degrees $p \geq 1$:

Check on stalks at $x \in X$. Say $x \in U_j$ $(j \in I)$.

Consider

$$\begin{tikzcd} h^p : \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})_x \arrow[r] & \check{\mathcal{C}}^{p-1}(\mathcal{U},\mathcal{F})_x \\ \alpha_x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \beta_x \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

w/ α_x represented by $\alpha \in \Gamma(V, \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F}))$, some $V \subset U_j$

$\rightsquigarrow$ $\beta \in \Gamma(V, \check{\mathcal{C}}^{p-1}(\mathcal{U},\mathcal{F}))$ defined by

$$\beta_{i_0,\dots,i_{p-1}} := \alpha_{j,i_0,\dots,i_{p-1}}$$

Here we use the convention

$$\beta_{j,i_0,\dots,i_{p-1}} := \begin{cases} 0 & \text{if } j \in \{i_0,\dots,i_{p-1}\} \\ (-1)^{\operatorname{sgn}\sigma} \cdot \beta_{\sigma(j),\sigma(i_0),\dots,\sigma(i_{p-1})} & \\ \text{if } \exists \text{ permutation} & \\ \sigma \in \operatorname{Sym}(\{j,i_0,\dots,i_{p-1}\}) & \\ \text{w/ } \sigma(j) < \sigma(i_0) < \cdots < \sigma(i_{p-1}) & \end{cases}$$

One computes for $p \geq 1$

$$h^{p+1} \circ d^p + d^{p-1} \circ h^p = \operatorname{id}_{\check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})_x}$$

$\Longrightarrow$ h chain homotopy

$\Longrightarrow$ $H^p(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})_x) = 0 \quad \forall\, p \geq 1$

$\Longrightarrow$ $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ exact in all degrees ≥ 1 $\square$

Cor If $\mathcal{F} \in Sh(X)$ is flabby, then

$$\check{H}^p(\mathcal{U},\mathcal{F}) = 0 \quad (= H^p(X,\mathcal{F}))$$

for all $p \geq 1$.

Pf. $\mathcal{F}$ flabby $\Rightarrow$ $j_*(\mathcal{F}|_U)$ flabby $\forall\, j : U \hookrightarrow X$

$$\Rightarrow \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ flabby } \forall p \geq 0$$

Prop $\Rightarrow$ flabby resolut^n $\mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$

$$\Rightarrow H^p(X, \mathcal{F}) \simeq H^p(\Gamma(X, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})))$$

$$\simeq H^p(\underbrace{\check{C}^\bullet(\mathcal{U}, \mathcal{F})}_{=0 \text{ in all degrees } \neq 0}) \qquad \square$$

Back to arbitrary $\mathcal{F} \in Sh(X)$:

Rem For any $\mathcal{F} \in Sh(X)$, all $p \geq 0$,

$\exists$ natural map $\check{H}(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$.

Pf. Pick an injective resolution $\mathcal{F} \to \mathcal{J}^\bullet$

Extension property for injective resolutions gives

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, equal] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[d, "\exists \text{ (unique up to homotopy)}"] \\ \mathcal{F} \arrow[r] & \mathcal{J}^\bullet \end{tikzcd}$$

Now apply $\Gamma(X, -)$. $\square$

Thm ("Leray's thm")

Suppose for all $q > 0$ & all $i_0, .., i_p \in I$ $(p \geq 0)$ that

$$H^q(U_{i_0, .., i_p}, \mathcal{F}) = 0.$$

Then the above morphism is an iso

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}) \quad \forall p \geq 0.$$

Pf. $p = 0$ known already. General case:

Pick embedding $\mathcal{F} \hookrightarrow \mathcal{G}$ w/ $\mathcal{G}$ flabby, and put $\mathcal{F}' := \mathcal{G} / \mathcal{F}$.

By assumpt^n we have for all $U = U_{i_0, .., i_p}$ w/ $i_0, .., i_p \in I$: $H^1(U, \mathcal{F}) = 0$

$\Rightarrow$ $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{F}'(U) \to 0$ exact

$\Rightarrow$ $0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{G}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}') \to 0$ exact

$\Rightarrow$ get long exact sequence of Čech cohomology

But $\mathcal{G}$ is flabby

$\Rightarrow \check{H}^p(\mathcal{U}, \mathcal{G}) = 0 \quad \forall p > 0$ by corollary

So we get exact sequences:

a)
$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{G}) \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}') \arrow[r] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] & 0 \end{tikzcd}$$

b) $\check{H}^p(\mathcal{U}, \mathcal{F}') \xrightarrow{\sim} \check{H}^{p+1}(\mathcal{U},\mathcal{F}) \quad \forall p \geq 1.$

Compare a) with long exact sequence in sheaf cohomology to get $\check{H}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F})$.

For higher degrees proceed by induction: Use b) and replace $\mathcal{F}$ by $\mathcal{F}'$.

Note: We still know $H^q(U_{i_0,\dots,i_p}, \mathcal{F}') = 0$ for all $q > 0$ since $H^q(-,\mathcal{F}') \simeq H^{q+1}(-,\mathcal{F})$ using that $\mathcal{G}$ is flabby.

□

<u>Cor</u> Let X be a separated scheme and $\mathcal{U} = (U_i)_{i \in I}$ a cover by affine open subsets $U_i \subset X$. Then for any $\mathcal{F} \in QCoh(X)$,

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^q(X, \mathcal{F}) \quad \forall q \geq 0.$$

Pf. X separated & all U_i affine

$\Rightarrow U_{i_0,\dots,i_p}$ affine $\forall i_0, \dots, i_p \in I$

$\Rightarrow H^q(U_{i_0,\dots,i_p}, \mathcal{F}) = 0 \quad \forall q > 0$

by Serre vanishing, so thm applies. □

<u>Ex</u> $H^\bullet(\mathbb{P}^1_k, \mathcal{F})$ for $\mathcal{F} = \Omega^1_{\mathbb{P}^1/k}$?

By the above, we only need to compute $\check{H}^\bullet(\mathcal{U}, \Omega^1)$ for the cover $\mathcal{U} = \{U_0, U_\infty\}$

w/ $U_0 = \operatorname{Spec} k[x]$
$U_\infty = \operatorname{Spec} k[y]$ glued via $x = \frac{1}{y}$.

- $\check{C}^0(\mathcal{U}, \mathcal{F}) = \Gamma(U_0, \mathcal{F}) \oplus \Gamma(U_\infty, \mathcal{F})$

  $= k[x]\,dx \oplus k[y]\,dy$

- $\check{C}^1(\mathcal{U}, \mathcal{F}) = \Gamma(U_0 \cap U_\infty, \mathcal{F})$

  $= k[x, x^{-1}]\,dx$

- $d : \check{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^1(\mathcal{U}, \mathcal{F})$

  $f(x)\,dx \longmapsto f(x)\,dx$

  $g(y)\,dy \longmapsto g(\frac{1}{x})\,d(\frac{1}{x})$

  $= -\frac{1}{x^2} g(\frac{1}{x})\,dx$

- $\ker(d) = \{ (f(x)dx, g(y)dy) \mid f(x) = \frac{1}{x^2} g(\frac{1}{x}) \} = 0$

- $\operatorname{im}(d) = \{ (f(x) - \frac{1}{x^2} g(\frac{1}{x}))\,dx \mid f, g \in k[x] \}$

$\Rightarrow \operatorname{coker}(d) \simeq k \cdot \frac{1}{x}\,dx$

Thus

$$H^i(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1/k}) \simeq \begin{cases} k & \text{for } i = 1 \\ 0 & \text{else} \end{cases}$$

Similarly

$$H^i(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \simeq \begin{cases} k & \text{for } i = 0 \\ 0 & \text{else} \end{cases}$$

$\leadsto$ as expected from Serre duality ...

6. Cohomology of line bdles on $\mathbb{P}^n$ and Serre vanishing on projective schemes

The following computation will be the key to most later thms about cohomology:

Thm Let $X = \mathbb{P}^n_A$ for a Noetherian ring A, and let $d \in \mathbb{Z}$. Then

a) $H^i(X, \mathcal{O}_X(d)) \simeq 0$ for all $i \notin \{0,n\}$.

b) $$H^0(X, \mathcal{O}_X(d)) \simeq \begin{cases} 0 & \text{for } d < 0 \\ S_d & \text{for } d \geq 0 \end{cases}$$

↑ (degree d part of S)

w/ $S := A[x_0, \dots, x_n]$.

c) $$H^n(X, \mathcal{O}_X(d)) \simeq \begin{cases} 0 & \text{for } d > 0 \\ M_d & \text{for } d \leq 0 \end{cases}$$

↑ (degree d part of M)

w/ $M := \frac{1}{x_0 \cdots x_n} \cdot A[\frac{1}{x_0}, \dots, \frac{1}{x_n}]$.

Pf. Leray's thm:

$$H^\bullet(X, \mathcal{O}_X(d)) \simeq \check{H}^\bullet(\mathcal{U}, \mathcal{O}_X(d))$$

for the standard cover

$$\mathcal{U} = (U_i)_{i=0,\dots,n} \quad \text{w/} \quad U_i = D_+(x_i) \subset X.$$

For $i_0 < \dots < i_p$ we have:

$$U_{i_0,\dots,i_p} = D_+(x_{i_0} \cdots x_{i_p})$$

$$\Rightarrow \quad \mathcal{O}_X(d)(U_{i_0,\dots,i_p}) \simeq \left(A[x_0, \dots, x_n, \tfrac{1}{x_{i_0} \cdots x_{i_p}}]\right)_d$$

$$\Rightarrow \quad \check{C}^p(\mathcal{U}, \mathcal{O}_X(d)) \simeq \prod_{i_0 < \dots < i_p} \left(A[x_0, \dots, x_n, \tfrac{1}{x_{i_0} \cdots x_{i_p}}]\right)_d$$

w/ $d : \check{C}^p \to \check{C}^{p+1}$ given by

$$(ds)_{i_0,\dots,i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k s_{i_0,\dots,\hat{i_k},\dots,i_{p+1}}$$

Each $\check{C}^p(\mathcal{U}, \mathcal{O}_X(d)) = \prod_{i_0<\dots<i_p} \left(A[x_0,\dots,x_n, \frac{1}{x_{i_0}\cdots x_{i_p}}]\right)_d$

is $\mathbb{Z}^{n+1}$- graded by the multidegree of Laurent polynomials:

$$\deg(x_0^{e_0}\cdots x_n^{e_n}) := (e_0,\dots,e_n) \in \mathbb{Z}^{n+1}$$

Since $d : \check{C}^p \to \check{C}^{p+1}$ preserves this grading, the Čech cplex decomposes as

$$\check{C}^\bullet = \bigoplus_{\substack{e\in\mathbb{Z}^{n+1}\\ e_0+\dots+e_n=d}} \check{C}^\bullet(e).$$

$$\Longrightarrow \quad \check{H}^p(\mathcal{U}, \mathcal{O}_X(d)) = \bigoplus_{\substack{e\in\mathbb{Z}^{n+1}\\ e_0+\dots+e_n=d}} H^p(\check{C}^\bullet(e))$$

$\Longrightarrow$ enough to compute cohomology of each $\check{C}^\bullet(e)$

Fix $e=(e_0,\dots,e_n)\in\mathbb{Z}^{n+1}$

Let $N(e) := \{\, i \mid e_i < 0 \,\}$ (indices of negative exponents),

then by definition $\check{C}^p(e) = \bigoplus_{\substack{i_0<\dots<i_p\\ N(e)\subset\{i_0,\dots,i_p\}}} A\cdot x_0^{e_0}\cdots x_n^{e_n}.$

Case 1: $N(e) = \{0,\dots,n\}$:

$$\Longrightarrow \quad \underset{\substack{\| \\ H^p(\check{C}^\bullet(e))}}{\check{C}^p(e)} = \begin{cases} 0 & \text{if } p\neq n \\ A\cdot\frac{1}{x_0^{-e_0}\cdots x_n^{-e_n}} & \text{if } p=n \end{cases}$$

Direct sum over all these gives

$$H^n(X,\mathcal{O}_X(d)) \supseteq \left(\frac{1}{x_0\cdots x_n}\cdot A[\tfrac{1}{x_0},\dots,\tfrac{1}{x_n}]\right)_d$$

Case 2: $N(e) = \emptyset$:

$$\Rightarrow \check{C}^p(e) = \bigoplus_{i_0 < \dots < i_p} A \cdot x_0^{e_0} \cdots x_n^{e_n} \simeq \bigoplus_{i_0 < \dots < i_p} A$$

$$\Rightarrow \check{C}^\bullet(e) \simeq \check{C}^\bullet(\mathcal{V}, \mathcal{O}_Y)$$

for $Y = \operatorname{Spec} A$

& $\mathcal{V} = (V_i)_{i=0,\dots,n}$ w/ $V_i := Y \ \forall i$

$$\Rightarrow H^p(\check{C}^\bullet(e)) \simeq H^p(\check{C}^\bullet(\mathcal{V}, \mathcal{O}_Y))$$

$$\underset{\substack{\uparrow \\ \text{Leray thm}}}{\simeq} H^p(Y, \mathcal{O}_Y) \underset{\substack{\uparrow \\ Y = \operatorname{Spec} A \text{ affine}}}{\simeq} \begin{cases} A, & p = 0 \\ 0, & p > 0 \end{cases}$$

More explicitly $H^0(\check{C}^\bullet(e)) = A \cdot x_0^{e_0} \cdots x_n^{e_n}$.

Direct sum over all these gives

$$H^0(X, \mathcal{O}(d)) \supseteq (A[x_0, \dots, x_n])_d$$

(as we knew before of course)

Case 3: $\emptyset \neq N(e) \neq \{0, 1, \dots, n\}$:

Pick $i_q \notin N(e)$ (q fixed)

Define $h : \check{C}^{p+1}(e) \longrightarrow \check{C}^p(e)$ by

$$h(s)_{i_0, \dots, i_p} := \begin{cases} 0 & \text{if } i_q \in \{i_0, \dots, i_p\} \\ \operatorname{sgn}(\sigma) \cdot s_{i_{\sigma(q)}, i_{\sigma(0)}, \dots, i_{\sigma(p)}} & \text{else,} \\ \text{w/ } \sigma \in \operatorname{Sym}\{\sigma(q), \sigma(0), \dots, \sigma(p)\} & \\ \text{sth } i_{\sigma(q)} < i_{\sigma(0)} < \cdots < i_{\sigma(p)}. & \end{cases}$$

Computation shows $hd + dh = id$

$$\Rightarrow H^p(\check{C}^\bullet(e)) = 0 \text{ for all } p$$

Upshot In steps 1 & 2 the inclusion is equality,

hence $H^0(X, \mathcal{O}_X(d)) \simeq \cdots$ $\square$

Cor ("Serre vanishing on projective schemes")

Let X be a proper scheme / Noeth ring A and $\mathcal{L} \in \mathrm{Pic}(X)$ an ample line bdle on X

$\Rightarrow \forall\, \mathcal{E} \in \mathrm{Coh}(X)\ \exists\, m_0 \in \mathbb{Z}\ \forall\, m \geq m_0$:

$$H^i(X, \mathcal{E} \otimes \mathcal{L}^{\otimes m}) = 0 \text{ for all } i > 0.$$

Pf. $\mathcal{L}$ ample & X proper over A

$\Rightarrow \exists\, n > 0\ \exists$ closed immersion

$$i : X \hookrightarrow \mathbb{P}^N_A \quad \text{w/ } \mathcal{L}^{\otimes n} \simeq i^*\mathcal{O}(1).$$

Wlog $n = 1$, ie $\mathcal{L} \simeq i^*\mathcal{O}(1)$ (replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$).

$\Rightarrow$ For any m,

$$H^i(X, \mathcal{E} \otimes \mathcal{L}^{\otimes m}) = H^i(X, \mathcal{E} \otimes i^*\mathcal{O}(m))$$

since i is an affine morphism

$$= H^i(\mathbb{P}^N_A, \underbrace{i_*(\mathcal{E} \otimes i^*\mathcal{O}(m))}_{\simeq\, i_*(\mathcal{E}) \otimes \mathcal{O}(m)})$$

(projection formula, see sheaf exercises)

$\Longrightarrow$ wlog $X = \mathbb{P}^N_A$ and $\mathcal{L} \simeq \mathcal{O}(1)$.

We show by descending induction on i that

$$\left.\begin{array}{l} \forall\, i > 0\ \ \forall\, \mathcal{E} \in \mathrm{Coh}(X)\ \exists\, m_0\ \forall\, m \geq m_0 : \\ \qquad H^j(X, \mathcal{E}(m)) = 0 \text{ for all } j \geq i \end{array}\right\} (*)_i$$

- For $i > N$ this is clear from the Cech cplex for the affine cover of $X = \mathbb{P}^N_A$ by $N+1$ charts.
- Now assume $(*)_{i+1}$ holds for some i.

  We must show that then also $(*)_i$ holds:

  Pick ℓ sth $\mathcal{E}(\ell)$ is globally generated, say

$$0 \to \underset{\ker\varphi}{\underset{\shortparallel}{\mathcal{G}}} \to \mathcal{O}^{\oplus r}_{\mathbb{P}^N_A} \xrightarrow{\exists \varphi} \mathcal{E}(\ell) \to 0$$

By induction $\exists\, n_0 \geq 0 \ \forall\, n \geq n_0$:

$$H^j(X, \mathcal{G}(n)) = 0 \quad \text{for all } j \geq i+1.$$

Looking at

$$0 \to \mathcal{G}(n) \to \mathcal{O}_X^{\oplus r}(n) \to \mathcal{E}(\ell+n) \to 0$$

we get

$$\underbrace{H^j(X, \mathcal{O}_X^{\oplus r}(n))}_{\substack{= 0 \text{ for } j \geq i > 0 \\ \text{by the theorem,} \\ \text{as } X = \mathbb{P}^N_A \ \& \ n > 0}} \to H^j(X, \mathcal{E}(\ell+n)) \to \underbrace{H^{j+1}(X, \mathcal{G}(n))}_{\substack{= 0 \text{ for } j \geq i \\ \text{by induction} \\ \text{assumption}}}$$

$\square$

$\Longrightarrow$ claim follows.

As in Serre's affinity criterion, $\exists$ also a sort of converse to this:

Prop Let X be a Noetherian scheme & $\mathcal{L} \in \operatorname{Pic}(X)$. If $\forall$ coherent $\mathcal{J} \trianglelefteq \mathcal{O}_X \ \exists\, m > 0 : H^1(X, \mathcal{J} \otimes \mathcal{L}^{\otimes m}) = 0$, then $\mathcal{L}$ is ample.

Pf. We show X can be covered by open affine subsets of the form $D(s)$ w/ $s \in \Gamma(X, \mathcal{L}^{\otimes m})$.

Let $p \in X$ be any closed pt (exists since X qcqs) w/ inclusion $i_p : \operatorname{Spec} \kappa(p) \hookrightarrow X$ & open nbhood $U \ni p$, $\quad Z := X \setminus U$.

From $0 \to \mathcal{J}_{Z \cup \{p\}} \to \mathcal{J}_Z \to i_{p*}\kappa(p) \to 0$

we get

$$\begin{tikzcd}
H^0(X, \mathcal{J}_Z \otimes \mathcal{L}^m) \arrow[r] \arrow[d, hook] & H^0(X, i_{p*}\kappa(p)) \arrow[r] \arrow[d, "\mathrm{ev}_p"] & \underbrace{H^1(X, \overbrace{\mathcal{J}_{Z \cup \{p\}}}^{\substack{\text{coherent} \\ \text{since } X \text{ Noeth.}}} \otimes \mathcal{L}^m)}_{\substack{= 0 \text{ for } m \gg 0 \\ \text{by assumption}}} \\
H^0(X, \mathcal{L}^m) \arrow[r] & \kappa(p) & \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
\Longrightarrow \exists\, s \arrow[r, mapsto] & 1 &
\end{tikzcd}$$

$\square$

Cor Let X be a proper scheme / Noetherian ring A.

Then for $\mathcal{L} \in \operatorname{Pic}(A)$ TFAE:

a) $\mathcal{L}$ is ample.

b) $\forall$ coherent $\mathcal{J} \trianglelefteq \mathcal{O}_X$ $\exists\, m > 0$:

$$H^1(X, \mathcal{L}^{\otimes m}) = 0.$$

Pf. a) $\Rightarrow$ b): Serre vanishing.

b) $\Rightarrow$ a): Previous proposition. □

Note Properness of X is needed for a) $\Rightarrow$ b),

eg for $X = \mathbb{A}^2_k \setminus \{(0,0)\}$ we have:

- $\mathcal{L} = \mathcal{O}_X$ ample as $X = D(x) \cup D(y)$
- $H^1(X, \mathcal{L}^m) = H^1(X, \mathcal{O}_X) \neq 0$!

7. Finiteness theorems

Motivation $X \hookrightarrow \mathbb{P}^n_A$ closed

$\Rightarrow$ X covered by $n+1$ affine charts

$\Rightarrow H^i(X, \mathcal{E}) = 0 \;\; \forall i > n \;\; \forall \mathcal{E} \in Coh(X)$

We can do much better:

Thm ("cohomological dimension of schemes")

Let X be a Noetherian scheme. Then for any sheaf of abelian gps $\mathcal{F} \in Sh(X)$, we have

$$H^i(X, \mathcal{F}) = 0 \quad \text{for all } i > \dim X.$$

Pf. Induction on $(d, m) := (\dim X, \#\text{irred cpt}^s \text{ of } X)$:

⓪ $(d, m) = (0, 1)$:

Then $X = \{pt.\}$ & claim is trivial.

① Induct^n step $(d, m-1) \to (d, m)$:

Say $X = Z_1 \cup \dots \cup Z_m$ w/ irred cpt^s $Z_\alpha \subset X$

By induct^n claim holds for $Z := Z_1$

and also for $X' := Z_2 \cup \dots \cup Z_m$.

Consider the inclusions $i : Z \hookrightarrow X$,

$j : U := X \setminus Z \hookrightarrow X$.

From $0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to i_* i^{-1} \mathcal{F} \to 0$

we get

$$\dots \to H^k(X, j_! j^{-1} \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(X, i_* i^{-1} \mathcal{F})$$

Now for any $\mathcal{G} \in Sh(Z)$:

$$H^k(X, i_* \mathcal{G}) \simeq H^k(Z, \mathcal{G})$$

(exercise, use that i is a closed immersion)

$$\simeq 0 \quad \text{for all } k > \dim Z$$

(by induction assumption the claim holds for Z)

$\Rightarrow$ $H^k(X, j_! j^{-1} \mathcal{F}) \twoheadrightarrow H^k(X, \mathcal{F})$ epi $\forall k > \dim Z$

$\Rightarrow$ enough to show $H^k(X, j_! j^{-1}\mathcal{F}) = 0 \ \forall k > \dim Z$

For this use $j_! j^{-1} \mathcal{F} \simeq i'_* i'^{-1} j_! j^{-1} \mathcal{F}$

w/ $i' : X' \hookrightarrow X$ closed:

$$\begin{tikzcd} U \arrow[rr, hook, "j"] \arrow[dr, hook] & & X \\ & X' \arrow[ur, hook, "i'"'] & \end{tikzcd}$$

Apply $i'_* i'^* (-)$ to the previous sequence:

$$0 \to i'_* i'^{-1} j_! j^{-1} \mathcal{F} \to i'_* i'^{-1} \mathcal{F} \to i'_* i'^{-1} i_* i^{-1} \mathcal{F} \to 0$$

$\|$

$i''_* i''^{-1} \mathcal{F}$ with

$i'' : Z' := Z \cap X' \hookrightarrow X$

We get

$$H^{k-1}(X, i''_* i''^{-1} \mathcal{F}) \to H^k(X, i'_* i'^{-1} j_! j^{-1} \mathcal{F}) \to H^k(X, i'_* i'^{-1} \mathcal{F})$$

$\|$ $H^{k-1}(Z', i''^{-1}\mathcal{F})$

vanishes for $k-1 > \dim Z'$
by induction on dimension,
because $\dim Z' < \dim Z$

$\|$ $H^k(X, j_! j^{-1} \mathcal{F})$

$\|$ $H^k(X', i'^{-1} \mathcal{F})$

vanishes for $k > \dim X'$
by induction on m,
since X' has only $m-1$
irreducible components

$\Rightarrow$ $H^k(X, j_! j^{-1} \mathcal{F}) = 0 \ \forall k > \dim X$

$\Rightarrow$ wlog X irreducible, ie $m = 1$

② Induct^n step $(d-1, ?) \to (d, 1)$:

Let X be irreducible of dimension $d = \dim X$.

Let $I := \{(U, s) \mid U \subset X \text{ open}, s \in \mathcal{F}(U)\}$.

Each (U, s) gives a morphism $j_! \mathbb{Z}_U \to \mathcal{F}$ by adjunction, and we get an epi

$$\bigoplus_{(U,s)\in I} j_{U!} \mathbb{Z} \twoheadrightarrow \mathcal{F}$$

$\Rightarrow$ $\mathcal{F} = \operatorname{colim}_S \mathcal{F}_S$

where S runs over all finite subsets of I

and $\mathcal{F}_S := \operatorname{im}\left(\bigoplus_{(U,s)\in S} j_{U!} \mathbb{Z} \to \mathcal{F}\right)$.

Cohomology commutes w/ filtered colim (exercise)

$$\Rightarrow \quad H^\bullet(X, \mathcal{F}) \simeq \operatorname*{colim}_{S} H^\bullet(X, \mathcal{F}_S)$$

$\Rightarrow$ wlog $\mathcal{F} = \mathcal{F}_S$ for some finite $S \subset I$.

Next we inductively reduce to $|S| = 1$:

Pick $s_0 \in S$ and consider

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{S \setminus \{s_0\}} \arrow[r] & \mathcal{F}_S \arrow[r] \arrow[d, equal] & \mathcal{F}' \arrow[r] \arrow[d, phantom, "\vdots"] & 0 \\ & & \mathcal{F} & \text{cokernel, generated by a single section } s_0\text{, ie } \mathcal{F}' = (\mathcal{F}')_{\{s_0\}} & \end{tikzcd}$$

We get

$$H^k(X, \mathcal{F}_{S \setminus \{s_0\}}) \to H^k(X, \mathcal{F}) \to H^k(X, \mathcal{F}')$$

$\Rightarrow$ By induction on $|S|$,

will be enough to deal w/ the case $|S| = 1$,

ie wlog $\exists$ epi $j_!\mathbb{Z} \twoheadrightarrow \mathcal{F}$ (some $j : U \hookrightarrow X$).

Consider then

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, phantom, "\vdots"] & j_!\mathbb{Z} \arrow[r, "\exists s"] & \mathcal{F} \arrow[r] & 0 \\ & \ker(s) & & & \end{tikzcd}$$

Recall X irreducible $\Rightarrow$ $\exists !$ generic pt $\eta \in X$

Let $d \in \mathbb{Z}$ with $\mathcal{G}_\eta = d\mathbb{Z} \subset (j_!\mathbb{Z})_\eta = \mathbb{Z}$.

If $d = 0$, then $\mathcal{G}_\eta = 0$, hence $\mathcal{G} = 0$

(since $\mathcal{G}$ embeds in constant sheaf on irreducible X).

If $d > 0$, we get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r, "\text{incl}"] & j_!(d\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{F}' \arrow[r] \arrow[d, phantom, "\vdots"] & 0 \\ & & j_!\mathbb{Z} & \operatorname{cok}(\text{incl}) & \end{tikzcd}$$

where incl is an iso on stalks at the generic pt η

$\Rightarrow$ $\mathcal{F}'_\eta = 0$, so $\mathcal{F}' \simeq i_* i^{-1} \mathcal{F}'$ for some

proper closed subset $i : Z \overset{\neq}{\hookrightarrow} X$.

$$\underset{\substack{\parallel \\ H^{k-1}(Z, \mathcal{F}') = 0 \ \forall k > d \\ \text{by induction on } d = \dim X}}{H^{k-1}(X, i_* i^{-1} \mathcal{F}')} \to H^k(X, \mathcal{G}) \to \underbrace{H^k(X, j_!\mathbb{Z})}_{\substack{\text{hence enough to show} \\ \text{this vanishes } \forall k > d}} \to \cdots$$

Upshot Suffices to prove $H^k(X, j_!\mathbb{Z}) = 0$

for X irred, $j: U \hookrightarrow X$ open & all $k > \dim X$.

For this consider $i: Z := X \setminus U \hookrightarrow X$.

From

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0$$

we get

$$\underset{\substack{\parallel \\ H^{k-1}(Z,\mathbb{Z})}}{H^{k-1}(X, i_*\mathbb{Z})} \to H^k(X, j_!\mathbb{Z}) \to \underset{\substack{\parallel \\ 0}}{H^k(X,\mathbb{Z})}$$

vanishes $\forall k > d$
by ind^n on $d = \dim X$
(note: $\mathbb{Z} \in Sh(Z)$
needn't be flabby
as Z need not be
irreducible ...)

because for X irred
the constant sheaf
$\mathbb{Z} \in Sh(X)$ is flabby

$\square$

Thm ("finiteness of cohomology")

Let X be a proper scheme / Noeth ring A.

$\Rightarrow$ For any $\mathcal{F} \in Coh(X)$, all $k \in \mathbb{N}_0$,

$H^k(X, \mathcal{F})$ is a fin. gen. A-module.

Pf. ① Assume first that X is projective / A,

ie $\exists$ closed immersion $i: X \hookrightarrow \mathbb{P}^n_A$.

Then $H^k(X, \mathcal{F}) \simeq H^k(\mathbb{P}^n_A, i_*\mathcal{F})$

$\Rightarrow$ wlog $X = \mathbb{P}^n_A$

We know $H^k(X, \mathcal{F}) = 0 \quad \forall k > n$

By decreasing induction, assume $H^k(X, \mathcal{F}) = 0$

$\forall k > k_0$ (some $k_0 > 1$).

We want to show vanishing also for $k = k_0$.

Pick $N \gg 0$ sth $\mathcal{F}(N)$ is globally generated.

$\Rightarrow$ $\exists$ exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_X(-N)^{\oplus r} \longrightarrow \mathcal{F} \to 0$$

We get

$$\underbrace{H^{k_0}(X, \mathcal{O}_X(-N))}_{\text{fin. gen. over } A \text{ by our computation in §6}} \to H^{k_0}(X, \mathcal{F}) \to \underbrace{H^{k_0+1}(X, \mathcal{G})}_{\text{fin. gen. over } A \text{ by induction on } k} \to \cdots$$

$\Longrightarrow$ $H^{k_0}(X, \mathcal{F})$ fin. gen. over A as required

② Reduction to the projective case:

Chow's lemma (§ III.5):

X proper / A $\Rightarrow$ $\exists\ \pi : X' \to X$
proper & iso over an open dense $U \subset X$
w/ X' projective over A,
ie $\exists$ closed imm. $i : X' \hookrightarrow \mathbb{P}^n_A$.

Consider the derived functors of π_*, ie. the functors

$$R^i\pi_* : \mathrm{Mod}(\mathcal{O}_{X'}) \to \mathrm{Mod}(\mathcal{O}_X).$$

Exercise Let $\pi : Y \to Z$ be a proper morphism of separated schemes. Given an ample line bdle $\mathcal{L} \in \mathrm{Pic}(Y)$, then $\forall$ open affine cover $Z = \bigcup_i U_i$ $\exists\ m > 0$:

a) $H^k(\pi^{-1}(U_i), \mathcal{L}^m) = 0 \quad \forall k > 0$

(use that $V_i := \pi^{-1}(U_i) \to U_i$ is proper and $\exists$ ample $\mathcal{L}|_{V_i}$, hence $V_i \to U_i$ is projective)

b) $\pi_*(\mathcal{L}^m) \in \mathrm{Coh}(Z)$ by ① and $R^k\pi_*(\mathcal{L}^m) = 0$ for all $k > 0$ (check on U_i)

c) $H^k(Z, \pi_*\mathcal{L}^m) \simeq H^k(Y, \mathcal{L}^m) \quad \forall k \geq 0$.

We apply this to our $\pi : Y = X' \to Z = X$.

Replace $\mathcal{L}$ by a power $\Longrightarrow$ wlog $m = 1$

Now pick $U \subset X$ w/ $\pi : U' = \pi^{-1} U \xrightarrow{\sim} U$.

Given $\mathcal{F} \in Coh(X)$, may shrink U sth

wlog $\exists$ epi $\mathcal{O}_U^{\oplus N} \twoheadrightarrow \mathcal{F}|_U$.

Can also assume (again shrinking $U' \simeq U$)

that $\mathcal{L}|_{U'} \simeq \mathcal{O}_{U'}$, hence $\pi_* \mathcal{L}|_U \simeq \mathcal{O}_U$.

$$\Longrightarrow \exists \; \varphi : (\pi_* \mathcal{L})|_U^{\oplus N} \twoheadrightarrow \mathcal{F}|_U.$$

Now:

$$\underbrace{\ker(\varphi)}_{\text{subsheaf}} \subset (\underbrace{\pi_* \mathcal{L}^{\oplus N}}_{\in Coh(X)})|_U$$

$\Longrightarrow$ $\exists$ coherent subsheaf

$\mathcal{K} \subset \pi_* \mathcal{L}^{\oplus N}$ w/ $\mathcal{K}|_U = \ker \varphi$.

Upshot:

$\mathcal{F}, \; \mathcal{G} := \pi_* \mathcal{L}^{\oplus N} / \mathcal{K}$ both $\in Coh(X)$

and $\exists$ open dense $U \subset X$ w/ $\mathcal{F}|_U \simeq \mathcal{G}|_U$.

Pick a coherent subsheaf $\mathcal{H} \subset \mathcal{F} \oplus \mathcal{G}$

w/ $\mathcal{H}|_U =$ graph $(\mathcal{F}|_U \xrightarrow{\sim} \mathcal{G}|_U)$

Put $Q := (\mathcal{F} \oplus \mathcal{G}) / \mathcal{H}$.

$\Longrightarrow$ $\varphi : \mathcal{F} \to Q$, $\psi : \mathcal{G} \to Q$ iso on U

$\Longrightarrow$ By ind$^{\text{n}}$ on support dim,

$\ker \varphi$, $\operatorname{cok} \varphi$, $\ker \psi$, $\operatorname{cok} \psi$

have $H^\bullet(X, -)$ fin gen $/ A$

$\Longrightarrow$ To show $H^\bullet(X, \mathcal{F})$ fin gen $/A$,

enough to show $H^\bullet(X, \mathcal{G})$ fin gen $/A$

But $\mathcal{G} \simeq (\pi_* \mathcal{L}^{\oplus N})/\mathcal{K}$

We get

$$\begin{tikzcd} \cdots \arrow[r] & H^k(X, \pi_* \mathcal{L}^{\oplus N}) \arrow[r] \arrow[d, equal, "\text{exercise part c)}"'] & H^k(X, \mathcal{G}) \arrow[r] & \underbrace{H^{k+1}(X, \mathcal{K})}_{\substack{\text{fin gen / A} \\ \text{by descending} \\ \text{induction on } k}} \arrow[r] & \cdots \\ & H^k(X', \mathcal{L}^{\oplus N}) & & & \end{tikzcd}$$

fin.gen. / A by ①
(the projective case)

$\Longrightarrow$ $H^k(X, \mathcal{G})$ fin gen / A □

8. A few remarks about $\mathcal{E}xt$ sheaves

Recall $\mathcal{A}$ abelian cat w/ enough inj.

For $M \in \mathcal{A}$, $i \in \mathbb{N}_0$ we define

$$\mathrm{Ext}^i_{\mathcal{A}}(M, -) := R^i \,\mathrm{Hom}_{\mathcal{A}}(M, -).$$

For Serre duality we want a sheaf version:

Def $(X, \mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$

For $i \in \mathbb{Z}$ we define:

- $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -) := R^i \,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -):$
  $$\mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\Gamma(X, \mathcal{O}_X))$$
- $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -) := R^i \,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -):$
  $$\mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{O}_X)$$

Lemma Let $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$.

a) For any open $U \subset X$,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_U \simeq \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

b) For $\mathcal{F} = \mathcal{O}_X$ we have

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \simeq \begin{cases} \mathcal{G}, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

and $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \simeq H^i(X, \mathcal{G})$.

c) Any exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

gives

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^i(\mathcal{F}'', \mathcal{G}) \arrow[r] & \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \arrow[r] & \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^{i+1}(\mathcal{F}'', \mathcal{G}) \arrow[r] & \cdots &
\end{tikzcd}$$

and similarly for $\mathcal{E}xt$.

Pf. a) clear for $i=0$.

For $i>0$ both sides are δ-functors in $\mathcal{G}$ and effaceable (for $\mathcal{G}$ injective also $\mathcal{G}|_U$ is injective), hence universal $\Rightarrow$ naturally isomorphic

b) Similar, w/ $\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{O}_X, -) = \mathrm{id}$,

$$\mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{O}_X, -) = H^0(X, -).$$

c) Pick an injective resolution $\mathcal{G} \to \mathcal{J}^\bullet$

Since each $\mathrm{Hom}(-, \mathcal{J}^k)$ is exact, we get an exact sequence of complexes

$$0 \to \mathrm{Hom}(\mathcal{F}'', \mathcal{J}^\bullet) \to \mathrm{Hom}(\mathcal{F}, \mathcal{J}^\bullet) \to \mathrm{Hom}(\mathcal{F}', \mathcal{J}^\bullet) \to 0$$

hence a long exact sequence on Ext groups.

Similarly for $\mathcal{E}xt$ sheaves. □

Note $\mathrm{Mod}(\mathcal{O}_X)$ usually doesn't have enough projectives, so our definition of right derived functors does NOT apply to

$$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{G}) : \mathrm{Mod}(\mathcal{O}_X)^{op} \longrightarrow \mathrm{Mod}(\Gamma(X, \mathcal{O}_X)).$$

But if X is quasiprojective / Noeth ring A, then every $\mathcal{F} \in \mathrm{Coh}(X)$ has a locally free resolution and that's enough to compute:

Lemma For any resolution $\cdots \to \mathcal{L}_{-1} \to \mathcal{L}_0 \to \mathcal{F}$ by loc. free sheaves $\mathcal{L}_k$ of finite rank,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq H^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_\bullet, \mathcal{G})).$$

Pf. True for $i=0$ by left exactness of $\mathcal{H}om(-, \mathcal{G})$.

For $i>0$ use that both sides are δ-functors in $\mathcal{G}$ and effaceable ($\mathcal{H}om(-, \mathcal{G})$ is exact for $\mathcal{G}$ injective). □

Exercise For $\mathcal{L} \in \mathrm{Mod}(\mathcal{O}_X)$ loc free of finite rk w/ dual $\mathcal{L}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ and any $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$ we have:

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}),$$
$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee$$

In passing to stalks we need some care:

Ex Let $i : \{p\} \overset{\ddagger}{\hookrightarrow} X$ a closed pt

and $j : U := X \setminus \{p\} \hookrightarrow X$.

Take $\mathcal{O}_X = \mathbb{Z}_X$, so $\mathrm{Mod}(\mathcal{O}_X) = \mathrm{Sh}(X)$.

$\Rightarrow$ $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_p \to \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{F}_p, \mathcal{G}_p)$ is

- not epi for $\mathcal{F} = i_{p*}\mathbb{Z}$ & $\mathcal{G} = \mathbb{Z}_X$
- not mono for $\mathcal{F} = \mathcal{G} = j_! \mathbb{Z}_U$.

Lemma Let X be a Noetherian scheme and $p \in X$.

For any $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$

and all $i \geq 0$ we have:

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_p \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{O}_{X,p}}(\mathcal{F}_p, \mathcal{G}_p)$$

Pf. Question local, so wlog X affine.

$\Rightarrow$ $\exists$ free resolution $\mathcal{L}_\bullet \twoheadrightarrow \mathcal{F}$

$\Rightarrow$ free resolution $\mathcal{L}_{\bullet,p} \twoheadrightarrow \mathcal{F}_p$ & claim follows. $\square$

Serre vanishing also shows:

Prop X projective scheme / Noeth ring A

w/ very ample line bdle $\mathcal{O}_X(1)$

Then $\forall\, \mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$ $\forall\, i \geq 0$

$\exists\, n_0 > 0$ sth $\forall\, n \geq n_0$:

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \simeq \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Pf. $i = 0$: Trivial for all $n \in \mathbb{Z}$.

So assume $i > 0$.

① For $\mathcal{F} = \mathcal{O}_X$ have

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) = H^i(X, \mathcal{G}(n)) \overset{\text{Serre vanishing}}{=} 0 \text{ for } n \gg 0,\ i > 0$$

and $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n)) = 0$ for all $n \in \mathbb{Z}$, $i > 0$.

② For $\mathcal{F} = \mathcal{L}$ loc. free of finite rank, reduce to case ① via

$$\mathrm{Ext}^i(\mathcal{L}, \mathcal{G}) \simeq \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{L}^\vee \otimes \mathcal{G}),$$
$$\mathcal{E}xt^i(\mathcal{L}, \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{O}_X, \mathcal{L}^\vee \otimes \mathcal{G}).$$

③ For arbitrary $\mathcal{F} \in \mathrm{Coh}(X)$, reduce to case ② by taking an epi $p : \mathcal{E} \twoheadrightarrow \mathcal{F}$ w/ $\mathcal{E}$ loc free of finite rank. Then

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{E} \arrow[r, "p"] & \mathcal{F} \arrow[r] & 0 \\
 & \ker p & & &
\end{tikzcd}$$

gives for $n \gg 0$ by step ②:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathcal{F}, \mathcal{G}(n)) \arrow[r] & \mathrm{Hom}(\mathcal{E}, \mathcal{G}(n)) \arrow[r] & \mathrm{Hom}(\mathcal{F}', \mathcal{G}(n)) \arrow[dll, rounded corners] \\
 & & \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}(n)) \arrow[r] & 0
\end{tikzcd}$$

and

$$\mathrm{Ext}^i(\mathcal{F}', \mathcal{G}(n)) \xrightarrow{\sim} \mathrm{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n)) \quad \text{for } i > 0.$$

$\Rightarrow$ Claim by induction on i. $\square$

9. Cohen-Macaulay schemes

Setup R Noetherian ring, $I \trianglelefteq R$ proper ideal

What is the "codimension" of

$Z := V(I) \subset X := \mathrm{Spec}(R)$? $\nearrow$ height $\searrow$ depth

Def 1 $\mathrm{codim}_X(Z) := \mathrm{codim}_R(I) := \min\limits_{\substack{\mathfrak{p} \supseteq I \\ \mathfrak{p} \in \mathrm{Spec} R}} \mathrm{ht}(\mathfrak{p})$

for the height

$$\mathrm{ht}_R(\mathfrak{p}) := \sup\left\{ n \geq 0 \;\middle|\; \begin{array}{l} \exists \text{ chain of primes} \\ \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \end{array} \right\} = \dim R_{\mathfrak{p}}$$

Rem For any $\mathfrak{p} \supseteq I$, $\mathrm{codim}_R(I) \leq \dim R_{\mathfrak{p}} \underset{\uparrow}{<} \infty$

($R_{\mathfrak{p}}$ local Noetherian ring)

Ex

- $I = (x^2) \trianglelefteq R = k[x]$ has codim $= 1$.
- $I = (zx, zy) \trianglelefteq R = k[x,y,z]$ has codim $= 1$.
- It can happen that $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \neq \dim(R)$, e.g. $\mathfrak{p} = (pt-1) \trianglelefteq R = \mathbb{Z}_p[t]$ has

$$\underset{\substack{\| \\ 1 \text{ by Krull}}}{\mathrm{ht}(\mathfrak{p})} + \dim(\underset{\hookrightarrow\, = \mathrm{Quot}(\mathbb{Z}_p)}{R/\mathfrak{p}}) = 1 + 0 \neq 2 = \dim(R).$$

Def 2 a) For $M \in \mathrm{Mod}(R)$, a sequence $x_1, \dots, x_n \in R$ is called an M-sequence or M-regular if

i) $(x_1, \dots, x_n) \cdot M \neq M$, and

ii) $\forall i$: x_i is not a zero divisor in $M/(x_1, \dots, x_{i-1})M$

For $M = R$ we call $x_1, \dots, x_n$ a regular sequence.

b) For $I \trianglelefteq R$ w/ $I \cdot M \neq M$ we put

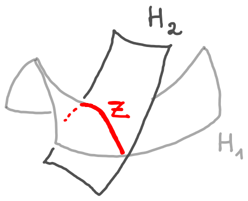
$$\mathrm{depth}_M(I) := \sup\left\{ n \;\middle|\; \begin{array}{l} \exists\, M\text{-sequence} \\ x_1, \dots, x_n \in I \end{array} \right\}$$

$$= \sup\left\{ n \;\middle|\; \begin{array}{l} \exists\, x_1, \dots, x_n \in I \\ \text{satisfying ii) above} \end{array} \right\}$$

Geometric interpretation:

$H_i := V(x_i) \subset X = \mathrm{Spec} R$ hypersurfaces

For $M := R$ we have:



i) $\Rightarrow$ $H_1 \cap \cdots \cap H_n \neq \emptyset$

ii) $\Rightarrow$ H_i does not contain any irred. component of $H_1 \cap \cdots \cap H_{i-1}$.

$x_i \in I \Rightarrow Z := V(I) \subset H_i$.

More precisely:

Lemma For $M \in \mathrm{Mod}(R)$ fin gen & $x_1, \dots, x_n \in R$ with (ii),

$$\dim \operatorname{Supp} M/(x_1, \dots, x_n) \le \dim \operatorname{Supp} M - n.$$

Pf. Wlog $n = 1$ by induction.

M fin gen / Noetherian ring R

$\Rightarrow \{\text{zerodivisors of } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$

for the set of associated primes

$\operatorname{Ass}(M) := \{ \mathfrak{p} \in \operatorname{Spec} R \mid \exists m \in M : \mathfrak{p} = \operatorname{Ann}(M) \}.$

$\Rightarrow x := x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$.

But $\operatorname{Supp} M = V(\operatorname{Ann} M) = \bigcup_{i=1}^{N} V(\mathfrak{p}_i)$

w/ $\mathfrak{p}_1, \dots, \mathfrak{p}_N \in \operatorname{Ass}(M)$ the minimal associated primes

(= minimal primes containing $\operatorname{Ann} M$)

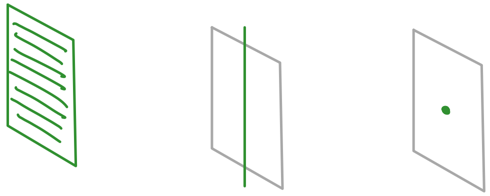
$\Rightarrow \operatorname{Supp} M/xM = \operatorname{Supp}(M) \cap V(x)$ has $\dim < \dim \operatorname{Supp} M$. $\square$

Ex For $IM \neq M$, have $\operatorname{depth}_M(I) \le \dim \operatorname{Supp} M$.

Rem The notion of regular sequence depends on the order:

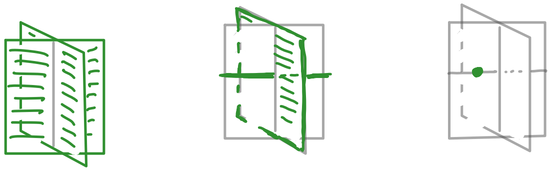
For $R = k[x,y,z]$ we have

a) $x,\ y(1-x),\ z(1-x)$ R-sequence



$V(x)$ $\quad$ $V(x, y(1-x)) = V(x,y)$ $\quad$ $V(x, y(1-x), z(1-x)) = V(x,y,z)$

b) $y(1-x),\ z(1-x),\ x$ NOT an R-sequence



$V(y(1-x))$ $\quad$ $V(y(1-x), z(1-x)) = V(1-x) \cup V(y,z)$ $\quad$ $V(y(1-x), z(1-x), x) = V(x,y,z)$

Lemma If R is local, then for $M \in \mathrm{Mod}(R)$ fin gen and any permutation $\sigma \in S_n$ TFAE:

a) $x_1, \dots, x_n \in \mathfrak{m}_R$ form an M-sequence

b) $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ —— // ——

Pf. S_n generated by transpositions of adjacent elements

$\Rightarrow$ wlog $\sigma = (i, i+1)$ for some i

Replace M by $M/(x_1, \dots, x_{i-1})M$ $\Rightarrow$ wlog $i = 1$, $n = 2$

Goal: If x_1, x_2 is an M-sequence, then so is x_2, x_1.

1) x_2 is not a zero divisor on M:

Let $N := \{ m \in M \mid x_2 \cdot m = 0 \} \subset M$

Since x_2 is a nonzerodivisor on $M/x_1 M$ but acts trivially on N, we must have $N/x_1 N = 0 \subset M/x_1 M$.

$\Rightarrow$ $N = 0$ by Nakayama

2) x_1 is not a zero divisor on $M/x_2 M$:

Let $[m] \in M/x_2 M$ w/ $x_1 \cdot [m] = 0$

$\Rightarrow$ $x_1 m = x_2 m'$ for some $m' \in M$

$\Rightarrow$ $x_2 \cdot [m'] = 0$ in $M/x_1 M$

$\Rightarrow$ $[m'] = 0$ since x_1, x_2 form an M-sequence

$\Rightarrow$ $\exists\, m'' \in M$: $m' = x_1 m''$

$\Rightarrow$ $x_1 \cdot (m - x_2 m'') = 0$ since $x_1 m = x_2 m'$

$\Rightarrow$ $m = x_2 m''$ since x_1 is a nonzerodivisor on M

$\Rightarrow$ $[m] = 0$ in $M/x_2 M$ □

Cor Assume R is local, $M \in \mathrm{Mod}(R)$ fin. gen. and $x_1, \dots, x_n \in R$ is an M-sequence.

$\Rightarrow$ For any $e_1, \dots, e_n \in \mathbb{N}$, $x_1^{e_1}, \dots, x_n^{e_n}$ form an M-sequence.

Pf. Enough to do the case where $\exists!\, i$: $e_i \neq 1$.

By the lemma we may assume $i = n$.

Then the claim is trivial:

x_n nonzerodivisor on $M/(x_1, \dots, x_{n-1})$

$\Rightarrow$ $x_n^{e_i}$ ————〃———— □

Ex $\mathrm{depth}_M(\sqrt{I}) = \mathrm{depth}_M(I)$ $\forall\, I \trianglelefteq R$ w/ $IM \neq M$

Recall $\operatorname{depth}_M(I) \leq \dim \operatorname{Supp} M$ for $I \trianglelefteq R$ w/ $IM \neq M$.

We may have $\dim \operatorname{Supp} M = \infty$. But using the above we can do better:

<u>Lemma</u> For $M \in \operatorname{Mod}(R)$ fin.gen & $I \trianglelefteq R$ w/ $IM \neq M$ we have

$$\operatorname{depth}_M(I) < \infty.$$

More precisely: For any prime $\mathfrak{p} \supseteq I$ and any maximal (ie non-refinable) chain of primes

$\mathfrak{p} = \mathfrak{p}_n \supsetneq \cdots \supsetneq \mathfrak{p}_0$ w/ $\mathfrak{p}_0 \in \operatorname{Ass}(M)$,

we have

$$\operatorname{depth}_M(I) \leq n.$$

Pf. Wlog $I = \mathfrak{p}$ (enlarge I if needed).

For $n = 0$, the claim is trivial as then $\mathfrak{p} \in \operatorname{Ass}(M)$, hence $\mathfrak{p} \subset \{$zero divisors on $M\}$ & $\operatorname{depth}_M(\mathfrak{p}) = 0$.

We now proceed by induction on n.

Any M-sequence in $\mathfrak{p}$ is also an $M_{\mathfrak{p}}$-sequence

(by flatness of $R \to R_{\mathfrak{p}}$)

$\Rightarrow$ wlog R local w/ max ideal $\mathfrak{p} = \mathfrak{p}_0$.

$\Rightarrow$ may permute elements in M-sequences in $\mathfrak{p}$ & replace them by arbitrary powers

Let $x_1, \ldots, x_m \in \mathfrak{p} = \mathfrak{p}_n$ be any M-sequence.

We must show that $m \leq n$.

Case 1: $x_1, \ldots, x_m \in \mathfrak{p}_{n-1}$ (not just $\in \mathfrak{p}_n$)

Then $m \leq \operatorname{depth}_M(\mathfrak{p}_{n-1}) \leq n-1$ by induction

Case 2: $\exists i$ with $x_i \in \mathfrak{p}_n \setminus \mathfrak{p}_{n-1}$

By permutation wlog $i = 1$.

Since $\mathfrak{p}_n \supsetneq \cdots \supsetneq \mathfrak{p}_0$ is a maximal chain, the prime $\mathfrak{p}_n$ is minimal $\supseteq \mathfrak{p}_{n-1} + (x_1)$

$\Rightarrow$ $\mathfrak{p}_n$ is the unique prime $\supseteq \mathfrak{p}_{n-1} + (x_1)$

$\Rightarrow \bar{\mathfrak{p}}_n := \mathfrak{p}_n \cdot \bar{R} \triangleleft \bar{R} := R/_{\mathfrak{p}_{n-1}+(x_1)}$ nilpotent

Replace $x_2, \dots, x_m$ by powers

$\Rightarrow$ wlog $x_2, \dots, x_m \in \mathfrak{p}_{n-1} + (x_1)$

Subtract from $x_2, \dots, x_m$ suitable R-multiples of x_1 (which still keeps $x_1, \dots, x_m$ an M-sequence)

$\Rightarrow$ wlog $x_2, \dots, x_m \in \mathfrak{p}_{n-1}$

Then $m-1 \leq \operatorname{depth}_M(\mathfrak{p}_{n-1}) \leq n-1$ by induction, so we're done. $\square$

In particular, for $R = M$ we get:

Cor For any $I \triangleleft R$ we have

$$\operatorname{depth}_R(I) \leq \operatorname{codim}_R(I)$$

Def A Noetherian ring R is Cohen-Macaulay if $\operatorname{depth}_R(m) = \operatorname{codim}_R(m) \quad \forall\, m \in \operatorname{Spm} R$.

Ex a) $\dim R = 0 \Rightarrow R$ is CM (trivially)

b) R regular local ring of $\dim R = n$

$\Rightarrow \exists\, x_1, \dots, x_n \in R$ w/ $m_R = (x_1, \dots, x_n)$ & these form an R-sequence

$\Rightarrow \operatorname{depth}_R(m_R) \geq n = \dim R \underset{\uparrow}{=} \operatorname{codim}_R(m_R)$

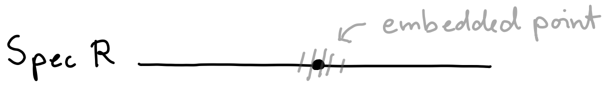
m_R unique max Ideal

$\Rightarrow$ equality by the corollary, so R is CM

c) $R = k[x,y]/_{(xy, y^2)}$ is NOT CM:

For $m := (\bar{x}, \bar{y}) \trianglelefteq R$ we have

- $\operatorname{depth}_R(m) = 0$ since $m \subset \{\text{zero divisors of } R\}$
- $\operatorname{codim}_R(m) = 1$ by taking $\mathfrak{p}_1 = m \supsetneq \mathfrak{p}_2 = (\bar{y})$.



Before giving more examples, it will be useful to understand the notion of depth more conceptually.

We start with the notion of zerodivisors:

<u>Key observation</u> For $0 \neq L, M \in \mathrm{Mod}(R)$ fin. gen.

TFAE:

a) $\mathrm{Hom}_R(L, M) = 0$.

b) $\exists\, x \in \mathrm{Ann}(L)$ which is NOT a zero divisor on M.

Pf. b) $\Rightarrow$ a): Take $0 \to M \xrightarrow{x\cdot} M \to M/_{xM} \to 0$

injective since x is not a zero divisor on M!

Apply $\mathrm{Hom}_R(L, -)$:

$$0 \to \mathrm{Hom}_R(L,M) \xrightarrow{x\cdot} \mathrm{Hom}_R(L,M) \longrightarrow \cdots$$

Zero morphism, since $x \in \mathrm{Ann}(L)$!

$\Rightarrow \mathrm{Hom}_R(L,M) = 0$

$\neg$b) $\Rightarrow$ $\neg$a):

By assumption

$$\mathrm{Ann}\, L \subset \{\text{zero divisors of } M\} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$$

Prime avoidance $\Rightarrow \exists\, \mathfrak{p} \in \mathrm{Ass}(M)$: $\mathrm{Ann}\, L \subset \mathfrak{p}$

L fin. gen $\Rightarrow \mathrm{Hom}_R(L,M)_{\mathfrak{p}} \simeq \mathrm{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, M_{\mathfrak{p}})$,

so enough to show RHS is $\neq 0$.

So wlog R local and $K := R/_{\mathfrak{p}} \hookrightarrow M$.

$L \neq 0 \Rightarrow L/_{\mathfrak{p}L} \neq 0$ (Nakayama)

$\Rightarrow \exists$ nonzero hom. $L/_{\mathfrak{p}L} \to R/_{\mathfrak{p}} \hookrightarrow M$

$\Rightarrow \mathrm{Hom}_R(L,M) \neq 0$ as required. $\square$

It follows that all maximal M-sequences in a given ideal I have the same length:

Thm Let $L, M \in \mathrm{Mod}(R)$ fin generated

a) For any M-sequence $x_1,\dots,x_n \in \mathrm{Ann}(L)$,

$$\mathrm{Ext}^i_R(L,M) \simeq \begin{cases} 0 & \text{for } i=0,1,\dots,n-1, \\ \mathrm{Hom}_R(L, M/(x_1,\dots,x_n)M) & \text{for } i=n. \end{cases}$$

b) We have

$$\mathrm{Ann}(L)\cdot M \neq M \iff \mathrm{Ann}(L)+\mathrm{Ann}(M) \neq R$$
$$\iff \mathrm{Ext}^{\bullet}_R(L,M) \neq 0$$

If these conditions hold, then

$$\mathrm{depth}_M(\mathrm{Ann}\,L) = \min\{ r \mid \mathrm{Ext}^r_R(L,M) \neq 0 \} \quad (<\infty)$$

and any M-sequence $x_1,\dots,x_s \in \mathrm{Ann}(L)$ can be extended to an M-sequence $x_1,\dots,x_n \in \mathrm{Ann}(L)$ of length $n = \mathrm{depth}_M(\mathrm{Ann}\,L)$.

Pf. a) Use induction on n.

n = 1: $x_1 \in \mathrm{Ann}(L)$ nonzerodivisor on M

$$\begin{tikzcd}
\Rightarrow \mathrm{Hom}_R(L,M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(L, M/x_1M) \arrow[d, "\text{hence iso!}"] & \\
0 & \mathrm{Ext}^1_R(L,M) \arrow[r, "x_1"] & \mathrm{Ext}^1_R(L,M)
\end{tikzcd}$$

by the key observation

zero since $x_1 \in \mathrm{Ann}(L)$

Induction step $n-1 \to n$: Let $\bar{M} := M/x_1M$.

Then $0 \to M \xrightarrow{x_1\cdot} M \to \bar{M} \to 0$ gives

$$\begin{tikzcd}
\mathrm{Ext}^{i-1}(L,M) \arrow[r, "0"] & \mathrm{Ext}^{i-1}(L,M) \arrow[r] & \mathrm{Ext}^{i-1}(L,\bar{M}) \arrow[dll, "\text{hence iso!}"] \\
\mathrm{Ext}^{i}(L,M) \arrow[r, "0"] & \cdots &
\end{tikzcd}$$

where by induction

$$\mathrm{Ext}^{i-1}_R(L,\bar{M}) = \begin{cases} 0, & i<n \\ \mathrm{Hom}_R(L, \underbrace{\bar{M}/(x_2,\dots,x_n)\bar{M}}_{M/(x_1,\dots,x_n)M}), & i=n \end{cases}$$

b) $\operatorname{Ann}(L) + \operatorname{Ann}(M) = R$

$$\Longrightarrow \operatorname{Ann}(L)\cdot M = (\operatorname{Ann}(L)+\operatorname{Ann}(M))\cdot M = M$$

Conversely:

$\operatorname{Ann}(L)\cdot M = M$

$\Longrightarrow \exists\, r \in \operatorname{Ann}(L) : (1-r)\cdot M = 0$ (Nakayama)

$\Longrightarrow s := 1-r \in \operatorname{Ann}(M)$ has $r+s = 1$

$\Longrightarrow \operatorname{Ann}(L) + \operatorname{Ann}(M) = R$

If these conditions hold, ie $\operatorname{Ann}(L) + \operatorname{Ann}(M) = R$, then $\operatorname{Ext}^{\bullet}_R(L,M) = 0$:

Indeed $\operatorname{Ext}^{\bullet}_R(L,M)$ is killed both by $\operatorname{Ann}(L)$ & $\operatorname{Ann}(M)$ since $\operatorname{Ext}^{\bullet}_R(-,-)$ is R-linear in both variables.

On the other hand, if $\operatorname{Ann}(L) + \operatorname{Ann}(M) \neq R$, then $\operatorname{Ann}(L)\cdot M \neq M$ by the above. Hence this case we have $\operatorname{depth}_M(\operatorname{Ann}(L)) < \infty$.

$\Longrightarrow$ any M-sequence $x_1, \dots, x_s \in \operatorname{Ann}(L)$ can be extended to $x_1, \dots, x_n \in \operatorname{Ann}(L)$ for some $n \geq s$ sth:

- $x_1, \dots, x_n$ is an M-sequence
- $x_1, \dots, x_{n+1}$ is NOT an M-sequence for any $x_{n+1} \in \operatorname{Ann}(L)$.

Since $\operatorname{Ann}(L)M \neq M$, condition (i) in the definition of an M-sequence is automatic for $x_1, \dots, x_{n+1} \in \operatorname{Ann}(L)$, so we get:

Every $x_{n+1} \in \operatorname{Ann}(L)$ is a zerodivisor on $\bar{M} := M/_{(x_1,\dots,x_n)M}$.

Our key observation then implies $\operatorname{Hom}_R(L, \bar{M}) \neq 0$

Part a) gives

$$\operatorname{Ext}^i_R(L,M) = \begin{cases} 0 & \text{for } i = 0, 1, \dots, n \\ \operatorname{Hom}_R(L, \bar{M}) \neq 0 & \text{for } i = n. \end{cases}$$

$\Longrightarrow n = \min\{ r \mid \operatorname{Ext}^r_R(L,M) \neq 0 \}$ independent of chosen M-sequence in $\operatorname{Ann}(L)$, hence also $= \operatorname{depth}_M(\operatorname{Ann} L)$. □

Cor Let $M \in \mathrm{Mod}(R)$ fin gen & $I \trianglelefteq R$ w/ $IM \neq M$.

a) $\forall \mathfrak{p} \in \mathrm{Spec}\, R$ w/ $I \subset \mathfrak{p}$,

$$\mathrm{depth}_M(I) \leq \mathrm{depth}_{M_\mathfrak{p}}(I_\mathfrak{p}).$$

Equality holds for at least one $\mathfrak{p} \in \mathrm{Spm}\, R$.

b) In particular $\mathrm{depth}_R(\mathfrak{m}) = \mathrm{depth}_{R_\mathfrak{m}}(\mathfrak{m}R_\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Spm}(R)$.

Pf. By the thm,

$$\mathrm{depth}_M(I) = \min\{r \mid \mathrm{Ext}^r_R(R/I, R) \neq 0\}$$

$$\mathrm{depth}_{M_\mathfrak{p}}(I_\mathfrak{p}) = \min\{r \mid \underset{\underset{\textstyle (\mathrm{Ext}^r_R(R/I, R))_\mathfrak{p}}{\|}}{\mathrm{Ext}^r_{R_\mathfrak{p}}(R_\mathfrak{p}/I_\mathfrak{p}, R_\mathfrak{p})} \neq 0\}$$

$\square$

Cor If R is CM, then

$$\mathrm{depth}_R(I) = \mathrm{codim}_R(I) \text{ for all } I \trianglelefteq R.$$

Pf. We must show $\mathrm{depth}_R(I) \geq \mathrm{codim}_R(I)$.

By the corollary, localize at a maximal $\mathfrak{m} \supset I$ without changing depth of I and of $\mathfrak{m}$

$\Longrightarrow$ wlog R local and $I \subset \mathfrak{m}$.

Recall $\mathrm{depth}_R(I) = \mathrm{depth}_R(\sqrt{I})$,

$\mathrm{codim}_R(I) = \mathrm{codim}_R(\sqrt{I})$.

$\Longrightarrow$ If $\sqrt{I} = \mathfrak{m}$ we're done since

$\mathrm{depth}_R(\mathfrak{m}) = \mathrm{codim}_R(\mathfrak{m})$ (as R is CM)

So assume $\sqrt{I} \neq \mathfrak{m}$.

$\Longrightarrow$ $\mathfrak{m}$ is not a minimal prime $\supset I$

(else $\mathfrak{m}$ would be the unique such prime and hence equal to the radical $\sqrt{I}$)

$\Longrightarrow$ By prime avoidance,

$\exists\, x \in \mathfrak{m}$ sth x does not lie in any minimal $\mathfrak{p} \supset I$

$$\Longrightarrow \quad \begin{array}{l} \mathrm{codim}_R(I + (x)) \geq \mathrm{codim}(I) + 1 \\ \qquad \wedge| \qquad \text{(by Noetherian induction)} \\ \mathrm{depth}_R(I + (x)) \\ \qquad \wedge| \leftarrow \\ \mathrm{depth}_R(I) + 1 \end{array}$$

$x \notin$ minimal prime above I

$\Rightarrow$ x nonzerodivisor on R/I

$\Rightarrow$ $x_1, .., x_n$ max. R-seq in I

iff $x_1, .., x_n, x$ $-\|-$ in $I + (x)$

and all max R-seq. have length $= \mathrm{depth}_R(-)$ by thm.

$\square$

Cor For any Noetherian ring R, TFAE:

a) R is CM.

b) $\forall \mathfrak{p} \in \operatorname{Spec} R$: $R_{\mathfrak{p}}$ is CM.

c) $\forall \mathfrak{m} \in \operatorname{Spm} R$: $R_{\mathfrak{m}}$ is CM.

Pf. a) $\Rightarrow$ b):

R CM, $\mathfrak{p} \in \operatorname{Spec} R$

$$\begin{aligned} \Rightarrow \operatorname{codim}_{R_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}) &= \operatorname{codim}_R(\mathfrak{p}) && \text{by def.} \\ &= \operatorname{depth}_R(\mathfrak{p}) && \text{since } R \text{ is CM } (I = \mathfrak{p} \text{ in previous cor.}) \\ &\leq \operatorname{depth}_{R_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}) && \text{by earlier cor.} \\ &\leq \operatorname{codim}_{R_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}) && \text{always} \end{aligned}$$

$\Rightarrow$ equality, hence $R_{\mathfrak{p}}$ is also CM.

b) $\Rightarrow$ c): trivial.

c) $\Rightarrow$ a): By earlier corollary, any $\mathfrak{m} \in \operatorname{Spm} R$ has $\operatorname{depth}_{R_{\mathfrak{m}}}(\mathfrak{m}R_{\mathfrak{m}}) = \operatorname{depth}_R(\mathfrak{m})$

$\Rightarrow$ claim since also $\operatorname{codim}_{R_{\mathfrak{m}}}(\mathfrak{m}R_{\mathfrak{m}}) = \operatorname{codim}_R(\mathfrak{m})$. $\square$

Def A scheme X is Cohen-Macaulay if $\forall p \in X$ $\exists$ affine open $p \in U \subset X$ sth $\mathcal{O}_X(U)$ is Noetherian & CM

($\Leftrightarrow$ X locally Noetherian & $\mathcal{O}_{X,p}$ CM $\forall p \in X$)

Ex a) Any 0-dim loc. Noeth scheme is CM.

b) Any regular scheme X is CM

c) Complete intersections:

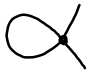
$Y = \operatorname{Spec} R$ regular scheme (more generally CM)

$X = V(f_1, \dots, f_n) \hookrightarrow Y$ w/ $f_1, \dots, f_n$ an R-sequence

$\Rightarrow$ X is CM

eg any hypersurface $X = V(f) \subset \mathbb{A}^n_k$ is CM, no matter how singular it is.

eg $\operatorname{Spec} k[x,y]/(y^2 - x^2(x+1))$



$\operatorname{Spec} k[x,y]/(y^2)$

Pf of c) $I = (f_1, \dots, f_n)$ generated by an R-sequence

Pick a max. ideal $\mathfrak{m} \supseteq I$.

Extend $f_1, \dots, f_n$ to a max. R-sequence $f_1, \dots, f_{n+\ell} \in \mathfrak{m}$.

$\Rightarrow$ $f_{n+1}, \dots, f_{n+\ell}$ is a max. R/I-sequence in $\mathfrak{m}$

$$\begin{aligned}
\Rightarrow \ell &= \operatorname{depth}_{R/I}(\mathfrak{m}/I) && \text{by the thm b)} \\
&\leq \operatorname{codim}_{R/I}(\mathfrak{m}/I) && \text{since depth} \leq \text{codim} \\
&\leq \operatorname{codim}_R(\mathfrak{m}) - \operatorname{codim}_R(I) && \text{by def of codim} \\
&= \operatorname{depth}_R(\mathfrak{m}) - \operatorname{depth}_R(I) && \text{since } R \text{ is CM} \\
&= (n+\ell) - n = \ell && \text{by the thm b)}
\end{aligned}$$

$\Rightarrow \operatorname{depth}_{R/I}(\mathfrak{m}/I) = \operatorname{codim}_{R/I}(\mathfrak{m}/I)$

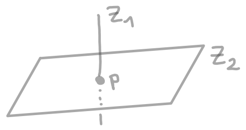
$\Rightarrow R/\mathfrak{m}$ is CM. $\square$

<u>Upshot</u> Any local complete intersection is CM.

(:= scheme X which is locally embeddable as a complete intersection in a regular scheme)

Non-examples:

$\operatorname{Spec} k[x,y]/(xy, y^2)$



$\operatorname{Spec} k[x,y,z]/(xz, yz)$

<u>Lemma</u> Let X be a CM scheme. Then

a) X has no embedded pts

b) X is locally equidimensional:

$\forall p \in X$ $\forall$ irred cpt^s $Z_1, Z_2 \subset X$ w/ $p \in Z_1 \cap Z_2$ we have
$$\dim_p Z_1 = \dim_p Z_2.$$

Pf. Wlog $X = \operatorname{Spec} R$. Have

$$\begin{aligned}
\operatorname{Min}(R) &:= \{\text{minimal primes } \mathfrak{p} \trianglelefteq R\} \\
&\cap\!| \\
\operatorname{Ass}(R) &:= \{\text{associated primes } \mathfrak{p} \trianglelefteq R\}
\end{aligned}$$

and

- irred cpt^s of X are given by primes $\mathfrak{p} \in \operatorname{Min}(R)$
- embedded pt^s $\longleftrightarrow$ $\mathfrak{p} \in \operatorname{Ass}(R) \setminus \operatorname{Min}(R)$

So we must show that if R is CM, then

a) $\operatorname{Ass}(R) = \operatorname{Min}(R)$

b) $\forall \mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Min}(R)$, $\mathfrak{p} \in \operatorname{Spec}(R)$ w/ $\mathfrak{q}_1 + \mathfrak{q}_2 \subseteq \mathfrak{p}$

we have $\dim R_\mathfrak{p}/\mathfrak{q}_1 R_\mathfrak{p} = \dim R_\mathfrak{p}/\mathfrak{q}_2 R_\mathfrak{p}$.

Since $\operatorname{Min}(R) \subseteq \operatorname{Ass}(R)$, it suffices to show:

b′) $\forall \mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Ass}(R)$, $\mathfrak{p} \in \operatorname{Spec}(R)$ w/ $\mathfrak{q}_1 + \mathfrak{q}_2 \subseteq \mathfrak{p}$

we have $\dim R_\mathfrak{p}/\mathfrak{q}_1 R_\mathfrak{p} = \dim R_\mathfrak{p}/\mathfrak{q}_2 R_\mathfrak{p}$.

(this implies a) by taking for any $\mathfrak{q}_1 \in \operatorname{Ass}(R)$
a minimal prime $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$ & taking $\mathfrak{p} := \mathfrak{q}_1$:
From b′) we get $\mathfrak{q}_1 = \mathfrak{q}_2$, hence $\mathfrak{p}$ is minimal)

To show c), let $\mathfrak{q} \in \operatorname{Ass}(R)$, $\mathfrak{q} \subseteq \mathfrak{p}$ (for us, $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2$).

Know

see lemma about depth $< \infty$

$$\underset{\substack{\| \\ \dim R_\mathfrak{p} \text{ as } R \text{ is CM}}}{\operatorname{depth}_R(\mathfrak{p})} \leq \max\{n \mid \exists \text{primes } \mathfrak{p} = \mathfrak{p}_n \supsetneq \cdots \supsetneq \mathfrak{p}_0 = \mathfrak{q}\}$$

$$= \dim R_\mathfrak{p}/_{\mathfrak{q} R_\mathfrak{p}} \leq \dim R_\mathfrak{p}$$

$\Rightarrow \dim R_\mathfrak{p}/_{\mathfrak{q} R_\mathfrak{p}} = \dim R_\mathfrak{p}$ independent of $\mathfrak{q} \in \operatorname{Ass}(R)$ w/ $\mathfrak{q} \subseteq \mathfrak{p}$

$\square$

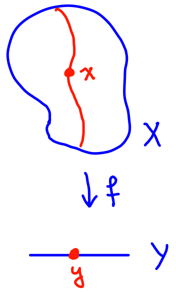
The CM property is also related to flatness.

Recall: $f: X \to Y$ morphism of schemes of finite type

$\Rightarrow \forall x \in X$, $y = f(x)$:

a) $\dim_x(X) \leq \dim_y(Y) + \dim_x(f^{-1}(y))$.

b) Equality holds if f is flat at x.



Pf. Locally $X = \operatorname{Spec} S$, $Y = \operatorname{Spec} R$, $x = \mathfrak{q}$, $y = \mathfrak{p}$.

a) Let $d = \dim R_\mathfrak{p} \Rightarrow \exists x_1, \dots, x_d \in R_\mathfrak{p}$

w/ $\dim R_\mathfrak{p}/_{(x_1, \dots, x_d)} = 0$

ie. $\sqrt{(x_1, \dots, x_d)} = \mathfrak{p} R_\mathfrak{p}$

Let $e = \dim S_\mathfrak{q}/\mathfrak{p} S_\mathfrak{q} \Rightarrow \exists y_1, \dots, y_e \in S_\mathfrak{q}$

w/ $\sqrt{(y_1, \dots, y_e) + \mathfrak{p} S_\mathfrak{q}} = \mathfrak{q} S_\mathfrak{q}$

Then $\sqrt{(x_1, \dots, x_d, y_1, \dots, y_e)} = \mathfrak{q} S_\mathfrak{q}$

$\Rightarrow \dim S_\mathfrak{q} \leq d + e = \dim R_\mathfrak{p} + \dim S_\mathfrak{q}/\mathfrak{p} S_\mathfrak{q}$

b) Flat ring homom. satisfy "going down":

Any prime chain $\mathfrak{p} = \mathfrak{p}_n \supsetneq \mathfrak{p}_{n-1} \supsetneq \cdots$ in R comes from S. $\square$

CM gives a converse to b):

<u>Thm ("miracle flatness")</u> Let $\varphi : R \to S$ be a local homom. between local Noetherian rings

sth 1) R is regular
2) S is CM
3) $\dim S = \dim R + \dim S/m_R S$.

Then φ is flat.

Pf. Induction on $\dim R$.

- $\dim R = 0$: trivial, then R is a field by 1).
- $\dim R > 0$: Then $\dim S > 0$ by 3)

Let $q_1, .., q_r$ be the minimal prime ideals in S.

$$\Rightarrow \dim S/q_i \underset{\substack{\uparrow \\ S\text{ CM}}}{=} \dim S \underset{\substack{\uparrow \\ 3)}}{>} \dim S/m_R S$$

$\Rightarrow p_i := q_i \cap R \neq m_R$

$\Rightarrow \exists\, x \in m_R \setminus (m_R^2 \cup p_i)$ (variant of prime avoidance)

Now $x \notin \bigcup_i q_i$, hence x is not a zero divisor on S (again since S is CM, so it has no embedded primes).

$\Rightarrow$ $S/_{xS}$ still CM & $\dim S/_{xS} = \dim S - 1$,

$R/_{xR}$ still regular & $\dim R/_{xR} = \dim R - 1$.

By induction then $R/_{xR} \to S/_{xS}$ is flat.

$$\Rightarrow \operatorname{Tor}_1^{R/xR}(R/m_R, S/xS) = 0$$

$$\begin{aligned} &\| \\ &\operatorname{Tor}_1^R(R/m_R, S) \quad \text{since } x \text{ is } \begin{array}{l} R\text{-regular} \\ \& \ S\text{-regular} \end{array} \quad (*) \end{aligned}$$

$\Rightarrow$ S flat over R by "local flatness criterion":

$M \in \mathrm{Mod}(R)$ flat $\Leftrightarrow \operatorname{Tor}_1^R(R/m_R, M) = 0$ □

For (*) we used:

<u>Rem</u> R any ring, $M, N \in \mathrm{Mod}(R)$, $R \ni x$ nonzerodivisor on R and on N, with $xM = 0$

$\Rightarrow$ For all $n \geq 0$

$$\operatorname{Tor}_n^R(M, N) \simeq \operatorname{Tor}_n^{R/x}(M, N/xN).$$

Pf. x is R-regular

$\Rightarrow$ $0 \to R \xrightarrow{x} R \to R/x \to 0$ free resolutⁿ

$\Rightarrow$ $\mathrm{Tor}^R_n(R/x, N) = 0 \quad \forall n > 0$

(for $n = 1$ using that x is N-regular)

$\Rightarrow$ For any free resolution $L_\bullet \to N$,

$L_\bullet \otimes_R R/x \longrightarrow N/x \to 0$ is still exact (**)

(since its homology computes $\mathrm{Tor}^R_n(R/x, N)$, $n > 0$)

$$\begin{aligned} \Rightarrow \mathrm{Tor}^R_n(M,N) &= H_n(M \otimes_R L_\bullet) \\ &= H_n(M \otimes_{R/x} R/x \otimes_R L_\bullet) \\ &= H_n(M \otimes_{R/x} (L_\bullet \otimes_R R/x)) \\ &= \mathrm{Tor}^{R/x}_n(M, N/xN) \quad \text{by } (**) \end{aligned} \qquad \square$$

<u>Cor</u> $f : X \to Y$ morph. of finite type between locally Noetherian schemes, $x \in X$, $y := f(x)$.

Assume X is CM at x & Y is regular at y.

Then f is flat at x iff

$$\underset{\substack{\| \\ \dim_x X}}{\dim(\mathcal{O}_{X,x})} = \underset{\substack{\| \\ \dim_y Y}}{\dim(\mathcal{O}_{Y,y})} + \underset{\substack{\| \\ \dim_x f^{-1}(y)}}{\dim(\mathcal{O}_{X,x} \otimes \kappa(y))}.$$

<u>Ex</u> Any finite dominant morphism between smooth irreducible varieties over a field is flat.

eg. X smooth scheme / $\mathbb{F}_p$

$\Rightarrow$ Frob : $X \to X$ is flat, etc.

<u>Note</u> In the miracle flatness corollary, Y needs to be regular, not just CM:

e.g. normalization of cuspidal cubic

$X = \mathbb{P}^1_k \longrightarrow Y = \{y^2 z = x^3\} \subset \mathbb{P}^2_k$

has X, Y CM of pure dim $= 1$,

all fibers of pure dim $= 0$,

yet is NOT flat

(since $k[t^2, t^3] \hookrightarrow k[t]$ not integral).

Appendix: Serre's conditions

Q: Relation between regular / normal / reduced / CM / ... ?

Def Let $k \geqslant 0$. A locally Noetherian scheme X satisfies

Serre's condition

$= ht_{\mathcal{O}_{X,p}}(m_p) = codim_X(\overline{\{p\}})$

(R_k) if $\forall p \in X$ w/ $\dim(\mathcal{O}_{X,p}) \leq k$,

the local ring $\mathcal{O}_{X,p}$ is regular

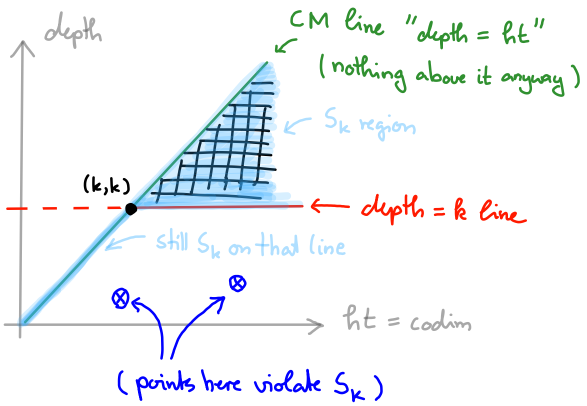
... ie if X is "regular in codim $\leq k$".

(S_k) if $\forall p \in X$,

$\operatorname{depth}(\mathcal{O}_{X,p}) \geqslant \min\{k, \dim(\mathcal{O}_{X,p})\}$

... ie if X is "CM in codim $\leq k$,

and has depth $\geqslant k$ in codim $> k$"



Ex Let X be a loc. Noetherian scheme. Then:

- X regular $\iff$ (R_k) for all $k \geqslant 0$.
- X CM $\iff$ (S_k) for all $k \geqslant 0$.

Prop X reduced $\iff$ (R_0) and (S_1).

Pf. Wlog $X = \operatorname{Spec} R$ w/ R Noetherian. Then:

$(S_1) \iff$ X has no embedded points, i.e.

$\underset{\hookrightarrow \operatorname{depth} = 0}{\text{every }\underline{\text{associated prime}}}$ in R is $\underset{\hookrightarrow ht = 0}{\underline{\text{minimal}}}$

Hence

(R_0) and (S_1) $\iff$ $\forall \wp \in \operatorname{Ass}(R)$,

$R_\wp$ is a field (= 0-dim regular local ring)

$\iff$ R is reduced

$\Longrightarrow$: Any Noetherian ring R has $R \overset{\varphi}{\hookrightarrow} \prod_{\wp \in \operatorname{Ass}(R)} R_\wp$ injective:

If $0 \neq a \in \ker(\varphi)$, take any $\wp \in \operatorname{Ass}(M)$ for $M := a \cdot R$,

then $\frac{a}{1} \neq 0$ in $R_\wp$ but also $\wp \in \operatorname{Ass}(R)$ since $M \subset R$ ↯

$\Longleftarrow$: Let $\wp = \operatorname{Ann}(r) \in \operatorname{Ass}(R)$ w/ $r \in R$. If R is reduced, then $r^2 \neq 0$

so $r \notin \wp$. So $\wp \cdot r = 0$ gives $\wp R_\wp = 0$, ie $R_\wp$ is a field. $\square$

Thm (Serre criterion) X normal $\Leftrightarrow$ (R_1) and (S_2)

Pf. Wlog $X = \operatorname{Spec} R$ w/ R Noetherian.

"$\Rightarrow$": Assume R normal.

For (S_2), let $\mathfrak{p} \in \operatorname{Spec} R$ w/ $\operatorname{depth}_R \mathfrak{p} = 1$. Goal: $\operatorname{ht}_R \mathfrak{p} = 1$.

$\operatorname{depth}_R \mathfrak{p} = 1 \Rightarrow \exists f \in R$ nonzerodivisor w/ $\mathfrak{p} \in \operatorname{Ass}_R(R/fR)$.

Localize $\Rightarrow$ wlog R local w/ max ideal $\mathfrak{p}$.

$\mathfrak{p} \in \operatorname{Ass}_R(R/fR) \Rightarrow \exists g \in R$ w/ $g \notin fR$:

$$\mathfrak{p} = \{x \in R \mid xg \equiv 0 \mod fR\}$$

"total ring of fractions"

Put $y := \frac{g}{f} \in K(R) := S^{-1}R$

($S :=$ nonzerodivisors $\subset R$)

If $y\mathfrak{p} \subseteq \mathfrak{p}$, then $\mathfrak{p}$ is a faithful f.gen module over $R[y]$,

hence y integral over R & thus $y \in R$ by normality,

but then $g \in fR$ ↯

$y\mathfrak{p} \trianglelefteq R$ by construction

So $y\mathfrak{p} = R$, ie $\mathfrak{p} = \frac{f}{g}R$ (hence $\frac{f}{g} \in R$), so $\operatorname{ht}_R \mathfrak{p} = 1$ by Krull.

For (R_1), let $\mathfrak{p} \in \operatorname{Spec} R$ w/ $\operatorname{ht}_R \mathfrak{p} = 1$. Goal: $R_{\mathfrak{p}}$ regular.

Localize $\Rightarrow$ wlog R local w/ max ideal $\mathfrak{p}$

Same argument as above shows $\mathfrak{p}$ is a principal ideal

$\Rightarrow$ R 1-dim local domain whose max. ideal is p'pal,

ie R is a regular local ring (of dim 1)

"$\Leftarrow$": Assume R satisfies (R_1) and (S_2).

In particular (R_0) and (S_1), so R is reduced.

Let $\mathfrak{p}_1, .., \mathfrak{p}_r$ be the min. primes in R.

Consider the total ring of fractions

$$K := S^{-1}R \quad \text{w/} \quad S := R \setminus \bigcup_{i=1}^{r} \mathfrak{p}_i.$$

$\Rightarrow$ $\mathfrak{p}_1 K \cap \dots \cap \mathfrak{p}_r K = 0$ (since R is reduced)

$\mathfrak{p}_i K + \mathfrak{p}_j K = K \quad \forall i \neq j$ (since $\mathfrak{p}_1 K, \dots, \mathfrak{p}_r K$ are all max. ideals of K)

$\Rightarrow$ $K \simeq \kappa(\mathfrak{p}_1) \times \dots \times \kappa(\mathfrak{p}_r)$ by CRT

w/ $\kappa(\mathfrak{p}_i) := K/\mathfrak{p}_i K \simeq \operatorname{Quot}(R/\mathfrak{p}_i) \simeq R_{\mathfrak{p}_i}$.

$\Rightarrow \exists\ e_i \in K(\mathfrak{p}_i) \subseteq K$ idempotent w/ $e_i e_j = 0\ \forall i \neq j$,

$e_1 + \cdots + e_r = 1$

Each $e_i \in K$ is integral over $R \subseteq K$ (being idempotent).

If we can show R is integrally closed in K, then $e_i \in R$ & hence $R \simeq \prod_i Re_i$ is a product of integrally closed domains, ie. normal as required.

Remains to show: R integrally closed in K.

For this let $f, g, a_i \in R$, g nonzerodivisor, sth.

$$\left(\frac{f}{g}\right)^n + a_1\left(\frac{f}{g}\right)^{n-1} + \cdots + a_n = 0.$$

We want to show that $f \in gR$.

- Each $\mathfrak{p} \in \mathrm{Ass}_R(R/gR)$ has $\mathrm{depth}_R(\mathfrak{p}) = 1$, hence by (S_2) it has $\mathrm{ht}_R(\mathfrak{p}) = 1$.
- By (R_1) then $R_\mathfrak{p}$ is a 1dim regular local ring $=$ DVR, hence $R_\mathfrak{p}$ is integrally closed
- Thus $\frac{f}{g} \in R_\mathfrak{p}$, i.e. $\frac{f}{1} \in gR_\mathfrak{p}$ $\forall\ \mathfrak{p} \in \mathrm{Ass}_R(R/gR)$.

Now take a primary decomposition

$$(\ast) \qquad gR = \bigcap_i \mathfrak{q}_i \quad \text{w/ } \mathfrak{q}_i \trianglelefteq R \text{ primary}$$

$\Rightarrow$ For any fixed i,

$\mathfrak{p} := \sqrt{\mathfrak{q}_i} \in \mathrm{Ass}_R(R/gR)$

By the above then $\frac{f}{1} \in g \cdot R_\mathfrak{p} \subseteq \mathfrak{q}_i R_\mathfrak{p}$

$\Rightarrow f \in \mathfrak{q}_i$ (use $\mathfrak{q}_i = R \cap (\mathfrak{q}_i R_\mathfrak{p})$ for $\mathfrak{p}$-primary $\mathfrak{q}_i$)

Since this works for all i, we get $f \in gR$ by $(\ast)$. $\square$

For instance, normal surfaces are CM:

<u>Ex</u> X loc. Noetherian scheme of dim ≤ 2

then: X is CM $\Longleftrightarrow$ X is normal.

10. Serre duality

Motivation Poincaré duality:

X cpact connected ~~oriented~~ $\mathbb{R}$-mfd of dim n

$\Rightarrow \exists$ canonical perfect pairing

$$H^i(X, \underset{\mathcal{F}}{\cancel{\mathbb{R}}}) \times H^{n-i}(X, \mathrm{or} \otimes_{\mathbb{R}} \underset{\mathcal{F}^\vee}{\cancel{\mathbb{R}}}) \xrightarrow{\cup} H^n(X, \mathrm{or} \otimes_{\mathbb{R}} \mathbb{R}) \xrightarrow{\sim} \mathbb{R}$$

$$(\alpha, \beta) \longmapsto \int_X \alpha \cup \beta$$

w/ or := "orientation bdle of X"

(fixing one of the two orientations gives an iso $\mathrm{or} \xrightarrow{\sim} \mathbb{R}_X$)

& any locally constant sheaf $\mathcal{F} \in \mathrm{Mod}(\mathbb{R}_X)$ of finite rk

w/ dual $\mathcal{F}^\vee := \mathcal{H}om_{\mathbb{R}_X}(\mathcal{F}, \mathbb{R}_X)$.

Goal: Analog for coherent sheaves:

$X \rightsquigarrow$ smooth proj scheme / k of pure dim = n

$\mathcal{F} \rightsquigarrow$ coherent sheaf on X

w/ dual $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$

$\mathrm{or}_X \rightsquigarrow$ canonical sheaf $\omega_X := \Lambda^n \Omega^1_{X/k}$

We start with the case $X = \mathbb{P}^n_k$ (any field k):

Thm Let $X = \mathbb{P}^n_k$ & $\omega_X := \Lambda^n \Omega^1_{X/k}$.

a) $\exists$ canonical iso $\mathrm{tr}: H^n(X, \omega_X) \xrightarrow{\sim} k$.

b) Fixing any such iso, we get for all $\mathcal{F} \in \mathrm{Coh}(X)$ and all $i \geq 0$ perfect pairings

$$\underbrace{\mathrm{Ext}^i(\mathcal{F}, \omega_X)} \times H^{n-i}(X, \mathcal{F}) \to k$$

($\simeq H^i(X, \mathcal{F}^\vee \otimes \omega)$ if $\mathcal{F}$ is loc. free of finite rank)

Pf. a) Try to stay coordinate-free:

Write $X = \mathbb{P}V = \mathrm{Proj}\, \mathrm{Sym}^\bullet(V^*)$ with $V := k^n$

$\Rightarrow$ Have natural iso $H^0(X, \mathcal{O}_X(1)) \xrightarrow{\sim} V^*$

For bases $x_0, .., x_n \in H^0(X, \mathcal{O}_X(1))$, we denote the induced basis of V^* and the dual basis of V by

$$dx_0, \dots, dx_n \in V^* \quad \& \quad \partial_0, \dots, \partial_n \in V$$

Euler sequence:

Contraction w/ the Euler field $\sum_i x_i \partial_i$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{\mathbb{P}V/k} \arrow[r, "\iota"] & \mathcal{O}_{\mathbb{P}V}(-1)\otimes V^* \arrow[r, "\varepsilon"] & \mathcal{O}_{\mathbb{P}V} \arrow[r] & 0 \\
 & & f\,dx_i \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & f\,x_i \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

Taking top wedge power $\Lambda^{n+1}(\dots)$ we get:

$$\begin{tikzcd}
\exists!\,\alpha \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & \frac{1}{x_0\cdots x_n}\cdot dx_0\wedge\cdots\wedge dx_n \arrow[d, phantom, sloped, "\in"] \\
\omega_{\mathbb{P}V} := \Lambda^n\Omega^1_{\mathbb{P}V/k} \arrow[r, "\sim", "\exists!\text{ iso}"'] & \mathcal{O}_{\mathbb{P}V}(-n-1)\otimes\Lambda^{n+1}(V^*) \\
\Lambda^n\Omega^1_{\mathbb{P}V/k}\otimes\mathcal{O}_{\mathbb{P}V} \arrow[u, equal] & \Lambda^{n+1}(\mathcal{O}_{\mathbb{P}V}(-1)\otimes V^*) \arrow[u, equal] \\
\Lambda^n\Omega^1_{\mathbb{P}V/k}\otimes(\mathcal{O}_{\mathbb{P}V}(-1)\otimes V^*) \arrow[u, "\mathrm{id}\otimes\varepsilon"'] \arrow[ur, "\iota\wedge\mathrm{id}"'] &
\end{tikzcd}$$

(One computes that the above α is the differential form

$\alpha = \frac{x_0^n}{x_1\cdots x_n}\, d\left(\frac{x_1}{x_0}\right)\wedge\cdots\wedge d\left(\frac{x_n}{x_0}\right)$ given in Hartshorne III.7)

For the standard cover $\mathcal{U} = (U_i)_{i=0,\dots,n}$ w/ $U_i = D_+(x_i)$ we have

$$\alpha \in \Gamma(U_0\cap\cdots\cap U_n,\ \omega_X) = \check{C}^n(\mathcal{U}, \omega_X)$$

Our computation of Čech cohom. of $X = \mathbb{P}^n_k$ more precisely shows $\exists!$ iso $\mathrm{tr}: \check{H}^n(X, \omega_X) \xrightarrow{\sim} k$ w/ $\mathrm{tr}(\alpha) = 1$.

Note: For the Čech computation we had to choose an open cover, corresponding to our choice of a basis of $V \simeq H^0(X, \mathcal{O}_X(1))$. But one can show that the obtained iso $H^1(X, \omega_X) \xrightarrow{\sim} k$ does not depend on the chosen basis of V, see [Hartshorne, Residues and Duality, cor. 10.2].

b) For $i = 0$ define

$$\begin{tikzcd}
H^0(X, \mathcal{F}^\vee\otimes\omega_X)\otimes H^n(X,\mathcal{F}) \arrow[r] & k \\
\mathrm{Hom}(\mathcal{F}, \omega_X)\otimes H^n(X,\mathcal{F}) \arrow[u, equal] \arrow[r] & H^n(X, \omega_X) \arrow[u, "\mathrm{tr}"'] \\
f\otimes\gamma \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & f_*(\gamma) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

We want to show the induced morphism

$$\varphi : \mathrm{Hom}(\mathcal{F}, \omega_X) \longrightarrow H^n(X, \mathcal{F})^\vee$$ is an iso.

If $\mathcal{F} = \mathcal{O}_X(d)$ for some $d \in \mathbb{Z}$, this follows from our Čech computation:

- $H^0(X, \mathcal{O}_X(d)) \simeq (k[x_0, .., x_n])_d = \mathrm{Sym}^d(V^*)$
- $H^n(X, \omega_X(-d)) \simeq \left(\frac{1}{x_0 \cdots x_n} \cdot k[\frac{1}{x_0}, .., \frac{1}{x_n}]\right)_{-n-1-d}$

$$= \mathrm{Sym}^d(V) \otimes \alpha$$

w/ $\alpha \mathrel{\hat{=}} \dfrac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n}$ as above

For arbitrary $\mathcal{F} \in \mathrm{Coh}(X)$, write $\mathcal{F} = \mathrm{cok}(\mathcal{E}_1 \to \mathcal{E}_0)$ for suitable $\mathcal{E}_i = \bigoplus_j \mathcal{O}_X(d_{ij})$ w/ $d_{ij} \in \mathbb{Z}$.

Left exactness of $\mathrm{Hom}(-, \omega_X)$ and $H^n(X, -)^\vee$ gives

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathcal{F}, \omega_X) \arrow[r] \arrow[d, "\varphi"'] & \mathrm{Hom}(\mathcal{E}_0, \omega_X) \arrow[r] \arrow[d, "S"'] & \mathrm{Hom}(\mathcal{E}_1, \omega_X) \arrow[d, "S"'] \\
0 \arrow[r] & H^n(X, \mathcal{F})^\vee \arrow[r] & H^n(X, \mathcal{E}_0)^\vee \arrow[r] & H^n(X, \mathcal{E}_1)^\vee
\end{tikzcd}$$

$\Rightarrow$ φ is an iso

It remains to show that for all $i > 0$, $\exists$ natural iso

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee.$$

Both sides are δ-functors & agree for $i = 0$ (by above), so we only need to show coeffaceability:

$\forall \mathcal{F}$ $\exists$ epi $\mathcal{G} \twoheadrightarrow \mathcal{F}$ from some $\mathcal{G} \in \mathrm{Coh}(X)$ w/

$$\mathrm{Ext}^i(\mathcal{G}, \omega_X) = H^{n-i}(X, \mathcal{G})^\vee = 0.$$

Indeed we can take $\mathcal{G} = \bigoplus_{i \in I} \mathcal{O}_X(-d)$ w/ $d \gg 0$:

- For $d \gg 0$ Serre vanishing gives

  $$\mathrm{Ext}^i(\mathcal{O}_X(-d), \omega_X) = H^i(X, \omega_X(d)) = 0$$

- For $i > 0$ and any $d > 0$,

  $$H^{n-i}(X, \mathcal{O}_X(-d))^\vee = 0. \qquad \square$$

To generalize this to arbitrary proper schemes X over k, we make the following definition:

<u>Def</u> Let X be a proper scheme of dim n over k. A <u>dualizing sheaf</u> on X is a coherent sheaf $\omega_X^\circ \in \mathrm{Coh}(X)$ together with a morphism $\mathrm{tr}: H^n(X, \omega_X^\circ) \to k$ sth $\forall\, \mathcal{F} \in \mathrm{Coh}(X)$ the pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{\mathrm{tr}} k$$

induces an iso

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee$$

<u>Rem</u> If it exists, a dualizing sheaf is unique: For any two such $(\omega_X^\circ, \mathrm{tr})$ and $(\tilde{\omega}_X^\circ, \tilde{\mathrm{tr}})$, $\exists!$ iso $\varphi: \omega_X^\circ \xrightarrow{\sim} \tilde{\omega}_X^\circ$ w/

$$\mathrm{tr} = \tilde{\mathrm{tr}} \circ H^n(\varphi).$$

Pf. By definition $(\omega_X^\circ, \mathrm{tr})$ represents the functor $\mathrm{Coh}(X) \to \mathrm{Vect}(k)$, $\mathcal{F} \longmapsto H^n(X, \mathcal{F})^\vee$. $\square$

For existence we need:

<u>Lemma</u> Let $X \hookrightarrow \mathbb{P} := \mathbb{P}_k^N$ be closed & $r := N - \dim(X)$.

a) $\mathcal{E}xt^i_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \simeq 0 \quad \forall\, i < r$.

b) For any $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ $\exists$ natural iso

$$\mathrm{Ext}^r_{\mathbb{P}}(\mathcal{F}, \omega_{\mathbb{P}}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{E}xt^r_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}))$$

Pf. a) Passing to stalks, this follows from the characterization of depth via $\mathcal{E}xt$ in §9, since $\mathbb{P}$ is CM so that codim = depth.

b) Use the spectral sequence

$$E_2^{pq} := \mathrm{Ext}^p_X(\mathcal{F}, \underbrace{\mathcal{E}xt^q_{\mathbb{P}}(\mathcal{O}_X, \omega_X)}_{= 0 \text{ for } q < r \text{ by a)}}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{P}}(\mathcal{F}, \omega_{\mathbb{P}})$$

Alternatively, argue by hand:

Let $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X) \subset \mathrm{Mod}(\mathcal{O}_{\mathbb{P}})$

$$\Rightarrow \mathrm{Hom}_{\mathbb{P}}(\mathcal{F}, -) = \mathrm{Hom}_X(\mathcal{F}, \mathcal{H}om(\mathcal{O}_X, -))$$

Plug in an injective resolution $0 \to \omega_{\mathbb{P}} \to J^\bullet$

$$\Rightarrow \mathcal{E}xt^i_{\mathbb{P}}(\mathcal{F}, \omega_{\mathbb{P}}) = H^i(\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}^\bullet))$$

$$\text{w/ } \mathcal{J}^\bullet = \mathcal{H}om(\mathcal{O}_X, J^\bullet)$$

By a) the cplex $\mathcal{J}^\bullet$ is exact in all degrees $< r$

Claim: $H^r(\mathcal{J}^\bullet) \simeq \mathcal{H}om(\mathcal{O}_X, H^r(J^\bullet))$

$$(=: \mathcal{E}xt^r_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}))$$

Indeed injectivity of J^{r-1} gives splitting $J^{r-1} \simeq \ker(d^{r-1}) \oplus \mathrm{im}(d^{r-1})$

$\Rightarrow J^\bullet \simeq J_1^\bullet \oplus J_2^\bullet$ w/ the "truncations"

$$J_1^\bullet := [\cdots \to J^{r-2} \to \ker(d^{r-1}) \to 0 \to \cdots]$$

$$J_2^\bullet := [\cdots \to 0 \to \mathrm{im}(d^{r-1}) \to J_2^r \to \cdots]$$

$$\Rightarrow \mathcal{J}^\bullet \simeq \underbrace{\mathcal{H}om(\mathcal{O}_X, J_1^\bullet)}_{\substack{H^r(\cdots) = 0 \\ \text{since } \mathcal{J}_1^r = 0}} \oplus \underbrace{\mathcal{H}om(\mathcal{O}_X, J_2^\bullet)}_{\substack{H^r(\cdots) = \mathcal{H}om(\mathcal{O}_X, H^r(J^\bullet)) \\ \text{by left exactness of } \mathcal{H}om(\mathcal{O}_X, -)}}$$

$\square$

<u>Prop</u> Any projective scheme X over k has a dualizing sheaf (ω_X°, tr).

Pf. Choose embedding $X \hookrightarrow \mathbb{P} := \mathbb{P}^N_k$

& put

$$\omega_X^\circ := \mathcal{E}xt^r_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \quad \text{w/} \quad \begin{matrix} r := N - n \\ n := \dim X. \end{matrix}$$

Then $\forall\, \mathcal{F} \in \mathrm{Coh}(X)$ get iso

$$\varphi : \mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \simeq \mathrm{Ext}^r_{\mathbb{P}}(\mathcal{F}, \omega_{\mathbb{P}}) \quad \text{(by the lemma)}$$

$$\simeq H^n(\mathbb{P}, \mathcal{F})^\vee \quad \text{(by Serre duality on } \mathbb{P})$$

The trace map is obtained by taking $\mathcal{F} := \omega_X^\circ$:

$$\begin{tikzcd} \mathrm{Hom}(\omega_X^\circ, \omega_X^\circ) \arrow[r, "\sim"] & H^n(\mathbb{P}, \omega_X^\circ)^\vee \\ & = \mathrm{Hom}_k(H^n(\mathbb{P}, \omega_X^\circ), k) \\ \mathrm{id} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, bend right] & tr \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

By naturality then $\varphi(f) = tr \circ H^n(X, f)$. $\square$

Thm (Serre duality) Let $X \subseteq \mathbb{P}^N_k$ w/ dim $X = n$ and fix a dualizing sheaf (ω_X°, tr) on X. Then:

a) $\forall \mathcal{F} \in Coh(X)$ $\forall i \geq 0$ $\exists$ natural maps

$$\theta^i : Ext^i(\mathcal{F}, \omega_X^\circ) \longrightarrow H^{n-i}(X, \mathcal{F})^\vee$$

b) the following are equivalent:

1) θ^i is an iso $\forall i$ $\forall \mathcal{F} \in Coh(X)$.
2) $\forall$ loc. free $\mathcal{F} \in Coh(X)$,
   $H^i(X, \mathcal{F}(-q))$ $\forall i < n$ $\forall q \gg 0$.
3) X is CM and equidimensional.

Pf. a) Pick an epi $\mathcal{E} := \bigoplus_{j=1}^m \mathcal{O}_X(-q) \twoheadrightarrow \mathcal{F}$ w/ $q \gg 0$

$\Longrightarrow$ $Ext^i(\mathcal{E}, \omega_X^\circ) \simeq Ext^i(\mathcal{O}_X, \mathcal{E}^\vee \otimes \omega_X^\circ)$ ($\mathcal{E}$ loc free)

$= H^i(X, \bigoplus_{j=1}^m \omega_X^\circ(q))$ (by def of $\mathcal{E}$)

$= 0$ for $q \gg 0$, $i > 0$ (Serre vanishing)

$\Longrightarrow$ $Ext^i(-, \omega_X^\circ)$ coeffacable for all $i > 0$

$\Longrightarrow$ universal δ-fctor & claim follows from the case $i = 0$

b) 1) $\Rightarrow$ 2): For $\mathcal{F} \in Coh(X)$ locally free,

$H^i(X, \mathcal{F}(-q)) \simeq Ext^{n-i}(\mathcal{F}(-q), \omega_X^\circ)^\vee$ by 1)

$\simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X(q))$ as $\mathcal{F}$ is loc. free

$= 0$ $\forall i < n$ $\forall q \gg 0$ by Serre vanishing

2) $\Rightarrow$ 1):

$Ext^i(-, \omega_X^\circ)$ form a universal δ-fctor by proof of a)

$\Longrightarrow$ only need to show the same for $H^{n-i}(X, -)^\vee$
(because of uniqueness of universal δ-functors)

$\Longrightarrow$ enough to show $H^{n-i}(X, -)^\vee$ coeffacable $\forall i > 0$,
but that's clear by our assumption 2)

3) $\Rightarrow$ 2): Recall $X \hookrightarrow \mathbb{P} := \mathbb{P}^N_k$

X CM of pure dim $= n$

$\Rightarrow$ $\forall$ loc free $\mathcal{F} \in Coh(X)$ $\forall$ closed pt $x \in X$:

depth $\mathcal{F}_x = n$

Here $\operatorname{depth} \mathcal{F}_x := \operatorname{depth}_{\mathcal{F}_x}(m_{X,x}) = \operatorname{depth}_{\mathcal{F}_x}(m_{\mathbb{P},x})$

$:=$ max length of an $\mathcal{F}_x$-sequence in $m_{X,x}$

... or, equivalently, in $m_{\mathbb{P},x}$

Now recall the Auslander-Buchsbaum formula:

$\forall$ Noetherian local ring R,

smallest length of a proj. resol. of M

$\forall$ fin gen $0 \neq M \in \operatorname{Mod}(R)$ w/ $\operatorname{pd}_R(M) < \infty$:

$$\operatorname{depth}(M) = \operatorname{depth}(R) - \operatorname{pd}_R(M)$$

(Using derived categories, this follows from

$$R\operatorname{Hom}_R(k, M) \simeq R\operatorname{Hom}_R(k, R) \otimes^{\mathbb{L}}_k (k \otimes^{\mathbb{L}}_R M)$$

by taking the smallest degree where cohomology is $\neq 0$.)

Apply to $R = \mathcal{O}_{\mathbb{P},x}$, $M = \mathcal{F}_x$:

$$\Longrightarrow \operatorname{pd}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{F}_x = \underbrace{\operatorname{depth} \mathcal{O}_{\mathbb{P},x}}_{\substack{= \dim \mathbb{P} = N \\ \text{since } \mathbb{P} \text{ is CM} \\ \text{of pure dim } N}} - \underbrace{\operatorname{depth} \mathcal{F}_x}_{= n \text{ by above}}$$

Upshot: $\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F} = N - n = r \quad \forall$ closed pt's $x \in X$

$$\Longrightarrow \mathcal{E}xt^i_{\mathbb{P}}(\mathcal{F}, -) = 0 \quad \forall i > r$$

Then 2) follows:

$$H^i(X, \mathcal{F}(-q)) \simeq \operatorname{Ext}^{N-i}_{\mathbb{P}}(\mathcal{F}, \omega_{\mathbb{P}}(q))^{\vee} \quad \text{by Serre duality on } \mathbb{P}$$

$$\simeq \Gamma(\mathbb{P}, \mathcal{E}xt^{N-i}_{\mathbb{P}}(\mathcal{F}, \omega_{\mathbb{P}}(q)))^{\vee} \quad \text{always true for } q >> 0$$

$$= 0 \quad \text{since } \mathcal{E}xt^{N-i}_{\mathbb{P}}(\mathcal{F}, -) = 0 \text{ for } i < n$$

2) $\Rightarrow$ 3) : Reading the above backwards for $\mathcal{F} := \mathcal{O}_X$,

from 2) we get $\mathcal{E}xt^{N-i}_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}) = 0 \quad \forall i < n$

$$\Longrightarrow \operatorname{Ext}^{N-i}_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P},x}) = 0 \quad \forall i < n$$

$$\Longrightarrow \operatorname{pd}_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}) \leq N - n$$

$$\Longrightarrow \operatorname{depth}(\mathcal{O}_{X,x}) \geq n \quad \text{by Auslander-Buchsbaum}$$

$$\Longrightarrow \operatorname{depth}(\mathcal{O}_{X,x}) = n \quad \text{since } \dim X = n$$

$$\Longrightarrow X \text{ is CM} \qquad \square$$

Cor (Serre duality for locally free sheaves)

For X projective / k, CM of pure dim n, and any locally free sheaf $\mathcal{F} \in \mathrm{Coh}(X)$, $\exists$ natural iso

$$H^i(X, \mathcal{F}^\vee \otimes \omega_X^\circ) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee \quad \forall i \geq 0.$$

Pf. $\mathcal{F}$ loc free

$$\Rightarrow \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \simeq H^i(X, \mathcal{F}^\vee \otimes \omega_X^\circ)$$ □

↗

NB This reformulation of Serre duality only works for $\mathcal{F}$ locally free: For instance, in the case $\dim \mathrm{Supp}\, \mathcal{F} < n = \dim X$ always $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = 0$ and hence $H^i(X, \mathcal{F}^\vee \otimes \omega_X^\circ) = 0$ for all i, even if $\mathcal{E}xt^i(\mathcal{F}, \omega_X^\circ) \simeq H^{n-i}(X, \mathcal{F}) \neq 0$!

Q: How to compute ω_X° explicitly?

Def For a smooth variety X of pure dim n over k we define its canonical bundle to be the line bundle $\omega_X := \Lambda^n \Omega^1_{X/k} \in \mathrm{Pic}(X)$.

Example ("adjunction formula on $\mathbb{P}^N_k$")

For any effective Cartier divisor $D \subset \mathbb{P} = \mathbb{P}^N_k$ we have:

a) $\omega_D^\circ \simeq \omega_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(D)|_D$.

b) If D is smooth, then $\omega_D^\circ \simeq \omega_D$.

Pf. Let $D = V_+(f)$ w/ $f \in \mathcal{O}_{\mathbb{P}}(d)$.

$\Rightarrow$ locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_D \to 0$$

$$\Rightarrow \omega_D^\circ := \mathcal{E}xt^1_{\mathbb{P}}(\mathcal{O}_D, \omega_{\mathbb{P}}) \simeq \mathrm{cok}(\omega_{\mathbb{P}} \xrightarrow{f} \omega_{\mathbb{P}}(d)) = \omega_{\mathbb{P}}(d)|_D.$$

If D is smooth, then also $\omega_D \simeq \omega_{\mathbb{P}}(d)|_D$ by the conormal sequence $0 \to \underset{\substack{\| \\ \mathcal{O}_{\mathbb{P}}(-d)|_D}}{\mathcal{J}/\mathcal{J}^2} \to \Omega^1_{\mathbb{P}/k}|_D \to \Omega^1_{D/k} \to 0$

⇝ dual gives $\mathcal{O}_{\mathbb{P}}(d)|_D$ □

More generally:

<u>Thm</u> Let $X = V(\mathcal{J}) \subset \mathbb{P} = \mathbb{P}^N_k$ be a local complete intersection of codimension r, then

a) $\omega^\circ_X \simeq \omega_{\mathbb{P}} \otimes \Lambda^r (\mathcal{J}/\mathcal{J}^2)^\vee$

b) If X is smooth, then $\omega^\circ_X \simeq \omega_X$.

Pf. Let $U = \operatorname{Spec} A \subset \mathbb{P}$ be any open affine sth $\exists$ regular sequence $f_1, .., f_r \in A$ w/

$$\mathcal{J}|_U = (f_1, \dots, f_r).$$

Free resolution $K_\bullet \to \mathcal{O}_X|_U$?

eg $r = 2$: $K_\bullet = \left[\mathcal{O}_U \xrightarrow{\binom{f_2}{-f_1}} \mathcal{O}_U^{\oplus 2} \xrightarrow{(f_1 f_2)} \mathcal{O}_U \right]$

For arbitrary r take "Koszul resolution":

$$0 \to K_r \to K_{r-1} \to \cdots \to K_1 \to K_0 \to \mathcal{O}_X|_U$$

w/ $K_0 := \mathcal{O}_U$

$K_1 := \mathcal{O}_U^{\oplus r}$

$K_p := \Lambda^p(K_1)$ for $p \in \{2, 3, \dots, r\}$

$$d: \begin{tikzcd} K_p \arrow[r] & K_{p-1} \\ e_{i_1} \wedge \dots \wedge e_{i_p} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \sum_{\nu=1}^{p} (-1)^\nu e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\nu}} \wedge \cdots \wedge e_{i_p} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Exercise: $K_\bullet$ is a resolution of $\mathcal{O}_X|_U$.

$$\begin{aligned} \Rightarrow \quad \omega^\circ_X|_U &:= \mathcal{E}xt^r_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}})|_U \\ &\simeq H^r(\mathcal{H}om_{\mathbb{P}}(K^\bullet, \omega_U)) \\ &\simeq \omega_U \otimes \underbrace{\operatorname{cok}(\Lambda^{k-1}(\mathcal{O}_U) \to \Lambda^k(\mathcal{O}_U))}_{\simeq \mathcal{O}_U/(f_1, .., f_r)\mathcal{O}_U = \mathcal{O}_X|_U} \end{aligned}$$

Note: The above iso depends on the choice of $f_1, \dots, f_r$. Passing to another regular sequence $\tilde{f}_1, \dots, \tilde{f}_r$ we can write $\tilde{f}_i = \sum_j c_{ij} f_j$ w/ $c_{ij} \in \mathcal{O}_U$. We get a morphism of their Koszul cplexes $K_\bullet \to \tilde{K}_\bullet$ given in degree $\bullet = r$ by $\det(c_{ij}) : \Lambda^r(\mathcal{O}_U^{\oplus r}) \to \Lambda^r(\mathcal{O}_U^{\oplus r})$

Compare w/ conormal sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}/\mathcal{J}^2|_U \arrow[r] & \Omega^1_{\mathbb{P}/k}|_{X\cap U} \arrow[r] & \Omega^1_{X\cap U/k} \arrow[r] & 0 \\
 & \mathcal{O}_U^{\oplus r} \arrow[u, "\wr"', "(f_1,\dots,f_r)"] & & &
\end{tikzcd}$$

(since $f_1,\dots,f_r$ form a regular sequence)

$\Longrightarrow$ get iso $\Lambda^r(\mathcal{O}_U^{\oplus r}) \xrightarrow{\sim} \Lambda^r(\mathcal{J}/\mathcal{J}^2)|_U$

which also transforms via $\det(c_{ij})$ under change of basis

$\Longrightarrow$ iso $\omega_X^\circ|_U \simeq \omega_{\mathbb{P}} \otimes \Lambda^r(\mathcal{J}/\mathcal{J}^2)^\vee|_U$ ← dualize!

independent of chosen regular sequence, hence glues to a global iso

$$\omega_X^\circ \xrightarrow{\sim} \omega_{\mathbb{P}} \otimes \Lambda^r(\mathcal{J}/\mathcal{J}^2)^\vee$$

Conormal sequence also shows

$$\omega_{\mathbb{P}} \simeq \Lambda^r(\mathcal{J}/\mathcal{J}^2) \otimes \underbrace{\Lambda^n(\Omega^1_{X/k})}_{=:\ \omega_X \text{ for } X \text{ smooth}}$$

$\Rightarrow$ $\omega_X \simeq \omega_{\mathbb{P}} \otimes \Lambda^r(\mathcal{J}/\mathcal{J}^2)^\vee \simeq \omega_X^\circ$ $\square$

Rem Same proof shows for any $X = V(\mathcal{J}) \subset \mathbb{P} = \mathbb{P}^N_k$:

If $U \subset \mathbb{P}$ is any open dense subset such that $U\cap X \subset U$ is a local complete intersection, then $\omega_X^\circ|_U \simeq \omega_{\mathbb{P}} \otimes \Lambda^r(\mathcal{J}/\mathcal{J}^2)^\vee|_U \simeq \omega_{X\cap U}$.

Cor Any projective variety X over k satisfies

$$\omega_X^\circ|_{\mathrm{Sm}(X)} \simeq \omega_{\mathrm{Sm}(X)}$$

Pf. If $X \subset \mathbb{P}^N_k$ is smooth at a point, then it is a complete intersection in a nbhood of that point e.g. by the Jacobian criterion. $\square$

Cor ("adjunction formula on smooth varieties")

Let X be a smooth proj. variety / k. Then for any effective Cartier divisor $D \subset X$ we have

$$\omega_D^\circ \simeq \omega_D \simeq \omega_X \otimes \mathcal{O}_X(D)|_D.$$

Pf. $X \subset \mathbb{P}^N_k$ lci $\Rightarrow$ $D \subset \mathbb{P}^N_k$ lci. Now use the thm. $\square$

Example X smooth proj surface $/ k$

$C \subset X$ smooth irreducible curve of genus g

$$\Rightarrow \quad 2g-2 = \deg(\omega_X(C)|_C)$$

e.g. for $X = \mathbb{P}^2_k$ and C of degree d

we recover the well-known formula:

$$2g-2 = \deg(\mathcal{O}_X(d-3)|_C) = d(d-3)$$

$$\Rightarrow \quad g = \frac{(d-1)(d-2)}{2}$$

Example For a smooth projective variety X over k we define the Hodge numbers

$:= \Lambda^p \Omega^1_{X/k}$

$$h^{p,q}(X) := \dim_k H^q(X, \Omega^p_X) .$$

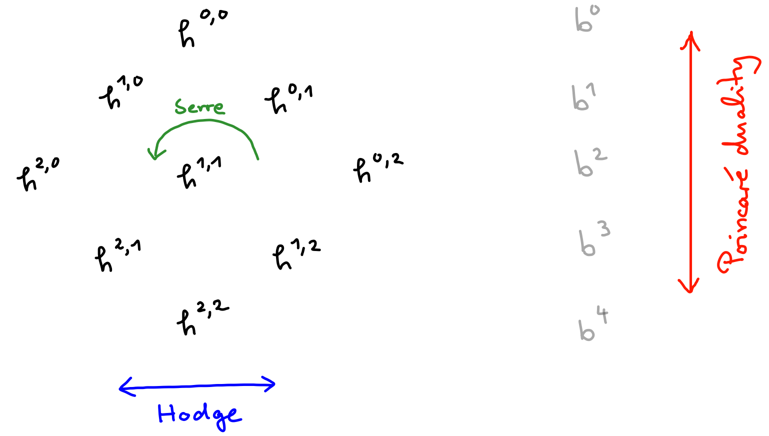
For X of pure dim n, Serre duality gives

$$h^{p,q}(X) = h^{n-p,n-q}(X)$$

since $\Omega_X^{n-p} \simeq (\Omega^p_X)^\vee$ (exercise).

They are often pictured by a "Hodge diamond",

e.g. for $n = 2$:



Over $k = \mathbb{C}$ they compute the topological Betti numbers via Hodge theory:

$$b^i := \dim_{\mathbb{Q}} H^i(X^{an}, \mathbb{Q}) = \sum_{p+q=i} h^{pq}$$

Basic symmetries:

- Serre duality $\widehat{=}$ rotation by π $(h^{pq} = h^{n-p,n-q})$
- For char $k = 0$ one also has:

  Hodge symmetry $\widehat{=}$ reflection at y-axis $(h^{pq} = h^{qp})$

Caution: Hodge symmetry can fail if char $k > 0$ (Serre '58)!

These are all the symmetries:

Thm All linear relations between $h^{p,q}(X)$'s that hold for all smooth proj. X over k are generated by

- Serre duality & Hodge symmetry if char $k = 0$ (Kotschick-Schreieder 2013)
- Serre duality if char $k > 0$ (van Dobben de Bruyn 2020)

Residues of differentials

We've constructed $\mathrm{tr} : H^n(X, \omega_X^\circ) \longrightarrow k$ abstractly via homological algebra of Ext. There's also a concrete approach via residues of differentials. We explain it here only for smooth projective curves $X = C$ over $k = \bar{k}$:

Let $K := k(C)$ be the function field of C. For $p \in C(k)$, fix a local parameter $x \in \mathcal{O}_{C,p}$ and define the residue of rational differential forms at p by

$$\begin{array}{rcl} \mathrm{Res}_{p,t} : \Omega^1_{K/k} = K \cdot dx & \longrightarrow & k \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \omega = f(x)\,dx & \longmapsto & c_{-1}(f) \end{array}$$

where

$$f(x) = \sum_{i<0} c_i(f) \cdot x^i + g(x) \quad \text{w/} \quad c_i(f) \in k \text{ almost all zero, and } g \in \mathcal{O}_{C,p}.$$

Over $k = \mathbb{C}$ we have

$$\mathrm{res}_{p,x}(f(x)dx) = \frac{1}{2\pi i}\oint_{|x|=\varepsilon} f(x)dx$$

& this does not depend on the choice of the local coordinate x at p (by Cauchy's thm / homotopy invariance of path integral).

For k of arbitrary characteristic we still have:

<u>Prop</u> For any two local parameters $x, y \in \mathcal{O}_{C,p}$, we have

$$\mathrm{Res}_{p,x} = \mathrm{Res}_{p,y}.$$

Pf. Enough to show:

$$\mathrm{Res}_{y}\left(\frac{1}{x^a}dx\right) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a > 1. \end{cases}$$

For the case $a = 1$, write $y = h \cdot x$ w/ $h \in \mathcal{O}^*_{C,p}$

$$\Rightarrow dy = h\,dx + x\,dh$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x} + \underbrace{\frac{x}{y}dh}_{\in \Omega^1_{C/k,p} \subset \Omega^1_{K/k}} \qquad \Rightarrow \mathrm{Res}_{p,y}\left(\frac{dx}{x}\right)$$

$$\Rightarrow \mathrm{Res}_{p,y}(\dots) = 0$$

$$= \mathrm{Res}_{p,y}\left(\frac{dy}{y}\right) = 1$$

For the case $a > 1$ note $\mathrm{Res}_{p,y}(df) = 0 \;\forall f \in K = k(C)$, so if $p := \mathrm{char}(k) \nmid (a-1)$, we're done by writing

$$\frac{dx}{x^a} = df \quad \text{w/} \quad f := -\frac{1}{a-1}\cdot\frac{1}{x^{a-1}}.$$

Hence assume $p \mid (a-1)$.

$\Rightarrow \exists$ iso ("inverse Cartier operator")

$$\begin{tikzcd} \varphi: \Omega^1_{K/k} \arrow[r, "\sim"] & \Omega^1_{K/k}/dK \\ \rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\ f(x)\,dx \arrow[r, mapsto] & f(x)^p \cdot x^{p-1} \cdot dx \end{tikzcd}$$

of v'spaces over K, where the RHS is regarded as a v'space over K via Frob: $K \to K$, $f \mapsto f^p$.

(check after passage to completion $\hat{\mathcal{O}}_{C,p} = k[[x]]$,

then $\hat{K} := K \otimes_{\mathcal{O}_{C,p}} \hat{\mathcal{O}}_{C,p} = \left\{ \sum_{i > i_0} c_i x^i \;\middle|\; i_0 \in \mathbb{Z}, c_i \in k \right\}$

and

$$\begin{aligned} \hat{K}\cdot dx &= \left\{ \sum_{i > i_0} c_i x^i \cdot dx \;\middle|\; i_0 \in \mathbb{Z}, c_i \in k \right\} \\ &\cup \\ d\hat{K} &= \left\{ \sum_{\substack{i > i_0 \\ i \equiv -1 \bmod p}} \cdots\cdots\cdots \right\} \end{aligned}$$

)

This iso φ is independent of the chosen parameter x, in fact it can be constructed alternatively as the unique group hom $\varphi : \Omega^1_{K/k} \to \Omega^1_{K/k} / dK$ sth $\forall a \in K, \omega \in \Omega^1_{K/k}$:

i) $\varphi(a \cdot \omega) = a^p \cdot \varphi(\omega)$

ii) $\varphi(da) = a^{p-1}\, da$

(Uniqueness is clear, for existence one has to note that

$$(ab)^{p-1} \cdot d(ab) = a^p \cdot b^{p-1} db + b^p \cdot a^{p-1} da \text{ and}$$

$$(a+b)^{p-1} \cdot d(a+b) \equiv a^{p-1} da + b^{p-1} db \mod dK \;)$$

difference is $d\left(\sum_{i=0}^{p-1} \frac{1}{p} \binom{p}{i} \cdot a^i\, b^{p-i} \right)$

The rules i), ii) imply $\varphi\left(\frac{dy}{y}\right) = \frac{dy}{y}$

$$\Rightarrow \mathrm{Res}_y(\varphi(\omega)) = (\mathrm{Res}_y(\omega))^p$$

Now for $a = pe+1$ w/ $e \in \mathbb{N}$, $\frac{dx}{x^a} = \varphi\left(\frac{dx}{x^{e+1}}\right)$

$$\Rightarrow \mathrm{Res}_y\left(\frac{dx}{x^a}\right) = \Big(\underbrace{\mathrm{Res}_y\left(\frac{dx}{x^{e+1}}\right)}_{= 0 \text{ by ind}^n \text{ on } a}\Big)^p = 0 \qquad \square$$

Upshot For each $p \in C(k)$, get well-defined residue map

$$\mathrm{Res}_p : \Omega^1_{K/k} / dK \longrightarrow k$$

sth for any local parameter x at p,

$$\mathrm{Res}_p \Big(\underbrace{\sum_{i<0} c_i \cdot x^i}_{\substack{\text{finite sum,} \\ c_i \in k}} + \underbrace{h(x)}_{\in \mathcal{O}_{C,p}} \Big) dx = c_{-1} .$$

More work shows the residue theorem:

$$\sum_{p \in C(k)} \mathrm{res}_p(\omega) = 0 \quad \forall\, \omega \in \Omega^1_{K/k} .$$

$\Longrightarrow$ get a well-defined "residue map"

diagonally embedded

$$\begin{tikzcd} \mathrm{tr} : \left(\bigoplus_{p \in C(k)} \Omega^1_{K/k} / \Omega^1_{C,p} \right) / \Omega^1_{K/k} \arrow[r] & k \\ (\omega_p)_{p \in C(k)} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \sum_p \mathrm{Res}_p(\omega_p) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Claim: This "is" our trace map from Serre duality!

More precisely consider

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] & \mathcal{K}_C \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{K}_C/\mathcal{O}_C \arrow[r] & 0 \\
& & \text{constant sheaf w/ stalks } K & &
\end{tikzcd}$$

This is a flabby resolution of $\mathcal{O}_C$, and

$$\mathcal{K}_C/\mathcal{O}_C \simeq \bigoplus_{p\in C(k)} i_{p*}(K/\mathcal{O}_{C,p}) \in \mathrm{Mod}(\mathcal{O}_C)$$

inclusion $i_p : \{p\} \hookrightarrow C$

Tensor w/ $\Omega^1_{C/k}$ to get a flabby resolution

$$0 \to \Omega^1_{C/k} \to \Omega^1_{C/k} \otimes \mathcal{K}_C \to \bigoplus_{p\in C(k)} i_{p*}(\Omega^1_{K/k}/\Omega^1_{C,p}) \to 0$$

Taking cohomology gives:

$$\begin{tikzcd}
\Omega_{K/k} \arrow[r] & \bigoplus_{p} \Omega^1_{K/k}/\Omega^1_{C,p} \arrow[r] \arrow[dr, "\sum_p \mathrm{Res}_p"'] & H^1(C, \Omega^1_{C/k}) \arrow[r] \arrow[d, "\mathrm{tr}"] & 0 \\
& & k &
\end{tikzcd}$$

IX. Direct images and base change

How does cohomology vary in families?

1. Higher direct images

Recall Any continuous map $f : X \to Y$ of top spaces gives a left exact functor

$$f_* : Sh(X) \to Sh(Y), \quad (f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}U).$$

Since $Sh(X)$ has enough injectives, we may consider for $i \geq 0$ its derived functors:

$$R^i f_* : Sh(X) \to Sh(Y).$$

Prop Let $\mathcal{F} \in Sh(X)$. Then for all $i \geq 0$,

$$R^i f_* \mathcal{F} = \text{sheafification of the presheaf}$$
$$U \mapsto H^i(f^{-1}U, \mathcal{F}|_{f^{-1}U})$$

Pf. Let

$$\tilde{R}^i f_* \mathcal{F} := \text{sheafification of } (U \mapsto H^i(f^{-1}U, \mathcal{F}|_{f^{-1}U})).$$

$\Longrightarrow$ both $R^i f_*, \tilde{R}^i f_* : Sh(X) \to Sh(Y)$ are δ-functors & agree for $i = 0$, so we only need to show effaceability:

For $\mathcal{J} \in Sh(X)$ injective and any $i > 0$,

- $R^i f_* \mathcal{J} = 0$ since $R^i f_*$ is a derived functor
- $\tilde{R}^i f_* \mathcal{J} = 0$ since $H^i(f^{-1}U, \underbrace{\mathcal{J}|_{f^{-1}U}}_{\text{injective!}}) = 0$.

$\square$

Cor a) For any open $U \subseteq Y$ and $f_U : f^{-1}U \to U$ we have

$$(R^i f_* \mathcal{F})|_U \simeq R^i f_{U*} \mathcal{F}|_{f^{-1}U}$$

b) If $\mathcal{F} \in Sh(X)$ is flabby, then

$$R^i f_* \mathcal{F} = 0 \text{ for all } i > 0.$$

$\square$

Rem For $i > 0$ sheafification is needed, the presheaf

$$\mathcal{G} : U \mapsto H^i(f^{-1}U, \mathcal{F})$$

need not be a sheaf. In fact, both sheaf axioms may fail:

a) $\mathcal{G}$ needn't be separated:

Take $f = id : X \to Y = X$ for any space X and $\mathcal{F} \in Sh(X)$ sth

- $H^i(X, \mathcal{F}) \neq 0$, but
- $\exists$ open cover $X = \bigcup_\alpha U_\alpha$

  w/ $H^i(U_\alpha, \mathcal{F}) = 0 \quad \forall \alpha$

$\Rightarrow \exists\ 0 \neq s \in \mathcal{G}(X)$

w/ $s|_{U_\alpha} = 0$ for all α

b) Gluing may fail:

Take the Hopf fibration

$$\begin{array}{rcl} f : S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} & \to & S^2 = \mathbb{P}^1(\mathbb{C}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (z,w) & \longmapsto & [z : w] \end{array}$$

For $\mathbb{P}^1(\mathbb{C}) = U_0 \cup U_\infty$ w/ $U_\alpha := \mathbb{P}^1(\mathbb{C}) \setminus \{\alpha\}$ one has

$$f^{-1} U_\alpha \simeq S^1 \times U_\alpha.$$

Pick $0 \neq s_\alpha \in H^1(f^{-1}U_\alpha, \mathbb{Z}) \simeq \mathbb{Z}$

w/ $s_0|_{U_0 \cap U_\infty} = s_\infty|_{U_0 \cap U_\infty}$ (use $\pi_1(S^2) = 0$)

Then $s_0 \in \mathcal{G}(U_0)$, $s_\infty \in \mathcal{G}(U_\infty)$ are sections of the presheaf $\mathcal{G} : U \mapsto H^1(f^{-1}U, \mathbb{Z})$ which are nonzero & agree on $U_0 \cap U_\infty$ but don't glue because

$$\mathcal{G}(S^2) = H^1(S^3, \mathbb{Z}) = 0.$$

Back from topology to geometry:

Prop $f : X \to Y$ morphism of ringed spaces

$\Longrightarrow$ The right derived functor $R^i f_*$ of $f_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$ is compatible with the previous one via the forget functors:

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{O}_X) \arrow[r, "R^i f_*"] \arrow[d] & \mathrm{Mod}(\mathcal{O}_Y) \arrow[d] \\
\mathrm{Sh}(X) \arrow[r, "R^i f_*"] & \mathrm{Sh}(\mathcal{O}_Y)
\end{tikzcd}$$

Pf. As in comparison between sheaf cohomology on ringed spaces & underlying top spaces:

- Injective objects in $\mathrm{Mod}(\mathcal{O}_X)$ are flabby
- Flabby sheaves are acyclic for $R^i f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

$\square$

For schemes we have:

Prop Let $f : X \to Y$ be a qcqs morphism of schemes and $\mathcal{F} \in \mathrm{QCoh}(X)$. Then $\forall\, i \geq 0$:

$$R^i f_*(\mathcal{F}) \in \mathrm{QCoh}(Y)$$

Pf. Claim is local on Y, so wlog $Y = \mathrm{Spec}\, A$ affine.

For $i = 0$ we know f_* preserves QCoh for f qcqs

(as we checked in § $\underline{\text{IV}}$.2 – this needed qcqs, e.g. $g : X := \coprod_{i \in \mathbb{N}} Y \to Y$ has $g_* \mathcal{O}_X \notin \mathrm{QCoh}(Y)$)

For $i > 0$ proceed in two steps:

a) If f is qc & separated:

Pick an open cover $X = \bigcup_{i=1}^{n} U_i$ w/ n minimal

(possible since f qc & Y affine $\Rightarrow$ X qc)

$n = 1$: Then X is affine

$\Rightarrow$ f is affine

$\Rightarrow$ $\forall$ open affine $U \subset Y$,

$f^{-1}(U)$ is affine

& hence $H^i(f^{-1}(U), \mathcal{F}) = 0$ for $i > 0$

(Serre vanishing for qcoh sheaves on affines)

$\Rightarrow$ $R^i f_*(\mathcal{F}) = 0$ $\forall i > 0$ by sheafification

$n > 1$: Use induction. Put

$$U := \bigcup_{i=1}^{n-1} U_i \xhookrightarrow{j_U} X \xhookleftarrow{j_V} V := U_n$$

$$U \cap V = \underbrace{(U_1 \cap V)}_{\text{affine}} \cup \dots \cup \underbrace{(U_{n-1} \cap V)}_{\text{affine}} \quad (\text{since } X \text{ separated})$$

$\Rightarrow$ by induction on n,

$R^i f_{U*} \mathcal{F}|_U$, $R^i f_{V*} \mathcal{F}|_V \in \mathrm{QCoh}(Y)$ $\forall i$

where $f_U := f \circ j_U$ and $f_V := f \circ j_V$.

Now use relative Mayer-Vietoris:

$$0 \to f_* \mathcal{F} \to f_{U*}\mathcal{F}|_U \oplus f_{V*}\mathcal{F}|_V \to f_{U\cap V *} \mathcal{F}|_{U \cap V} \to R^1 f_* \mathcal{F} \to \dots$$

(exercise – works for any morphism of ringed spaces, using injective resolutⁿ $\mathcal{F} \to \mathcal{J}^\bullet$ & $\mathcal{F}|_U \to \mathcal{J}^\bullet|_U$ etc)

$\Rightarrow$ $R^i f_* \mathcal{F} \in \mathrm{QCoh}(X)$ for all i

b) Now assume f qcqs (maybe not separated).

Again write $X = \bigcup_{i=1}^{n} U_i$ w/ U_i open affine, n minimal.

$n = 1$: Then f is affine, hence separated.

$n > 1$: As before let $U := \bigcup_{i=1}^{n-1} U_i$ & $V := U_n$.

By indⁿ on n, claim holds for U & V

It also holds on $U \cap V$ by a), since $U \cap V$ is separated (open in the affine U)

$\Rightarrow$ Done by relative Mayer-Vietoris. $\square$

Cor Let $f : X \to Y$ be qcqs & $\mathcal{F} \in \mathrm{QCoh}(X)$.

a) For $U = \operatorname{Spec} A \subset Y$ open affine,

$$(R^i f_* \mathcal{F})|_U \simeq \widetilde{M}$$

is the quasicoh. sheaf associated to

$M := H^i(f^{-1}U, \mathcal{F}) \in \mathrm{Mod}(A)$.

b) In particular, if $Y = \operatorname{Spec} A$ is affine,

then $H^i(X, \mathcal{F}) = H^0(Y, R^i f_* \mathcal{F})$.

Pf. b) Leray spectral sequence:

$$E_2^{pq} = \underbrace{H^p(Y, \underbrace{R^q f_* \mathcal{F}}_{\substack{\text{qcoh by} \\ \text{proposit}^n}})}_{\substack{= 0 \text{ for all } p > 0 \\ \text{by Serre vanishing}}} \Rightarrow H^{p+q}(X, \mathcal{F})$$

a) claim local $\Rightarrow$ wlog $Y = U = \operatorname{Spec} A$.

Since $R^i f_* \mathcal{F} \in \mathrm{QCoh}(Y)$, we're done by b). $\square$

For coherence of direct images we need stronger assumptions, generalizing the finiteness thm for cohomology:

Thm $f : X \to Y$ proper morphism w/ Y Noetherian & $\mathcal{F} \in \mathrm{Coh}(X)$, then

a) $R^i f_* \mathcal{F} \in \mathrm{Coh}(Y)$ for all $i \geq 0$.

("finiteness thm for direct images")

b) If f is projective, ie factors ${}^{\text{locally on } Y}$ over $X \xhookrightarrow[\text{closed}]{\exists i} \mathbb{P}^n_Y$,

then the twists $\mathcal{F}(n) := \mathcal{F} \otimes i^* \mathcal{O}(n)$ satisfy for $n \gg 0$:

- $f^* f_* (\mathcal{F}(n)) \twoheadrightarrow \mathcal{F}(n)$ epi
- $R^i f_* (\mathcal{F}(n)) = 0$ for all $i > 0$

("relative Serre vanishing")

Pf. Wlog $Y = \operatorname{Spec} A$ w/ A Noetherian. Then by the above, the claim holds by the corresponding result for $H^i(X, -)$. $\square$

2. Flat base change

Next we want to discuss "base change properties" for direct images. For a morphism $f: X \to Y$ take its base change under a morphism $g: Y' \to Y$ as in the Cartesian diagram below:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

w/ $X' := Y' \times_Y X$.

Lemma For $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ $\exists$ natural morphisms

$$\varphi_i : g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* g'^* \mathcal{F} \quad \text{for all } i \geq 0.$$

Pf. Pick injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$

$$g'^* \mathcal{F} \to \mathcal{J}^\bullet$$

Consider

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\text{adjunction}"] \arrow[d] & g'_* g'^* \mathcal{F} \arrow[d] \\ \mathcal{I}^\bullet \arrow[r, dashed] & g'_* \mathcal{J}^\bullet \end{tikzcd}$$

$\exists \alpha$ since $g'_* \mathcal{J}^\bullet$ is a cplex of injectives and $\mathcal{F} \to \mathcal{I}^\bullet$ is exact

We get

$$\beta : g^* f_* \mathcal{I}^\bullet \xrightarrow{g^* f_*(\alpha)} g^* f_* g'_* \mathcal{J}^\bullet \simeq g^* g_* f'_* \mathcal{J}^\bullet \xrightarrow{\text{adjunction for } g} f'_* \mathcal{J}^\bullet$$

Apply $H^i(\dots)$ to define φ_i:

$$\begin{tikzcd} H^i(g^* f_* \mathcal{I}^\bullet) \arrow[r, "H^i(\beta)"] & H^i(f'_* \mathcal{J}^\bullet) \arrow[d, equal, "\sim \mathcal{J}^\bullet \text{ inj res of } g'^* \mathcal{F}"] \\ g^* H^i(f_* \mathcal{I}^\bullet) \arrow[u, "\exists \gamma"] \arrow[d, equal, "\sim \mathcal{I}^\bullet \text{ inj. res. of } \mathcal{F}"] & R^i f'_* g'^* \mathcal{F} \\ g^* R^i f_* \mathcal{F} \arrow[ur, bend right, "\varphi_i"'] & \end{tikzcd}$$

Here γ comes from the "FHHF thm":

F right exact fctor $\Rightarrow$ $\exists$ natural morphism $F H^i C^\bullet \to H^i F C^\bullet$

(& same for F left exact w/ morphism in opposite direct^n) □

Ex a) For f affine, φ_i is an iso:

For $i > 0$ both sides are zero, so wlog $i = 0$.

Wlog $Y = \operatorname{Spec} A$ & then $X = \operatorname{Spec} B$ (since f affine)

Wlog $Y' = \operatorname{Spec} A'$ & then $X' = \operatorname{Spec} B'$ w/ $B' = A' \otimes_A B$.

$\Rightarrow$ For $M \in \operatorname{Mod}(B)$ we have as A'-modules:

$$\varphi : B' \otimes_B M = (A' \otimes_A B) \otimes_B M \xrightarrow{\sim} A' \otimes_A M$$

b) For f not affine usually $g^* f_* \mathcal{F} \not\cong f'_* g'^* \mathcal{F}$ even for $\mathcal{F} = \mathcal{O}_X$:

e.g.

disconnected!

$$\begin{tikzcd} X' = f^{-1}(0) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X = V(xy - z) \setminus \{(0,0,0)\} \subset \operatorname{Spec} k[x,y,z] \arrow[d, "f"] \\ Y' = \{0\} \arrow[r, hook] & Y = \operatorname{Spec} k[z] \end{tikzcd}$$

The reason for problems is:

- f_*, f'_* are only left exact (unless f affine) ↱ see above
- g^*, g'^* are only right exact (unless g flat) ↳ see below

Thm ("flat base change")

Let $f : X \to Y$ be qcqs & $g : Y' \to Y$ flat.

Then for all $\mathcal{F} \in \operatorname{QCoh}(X)$ we have:

a) The morphism

$$\varphi_i : g^* R^i f_* \mathcal{F} \xrightarrow{\sim} R^i f'_* g'^* \mathcal{F}$$

is an isomorphism for all $i \geq 0$.

b) In particular, if $Y' = \operatorname{Spec} A' \to Y = \operatorname{Spec} A$ is a morphism between affine schemes, then

$$H^i(X, \mathcal{F}) \otimes_A A' \xrightarrow{\sim} H^i(X', g'^* \mathcal{F})$$

Pf. Claim is local on Y'

$\Rightarrow$ wlog $Y' = \operatorname{Spec} A' \to Y = \operatorname{Spec} A$

$\Rightarrow$ enough to show b)

Recall $f : X \to Y = \operatorname{Spec} A$ is qcqs. $\Rightarrow$ X qcqs

- If X is separated:

  Pick an affine open cover $\mathcal{U}$: $X = \bigcup_{\alpha=1}^{n} U_\alpha$

  $\Longrightarrow \check{H}^i(\mathcal{U}, \mathcal{F}) \simeq H^i(X, \mathcal{F})$

  Moreover $X' = X \times_Y Y'$ is still separated

  w/ open affine cover $\mathcal{U}'$: $X' = \bigcup_{\alpha=1}^{n} U_\alpha \times_Y Y'$

  $\Longrightarrow \check{H}^i(\mathcal{U}', g'^* \mathcal{F}) \simeq H^i(X', g'^* \mathcal{F})$

  Now $\check{C}^\bullet(\mathcal{U}', g'^* \mathcal{F}) \simeq \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A'$.

  Take $H^\bullet(\ldots)$ & use that A' is flat over A

  $\Rightarrow \check{H}^i(\mathcal{U}', g'^* \mathcal{F}) \simeq \check{H}^i(\mathcal{U}, \mathcal{F}) \otimes_A A'$

- If X is not separated:

  Use Mayer-Vietoris & induction on n. $\square$

<u>Ex</u> X scheme / k and $k \hookrightarrow K$ field extension

$\Rightarrow H^i(X, \mathcal{F}) \otimes_k K \simeq H^i(X_K, \mathcal{F}_K)$.

3. Semicontinuity & Grauert's thm

Q Let $f : X \to Y$ be proper & $\mathcal{F} \in Coh(X)$.

How does $H^i(X_y, \mathcal{F})$ vary with $y \in Y$?

$\hookrightarrow := X \times_Y \operatorname{Spec} \kappa(y)$

In general $\varphi_i : R^i f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \kappa(y) \to H^i(X_y, \mathcal{F})$

need not be an isomorphism, e.g. let E be an

elliptic curve & take

$f = pr_1 : X = E \times E \longrightarrow Y = E$

$\mathcal{F} := \mathcal{O}_X(\Delta - Z) \in Pic(X)$

w/ $\Delta \subset E \times E$ diagonal,

$Z = 0 \times E$ zero section.

$\Rightarrow H^0(X_y, \mathcal{F}) \neq 0$ for $y = 0$,

but $f_* \mathcal{F} \simeq 0$ (exercise)!

Recall Given $f : X \to Y$, we say $\mathcal{F} \in QCoh(X)$ is f-flat or flat over Y if $\forall\, x \in X$, $\mathcal{F}_x$ is flat as a module over $\mathcal{O}_{Y,y}$.

The key to understanding cohomology of fibers will be:

Thm ("base change & cohomology")

Let $f : X \to Y$ be a proper morphism of Noetherian schemes w/ $Y = \operatorname{Spec} A$.

Then for any f-flat $\mathcal{F} \in Coh(X)$, $\exists$ finite cplex $K^\bullet = [0 \to K^1 \to \dots \to K^n \to 0]$ of fin. gen. projective A-modules w/ isomorphisms

$H^p(X_B, \mathcal{F}_B) \xrightarrow{\sim} H^p(K^\bullet \otimes_A B)$

functorial in the A-algebra B.

Pf. ① Let $\mathcal{U} = (U_i)_{i \in I}$ be a finite open affine cover of X. Then $C^\bullet := \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a finite cplex of flat A-modules. For any A-algebra B we have $C^\bullet \otimes_A B = \check{C}^\bullet(\mathcal{U}_B, \mathcal{F}_B)$ and hence

$$H^p(X_B, \mathcal{F}_B) \simeq H^p(C^\bullet \otimes_A B)$$

functorially in B. But: C^i needn't be fingen $/A$ ...

② Claim: Let $C^\bullet$ be any cplex of A-modules sth

* $C^p \neq 0$ at most for $0 \leq p \leq n$
* $H^p(C^\bullet)$ fin. gen. $/A$ for all p.

$\Rightarrow$ $\exists$ quasi-iso $K^\bullet \longrightarrow C^\bullet$ where $K^\bullet$ is a cplex of fingen A-modules sth

* $K^p \neq 0$ at most for $0 \leq p \leq n$,
* K^p free A-module for $1 \leq p \leq n$,

and if all C^p are flat, then so is K^0.

We construct $K^\bullet$ by descending induction on degrees:

Put $K^p := 0$ for $p > n$.

Inductively consider diagrams

$$\begin{tikzcd}
 & K^m \arrow[r, "\partial^m"] \arrow[d, "\varphi^m"'] & K^{m+1} \arrow[r, "\partial^{m+1}"] \arrow[d, "\varphi^{m+1}"] & K^{m+2} \arrow[r, "\partial^{m+2}"] \arrow[d, "\varphi^{m+2}"] & \cdots \\
\cdots \arrow[r] & C^m \arrow[r, "\partial^m"'] & C^{m+1} \arrow[r, "\partial^{m+1}"'] & C^{m+2} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose we have constructed $(K^p, \partial^p, \varphi^p)$ for all $p \geqslant m+1$ in such a way that $\forall p \geqslant m+1$:

i) $\partial^p \varphi^p = \varphi^{p+1} \partial^p$

ii) $\partial^{p+1} \circ \partial^p = 0$

iii) K^p free of finite rank $/ A$

and moreover

iv) $H^q(K^\bullet) \xrightarrow{\sim} H^q(C^\bullet)$ iso $\forall q > m+1$,

and $\ker(\partial^{m+1}) \twoheadrightarrow H^{m+1}(C^\bullet)$ epi.

<u>Goal</u>: Find $(K^m, \partial^m, \varphi^m)$ w/ i), ii), iii) for $p = m$.

For $m \geqslant 0$:

Pick surjections π', π'':

$$K' := A^{\oplus n'} \overset{\pi'}{\twoheadrightarrow} B^{m+1} := \ker(\ker \partial^{m+1} \to H^{m+1}(C^\bullet)) \subseteq K^{m+1}$$

$$\begin{tikzcd}
K'' := A^{\oplus n''} \arrow[r, two heads, "\pi''"] \arrow[dr, dashed, "\exists \text{ lift } \lambda \text{ (not unique)}"'] & H^m(C^\bullet) \\
& Z^m(C^\bullet) := \ker \partial^m \subseteq C^m \arrow[u, two heads]
\end{tikzcd}$$

Let $\varphi'' := \text{inclusion} \circ \lambda : K'' \to C^m$

Put $K^m := K' \oplus K''$

$$\partial^m := \pi' \oplus 0 : K^m = K' \oplus K'' \to K^{m+1}$$

By definition $\varphi^{m+1}(B^{m+1}) \subseteq \partial C^m$ & we pick φ^m as shown:

$$\begin{tikzcd}
K' \arrow[r, dashed, "\exists \varphi^m"] \arrow[d, "\pi'"'] & C^m \arrow[d, "\partial"] \\
B^{m+1} \arrow[r] \arrow[d, phantom, "\cap"] & \partial C^m \arrow[d, phantom, "\cap"] \\
K^{m+1} \arrow[r, "\varphi^{m+1}"] & C^{m+1}
\end{tikzcd}$$

$\Longrightarrow$ $(K^m, \partial^m, \varphi^m)$ satisfies i), ii), iii).

For $m = -1$:

Have $(K^p, \partial^p, \varphi^p)$ w/ i), ii), iii) $\forall p \geqslant 0$

Replace K^0 by $K^0 / \ker(\partial^0) \cap \ker(\varphi^0)$

Get $K^\bullet = [0 \to K^0 \to \cdots \to K^n \to 0]$

w/ quasi-iso $\varphi : K^\bullet \to C^\bullet$.

Remains to check: K^0 is flat if all C^p are flat.

Take the cone $C(\varphi)^\bullet$ defined by

$$C(\varphi)^p := K^{p+1} \oplus C^p \quad \text{w/} \quad \partial^p := \begin{pmatrix} -\partial^p_K & 0 \\ \varphi^p & \partial^p_C \end{pmatrix}$$

$\Longrightarrow$ exact sequence of complexes

$$0 \to C^\bullet \to C(\varphi)^\bullet \to K[1]^\bullet \to 0$$

$\Longrightarrow$ Since $\varphi : H^p(K^\bullet) \xrightarrow{\sim} H^p(C^\bullet)$ iso $\forall p$,

long exact sequence gives $H^p(C(\varphi)^\bullet) \simeq 0 \ \forall p$.

$$\Longrightarrow \quad 0 \to \underset{\underset{K^0}{\|}}{C(\varphi)^0} \to \underbrace{C(\varphi)^1 \to \cdots \to C(\varphi)^n}_{\text{all flat } A\text{-modules}} \to 0$$

$\Longrightarrow$ K^0 flat A-module (use induction on n)

③ Back to our application:

Get $K^\bullet$ cplex of fin gen proj. A-modules K^p w/ $K^p \neq 0$ at most for $0 \leq p \leq n$, and a qiso $K^\bullet \to C^\bullet := \check{C}^\bullet(\mathcal{U}, \mathcal{F})$,

so $H^p(K^\bullet) \xrightarrow{\sim} H^p(X, \mathcal{F}) \quad \forall\, p \geq 0$.

Claim: For any A-algebra B,

$$H^p(K^\bullet \otimes_A B) \xrightarrow{\sim} H^p(X_B, \mathcal{F}_B) \quad \forall p \geq 0.$$

Indeed:

$C(\varphi)^\bullet$ exact finite cplex of flat A-modules

$\Rightarrow Z^p := \ker\left(C(\varphi)^p \xrightarrow{\partial} C(\varphi)^{p+1}\right)$ also flat $/ A$

$\Rightarrow C(\varphi)^\bullet \otimes_A B$ still exact

$\quad\quad ||$

$\quad C(\varphi_B)^\bullet$ w/ $\varphi_B := \varphi \otimes \mathrm{id} : K^\bullet \otimes_A B \to C^\bullet \otimes_A B$

$\Rightarrow H^p(K^\bullet \otimes_A B) \xrightarrow{\sim} H^p(C^\bullet \otimes_A B) = H^p(X_B, \mathcal{F}_B)$

$\square$

Cor 1 ("semicontinuity thm")

Let $f : X \to Y$ be a proper morphism of Noetherian schemes & $\mathcal{F} \in \mathrm{Coh}(X)$ flat over Y.

a) For all $p \geq 0$ the fct^n

$$h^p_{\mathcal{F}} : Y \to \mathbb{Z},$$
$$y \mapsto h^p_{\mathcal{F}}(y) := \dim_{\kappa(y)} H^p(X_y, \mathcal{F}_y)$$

is upper semicontinuous, i.e. $\forall\, c \in \mathbb{R}$ the set $\{ y \in Y \mid h^p_{\mathcal{F}}(y) \geq c \} \subseteq Y$ is closed.

b) The Euler characteristic fct^n

$$\chi_{\mathcal{F}} : Y \to \mathbb{Z},$$
$$y \mapsto \chi_{\mathcal{F}}(y) := \sum_{p \geq 0} (-1)^p h^p_{\mathcal{F}}(y)$$

is locally constant on Y.

Pf. Claim local on Y $\Rightarrow$ wlog $Y = \mathrm{Spec}\, A$

$\Rightarrow \exists$ finite cplex $K^\bullet$ as in the thm, w/ $K^p \in \mathrm{Mod}(A)$ projective $\rightsquigarrow$ wlog free (shrink Y)

Consider $d^p : K^p \longrightarrow K^{p+1}$ & put $\dim := \dim_{\kappa(y)}$

$$\Rightarrow \dim H^p(X_y, \mathcal{F}|_{X_y})$$

$$= \underbrace{\dim \ker(d^p \otimes \kappa(y))}_{= \dim K^p \otimes \kappa(y) - \dim \operatorname{im}(d^p \otimes \kappa(y))} - \dim \operatorname{im}(d^{p-1} \otimes \kappa(y))$$

$(\ast)$

$$\Rightarrow \chi_{\mathcal{F}}(y) = \sum_{p=0}^{n} (-1)^p \dim H^p(X_y, \mathcal{F}|_{X_y})$$

$$= \sum_{p=0}^{n} (-1)^p \dim K^p \otimes \kappa(y)$$

independent of y !

For upper semicontinuity of $\dim H^p(X_y, \mathcal{F}|_{X_y})$, suffices by $(\ast)$ to show $y \longmapsto \dim \operatorname{im}(d^p \otimes \kappa(y))$ is lower semicontinuous for all p.

For this note for $r \in \mathbb{N}$:

$$S_r := \{ y \in Y \mid \operatorname{rk}(d^p \otimes \kappa(y)) < r \}$$

$$= \{ y \in Y \mid \Lambda^r(d^p) \otimes \kappa(y) = 0 \}$$

for $\Lambda^r(d^p) : \Lambda^r K^p \longrightarrow \Lambda^r K^{p+1}$

$\Rightarrow S_r \subset Y$ closed

(given by vanishing of entries of matrix describing $\Lambda^r(d^p)$) $\square$

<u>Non-example</u> $f : X = Bl_0(\mathbb{A}^2_k) \longrightarrow Y = \mathbb{A}^2_k$

$\mathcal{F} := \mathcal{O}_X(E)$ w/ $E := f^{-1}(0)$

$\Rightarrow h^0(X_y, \mathcal{F}|_{X_y}) = h^0(X_y, \mathcal{O}_{X_y}) = 1 \quad \forall y \neq 0$

but $h^0(X_0, \mathcal{F}|_{X_0}) = h^0(E, \mathcal{O}_E(-1)) = 0$

$\Rightarrow y \mapsto h^0(X_y, \mathcal{F}|_{X_y})$ not upper semicont.

(however, here $\mathcal{F}$ is not flat over Y)

Cor 2 ("Grauert's thm")

Let $f : X \to Y$ be a proper morphism of Noetherian schemes, w/ Y reduced and connected. Let $\mathcal{F} \in Coh(X)$ be flat over Y. Then $\forall\, p \geq 0$,

TFAE :

a) $h^p_{\mathcal{F}} : Y \to \mathbb{Z}$, $y \mapsto \dim_{\kappa(y)} H^p(X_y, \mathcal{F}|_{X_y})$

is a constant function.

b) $\mathcal{E} := R^p f_* \mathcal{F}$ is locally free on Y

and for all $y \in Y$,

$\mathcal{E} \otimes_{\mathcal{O}_Y} \kappa(y) \xrightarrow{\sim} H^p(X_y, \mathcal{F}|_{X_y})$

Moreover, if these hold, then $R^{p-1} f_*(\mathcal{F})$ is also locally free and for all $y \in Y$,

$R^{p-1} f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \kappa(y) \xrightarrow{\sim} H^{p-1}(X_y, \mathcal{F}|_{X_y})$.

Pf. b) $\Rightarrow$ a) trivial.

a) $\Rightarrow$ b) : Wlog $Y = \operatorname{Spec} A$.

Let $K^\bullet$ be the cplex given by the thm. Since

$$h^p_{\mathcal{F}}(y) = \dim \ker d^p \otimes \kappa(y) - \dim \operatorname{im} d^{p-1} \otimes \kappa(y)$$

$$= \dim K^p \otimes \kappa(y) - \dim \operatorname{im} d^p \otimes \kappa(y) - \dim \operatorname{im} d^{p-1} \otimes \kappa(y)$$

is constant by assumption, semicontinuity of the summands implies

$\dim \operatorname{im} d^p \otimes \kappa(y)$ & $\dim \operatorname{im} d^{p-1} \otimes \kappa(y)$ are constant.

Fact : $Y = \operatorname{Spec} A$ reduced Noetherian affine

$\varphi : \mathcal{F} \to \mathcal{G}$ hom. of loc. free coherent $\mathcal{O}_Y$-mod

w/ $\dim_{\kappa(y)} \operatorname{im}(\varphi \otimes \kappa(y))$ locally constant

$\Rightarrow \exists$ splittings $\mathcal{F} \simeq \mathcal{F}_1 \oplus \mathcal{F}_2$, $\mathcal{G} \simeq \mathcal{G}_1 \oplus \mathcal{G}_2$

sth

$\varphi = \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix}$ w/ $\psi : \mathcal{F}_2 \xrightarrow{\sim} \mathcal{G}_1$.

(use $\dim_{\kappa(y)} \operatorname{cok}(\varphi) \otimes \kappa(y)$ loc. cst & Y reduced $\Rightarrow$ $\operatorname{cok}(\varphi)$ loc free, hence projective on affine charts ... see Mumford, AV, p. 51/52)

In our case, get

$$\begin{tikzcd}
K^{p-1} \arrow[r, "d^{p-1}"] \arrow[d, phantom, sloped, "\simeq"] & K^p \arrow[r, "d^p"] \arrow[d, phantom, sloped, "\simeq"] & K^{p+1} \arrow[d, phantom, sloped, "\simeq"] \\
Z^{p-1} \oplus \bar{K}^{p-1} & B^p \oplus H^p \oplus \bar{K}^p & B^{p+1} \oplus \bar{K}^{p+1}
\end{tikzcd}$$

w/ $Z^{p-1} = \ker(d^{p-1})$, $\quad B^p \oplus H^p = \ker(d^p)$,

$d^{p-1} : \bar{K}^{p-1} \xrightarrow{\sim} B^p$, $\quad d^p : \bar{K}^p \xrightarrow{\sim} B^{p+1}$.

$$\Longrightarrow \quad H^p(K^\bullet \otimes_A B) \simeq H^p \otimes_A B \simeq H^p(K^\bullet) \otimes_A B \quad \forall B$$

and $H^{p-1}(K^\bullet \otimes_A B) \simeq Z^{p-1} \otimes_A B / \operatorname{im}(d^{p-2} \otimes B)$

$$\underset{\substack{\uparrow \\ \otimes \text{ right exact}}}{\simeq} H^{p-1}(K^\bullet) \otimes_A B \qquad \square$$

Ex ① $X = \mathbb{P}^1_Y \xrightarrow{f} Y = \operatorname{Spec} k[t]$

Define $\mathcal{F} \in \mathrm{Coh}(X)$ by

$$\begin{tikzcd}
\mathrm{Ext}^1_X(\mathcal{O}_X(-2), \mathcal{O}_X) \arrow[d, phantom, sloped, "\ni"] \simeq H^1(X, \mathcal{O}_X(-2)) \simeq H^0(X, \mathcal{O}_X)^\vee \simeq & k[t] \arrow[d, phantom, sloped, "\ni"] \\
(0 \to \mathcal{O}_X(-2) \to \mathcal{F} \to \mathcal{O}_X \to 0) \arrow[r, mapsto] & t
\end{tikzcd}$$

Then $\mathcal{F}$ is loc. free on X, hence flat over Y. We have

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{E}) \arrow[r] & H^0(X, \mathcal{O}_X(-2)) \arrow[r] \arrow[d, equal] & H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, equal] & \cdots \\
& & k[t] \arrow[r, "\cdot t"] & k[t] &
\end{tikzcd}$$

& similarly for $X_y = X \times_{k[t]} k[t]/(t-y)$:

$$H^0(X_y, \mathcal{E}|_{X_y}) = \ker\left(k[t]/(t-y) \xrightarrow{\cdot t} k[t]/(t-y) \right)$$

By a) $\Rightarrow$ b) we see $f_*(\mathcal{F})|_U$ locally free on $U = Y \setminus \{0\}$

(but not locally free at $0 \in Y$)

② Reducedness of Y is needed: Take

$X = \mathbb{P}^1_Y \xrightarrow{f} Y = \operatorname{Spec} k[t]/(t^2)$

$\mathcal{F} \triangleq t \in \mathrm{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_X(-2)) \simeq k[t]/(t^2)$

$\Longrightarrow$ $\mathcal{F}$ loc free on X, hence flat over Y

$$f_*(\mathcal{F}) = \ker\left(k[t]/(t^2) \xrightarrow{\cdot t} k[t]/(t^2) \right) \simeq t \cdot k[t]/(t^2)$$

NOT locally free although $h^0_{\mathcal{F}} : Y \to \mathbb{Z}$ is constant (since Y is a singleton), equal to zero.

4. Hilbert polynomials

Def Let $X \subset \mathbb{P}^n_k$ be a projective scheme over a field k. For $\mathcal{F} \in Coh(X)$ we call

$$p_{\mathcal{F}} : \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$d \longmapsto \chi(X, \mathcal{F}(d)) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F}(d))$$

the Hilbert polynomial of $\mathcal{F} \in Coh(X)$.

We call $p_X := p_{\mathcal{O}_X}$ the Hilbert polynomial of X.

Lemma $p_{\mathcal{F}}$ is a polynomial $\in \mathbb{Q}[x]$ of degree $\leq \dim X$.

Pf. Induction on n with $X \subset \mathbb{P}^n_k$:

$n = 0$: $X = \operatorname{Spec} k$ & $\mathcal{F}(d) = \mathcal{F}$

$n > 0$: Consider $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \underbrace{i_* \mathcal{G}(d)}_{\in Coh(\mathbb{P}^{n-1}_k)} \to 0$

w/ $i : X \cap \mathbb{P}^{n-1}_k \hookrightarrow \mathbb{P}^n_k$.

$\Rightarrow$ $p_{\mathcal{F}}(d) - p_{\mathcal{F}}(d-1)$ polynomial of degree $\leq \dim X - 1$ in d by induct^n

Claim: Then $p_{\mathcal{F}}(d)$ is a polynomial in d ...

Claim: Then $p_{\mathcal{F}}(d)$ is a polynomial in d of degree $\leq \dim X$:

Indeed:

$p(x) - p(x-1)$ is a numerical polynomial, ie polynomial taking integer values at all integers

Exercise: The numerical polynomials form a free abelian grp with $\mathbb{Z}$-basis given by the polynomials

$$x \mapsto \binom{x}{i} := \frac{1}{i!} \cdot x(x-1) \cdots (x-i), \quad i \in \mathbb{N}_0.$$

Our case:

Let $p(x) - p(x-1) = \sum_{i=0}^{\overset{r \leq \dim X - 1}{r}} a_i \cdot \binom{x}{i}$ $\quad (a_i \in \mathbb{Z})$

Put $q(x) := p(x) - \sum_{i=0}^{r} a_i \cdot \binom{x+1}{i+1}$

$\Rightarrow q(x) - q(x-1) = 0$

$\Rightarrow q(x) = \text{const} =: a_{-1}$

$\Rightarrow p(x) = a_{-1} + \overbrace{\sum_{i=0}^{r} a_i \binom{x+1}{i+1}}^{\text{polynomial of degree} \leq \dim X} \in \mathbb{Q}[x]$ $\qquad \square$

Ex a) For $X = \mathbb{P}^n_k$ we get

$$p_X(m) = \binom{m+n}{n} = \frac{1}{n!} \cdot m^n + \dots$$

enough to check this for $m \gg 0$, then

$$\chi(X, \mathcal{O}(m)) = \dim_k H^0(X, \mathcal{O}(m))$$

$$= \dim_k k[x_0, .., x_n]_m = \binom{m+n}{n}$$

b) For hypersurfaces $X = V_+(f) \subset \mathbb{P}^n_k$, $\deg f = d$, we get

$$p_X(m) = \binom{m+n}{n} - \binom{m+n-d}{n}$$

$$= \frac{d}{(n-1)!} \cdot m^{n-1} + \dots$$

from $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$.

Def The *degree* of a subscheme $X \subset \mathbb{P}^n_k$ is

$$\deg(X) := (\dim X)! \cdot \text{leading coeff of } p_X.$$

($\to$ Bezout's thm: $\deg(X \cap V_+(f)) = \deg(X) \deg(f) \dots$)

Next we want to relate Hilbert polynomials to flatness.

Prop $f : X \to Y$ projective morphism of Noeth. schemes then for $\mathcal{F} \in Coh(X)$ TFAE:

a) $\mathcal{F}$ is flat over Y

b) $\forall m \gg 0$, $f_*(\mathcal{F}(m))$ is locally free.

Pf. Wlog $Y = \operatorname{Spec} A$ and $X = \mathbb{P}^n_Y = \operatorname{Proj} A[x_0, .., x_n]$.

"$\Rightarrow$": Assume $\mathcal{F}$ flat / Y

Relative Serre vanishing:

$$R^i f_*(\mathcal{F}(m)) = 0 \quad \forall m \gg 0 \quad \forall i > 0$$

$\Rightarrow$ Pick an open affine cover $\mathcal{U} = (U_i)_{i \in I}$ of X then

$$f_* \check{C}^0(\mathcal{U}, \mathcal{F}(m)) \to \dots \to f_* \check{C}^n(\mathcal{U}, \mathcal{F}(m)) \to 0$$

is a resolution of $f_*(\mathcal{F}(m))$ $\forall m \gg 0$.

Resolution is finite & all terms are flat over A

$\Rightarrow$ $f_*(\mathcal{F}(m))$ flat over A

(ie locally free on $Y = \operatorname{Spec} A$)

"⇐" : Assume $f_*(\mathcal{F}(m))$ locally free $\forall\, m \gg 0$

Write $\mathcal{F} = \tilde{M}$ w/ $M := \bigoplus_{d \gg 0} \underbrace{H^0(X, \mathcal{F}(d))}_{\substack{\text{flat over } A \\ \text{by assumpt}^n}}$

$\Rightarrow$ M flat A-module

$\Rightarrow$ M_{x_i} flat A-module $\forall i$

$\Rightarrow$ $M_{x_i,0} = \mathcal{F}(D_+(x_i))$ flat $/A$ $\forall\, i$

(a direct summand in a flat A-module) □

<u>Cor</u> $f : X \to Y$ proj morphism of Noetherian schemes, with Y reduced. Then for $\mathcal{F} \in Coh(X)$ TFAE:

a) $\mathcal{F}$ is flat over Y

b) The Hilbert polynomial $p_{\mathcal{F}|_{X_y}}(t) \in \mathbb{Q}[t]$ is locally constant on Y.

Pf. a) $\Rightarrow$ b) : Base change (see previous section)

b) $\Rightarrow$ a) : $p_{\mathcal{F}|_{X_y}}(m) = \chi(X_y, \mathcal{F}|_{X_y}(m)) \underset{\substack{\uparrow \\ \text{for } m \gg 0 \text{ (Serre vanishing)}}}{=} h^0(X_y, \mathcal{F}|_{X_y}(m))$

If this is locally constant, then $f_* \mathcal{F}(m)$ is loc. free by Grauert's thm (as Y is reduced)

$\Rightarrow$ $\mathcal{F}$ flat over Y by previous proposition. □

<u>Outlook : Hilbert schemes</u>

Given $f : X \to S$ fin. type w/ S Noetherian

parametrize closed subschemes $Y \subset X$ flat over S.

More generally : Given $\mathcal{E} \in Coh(X)$,

parametrize quotients $\mathcal{E} \twoheadrightarrow \mathcal{F}$ w/ $\mathcal{F}$ flat over S

(for the case of closed subschemes take $\mathcal{E} := \mathcal{O}_X$).

loc. Noeth. schemes / S

<u>Def</u> A <u>family of quotients</u> of $\mathcal{E}$ parametrized by $T \in Sch_S$ is a pair $(\mathcal{F}, q)$ w/

- $\mathcal{F} \in Coh(X_T)$ flat over T w/ $\operatorname{Supp} \mathcal{F} \to T$ proper
- $q : \mathcal{E}_T \twoheadrightarrow \mathcal{F}$ epi of $\mathcal{O}_{X_T}$-modules.

We write $(\mathcal{F}, q) \simeq (\mathcal{F}', q')$ if $\ker(q) = \ker(q')$.

⇝ contravariant functor

$$\begin{aligned} Quot_{X/S,\mathcal{E}} : Sch_S^{op} &\longrightarrow Sets \\ T &\longmapsto \{\text{families of quotients of } \mathcal{E} \text{ over } T\}/\simeq \end{aligned}$$

Special case: $\mathcal{H}ilb_{X/S} := \mathrm{Quot}_{X/S, \mathcal{O}_X}$.

Fixing $\mathcal{L} \in \mathrm{Pic}(X)$, we get for any $s \in S$ a Hilbert polynomial $p_{\mathcal{F}|_{X_s}, \mathcal{L}}(m) := \chi(X_s, \mathcal{F} \otimes \mathcal{L}^{\otimes m}|_{X_s})$ which is loc cst in flat families

$$\Rightarrow \mathrm{Quot}_{X/S, \mathcal{E}} = \coprod_{p \in \mathbb{Q}[x]} \mathrm{Quot}^{p, \mathcal{L}}_{X/S, \mathcal{E}}$$

$$\shortparallel$$

$$\{ (\mathcal{F}, q) \mid p_{\mathcal{F}|_{X_s}, \mathcal{L}}(m) = p(m) \ \forall m \}$$

Ex a) $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Quot}^{1, \mathcal{O}_S}_{S/S, \mathcal{O}_S^{n+1}}$ w/ $S := \mathrm{Spec}\, \mathbb{Z}$:

$$\mathbb{P}^n_{\mathbb{Z}}(T) = \{ \mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{F} \text{ w/ } \mathcal{F} \text{ line bdle} \} / \simeq$$

$\mathcal{F} \in \mathrm{Coh}(T)$ line bdle $\Leftrightarrow$ flat over T w/ 1-dim fibers

$\Leftrightarrow$ $p_{\mathcal{F}_s, \mathcal{L}}(x) \equiv 1$ for $\mathcal{L} = \mathcal{O}_S$

b) More generally, Grassmannians:

$$G(r, d) = \mathrm{Quot}^{r, \mathcal{O}_S}_{S/S, \mathcal{O}_S^d} \quad \text{w/ } S := \mathrm{Spec}\, \mathbb{Z}.$$

Thm (Grothendieck) Let S be Noetherian, $f : X \to S$ projective & $\mathcal{L} \in \mathrm{Pic}(X)$ relatively very ample. Then $\forall\, \mathcal{E} \in \mathrm{Coh}(X)$ $\forall\, p \in \mathbb{Q}[x]$, the functor $\mathrm{Quot}^{p, \mathcal{L}}_{X/S, \mathcal{E}}$ is representable by a projective S-scheme.