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Algebraic $\mathcal{D}$-Modules

0. Introduction

X smooth alg. variety $/ \mathbb{C}$ (or cplex mfd)

What can we say about the topology of X?

$\to$ Consider $\pi_1(X, x)$,

or its rep^s $\rho : \pi_1(X, x) \to Gl_n(\mathbb{C})$.

Such rep^s arise by analytic continuation of solutions to linear diff eq^{ns}:

Ex 1 Let $X = \mathbb{C}^*$ w/ coordinate z. Fix $\alpha \in \mathbb{C}$.

For $U \subseteq X$ open, put

$$\mathcal{L}_\alpha(U) := \{ f \in \mathcal{O}(U) \mid f'(z) - \tfrac{\alpha}{z} f(z) = 0 \}$$

(holomorphic fct^s $U \to \mathbb{C}$)

$\Rightarrow$ For any simply connected U,

$$\mathcal{L}_\alpha(U) = \mathbb{C} \cdot z^\alpha \quad \text{w/} \quad z^\alpha := e^{\alpha \log z}$$

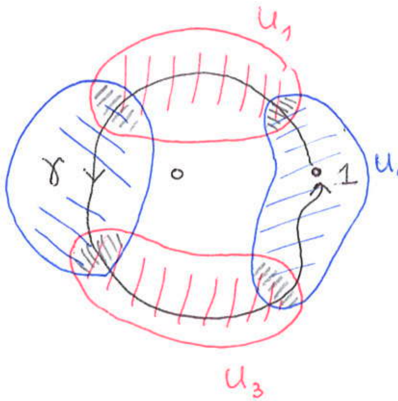
$\log : U \to \mathbb{C}$ any branch of the logarithm (well-def mod $2\pi i \mathbb{Z}$).

Given a continuous loop $\gamma : [0,1] \to X$
w/ $\gamma(0) = \gamma(1) = 1$,

pick $U_0, \dots, U_N \subset X$ simply connected open

w/

- $\mathrm{Image}(\gamma) \subset \bigcup_{i=0}^{N} U_i$,
- $U_i \cap U_{i+1} \neq \emptyset$ & connected $\forall\, i = 0, 1, \dots, N$ (put $U_{N+1} := U_0$)
- $1 \in U_0 \cap U_N$.



Start w/ any nonzero solution $f_0 \in \mathcal{L}_\alpha(U_0)$

$\Rightarrow \exists!\, f_1 \in \mathcal{L}_\alpha(U_1)$
w/ $f_1|_{U_0 \cap U_1} = f_0|_{U_0 \cap U_1}$

$\vdots$

$\Rightarrow \exists!\, f_N \in \mathcal{L}_\alpha(U_N)$ w/ $f_N|_{U_{N-1} \cap U_N} = f_{N-1}|_{U_{N-1} \cap U_N}$

Exercise a) $\exists!\, \rho(\gamma) \in \mathbb{C}^*$ sth $f_N|_{U_0 \cap U_N} = \rho(\gamma) \cdot f_0|_{U_0 \cap U_N}$.

b) $\rho(\gamma)$ only depends on the homotopy class of γ & this gives a rep^n $\rho : \pi_1(X, 1) \longrightarrow Gl_1(\mathbb{C}) = \mathbb{C}^*$

c) Explicitly, $\rho(\circlearrowleft) = e^{2\pi i \alpha}$.
$\uparrow$ counterclockwise generator of $\pi_1(X,1) \simeq \mathbb{Z}$.

Rem The main point was that $\mathcal{L}_\alpha$ is a "local system" ie a locally constant sheaf of $\mathbb{C}$-vspaces on X.

In general we have an equivalence of categories

$$\left\{ \begin{array}{l} \text{representations} \\ \rho : \pi_1(X,x) \to Gl_n(\mathbb{C}) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{local systems} \\ \text{on } X \end{array} \right\}$$

$$\rho \longmapsto \mathcal{L}_\rho$$

w/ $\mathcal{L}_\rho(U) := \left\{ \begin{array}{l} \text{sections over } U \\ \text{of the } \mathbb{C}^n\text{-bdle} \\ (\tilde{X} \times \mathbb{C}^n) / \text{diagonal } \pi_1\text{-action} \\ \quad\downarrow \\ \tilde{X}/\pi_1 = X \end{array} \right\}$

($\tilde{X} \to X$ universal cover)

whose quasinverse is given by the above "monodromy construction."

(exercise)

Note $\pi_1(X,x)$ & $\tilde{X} \to X$ are topological objects, and the local analytic branches of $\log z$ are transcendental functions. Nevertheless the differential operator $D := \frac{d}{dz} - \frac{\alpha}{z}$ has as coefficients rational functions of z (namely 1 and α/z)!

topology $\rightsquigarrow$ analysis $\rightsquigarrow$ algebra

Drawback No intrinsic choice of diff eq^n for a given rep^n of $\pi_1(X,x)$!

Ex 2 a) For the diffeq^n in ex 1 we have:

$$(z\tfrac{d}{dz}-\alpha)f=0 \iff (z\cdot\tfrac{d}{dz}-(\alpha+n))\,g=0$$

where $g(z):=z^n f(z)$, $n\in\mathbb{Z}$.

Thus the diffeq^{ns} for parameter α and $\alpha+n$ are equivalent via the substitution $f\longleftrightarrow g=z^n f$.

No surprise: Have Iso $\mathcal{L}_\alpha\simeq\mathcal{L}_{\alpha+n}$ of local systems, since the monodromy of $z^\alpha=e^{\alpha\log z}$ only depends on $\alpha \bmod \mathbb{Z}$.

b) Consider on $X=\mathbb{C}$ the diffeq^n

$$\boxed{z(z+1)\cdot h'(z) = -h(z)} \qquad (*)$$

Substituting $f(z):=-z\cdot h'(z)$ we get:

$$(z+1)f(z) = h(z)$$

$$\overset{z\frac{d}{dz}(\dots)}{\Longrightarrow} \quad z(z+1)f'(z)+zf(z) = zh'(z) =: -f(z)$$

$$\Longrightarrow \quad z(z+1)f'(z) = -(z+1)f'(z)$$

Thus we get

$$\boxed{zf'(z) = -f'(z)} \qquad (**)$$

(whose solution we know to be $\sim 1/z$)

Conversely, $(**)$ implies $(*)$ for $h(z):=(1+z)\cdot f(z)$.

Q: More intrinsic description without explicit equations?

Kashiwara, Malgrange, Bernstein ... (1970's):

Replace (systems of) linear ODE's on $X=\mathbb{C}$ by modules M under the Weyl algebra

$$\mathcal{D}_X := \{\text{diff operators } P=\sum_{i=0}^n p_i(z)\cdot\partial^i \mid p_i\in\mathbb{C}[z]\}$$

Note: This algebra is non-commutative, it is generated as a $\mathbb{C}$-algebra by z and $\partial:=\frac{d}{dz}$ w/ the commutator relation $[\partial,z]:=\partial z-z\partial=1$.

By convention "module" means "left module" ie $(PQ)\cdot m = P\cdot(Q\cdot m)$ for $P,Q\in\mathcal{D}_X$, $m\in M$.

Ex 3 For any open $U \subseteq X = \mathbb{C}$,

$$\mathcal{O}(U) := \{ f : U \to \mathbb{C} \text{ holomorphic} \}$$

is a $\mathcal{D}_X$-module via $z \cdot f := zf$ (product of fct^s)

$\partial \cdot f := f'$ (cplex derivative)

Note: $\partial \cdot (z \cdot f) = (zf)' = f + zf' = f + z \cdot (\partial \cdot f)$,

so the relation $[\partial, z] = 1$ expresses the product rule.

Lemma 4. For $P \in \mathcal{D}_X$ consider the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X \cdot P$

(quotient by the left ideal $\mathcal{D}_X \cdot P$ is still a left $\mathcal{D}_X$-module)

$\Longrightarrow$ For any open $U \subseteq X$, $\exists$ iso of $\mathbb{C}$-vector spaces

$$\{ f \in \mathcal{O}(U) \mid P(f) = 0 \} \xrightarrow[\varphi]{\sim} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}(U)).$$

(homomorphisms of left $\mathcal{D}_X$-modules)

Pf. Given $f \in \mathcal{O}(U)$ w/ $P(f) = 0$,

define $F = \varphi(f) : \mathcal{M} \longrightarrow \mathcal{O}(U)$

by $F(Q \bmod \mathcal{D}_X \cdot P) := Q(f) \in \mathcal{O}(U)$

(well-defined since $(R \cdot P)(f) = R(\underbrace{P(f)}_{=0}) = 0 \ \forall R \in \mathcal{D}_X$).

Conversely, given $F \in \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}(U))$,

put $f := F(1 \bmod \mathcal{D}_X \cdot P) \in \mathcal{O}(U)$.

Then $P(f) = P \cdot F(1)$

$= F(P \cdot 1) = F(0) = 0.$ $\square$

(since F is $\mathcal{D}_X$-linear) (since $P \cdot 1 = P \equiv 0 \bmod \mathcal{D}_X \cdot P$)

Note An iso $\mathcal{M} \simeq \mathcal{N}$ of $\mathcal{D}_X$-modules clearly induces an iso of solution spaces

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}(U)) \simeq \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}(U)) \quad \forall U \subseteq X,$$

or better an iso of sheaves

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

$\Longrightarrow$ Conceptual view on when two diff eq^{ns} are "equivalent": The intrinsic object is not the eq^n but the $\mathcal{D}_X$-module!

Ex 5 In ex. 2(b) consider $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X \cdot (z(z+1)\partial + 1)$

$$\mathcal{N} := \mathcal{D}_X / \mathcal{D}_X \cdot (z\partial + 1).$$

Exercise: We have an iso of $\mathcal{D}_X$-modules $\mathcal{M} \simeq \mathcal{N}$.

More precisely one computes

$$(z(z+1)\partial + 1) \circ (z+1) = \cdots = (z+1)^2 \circ (z\partial + 1) \quad \text{in } \mathcal{D}_X$$

so the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{D}_X \arrow[r, "\circ(z\partial+1)"] & \mathcal{D}_X \arrow[r] & \mathcal{N} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{D}_X \arrow[r, "\circ(z(z+1)\partial+1)"] \arrow[u, "\cdot(z+1)^2"] & \mathcal{D}_X \arrow[r] \arrow[u, "\cdot(z+1)"] & \mathcal{M} \arrow[r] \arrow[u, dashed, "\exists!\varphi"] & 0 \end{tikzcd}$$

commutes (w/ exact rows).

$\Longrightarrow$ get a unique $\varphi \in \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$ as shown.

Now check that φ is an iso

(hint: for surjectivity note that

$$\underbrace{z\partial \cdot (z+1)}_{\varphi(1)} = z(z+1)\partial + z = z\underbrace{(z\partial + 1)}_{\equiv 0 \text{ in } \mathcal{N}} + z\partial \equiv z\partial \equiv -1 \quad \text{in } \mathcal{N} \ldots)$$

Conclusion The two diff eq^{ns} (*) & (**) from ex 2(b) correspond to two different presentations of "the same" abstract $\mathcal{D}_X$-module.

More generally, can do several variables & several eq^{ns}:

On $X = \mathbb{C}^n$ consider the Weyl algebra

$$\mathcal{D}_X := \Big\{ \sum_I f_I(z)\, \partial^I \;\Big|\; f_I(z) \in \mathbb{C}[z_1, \ldots, z_n] \Big\}$$

$$\text{where } \partial^I := \partial_1^{i_1} \cdots \partial_n^{i_n}$$

$$= \mathbb{C}[z_1, \ldots, z_n]\langle \partial_1, \ldots, \partial_n \rangle$$

generated by $z_1, \ldots, z_n, \partial_1, \ldots, \partial_n$

w/ relations
- $[z_i, z_j] = [\partial_i, \partial_j] = 0 \quad \forall i, j$
- $[\partial_i, z_j] = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Lemma 6 Consider the system of linear PDE's $D(f) = 0$ for a given matrix $D \in \mathrm{Mat}_{r \times s}(\mathcal{D}_X)$, $X = \mathbb{C}^n$, and an unknown solution vector $f \in (\mathcal{O}(U))^{\oplus s}$ ($U \subset X$ open).

$\Longrightarrow$ $\exists$ Iso of $\mathbb{C}$-v'spaces

$$\{ f \in (\mathcal{O}(U))^s \mid D(f) = 0 \} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}(U))$$

for $$\mathcal{M} := \mathrm{coker}\left(\mathcal{D}_X^{\oplus r} \xrightarrow{\cdot D} \mathcal{D}_X^{\oplus s} \right).$$

$\square$

Pf. Same as for lemma 4.

Still more generally:

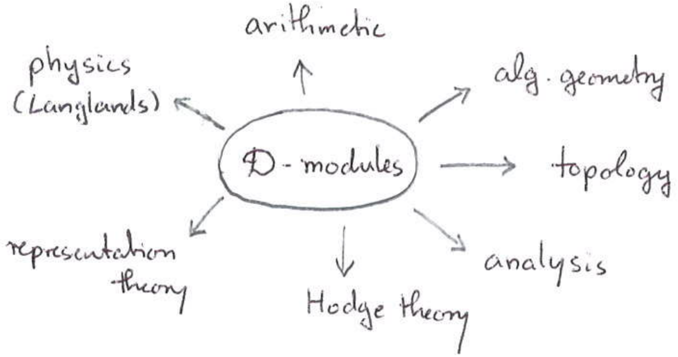
Can do the same on any smooth alg variety $/\mathbb{C}$, using the <u>sheaf</u> $\mathcal{D}_X$ of algebraic differential operators.

(later in the course)

$\Longrightarrow$ Algebraic geometer's view on linear PDE's via homological algebra & sheaf theory

(6 functors ...)

<u>Many facets:</u>



Plan of the lecture:

- $\mathcal{D}$-modules on affine space $\mathbb{A}^n$
- Some homological algebra (derived categories ...)
- $\mathcal{D}$-modules on arbitrary varieties & the 6 functors
- Outlook: The Riemann-Hilbert correspondence

"topology" $\{\text{perverse sheaves}\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{regular holonomic} \\ \mathcal{D}\text{-modules}\end{array}\right\}$ "alg. geometry"

(Kashiwara-Mebkhout 1980's)

<u>Literature:</u>

- Coutinho, A primer ...
- HTT, $\mathcal{D}$-modules, perverse sheaves and rep theory
- Kashiwara, $\mathcal{D}$-modules and microlocal calculus
- Borel, Algebraic $\mathcal{D}$-modules
- Bernstein
- Braverman / Chmutova / Etingof / Yang

} unpublished notes (online)

I. $\mathcal{D}$-modules on affine space

1. Motivation: Bernstein-Sato polynomials

Let $f \in \mathbb{R}[x_1, \dots, x_n]$.

Q (Gelfand '63). What's the meaning of the complex power f^s for $s \in \mathbb{C}$?

More precisely: For $\mathrm{Re}(s) > 0$ the fct^n

$$f_+^s(x) := \begin{cases} f(x)^s & \text{if } f(x) > 0 \\ 0 & \text{else} \end{cases}$$

is locally integrable (:= integrable on any compact subset of $\mathbb{R}^n$), hence defines a distribution via

$$\langle f_+^s, \varphi \rangle := \int_{\mathbb{R}^n} f_+^s(x)\, \varphi(x)\, \underbrace{dx}_{\text{Lebesgue measure}} \quad \forall\, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$$

"test fct^n"

Exercise a) These distributions are tempered, ie we can also take

$$\varphi \in \mathcal{S}(\mathbb{R}^n) := \{ g \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \ \ \forall \text{ multiindices } \alpha, \beta \}$$

"Schwartz space of rapid decay test fct^{ns}"

b) they depend holomorphically on $s \in \mathbb{C}$ w/ $Re(s) > 0$, ie the function

$$F_{f,\varphi} : \{ s \in \mathbb{C} \mid Re(s) > 0 \} \to \mathbb{C}$$

$$s \longmapsto \langle f_+^s, \varphi \rangle$$

is holomorphic for any fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

<u>Gelfand's question:</u> Do these $F_{f,\varphi}$ extend meromorphically to the whole complex plane? If so, are their poles independent of φ?

<u>Ex 1</u> Take $n = 1$ & $f(x) = x$

<u>Claim:</u> For any $\varphi \in \mathcal{S}(\mathbb{R})$, the fct^n

$$F(s) := \int_0^\infty x^s \cdot \varphi(x)\, dx \qquad (s \in \mathbb{C},\ Re(s) > 0)$$

extends to a meromorphic fct^n on the cplex plane w/ poles in $s = -1, -2, -3, \ldots$

(eg. for $\varphi(x) = e^{-x}$ we get the Γ-function $\Gamma(s+1) = \int_0^\infty x^s e^{-x} dx \ldots$)

Proof of the claim: $\frac{d}{dx}(x^{s+1}) = (s+1)\, x^s$ for $x > 0$, $Re(s) > 0$

$$\Rightarrow \underbrace{\int_0^\infty x^s \varphi(x)\, dx}_{\text{(defined for } Re(s)>0)} = \frac{1}{s+1} \int_0^\infty \frac{d}{dx}(x^{s+1})\, \varphi(x)\, dx$$

$$= \frac{1}{s+1} \int_0^\infty x^{s+1} \underbrace{\frac{d}{dx}\varphi(x)}_{\in \mathcal{S}(\mathbb{R})} dx \qquad \text{(defined for } Re(s) > -1)$$

$$= \ldots$$

$$= \frac{1}{(s+1)(s+2)\cdots(s+k)} \int_0^\infty x^{s+k} \varphi^{(k)}(x)\, dx$$

(defined for $Re(s) > k$, $s \neq -1, -2, -3, \ldots$)

$\square$

via integration by parts.

Back to the general case:

<u>Thm 2</u> ($\underbrace{\text{Bernstein-Gelfand '69, Atiyah '70}}_{\text{using resolution of singularities (complicated...)}}$, $\underbrace{\text{Bernstein '72}}_{\text{using } \mathcal{D}\text{-modules (very easy!)}}$)

For any $f \in \mathbb{R}[x_1, \ldots, x_n]$ $\exists\ s_1, \ldots, s_N \in \mathbb{C}$ s.th. for all test fct^s $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the fct^n $F_{f,\varphi}(s) := \langle f_+^s, \varphi \rangle$ extends meromorphically to $\mathbb{C}$ w/ poles in $\bigcup_{i=1}^N \underbrace{(s_i - \mathbb{N})}_{\text{"arithmetic progression to the left"}}$.

Bernstein's proof is purely algebraic, generalizing the identity

$$\frac{d}{dx}(x^{s+1}) = (s+1)\, x^s \quad \text{from example 1.}$$

$\frac{d}{dx}$: diff operator

$(s+1)$: polynomial in $\mathbb{C}[s]$

Setup
- k a field w/ $\mathrm{char}(k) = 0$.
- $\mathcal{D}_{n,k} := \mathcal{D}_{\mathbb{A}^n_k} := k[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$

  "Weyl algebra" w/ relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$,
  $$[\partial_i, x_j] = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$
- $f \in k[x_1, \ldots, x_n]$ non-constant polynomial.
- take a "dummy variable" s

  & extend the base field: $k \rightsquigarrow K := k(s) := \mathrm{Quot}(k[s])$

$$\mathcal{D}_{n,k} \rightsquigarrow \mathcal{D}_{n,K}$$

Def Let $\mathcal{M}_{f^s} := K[x_1, \ldots, x_n, \frac{1}{f}] \cdot f^s$

f^s: formal basis vector (just notation!)

denote the free $K[x_1, \ldots, x_n, 1/f]$-module of rank 1, viewed as a $\mathcal{D}_{n,K}$-module via

$$\partial_i(g \cdot f^s) := \underbrace{\left(\partial_i(g) + s \cdot g \cdot \frac{\partial_i(f)}{f}\right)}_{\in K[x_1, \ldots, x_n, 1/f]} \cdot f^s \in \mathcal{M}_{f^s}$$

for $g \in K[x_1, \ldots, x_n, 1/f]$.

To simplify notation we put $f^{s+m} := f^m \cdot f^s \in \mathcal{M}_{f^s}$ for $m \in \mathbb{Z}$.

Key point for Bernstein's proof of thm 2:

Thm 3 $\exists$ non-zero polynomial $b(s) \in k[s]$ sth

$$b(s) \cdot f^s = P(f^{s+1})$$

for some diff'operator

$$P \in \mathcal{D}_{n,k}[s] := k[s, x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle \subseteq \mathcal{D}_{n,K}.$$

Ex a) $n = 1$, $f(x) = x \Rightarrow \partial x^{s+1} = (s+1)\, x^s$

$\Rightarrow b(s) := s+1$ works.

b) $f(x) = x_1^2 + \cdots + x_n^2 \Rightarrow \partial_i f^{s+1} = 2(s+1)\, x_i f^s$

$\Rightarrow \partial_i^2 f^{s+1} = 4s(s+1)\, x_i^2 f^{s-1} + 2(s+1) f^s$

$\Rightarrow P := \partial_1^2 + \cdots + \partial_n^2$ has $P(f^{s+1}) = \underbrace{(4s(s+1) + 2n(s+1))}_{= 4(s+1)(s+n/2)} f^s$

$\Rightarrow b(s) := 4(s+1)(s+\frac{n}{2})$ works.

c) Exercise: Find a suitable $b(s)$ for monomials

$$f(x) = x_1^{e_1}\cdots x_n^{e_n}$$

(by applying $P := \partial_1^{e_1}\cdots\partial_n^{e_n}$).

d) Exercise: Show that for $f(x) = x_1^2 - x_2^3$ $(n=2)$,

$$b(s) := (s+1)(s+\tfrac{5}{6})(s+\tfrac{7}{6}) \quad \text{works.}$$

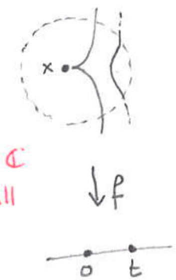
Rem The set of all $b(s)\in k[s]$ that work for a given $f\in k[x]$ form an ideal $I \trianglelefteq k[s]$. Its unique monic generator is denoted $b_f(s)\in k[s]$ and called the

"Bernstein-Sato polynomial of f."

These are subtle to compute and deeply linked to singularity theory. A thm by Malgrange & Kashiwara says that its zeroes are in $\mathbb{Q}_{<0}$ and

$$\exp(\text{zeroes of } b_f) = \bigcup_{\substack{x\in f^{-1}(0)\\ \text{close to } 0}} \left\{ \begin{matrix}\text{monodromy eigenvalues}\\ \text{on } H^\bullet(F_x,\mathbb{C})\end{matrix} \right\}$$

for the "Milnor fibre" $F_x := \underbrace{B_\varepsilon(x)}_{\text{small ball around } x} \cap \underbrace{f^{-1}(t)}_{\text{fibre over } t\in\mathbb{C} \text{ w/ } |t| \text{ small}}$



Pf of thm 2, assuming thm 3:

Let $b(s)\, f^s = P(f^{s+1})$ as in thm 3, w/ $b\in\mathbb{C}[s]$, $P\in D_{n,\mathbb{C}}[s]$.

$$\Rightarrow b(s)\cdot\langle f_+^s,\varphi\rangle = b(s)\cdot\int_{\mathbb{R}^n} f_+^s(x)\,\varphi(x)\,dx$$

$$\overset{(\text{thm } 2)}{=} \int_{\{f>0\}} \varphi(x)\cdot P(f^{s+1})\,dx$$

$$= \int_{\mathbb{R}^n} P^*(\varphi(x))\cdot f_+^{s+1}\,dx$$

(for the adjoint operator $P^* := \sum_I (-1)^{|I|}\,\partial^I c_I(x,s)$ of $P = \sum_I c_I(x,s)\cdot\partial^I$)

$$= \langle f_+^{s+1}, P^*(\varphi)\rangle \quad \text{for } \mathrm{Re}(s) > 0$$

$\Rightarrow$ For $\mathrm{Re}(s) > 0$,

$$\langle f_+^s,\varphi\rangle = \underbrace{\frac{1}{b(s)}\cdot\langle f_+^{s+1}, P^*(\varphi)\rangle}_{\substack{\text{well-defined meromorphic fct}^n \\ \text{on } \mathrm{Re}(s) > -1, \text{ w/ poles only} \\ \text{in the set } \{s_1,\dots,s_N\} \text{ of zeroes of } b(s).}}$$

$\Rightarrow$ Claim by induction. $\square$

2. Reformulation as a finiteness condition

Recall k field of char $k = 0$, $K := k(s)$,

$f \in k[x_1, \dots, x_n]$ non-constant,

$$M_{f^s} := K[x_1, \dots, x_n, \tfrac{1}{f}] \cdot f^s$$

viewed as a left module under $D_{n,K}$ via

$$\partial_i(g \cdot f^s) := \left(\partial_i(g) + s g \frac{\partial_i(f)}{f}\right) \cdot f^s$$

for $g \in K[x_1, \dots, x_n, \tfrac{1}{f}]$.

Prop The following are equivalent:

a) $\exists\, b \in k[s] \setminus \{0\}$, $P \in D_{n,k}[s]$:

$$P(f^{s+1}) = b(s) \cdot f^s \quad \text{in } M_{f^s}.$$

b) M_f is finitely generated as a $D_{n,K}$-module.

shorthand: $x = (x_1, \dots, x_n)$
$\partial = (\partial_1, \dots, \partial_n)$

Pf. a) $\Rightarrow$ b): Take $P = P(s, x, \partial) \in D_{n,k}[s]$
and $b = b(s) \in k[s]$ as in a)

For $m \in \mathbb{N}$ put $P_m := P(s-m, x, \partial) \in D_{n,k}[s]$
$b_m := b(s-m) \in k[s]$

$$\Longrightarrow \quad P_m(f^{s+1-m}) = b_m(s) \cdot f^{s-m} \quad \text{in } M_{f^s}$$

since the variable transformation $s \mapsto t = s - m$ yields a diagram

$$\begin{tikzcd}
D_{n,k}[t] \arrow[r, "\sim", "t \mapsto s-m"'] \arrow[d, phantom, "\circlearrowright"] & D_{n,k}[s] \arrow[d, phantom, "\circlearrowright"] \\
M_{f^t} \arrow[r, "\sim"] & M_{f^s}
\end{tikzcd}$$

$$g(t,x) \cdot f^t \longmapsto g(s-m, x) \cdot f(x)^{-m} \cdot f^s$$
$$=: g(s-m, x) \cdot f^{s-m}$$

$$P(f^{t+1}) = b(t) \cdot f^t \quad \Longleftrightarrow \quad P_m(f^{s+1-m}) = b_m(s) f^{s-m}$$

by assumption a)

Upshot: $f^{s-m} \in D_{n,K} \cdot f^{s+1} \quad \forall\, m \in \mathbb{Z}$

$\Longrightarrow$ As a $D_{n,K}$-module,

M_{f^s} is generated by f^{s+1} (or by any other f^{s+p} with $p \in \mathbb{Z}$ fixed)

$\Longrightarrow$ In particular, b) holds.

b) $\Longrightarrow$ a):

Put $F_i M_{f^s} := \mathcal{D}_{n,K} \cdot f^{s+1-i} \subseteq M_{f^s}$

(the submodule generated by f^{s+1-i})

Then $F_i M_{f^s} \subseteq F_{i+1} M_{f^s} \subseteq \cdots \subseteq M_{f^s} = \bigcup_{j \in \mathbb{N}} F_j M_{f^s}$

(increasing, exhaustive filtration).

Assuming b), $\exists$ finite set of elements generating M_{f^s} as a $\mathcal{D}_{n,K}$-module.

Pick $m \in \mathbb{N}$ sth $F_m M_{f^s}$ contains all these generators

$\Longrightarrow$ $M_{f^s} = F_m M_{f^s}$

$\Longrightarrow$ $\exists\, Q_m \in \mathcal{D}_{n,K}$ sth $Q_m(f^{s+1-m}) = f^{s-m}$

As before, by a variable transformation $s \longmapsto s-m$ we get $Q \in \mathcal{D}_{n,K}$ sth $Q(f^{s+1}) = f^s$.

Writing $Q(s,x,\partial) = \frac{1}{b(s)} \cdot P(s,x,\partial)$ w/ $P \in \mathcal{D}_{n,k}[s]$, $b \in k[s] \setminus \{0\}$

we get a) as claimed. $\square$

<u>Conclusion</u> For Bernstein's thm 1.2 & 1.3 we only need to prove M_{f^s} is finitely generated as a $\mathcal{D}_{n,K}$-module. This will require some mildly non-commutative algebra (see below).

3. Filtered & graded algebras

Let $\mathcal{D}$ be a k-algebra (associative with 1 but maybe non-commutative), endowed w/ an increasing filtration by k-subspaces

$$\cdots \subseteq F_i \mathcal{D} \subseteq F_{i+1} \mathcal{D} \subseteq \cdots \subseteq \mathcal{D} \qquad (i \in \mathbb{Z})$$

sth

① $\mathcal{D} = \bigcup_i F_i \mathcal{D}$,

② $F_i \mathcal{D} = 0 \ \forall i < 0$ and $1 \in F_0 \mathcal{D}$,

③ $F_i \mathcal{D} \cdot F_j \mathcal{D} \subseteq F_{i+j} \mathcal{D} \ \forall i,j \in \mathbb{Z}$

Def The <u>associated graded algebra</u> is

$$\mathrm{gr}^F_\bullet \mathcal{D} := \bigoplus_{i \in \mathbb{N}_0} \mathrm{gr}^F_i \mathcal{D} \quad \text{w/} \quad \mathrm{gr}^F_i \mathcal{D} := F_i\mathcal{D} / F_{i-1}\mathcal{D}$$

Rem 1 a) This is a graded k-algebra:

For $a \in F_i \mathcal{D}$, $b \in F_j \mathcal{D}$ w/ classes $[a] \in \mathrm{gr}^F_i \mathcal{D}$, $[b] \in \mathrm{gr}^F_j \mathcal{D}$

we put

$$[a]\cdot[b] := [ab] \in \mathrm{gr}^F_{i+j} \mathcal{D}.$$

The unit of this algebra is $[1] \in \mathrm{gr}^F_0 \mathcal{D}$ and the grading is compatible w/ multiplication, ie

$$\mathrm{gr}^F_i \mathcal{D} \cdot \mathrm{gr}^F_j \mathcal{D} \subseteq \mathrm{gr}^F_{i+j} \mathcal{D} \quad \forall i,j \in \mathbb{Z}.$$

commutator

b) In our later applications $[F_i\mathcal{D}, F_j\mathcal{D}] \subseteq F_{i+j-1}\mathcal{D}$, and then $\mathrm{gr}^F_\bullet \mathcal{D}$ will be a commutative algebra.

$\Longrightarrow$ Reduction to commutative algebra!

Ex 2 On the Weyl algebra $\mathcal{D} := \mathcal{D}_{n,k}$, $\exists$ two important filtrations:

a) the <u>order filtration</u>:

$$F_i \mathcal{D} := \left\{ P = \sum_I a_I(x)\, \partial^I \;\middle|\; a_I = 0 \text{ for } |I| > i \right\}$$

multiindices: $I = (i_1, \dots, i_n)$, $\partial^I = \partial_1^{i_1} \cdots \partial_n^{i_n}$, $|I| = i_1 + \dots + i_n$

ie
- $F_0 \mathcal{D} := k[x_1, \dots, x_n]$
- $F_1 \mathcal{D} := F_0\mathcal{D} \oplus \bigoplus_{i=1}^n F_0\mathcal{D}\cdot\partial_i$
- $F_i \mathcal{D} := \mathrm{image}\left((F_1\mathcal{D})^{\otimes i} \xrightarrow[\text{multiplication map}]{} \mathcal{D} \right)$

⚠ The order filtration is independent of the chosen coordinates, so it will glue to a filtration on $\mathcal{D}_X$ for any smooth variety X (see later).

However, $\dim_k F_i \mathcal{D} = \infty \quad \forall i \in \mathbb{N}_0$!

b) If we only work on $X = \mathbb{A}^n_k$ w/ a fixed coordinate system, we can instead use the <u>Bernstein filtration</u>:

Here one puts

- $F_0 \mathcal{D} := k$
- $F_1 \mathcal{D} := \langle 1, x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle_k$ ← "k-span"
- $F_i \mathcal{D} := \operatorname{image}\left((F_1 \mathcal{D})^{\otimes i} \xrightarrow[\text{mult}]{} \mathcal{D} \right)$

⚠ Here $\dim_k F_i \mathcal{D} < \infty \ \forall i \in \mathbb{Z}$

but $F_\bullet \mathcal{D}$ is NOT preserved under non-linear coordinate changes.

__Lemma 2__ Both the order & the Bernstein filtration on $\mathcal{D} = \mathcal{D}_{n,k}$ satisfy

$$\operatorname{gr}^F_\bullet \mathcal{D} \simeq k[x_1, \dots, x_n, \xi_1, \dots, \xi_n].$$

Pf. Put

$$x_i := [x_i] \in \operatorname{gr}^F_\nu \mathcal{D}, \quad \nu = \begin{cases} 0 & \text{for order filtrat}^n \\ 1 & \text{for Bernstein filtrat}^n \end{cases}$$

abuse of notation...

$$\xi_i := [\partial_i] \in \operatorname{gr}^F_1 \mathcal{D} \quad \text{for both filtrations.}$$

Since $\mathcal{D}$ is generated as a k-algebra by the x_i and ∂_j w/ the only relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0,$$

$$[\partial_i, x_j] = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else}, \end{cases}$$

it only remains to check $[\xi_i, x_i] = 0$ in $\operatorname{gr}^F_{\nu+1} \mathcal{D}$ (ν as above).

This follows from

$$[\partial_i, x_i] = -1 \in F_0 \mathcal{D} \subseteq F_\nu \mathcal{D} \text{ in both cases.} \quad \square$$

__Rem 3__ For Bernstein's filtration $\nu = -1$, so we even have $[F_i \mathcal{D}, F_j \mathcal{D}] \subseteq F_{i+j-1-\nu} \mathcal{D}$.

4. Filtered and graded modules

__Recall:__ $(\mathcal{D}, F_\bullet \mathcal{D})$ filtered k-algebra, w/ $F_\bullet \mathcal{D}$ an increasing filtration by k-subspaces sth

1. $\mathcal{D} = \bigcup_i F_i \mathcal{D}$
2. $F_i \mathcal{D} = 0 \ \forall i < 0$ and $1 \in F_0 \mathcal{D}$
3. $F_i \mathcal{D} \cdot F_j \mathcal{D} \subseteq F_{i+j} \mathcal{D} \ \forall i, j \in \mathbb{Z}$

From now on we also assume:

4. $\operatorname{gr}^F_\bullet \mathcal{D}$ is commutative, and generated as a $\operatorname{gr}^F_0 \mathcal{D}$-algebra by finitely many $\xi_1, \dots, \xi_n \in \operatorname{gr}^F_1 \mathcal{D}$.

(like the Weyl algebra $\mathcal{D} = \mathcal{D}_{n,k}$)

Def Let $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}) := \{\text{left } \mathcal{D}\text{-modules}\}$.

A filtration by k-subspaces $\cdots \subseteq F_i\mathcal{M} \subseteq F_{i+1}\mathcal{M} \subseteq \cdots \subseteq \mathcal{M}$ $(i \in \mathbb{Z})$

is
- exhaustive if $\bigcup_i F_i\mathcal{M} = \mathcal{M}$
- separated if $\bigcap_i F_i\mathcal{M} = \{0\}$
- compatible if it is exhaustive, separated, and

$$F_i\mathcal{D} \cdot F_j\mathcal{M} \subseteq F_{i+j}\mathcal{M} \quad \forall i,j \in \mathbb{Z}.$$

$\Rightarrow$ $gr^F_\bullet\mathcal{M} := \bigoplus_{i\in\mathbb{Z}} gr^F_i\mathcal{M}$ is a module over $gr^F_\bullet\mathcal{D}$,

in fact a graded module over the graded ring $gr^F_\bullet\mathcal{D}$:

$$gr^F_i\mathcal{D} \cdot gr^F_j\mathcal{M} \subseteq gr^F_{i+j}\mathcal{M} \quad \forall i,j \in \mathbb{Z}.$$

Def A compatible filtration $F_\bullet\mathcal{M}$ is called good if $gr^F_\bullet\mathcal{M}$ is finitely generated as a module over $gr^F_\bullet\mathcal{D}$.

Prop 1 For a compatible filtration $F_\bullet\mathcal{M}$, the following are equivalent:

a) $F_\bullet\mathcal{M}$ is good

b)
- each $F_i\mathcal{M}$ is fingen over the ring $F_0\mathcal{D}$,
- $F_i\mathcal{M} = \{0\}$ $\forall i \ll 0$ $\leftarrow$ (this is more than separatedness!)
- $\exists j_0 \in \mathbb{Z}$ sth $\forall j \geq j_0$, $F_i\mathcal{D} \cdot F_j\mathcal{M} = F_{i+j}\mathcal{M}$ for all $i \geq 0$.

(equality, not just "$\subseteq$"!)

Pf. b) $\Rightarrow$ a): By assumption,

- $\forall j \geq j_0$ $\forall i \geq 0$, $gr^F_i\mathcal{D} \cdot gr^F_j\mathcal{M} = gr^F_{i+j}\mathcal{M}$

$$\Rightarrow \quad gr^F_\bullet\mathcal{M} = gr^F_\bullet\mathcal{D} \cdot \Big(\bigoplus_{j \leq j_0} gr^F_j\mathcal{M}\Big)$$

- $F_j\mathcal{M} = 0$ $\forall j \ll 0$ $\Rightarrow$ Only finitely many direct summands enter
- each $F_j\mathcal{M}$ is fingen over $F_0\mathcal{D}$ $\Rightarrow$ so is each $gr^F_j\mathcal{M}$

Altogether then $gr^F_\bullet\mathcal{M}$ is fingen over $gr^F_\bullet\mathcal{D}$.

a) $\Rightarrow$ b):

- $gr^F_\bullet\mathcal{M}$ fingen over $gr^F_\bullet\mathcal{D}$ and $gr^F_i\mathcal{D} = 0$ $\forall i < 0$

$$\Rightarrow \quad gr^F_i\mathcal{M} \begin{cases} \text{fingen over } F_0\mathcal{D} \ \forall i \in \mathbb{Z}, \\ = \{0\} \ \forall i \ll 0 \end{cases} \quad \text{(exercise).}$$

Key point: Pick homogenous generators for $gr^F_\bullet\mathcal{M}$ & use that $\exists$ only fin. many monomials of given degree in the fin. many elements from (4)

$\Rightarrow$ by separatedness $F_i M = \{0\}$ $\forall i \ll 0$

& induction implies all $F_i M$ are fin gen over $F_0 D$

(using the exact sequences $0 \to F_{i-1} M \to F_i M \to gr_i^F M \to 0$)

- Now pick $j_0 \in \mathbb{Z}$ s.th. $gr_\bullet^F M = gr_\bullet^F D \left(\bigoplus_{j \le j_0} gr_j^F M \right)$.

For $\ell \geq j_0$ then

$$gr_{\ell+1}^F M = \sum_{j \le j_0} gr_{\ell+1-j}^F D \cdot gr_j^F M$$

$$\subseteq gr_1^F D \cdot gr_\ell^F M \subseteq gr_{\ell+1}^F M$$

$$\Rightarrow gr_{\ell+1}^F M = gr_1^F D \cdot gr_\ell^F M \quad \forall \ell \geq j_0$$

$$\Rightarrow F_{\ell+1} M = F_1 D \cdot F_\ell M + F_\ell M$$

$$= F_1 D \cdot F_\ell M \qquad \forall \ell \geq j_0$$

$\Rightarrow$ By induction,

$$F_{i+j} M = \underbrace{F_1 D \cdots F_1 D}_{i \text{ factors}} \cdot F_j M \subseteq F_i D \cdot F_j M$$

$$\subseteq F_{i+j} M \qquad \forall j \geq j_0, \; i \geq 0$$

$$\Rightarrow F_{i+j} M = F_i D \cdot F_j M \quad \forall j \geq j_0, \; \forall i \geq 0.$$

$\square$

__Exercise__ It suffices to assume b) for $j = j_0$ fixed (a priori a weaker condit^n)

__Cor 2__ For any D-module M, the following are equivalent:

a) M admits a (compatible, exhaustive, separated and) good filtration.

b) M is fingen as a D-module.

Pf. a) $\Rightarrow$ b): By prop 1,

$$M = \bigcup_i F_i M \quad \text{and} \quad F_{i+j_0} M = F_i D \cdot F_{j_0} M$$

for $j_0 \gg 0$

$\Rightarrow$ $F_{j_0} M$ generates M as a D-module

fin.gen. as an $F_0 D$-module

$\Rightarrow$ M fingen as a D-module

b) $\Rightarrow$ a):

Pick a fingen $F_0 D$-submodule $M_0 \subseteq M$ that generates M as a D-module (using assumption b)).

Put

$$F_i M := \begin{cases} 0 & \text{for } i < 0 \\ F_i D \cdot M_0 & \text{for } i \geq 0 \end{cases}$$

$\Rightarrow$ $F_\bullet M$ separated + exhaustive

& satisfies the conditions in prop 1 (b) (use ④ for finite generation of $F_i M$ over $F_0 D$)

$\square$

Thus we've passed to commutative algebra:

$$\underset{\text{(non-commutative algebra)}}{M \in \mathrm{Mod}(\mathcal{D}) \text{ fingen}} \iff \exists \text{ good } F_\bullet M \iff \underset{\text{(commutative algebra)}}{\mathrm{gr}^F_\bullet M \in \mathrm{Mod}(\mathrm{gr}^F_\bullet \mathcal{D}) \text{ fingen.}}$$

Good filtrations are well-behaved:

Lemma 3 Let $M \in \mathrm{Mod}(\mathcal{D})$ fingen w/ a good filtration $F_\bullet M$. Then $F_\bullet M$ is "finer" than any other compatible filtratⁿ $G_\bullet M$,

ie $\exists\, \delta \in \mathbb{Z}$: $\quad F_i M \subseteq G_{i+\delta} M \quad \forall i \in \mathbb{Z}$.

Pf. Let $j_0 \in \mathbb{Z}$ sth $F_i \mathcal{D} \cdot F_{j_0} M = F_{i+j_0} M \ \forall i \geq 0$

(exercise after prop 1)

Let $G_\bullet M$ be any compatible filtration.

$\underset{\substack{F_{j_0}M \text{ fin.gen. over } F_0\mathcal{D} \\ \text{(prop 1b)}}}{\Longrightarrow}$ $\exists\, j_1 \in \mathbb{Z}$ sth $F_{j_0} M \subseteq G_{j_1} M$ because $G_\bullet$ is exhaustive

$\Longrightarrow$ For all $i \geq j_0$,

$$F_i M \underset{\text{choice of } j_0}{=} F_{i-j_0}\mathcal{D} \cdot F_{j_0} M \underset{\text{choice of } j_1}{\subseteq} F_{i-j_0}\mathcal{D} \cdot G_{j_1} M \overset{\text{compatibility of } G_\bullet}{\subseteq} G_{i-j_0+j_1} M$$
$$\subseteq G_{i+\delta} M$$

for any $\delta \geq j_1 - j_0$.

The finitely many $i < j_0$ can be taken care of by enlarging δ as needed. $\square$

Cor 4 Any two good filtrations $F_\bullet M$, $G_\bullet M$ are "equivalent"

ie each is finer than the other:

$\exists\, \varepsilon, \delta \in \mathbb{Z}$ sth $\quad G_{i-\varepsilon} M \subseteq F_i M \subseteq G_{i+\delta} M \quad \forall i \in$

⚠ In general the $\mathrm{gr}^F_\bullet \mathcal{D}$-module $\mathrm{gr}^F_\bullet M$ depends on the chosen good filtration $F_\bullet M$, there's no canonical choice.

Q: $\exists$ more intrinsic invariant? ..

5. A reminder on Hilbert polynomials

Setup: $A = \bigoplus_{i \in \mathbb{N}_0} A_i$ a graded k-algebra

which is
- commutative w/ $1 \in A_0$,
- finitely generated as a k-algebra,
- not too big: $\dim_k A_i < \infty \ \forall i$.

Ex
- $A = k[y_1, \dots, y_m]$ polynomial ring, graded by any choice of degrees $\deg(y_i) := d_i \in \mathbb{N}_0$.
- In particular, we'll later take $A = \mathrm{gr}^F_\bullet \mathcal{D}_{n,k} \simeq k[x_1, .., x_n, \xi_1, .., \xi_n]$ for the Bernstein filtration $F_\bullet$ on the Weyl algebra.

  Here $\deg(x_i) = \deg(\xi_i) = 1$ (unlike for the order filtration where $\deg(x_i) = 0$ & $\dim_k A_0 = \infty$

Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a <u>graded A-module</u>,

ie

$M \in \mathrm{Mod}(A)$ is the direct sum of k-subspaces $M_i \subseteq M$

sth $A_i \cdot M_j \subseteq M_{i+j}$ $\forall i \in \mathbb{N}_0, j \in \mathbb{Z}$.

(later $M = gr^F_\bullet \mathcal{M}$ for a good

filtration $F_\bullet$ on $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_{n,k})$)

Rem 5 If M is fingen as an A-module,

then a) $M_i = 0$ $\forall i \ll 0$,

b) $\dim_k M_i < \infty$ $\forall i \in \mathbb{Z}$.

Pf. Pick a set of generators $m_1, \dots, m_n \in M$.

Replace them by all their "homogenous summands"

$\Rightarrow$ wlog all m_ν homogenous, say $m_\nu \in M_{i(\nu)}$ w/ $i(\nu) \in \mathbb{Z}$.

$\Rightarrow$ $M_i = 0$ $\forall i < \min\{i(1), \dots, i(n)\}$, so a) holds.

For b) consider the "<u>shifted modules</u>" $N(d) := \bigoplus_{i \in \mathbb{Z}} N(d)_i$

w/ $N(d)_i := N_{d+i}$

$\Rightarrow$ $\exists$ degree-preserving epi for any graded $N \in \mathrm{Mod}(A)$.

$$\begin{array}{ccc} \bigoplus_{\nu=1}^{n} A(i(\nu)) & \longrightarrow\!\!\!\!\!\rightarrow & M \\ \uplus & & \uplus \\ a_\nu & \longmapsto & a_\nu \cdot m_\nu \end{array}$$

$\Rightarrow$ Claim b) since $\dim_k A_i < \infty$ $\forall i$. $\square$

Def For a fingen graded $M \in \mathrm{Mod}(A)$,

we consider the <u>Hilbert function</u>

$$h_M : \mathbb{Z} \longrightarrow \mathbb{N}_0$$
$$i \longmapsto \dim_k(M_i).$$

Write $A = k[a_1, \dots, a_n]$ w/ $a_i \in A$ homogenous of degree $d_i \in \mathbb{N}_0$.

→ the "<u>Hilbert polynomial</u> of M"

Thm 6 $\exists!$ polynomial $p_M(t) \in \mathbb{Q}[t]$ of degree $\deg p_M \leq n-1$

sth $p_M(i) = h_M(i)$ for all sufficiently large $i \in \mathbb{Z}$.

Pf.

$n = 0$ (or $d_i = 0$ $\forall i$): Trivial since then $A = A_0$,

so $M_i = 0$ for almost all i & we can take $p_M \equiv 0$.

Induction step: Assume claim for $n-1$ instead of n. Wlog $d_n > 0$

Put $K := \ker\left(M(-d_n) \xrightarrow{a_n \cdot} M \right)$.

$$\Rightarrow \quad 0 \to K \to M(-d_n) \to M \to M/a_n M \to 0$$

exact sequence of graded modules

$$\Rightarrow \quad h_M(i) - h_M(i-d_n) = \underbrace{h_{M/a_n M}(i) - h_K(i)}_{\text{polynomials of degree } \leq n-2 \text{ for } i \gg 0}$$

since M/a_nM and K are graded modules for $k[a_1, \dots, a_{n-1}]$.

$\Rightarrow$ claim by the auxiliary lemma below.

(up to rescaling $t \leftrightarrow t/d_n$) $\square$

Here we've used:

<u>Auxiliary lemma</u> Let $h : \mathbb{Z} \to \mathbb{N}_0$, and put $\Delta_h(i) := h(i) - h(i-1)$

If $\Delta_h(i) = q(i)$ for some $q \in \mathbb{Q}[t]$, all $i \gg 0$,

then also $h(i) = p(i)$ for some $p \in \mathbb{Q}[t]$, all $i \gg 0$,

and $\deg p = \deg q + 1$.

Pf. Assume $\Delta_h(i) = q(i)$ $\forall i \geq i_0$.

Put $p(i) := h(i_0) + \sum_{j \in \{i_0+1, \dots, i\}} q(j)$

regardless of whether $i \geq i_0 + 1$ or not

$\Longrightarrow$ ① $p(i) = h(i)$ $\forall i \geq i_0$

② $\Delta_p(i) := p(i) - p(i-1) = q(i)$ $\forall i \in \mathbb{Z}$,

hence $\Delta_p \in \mathbb{Q}[t]$

Exercise Conclude from ② that $p(t) \in \mathbb{Q}[t]$ is a polynomial fct^n and $\deg p = \deg q + 1$.

Hint: • Enough to show $\sum_{i=0}^{t} q(i)$ is polynomial in t of degree $\deg q + 1$ $\forall q \in \mathbb{Q}[x]$.

• Reduce to the case $q(x) = \binom{x}{m}$ w/ $m \in \mathbb{N}_0$

(these form a $\mathbb{Q}$-basis for the space $\mathbb{Q}[t]$)

$\square$

6. Bernstein's inequality

Let's come back to non-commutative algebra:

<u>Setup</u> $\mathcal{D} = \mathcal{D}_{n,k}$ Weyl algebra

$F_\bullet \mathcal{D}$ Bernstein's filtration,

ie $F_i \mathcal{D} := \langle x^I \partial^J \mid |I| + |J| \leq i \rangle_k$

multiindex notation in $x = (x_1, \dots, x_n)$ & $\partial = (\partial_1, \dots, \partial_n)$

$M \in \mathrm{Mod}(\mathcal{D})$ a left $\mathcal{D}$-module w/ a good filtration $F_\bullet M$

Def The <u>Hilbert function</u> of $(M, F_\bullet)$ is defined by

$$h_{M,F_\bullet} : \mathbb{Z} \to \mathbb{N}_0$$
$$i \mapsto \dim_k F_i M.$$

Rem. Each $F_i M$ is fin. gen. as a module over $F_0 \mathcal{D}$, and $\dim_k F_0 \mathcal{D} < \infty$

$\Rightarrow \dim_k F_i M < \infty$ $\forall i$.

Lemma 1 $\exists!$ polynomial $p_{\mathcal{M},F_\bullet}(t) \in \mathbb{Q}(t)$ of degree $\leq 2n$ sth

$$p_{\mathcal{M},F_\bullet}(i) = h_{\mathcal{M},F_\bullet}(i) \quad \forall\, i \gg 0.$$

Pf. The previous section applies to $M := \mathrm{gr}^F_\bullet \mathcal{M}$ as a graded module over $A := \mathrm{gr}^F_\bullet \mathcal{D} \simeq k[\underbrace{x_1,\dots,x_n,\xi_1,\dots,\xi_n}_{\text{2n generators (all in degree 1)}}]$

By def^n

$$\begin{aligned} h_{\mathcal{M},F_\bullet}(i) &= \dim_k F_i\mathcal{M} \\ &= \sum_{j\leq i} \dim_k \mathrm{gr}^F_j \mathcal{M} = \sum_{j\leq i} \underbrace{h_M(j)}_{\substack{\text{polynomial in } j \\ \forall j \geq j_0 \text{ (w/ } j_0 \text{ fixed)}}} \end{aligned}$$

$\Rightarrow$ By the auxiliary lemma from the previous section we're done. $\square$

Def We call $p_{\mathcal{M},F_\bullet}$ the <u>Hilbert polynomial</u> of $(\mathcal{M}, F_\bullet)$ and write

$$p_{\mathcal{M},F_\bullet}(t) = c \cdot \frac{t^d}{d!} + \text{lower order terms}$$

w/ $c = c(\mathcal{M}) \in \mathbb{N}$ (a priori $c \in \mathbb{Q}_{>0}$. But $p_{\mathcal{M},F}(t) \in \mathbb{Q}[t]$ takes integer values at all large integers $t = n \gg 0$, so it is in the $\mathbb{Z}$-span of the polynomials $p_m(t) = \binom{t}{m}$, $m \in \mathbb{N}_0$, which implies $c \in \mathbb{Z}$.)

$d = d(\mathcal{M}) \in \{0, 1, \dots, 2n\}$.

Lemma 2 The numbers $c(\mathcal{M})$ & $d(\mathcal{M})$ only depend on $\mathcal{M}$ (not on the good filtrat^n $F_\bullet\mathcal{M}$).

Pf. Let $F_\bullet, G_\bullet$ be two good filtrations on $\mathcal{M}$.

By Cor. 4.4, $\exists\, \varepsilon, \delta \in \mathbb{Z}$ w/ $G_{i-\varepsilon} \subseteq F_i \subseteq G_{i+\delta}$ $\forall i \in \mathbb{Z}$.

$\Rightarrow$ $h_{\mathcal{M},G_\bullet}(i-\varepsilon) \leq h_{\mathcal{M},F_\bullet}(i) \leq h_{\mathcal{M},G_\bullet}(i+\varepsilon)$ $\forall i \in \mathbb{Z}$

$\Rightarrow$ $p_{\mathcal{M},G_\bullet}(i-\varepsilon) \leq p_{\mathcal{M},F_\bullet}(i) \leq p_{\mathcal{M},G_\bullet}(i+\varepsilon)$ $\forall i \gg 0$

$\Rightarrow$ $p_{\mathcal{M},G_\bullet}(t)$ & $p_{\mathcal{M},F_\bullet}(t)$ must have the same leadin term. $\square$

Ex 3 a) $\mathcal{M} := k[x_1, \dots, x_n]$

$\Rightarrow$ $F_i\mathcal{M} := \{\text{polynomials of degree} \leq i\}$ is a good filtration (exercise)

Here

$$h_{\mathcal{M},F_\bullet}(i) = \binom{n+i}{n} = \frac{(n+i)(n+i-1)\cdots(i+1)}{n!} = p_{\mathcal{M},F}(i)$$

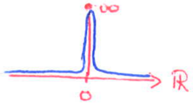
$$= \frac{i^n}{n!} + \dots$$

$\Rightarrow$ $p_{\mathcal{M},F_\bullet}(t) = \dfrac{t^n}{n!} + \dots$

$\Rightarrow$ $c(\mathcal{M}) = 1$

$d(\mathcal{M}) = n$.

b) The "Dirac δ distribution":



$$\mathcal{M} := \bigoplus_{I \in \mathbb{N}_0^n} k \cdot \delta^I$$

(intuitively, $\delta^{\underline{0}}$ = Dirac δ fct^n and $\delta^I = \partial^I(\delta^{\underline{0}}) \ldots$)

viewed as a $\mathcal{D}$-module via

$$\partial_\alpha(\delta^I) := \delta^{I+e_\alpha} \quad \text{w/ } e_\alpha := (0,\ldots,0,\overset{\text{position}}{\overset{\downarrow}{1}},0,\ldots,0)$$

$$x_\alpha(\delta^I) := \begin{cases} (-1)^{i_\alpha} \cdot i_\alpha \cdot \delta^{I-e_\alpha} & \text{if } i_\alpha > 0 \\ 0 & \text{if } i_\alpha = 0 \end{cases}$$

(intuitively, $x_\alpha \cdot \delta^{\underline{0}} = 0 \ldots$

exercise: For $n = 1$, one has $[x, \partial^i] = \begin{cases} (-1)^i \cdot i \cdot \partial^{i-1} & \text{if } i > 0 \\ 0 & \text{else} \end{cases}$)

Here $F_i\mathcal{M} := \bigoplus_{|I| \leq i} k \cdot \delta^I$ is a good filtration

$$\Longrightarrow \quad p_{\mathcal{M},F_\bullet}(i) = h_{\mathcal{M},F_\bullet}(i) = \#\{I \in \mathbb{N}_0^n \mid |I| \leq i\}$$

$\Longrightarrow$ $c(\mathcal{M}) = 1$

$d(\mathcal{M}) = n$ as in previous example!

c) $\mathcal{M} := \mathcal{D}$ viewed as a $\mathcal{D}$-module via left multiplication

$\Longrightarrow$ the Bernstein filtration $F_\bullet$ on $\mathcal{M}$ is good

Here $F_i\mathcal{M} = \langle x^I \partial^J \mid |I| + |J| \leq i \rangle_k$

$$\Longrightarrow \quad h_{\mathcal{M},F_\bullet}(i) = p_{\mathcal{M},F_\bullet}(i) = \#\{(I,J) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid |I|+|J| \leq i\} = \frac{i^{2n}}{(2n)!} + \ldots$$

$\Longrightarrow$ $c(\mathcal{M}) = 1$

$d(\mathcal{M}) = 2n$

Rem In general $c(\mathcal{M})$ can be any natural number, for instance one trivially has $c(\mathcal{M}^{\oplus m}) = m \cdot c(\mathcal{M})$ for $m \in \mathbb{N}$.

→ See lemma 4 (end of this section, p. 23)

Q What are the possibilities for $d(\mathcal{M})$?

If $d(\mathcal{M}) = 0$, then $\dim_k \mathcal{M} < \infty$,

hence $\dim_k \mathcal{M} = tr(id_{\mathcal{M}}) \underset{[\partial_i, x_i] = 1}{=} tr\,[\partial_i, x_i] \underset{\text{commutator of endomorphisms have trace zero}}{=} 0 \Longrightarrow \mathcal{M} \cong 0.$

Can do much better:

<u>Thm 5</u> (Bernstein's inequality) Any fin.gen. $\mathcal{D}$-module $\mathcal{M}$ satisfies $d(\mathcal{M}) \geq n$.

Pf. Pick a good filtration $F_\bullet \mathcal{M}$.

① Key point: $\forall i \in \mathbb{Z}$, the map

$$F_i \mathcal{D} \hookrightarrow \mathrm{Hom}_k(F_i\mathcal{M}, F_{2i}\mathcal{M})$$
$$P \longmapsto (m \mapsto P \cdot m)$$

is <u>injective</u>.

This is shown by induction on i:

- $i=0$: Trivial since $F_0\mathcal{D} = 0$ for Bernstein filtration.
- Let $i>0$ & assume the claim $\forall j<i$.

Let $P = \sum_I p_I(x) \cdot \partial^I \in F_i \mathcal{D}$, wlog $P \notin k$

$\Rightarrow \exists \alpha \in \{1,\dots,n\}$ sth $[P, x_\alpha] \neq 0$ ← if $p_I \neq 0$ for some $I=(i_1,\dots,i_n)$ with $i_\alpha > 0$

or $[P, \partial_\alpha] \neq 0$ ← if $\deg_{x_\alpha}(p_I) > 0$ for some I

(or both).

Note: $[P, x_\alpha], [P, \partial_\alpha] \in F_{i-1}\mathcal{D}$ (not just $\in F_i \mathcal{D}$) by the properties of the Bernstein filtrat^n (remark 3.3).

Assume for instance $[P, x_\alpha] \neq 0$ (the other case is similar)

$\Rightarrow$ by induction $\exists\, m \in F_{i-1}\mathcal{M}$ sth $[P, x_\alpha]\cdot m \neq 0$.

$$\Rightarrow P \cdot \underbrace{x_\alpha \cdot m}_{\in F_i \mathcal{M}} - x_\alpha \cdot P \cdot \underbrace{m}_{\in F_{i-1}\mathcal{M} \subseteq F_i\mathcal{M}} \neq 0$$

$\Rightarrow P \cdot F_i\mathcal{M} \neq \{0\}$, ie the claim holds for i as well.

② By part ① we have

$$\underbrace{\dim_k F_i \mathcal{D}}_{= h_{\mathcal{D},F_\bullet}(i)} \leq \underbrace{\dim_k \mathrm{Hom}_k(F_i\mathcal{M}, F_{2i}\mathcal{M})}_{= h_{\mathcal{M},F_\bullet}(i) \cdot h_{\mathcal{M},F_\bullet}(2i)}$$

$\Rightarrow$ For $i \gg 0$,

$$\frac{i^{2n}}{(2n)!} + \dots \leq c^2 \cdot \frac{i^d \cdot (2i)^d}{(d!)^2} + \dots \quad \Big| \; \text{w/ } c = c(\mathcal{M}),\ d = d(\mathcal{M})$$

lower order terms

$\Rightarrow n \leq d$. $\square$

[Addendum: Extending the remark on p. 21, we have:]

Lemma 4 Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be a short exact sequence of $\mathcal{D}$-modules.

If $\mathcal{M}$ is fin.gen, then so are $\mathcal{M}', \mathcal{M}''$ and we have:

a) $d(\mathcal{M}) = \max\{d(\mathcal{M}'), d(\mathcal{M}'')\}$

b)
$$c(\mathcal{M}) = \begin{cases} c(\mathcal{M}') & \text{if } d(\mathcal{M}') > d(\mathcal{M}'') \\ c(\mathcal{M}') + c(\mathcal{M}'') & \text{if } d(\mathcal{M}') = d(\mathcal{M}'') \\ c(\mathcal{M}'') & \text{if } d(\mathcal{M}') < d(\mathcal{M}'') \end{cases}$$

Pf. Pick a good filtration $F_\bullet \mathcal{M}$.

Define $F_i\mathcal{M}' := \mathcal{M}' \cap F_i\mathcal{M}$

$F_i\mathcal{M}'' := \text{image}(F_i\mathcal{M} \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{M}'')$

$\Longrightarrow$ compatible filtrations $F_\bullet$ on $\mathcal{M}'$ and $\mathcal{M}''$ w/ exact sequence of $gr^F_\bullet\mathcal{D}$-modules

$$0 \to gr^F_\bullet\mathcal{M}' \to \underbrace{gr^F_\bullet\mathcal{M}}_{\substack{\text{fin.gen over the} \\ \text{Noetherian ring } gr^F_\bullet\mathcal{D} \simeq k[x_1,\dots,x_n,\xi_1,\dots,\xi_n]}} \to gr^F_\bullet\mathcal{M}'' \to 0$$

$\Longrightarrow$ $gr^F_\bullet\mathcal{M}'$ & $gr^F_\bullet\mathcal{M}''$ fin gen, ie $F_\bullet\mathcal{M}'$ & $F_\bullet\mathcal{M}''$ are good

$\Longrightarrow$ $\mathcal{M}'$ & $\mathcal{M}''$ fin gen as $\mathcal{D}$-modules

Furthermore $h_{\mathcal{M},F_\bullet}(i) = h_{\mathcal{M}',F_\bullet}(i) + h_{\mathcal{M}'',F_\bullet}(i) \quad \forall i \in \mathbb{Z}$

$\Longrightarrow$ claim a) & b) $\square$

7. Holonomic $\mathcal{D}$-modules ($\mathcal{D} = \mathcal{D}_{n,k}$)

Recall: Any fin.gen. $\mathcal{D}$-module $\mathcal{M}$ satisfies $d(\mathcal{M}) \in \{n, n+1, \dots, 2n\}$

(Bernstein's inequality).

Def $\mathcal{M}$ is called holonomic if $d(\mathcal{M}) = n$.

[Intuitively, $\mathcal{M} \longleftrightarrow$ system of linear PDE's

eg. $\mathcal{M} \simeq \mathcal{D}/\mathcal{D}P_1 + \dots + \mathcal{D}P_m \longleftrightarrow P_1(f) = \dots = P_m(f) = 0$

If $P_1, \dots, P_m$ are "independent equations"

we hope $gr^F_\bullet\mathcal{M} \simeq gr^F_\bullet\mathcal{D}/(\sigma(P_1), \dots, \sigma(P_m))$ ⟵ Caution: Not always true!

for the "leading terms" $\sigma(P_i) \in gr^F_\bullet\mathcal{D}$

$\Longrightarrow d(\mathcal{M}) \geq 2n - m$

$\Longrightarrow$ holonomic modules $\hat{=}$ systems of linear PDE's w/ a maximum number of independent eq^{ns} allowing for a solution $\neq 0$ ...]

("maximally over-determined system of PDE's" in particular $\dim_k(\text{solutions}) < \infty$ ...)

Ex 1 a) For $n=1$ & any left ideal $J \trianglelefteq \mathcal{D}$,

the module $M = \mathcal{D}/J$ is holonomic

iff $J \neq 0$ (exercise).

b) $M := k[x_1, \dots, x_n] \simeq \mathcal{D}/\mathcal{D}\cdot\partial_1 + \dots + \mathcal{D}\cdot\partial_n$

is holonomic (example 6.3 a)

c) The Dirac module $M = \bigoplus_{I \in \mathbb{N}_0^n} \delta^I$

is holonomic (example 6.3 b)

d) $M := \mathcal{D}$ viewed as a module over itself

is NOT holonomic.

(example 6.3 c)

Holonomicity is a "finiteness condition".
Let's make this more precise:

Def An abelian category $\mathcal{A}$ is called

- Noetherian if $\forall M \in \mathcal{A}$, (ie M is a "Noetherian object of $\mathcal{A}$")

  every ascending chain $M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \dots \hookrightarrow M$

  stabilizes.

- Artinian if $\forall M \in \mathcal{A}$,

  every descending chain $M \hookleftarrow M_0 \hookleftarrow M_1 \hookleftarrow \dots$

  stabilizes.

  (ie M is an "Artinian object of $\mathcal{A}$")

Exercise a) $\mathcal{A}$ is Noetherian / Artinian iff $\forall M \in \mathcal{A}$,

any set of subobjects $(M_i \hookrightarrow M)_{i \in I}$ has

a maximal / minimal element

↳ ie an M_i which is NOT properly contained in / doesn't properly contain any other M_j with $j \in I$.

b) If $\mathcal{A}$ is both Noetherian and Artinian, then $\forall M \in \mathcal{A}$

$\exists$ "composition series" $0 = M_0 \overset{\neq}{\hookrightarrow} M_1 \overset{\neq}{\hookrightarrow} \dots \overset{\neq}{\hookrightarrow} M_\ell = M$

of subobjects sth $\forall i$,

$$Q_i := M_i / M_{i-1} \text{ is a simple object of } \mathcal{A}.$$

↳ ie without subobjects other than 0 & itself.

c) In general, if $M \in \mathcal{A}$ admits a composition series, then the length $\ell = \ell(M)$ only depends on M but not on the chosen series. Ditto for the quotients Q_i (up to permutation and iso).

Def We then say M has finite length and call $Q_1, \dots, Q_\ell \in \mathcal{A}$ its composition factors.

Back to $\mathcal{D}$-modules:

Prop 2 The Noetherian objects of $\mathrm{Mod}(\mathcal{D})$ are precisely the fingen $\mathcal{D}$-modules. In particular, $\mathcal{D}$ is a (left and right) Noetherian ring.

Pf.

- $\mathcal{M} \in \mathrm{Mod}(\mathcal{D})$ Noetherian $\Rightarrow$ pick $m_1 \in \mathcal{M}$
  if $\mathcal{M} \neq \mathcal{D}m_1$, pick $m_2 \in \mathcal{M} \setminus \mathcal{D}m_1$
  $\vdots$
  if $\mathcal{M} \neq \mathcal{D}m_1 + \dots + \mathcal{D}m_\nu =: \mathcal{M}_\nu$,
  pick $m_\nu \in \mathcal{M} \setminus \mathcal{M}_\nu$

  $\Rightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \hookrightarrow \dots \hookrightarrow \mathcal{M}$

  By assumption this chain must stabilize, thus $\mathcal{M} = \mathcal{M}_{\nu_0}$ fin. gen.

- $\mathcal{M} \in \mathrm{Mod}(\mathcal{D})$ fingen $\Rightarrow$ any submodule $\mathcal{M}' \hookrightarrow \mathcal{M}$ fingen (Lemma 6.4)

  (recall the key points:
  - good filtration on $\mathcal{M}$ induces one on $\mathcal{M}'$
  - $\mathrm{gr}^F \mathcal{D} \simeq k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ Noetherian
  - $\mathrm{gr}^F \mathcal{M}' \hookrightarrow \mathrm{gr}^F \mathcal{M}$ ...)

  $\Rightarrow$ Given a chain $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \hookrightarrow \dots \hookrightarrow \mathcal{M}$ of submodules, it must stabilize because $\mathcal{M}' := \bigcup_{i \in \mathbb{N}} \mathcal{M}_i \hookrightarrow \mathcal{M}$ is fingen.

- Apply to $\mathcal{M} := \mathcal{D}$ as a left $\mathcal{D}$-module

  $\Rightarrow \mathcal{D}$ left Noetherian

- "right" $\leftrightarrow$ "left" via $\mathcal{D} \xrightarrow{\sim} \mathcal{D}^{op}$
  $$x_i \longmapsto \partial_i$$
  $$\partial_j \longmapsto -x_j \quad \text{(exercise)}. \qquad \square$$

Conclusion. The finitely generated $\mathcal{D}$-modules form a Serre subcategory $\mathrm{Mod}_{fg}(\mathcal{D}) \subset \mathrm{Mod}(\mathcal{D})$

↳ (ie stable under subobjects, quotients and extensions)

and this subcategory is Noetherian.

⚠ Obviously NOT Artinian, e.g. $\mathcal{D} \supsetneq \mathcal{D}\cdot\partial_1 \supsetneq \mathcal{D}\cdot\partial_1^2 \supsetneq \dots$

Thm 3 a) The holonomic $\mathcal{D}$-modules form a Serre subcategory $\mathrm{Hol}(\mathcal{D}) \subset \mathrm{Mod}_{fg}(\mathcal{D})$.

b) $\mathrm{Hol}(\mathcal{D})$ is Artinian & Noetherian, and the length of $\mathcal{M} \in \mathrm{Hol}(\mathcal{D})$ is bounded by
$$\ell(\mathcal{M}) \leq c(\mathcal{M}).$$

Pf. a) Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be exact in $\mathrm{Mod}_{fg}(\mathcal{D})$.

Lemma 6.4 a: $d(\mathcal{M}) = \max\{d(\mathcal{M}'), d(\mathcal{M}'')\}$

Bernstein inequality thus gives: $d(\mathcal{M}) = n$ iff $d(\mathcal{M}') = d(\mathcal{M}'') = n$.

b) Induction: Recall the Hilbert polynomial $P_{M,F.}(t) = c \cdot \frac{t^d}{d!} + \dots$

- $\mathcal{M}$ simple $\Rightarrow \ell(\mathcal{M}) = 1$
  $\Rightarrow$ trivially $\ell(\mathcal{M}) \le c(\mathcal{M})$

  w/ $c = c(\mathcal{M}) \in \mathbb{N}$
  $d = d(\mathcal{M}) = n$ for $\mathcal{M}$ holonomic

- $\mathcal{M}$ not simple:

  Write $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ w/ $\mathcal{M}', \mathcal{M}'' \neq 0$

  By a) we have $d(\mathcal{M}') = d(\mathcal{M}'')$ $(= n)$,

  so $c(\mathcal{M}) = c(\mathcal{M}') + c(\mathcal{M}'')$ (lemma 6.4 b)

  Compare w/ $\ell(\mathcal{M}) = \ell(\mathcal{M}') + \ell(\mathcal{M}'') \in \mathbb{N} \cup \{\infty\}$

  By induction on $c(\mathcal{M})$ we have

$$\left.\begin{array}{l} \ell(\mathcal{M}') \le c(\mathcal{M}') < \infty \\ \ell(\mathcal{M}'') \le c(\mathcal{M}'') < \infty \end{array}\right\} \text{ thus } \ell(\mathcal{M}) \le c(\mathcal{M}) < \infty.$$

$\square$

Ex 4 a) $\mathcal{M} = k[x_1, \dots, x_n]$ has $\ell(\mathcal{M}) \le c(\mathcal{M}) = 1$, hence $\mathcal{M}$ is simple. (example 6.3a)

We can see this by hand:

Let $\mathcal{N} \subseteq \mathcal{M}$ be a $\mathcal{D}$-submodule $\neq 0$.

Pick $0 \neq f(x) = \sum_I c_I \cdot x^I \in \mathcal{N}$

Put $j_1 := \max\{i_1 \mid \exists I = (i_1, i_2, \dots, i_n) \text{ w/ } c_I \neq 0\}$.

$\Rightarrow 0 \neq \partial^{j_1}(f) \in k[x_2, \dots, x_n] \cap \mathcal{N} \subset \mathcal{M}$

(we assume char $k = 0$) (no dependence on x_1)

$\vdots$

$\Rightarrow 0 \neq \partial^{j_n} \cdots \partial^{j_1}(f) \in k \cap \mathcal{N} \subset \mathcal{M}$

$\Rightarrow 1 \in \mathcal{N}$

$\Rightarrow \mathcal{N} = \mathcal{M}$.

b) The "Dirac module" $\mathcal{M} = \bigoplus_{I \in \mathbb{N}_0^n} \delta^I$ $(\simeq \mathcal{D}/\mathcal{D}\partial_1 + \dots + \mathcal{D}\partial_n)$ has $\ell(\mathcal{M}) \le c(\mathcal{M}) = 1$, hence is simple. (example 6.3b)

Exercise: Check this directly by hand!

c) Caution: $\mathcal{M}$ simple $\not\Rightarrow$ $c(\mathcal{M}) = 1$ in general:

Exercise Let $n = 1$.

For $s \in k$ put $\mathcal{M} := \mathcal{M}_{x^s} = k[x, x^{-1}] \cdot x^s$ formal basis vector

$$= \bigoplus_{m \in \mathbb{Z}} k \cdot x^{s+m}$$

w/ $\partial(x^{s+m}) := (s+m) \cdot x^{s+m-1}$
$x(x^{s+m}) := x^{s+m+1}$ for $m \in \mathbb{Z}$.

Verify that a) $c(\mathcal{M}) = 2$
b) $\mathcal{M}$ is simple iff $s \notin \mathbb{Z}$.

Rem The full subcategory $\mathrm{Mod}_{fe}(\mathcal{D}) \subset \mathrm{Mod}_{fg}(\mathcal{D})$ of finite length $\mathcal{D}$-modules is also a Serre subcategory which is Noetherian & Artinian. We have:

Lemma 5 a) The algebra $\mathcal{D}$ is simple (ie has no proper 2 sided ideals).
b) Hence any $\mathcal{M} \in \mathrm{Mod}_{fe}(\mathcal{D})$ is cyclic,
ie $\mathcal{M} \simeq \mathcal{D}/\mathcal{J}$ for some left ideal $\mathcal{J} \trianglelefteq \mathcal{D}$.

Pf. a) Let $0 \neq P \in \mathcal{J} \subsetneq \mathcal{D}$, say $P \in F_i \mathcal{D}$
proper 2-sided ideal (for the Bernstein filtrat^n)

Wlog $P \notin k = Z(\mathcal{D})$ (else $1 \in \mathcal{J}$, so $\mathcal{J} = \mathcal{D}$).
centre of $\mathcal{D}$

$\Rightarrow$ $\exists j$ sth $[x_j, P] \neq 0$ or $[\partial_j, P] \neq 0$

But $[x_j, P], [\partial_j, P] \in \mathcal{J}$ (because $\mathcal{J}$ is a 2sided ideal)
and $\qquad$ // $\qquad \in F_{i-1}\mathcal{D}$ (property of Bernstein filtrat^n)

$\Rightarrow$ By induction on i we arrive at the case $i = 0$,
ie. $P \in k$ $\lightning$

b) induction on $\ell(\mathcal{M})$.

$\ell(\mathcal{M}) = 1$: $\mathcal{M}$ simple $\Rightarrow$ $\mathcal{M} = \mathcal{D} \cdot m \simeq \mathcal{D}/\mathcal{J}$
$\forall m \in \mathcal{M} \setminus \{0\}$ & $\mathcal{J} := \mathrm{Ann}_{\mathcal{D}}(m)$.

$\ell(\mathcal{M}) > 1$: Pick an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \xrightarrow{p} \mathcal{M}'' \to 0 \quad \text{w/ } \mathcal{M}' \neq 0 \text{ simple.}$$

By induction $\mathcal{M}'' = \mathcal{D} \cdot m'' \simeq \mathcal{D}/\mathcal{J}''$ w/ $m'' \in \mathcal{M}'' \setminus \{0\}$
$\mathcal{J}'' := \mathrm{Ann}_{\mathcal{D}}(m'')$.

Assume $\mathcal{M}$ is NOT cyclic.

$\Rightarrow \quad \mathcal{D}\cdot m \neq \mathcal{M} \quad \forall\, m \in p^{-1}(m'')$.

Then

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M}' \cap \mathcal{D}\cdot m \arrow[r] \arrow[d, hook] & \mathcal{D}\cdot m \arrow[r] \arrow[d, hook, "\neq"'] & \mathcal{D}\cdot m'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{M}' \arrow[r, hook] & \mathcal{M} \arrow[r] & \mathcal{M}'' \arrow[r] & 0 \end{tikzcd}$$

implies $\mathcal{M}' \cap \mathcal{D}\cdot m = \{0\}$ because $\mathcal{M}'$ is simple.

$\Rightarrow \quad \mathcal{D}\cdot m \simeq \mathcal{M}'' \quad \forall\, m \in p^{-1}(m'')$

$\Rightarrow \quad \mathrm{Ann}_{\mathcal{D}}(m) = \mathcal{J}'' \quad \forall\, m \in p^{-1}(m'')$

But $p^{-1}(m'')$ is a full coset of $\mathcal{M}' \subset \mathcal{M}$

$\Rightarrow \quad \mathcal{J}'' \cdot \mathcal{M}' = \{0\}$

$\Rightarrow \quad \mathcal{J}'' \subseteq \mathrm{Ann}_{\mathcal{D}}(\mathcal{M}') := \{P \in \mathcal{D} \mid P\cdot m' = 0 \ \forall\, m' \in \mathcal{M}'\} \subset \mathcal{D}$

(a 2-sided ideal, & proper because $\mathcal{M}' \neq 0$!)

$\mathcal{D}$ simple by a)

$\Rightarrow \quad \mathcal{J}'' = \{0\}$

$\Rightarrow \quad \mathcal{M}'' \simeq \mathcal{D}$ as a left $\mathcal{D}$-module

but this doesn't have finite length! $\lightning$

$\square$

So why consider $\mathrm{Hol}(\mathcal{D}) \subset \mathrm{Mod}_{fl}(\mathcal{D})$ at all?

- For $n \geq 2$, $\exists$ simple modules $\mathcal{M} \in \mathrm{Mod}(\mathcal{D})$ that are NOT holonomic, in fact $\mathcal{M} := \mathcal{D}/_{\mathcal{D}\cdot P}$ is an example for "generic" $P \in \mathcal{D}$

(Bernstein-Lunts 1988)

(for a specific example see Stafford 1985).

Such modules are VERY NASTY,

e.g. $\dim_{k} \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M},\mathcal{M})$ can be ∞,

$\mathcal{M} \otimes_{k[x]} k[x, 1/x_1]$ can be non-finitely generated / $\mathcal{D}$,

etc.

- Holonomic modules have MUCH NICER homological properties (see next chapters).

- Sometimes Holonomicity is even easier to check than finite length / finite generation (see below)!

8. Proof of Bernstein's thm

Holonomicity is easy to check:

Prop 1 Let $\mathcal{M} \in \mathrm{Mod}(\mathcal{D})$ ← a priori not finitely gen.

w/ a compatible filtration $F_\bullet \mathcal{M}$ ← a priori not good.

Assume $\exists\, h(t) = c \cdot \frac{t^n}{n!} + \underbrace{\cdots}_{\text{(lower order terms)}} \in \mathbb{Q}[t]$ w/ $c \geq 0$

s.th.

$$\dim_k F_i\mathcal{M} \leq h(i) \quad \forall i \gg 0.$$

Then $\mathcal{M} \in \mathrm{Hol}(\mathcal{D})$ and $\ell(\mathcal{M}) \leq c$.

Pf. Enough to show:

Every (!) fin.gen. $\mathcal{D}$-submodule $\mathcal{N} \subseteq \mathcal{M}$ is holonomic w/ $\ell(\mathcal{N}) \leq c$.

So let $\mathcal{N} \subseteq \mathcal{M}$ be fin.gen.

Put $F_i\mathcal{N} := \mathcal{N} \cap F_i\mathcal{M}$ for $i \in \mathbb{Z}$.

$\Longrightarrow$ $F_\bullet\mathcal{N}$ compatible filtration.

Fix any good filtration $G_\bullet\mathcal{N}$ (exists since $\mathcal{N}$ is fin.gen)

Wlog $G_i\mathcal{N} \subseteq F_i\mathcal{N}$ $\forall i$ (e.g. by lemma 5.3)

↳ good filtrations are finer than any other compatible filtrat^n

$$\Longrightarrow h_{\mathcal{N},G_\bullet}(i) := \dim_k G_i\mathcal{N} \leq \dim_k F_i\mathcal{N} \leq \dim_k F_i\mathcal{M} \underset{\substack{\uparrow \\ \text{by assumpt}^n}}{\leq} h(i) \quad \forall i \gg 0$$

$\Longrightarrow$ $p_{\mathcal{N},G_\bullet}(t) \leq h(t) = c\frac{t^n}{n!} + \cdots$ $\forall t \gg 0$

$\Longrightarrow$ $d(\mathcal{N}) \leq n$, hence "=" by Bernstein's inequality

$\Longrightarrow$ $c(\mathcal{N}) \leq c$, hence $\ell(\mathcal{N}) \leq c$ by thm 8.3b. $\square$

Apply this to

$$\mathcal{M} = \mathcal{M}_{f^s} := k[x_1, \ldots, x_n, 1/f] \cdot f^s \qquad (\text{for } s \in k)$$

w/ $\partial(g(x) \cdot f^s) := \partial(g(x)) \cdot f^s + s \cdot \frac{g(x)}{f(x)} \cdot \partial(f(x)) \cdot f^s$

for $g \in k[x_1, \ldots, x_n, 1/f]$.

(in our previous notation we had $k \leftrightarrow K = k(s) \ldots$)

Thm 2 $\mathcal{M}_{f^s} \in \mathrm{Hol}(\mathcal{D})$.

Pf. Put

$$F_i\mathcal{M}_{f^s} := \{ g(x) \cdot f^{s-i} \mid \deg(g) \leq i \cdot (\deg(f) + 1) \}, \quad i \geq 0.$$

($f^{s-i} := 1/f^i \cdot f^s$)

This is a compatible filtration:

Let $g f^{s-i} \in F_i\mathcal{M}_{f^s}$.

- $x_\alpha \cdot g f^{s-i} = (x_\alpha g f) \cdot f^{s-(i+1)} \in F_{i+1} M_{f^s}$

since $\deg(x_\alpha g f) = \deg(g) + \deg(f) + 1$
$\leq (i+1) \cdot (\deg(f)+1)$

- $\partial_\alpha \cdot g f^{s-i} = \partial_\alpha(g) \cdot f^{s-i} + (s-i) \cdot g \cdot \partial_\alpha(f) \cdot f^{s-i-1}$
$= (f \partial_\alpha(g) + (s-i) g \cdot \partial_\alpha(f)) \cdot f^{s-i-1} \in F_{i+1} M_{f^s}$

since $\deg(\cdots) \leq \deg(f) + \deg(g) - 1$
$\leq (i+1) \cdot (\deg(f)+1) \qquad (i \geq 0)$
$(\deg(g) \leq i \cdot (\deg(f)+1))$

$\Rightarrow F_1 \mathcal{D} \cdot F_i M_{f^s} \subseteq F_{i+1} M_{f^s}$, ie $F_\bullet M_{f^s}$ compatible.

Now compute

$$\dim_k F_i M_{f^s} = \dim_k \{\text{polynomials of degree} \leq i \cdot (\deg f + 1) \text{ in } n \text{ variables}\}$$

$$= \binom{i \cdot (\deg f + 1) + n}{n}$$

$$= c \cdot \frac{i^n}{n!} + \cdots \quad \text{w/ } c > 0.$$

$\Rightarrow$ prop 1 applies. $\square$

9. Characteristic varieties

For $M \in \mathrm{Mod}_{\text{fingen}}(\mathcal{D})$ w/ any good filtration, (wrt Bernstein's $F_\bullet \mathcal{D}$)
we had the Hilbert polynomial

$$p_{M,F}(t) = c \cdot \frac{t^d}{d!} + \cdots \qquad \text{w/ } d = d(M) \in \{n, \ldots, 2n\}, \; c = c(M) \in \mathbb{N}.$$

Geometric meaning?

Recall $\mathrm{gr}^F_\bullet M$ is a fingen graded module / $\mathrm{gr}^F_\bullet \mathcal{D}$ (wrt Bernstein's $F_\bullet \mathcal{D}$)

w/ Hilbert polynomial

$$p_{\mathrm{gr} M}(t) = p_{M,F}(t) - p_{M,F}(t-1)$$
$$= c \cdot \frac{t^d - (t-1)^d}{d!} + \cdots$$
$$= c \cdot \frac{t^{d-1}}{(d-1)!} + \text{lower order terms.}$$

Algebraic Geometry:

On $\mathbb{P}^m = \mathrm{Proj}(S)$, $S = k[y_0, \ldots, y_m]$,

$\exists$ equiv. of ab. cat.

$$\left\{\begin{matrix}\text{fin. gen. graded} \\ S\text{-modules}\end{matrix}\right\} \Big/ \text{negligibles} \xrightarrow{\sim} \mathrm{Coh}(\mathbb{P}^m) \quad \text{(coherent sheaves)}$$

$$M_\bullet = \bigoplus_{i \in \mathbb{Z}} M_i \longmapsto \tilde{M}_\bullet$$

w/ $M_i = H^0(\mathbb{P}^n, \tilde{M}_\bullet(i)) \; \forall i \gg 0.$

Here $M_\bullet$ is called <u>negligible</u> if $M_i = 0 \ \forall i \gg 0$, these form a Serre subcategory in the abelian category of graded S-modules.

Note: $\underbrace{\dim \operatorname{Supp}}_{\substack{= Z(\operatorname{Ann}_S \tilde{M}_\bullet) \\ \subseteq \mathbb{P}^n \\ \text{(Zariski-closed subset)}}} \tilde{M}_\bullet = \deg(p_{M_\bullet}(t))$

Hilbert polynomial of the graded S-module $M_\bullet$

[Hartshorne, thm I.7.5]

Back to our case:

$\mathcal{M} \in \operatorname{Mod}_{\text{fingen}}(\mathcal{D})$ w/ good $F_\bullet \mathcal{M}$

$\rightsquigarrow M_\bullet := \operatorname{gr}^F_\bullet \mathcal{M} \in \operatorname{Mod}^{\text{graded}}_{\text{fingen}}(S)$, $S = \operatorname{gr}^F \mathcal{D}$
$= k[x_1, .., x_n, \xi_1, .., \xi_n]$

$\rightsquigarrow \tilde{M}_\bullet \in \operatorname{Coh}(\mathbb{P}^{n-1})$

w/ $\dim \operatorname{Supp} \tilde{M}_\bullet = d(\mathcal{M}) - 1$

Passing to $\mathbb{A}^{2n} = \operatorname{Spec}(S) \xrightarrow{q} \mathbb{P}^{2n-1} = \operatorname{Proj}(S)$, we get:

<u>Lemma 1</u> $\quad d(\mathcal{M}) = \dim \operatorname{Supp} \tilde{M}$

where $\tilde{M} = q^* \tilde{M}_\bullet \in \operatorname{Coh}(\mathbb{A}^{2n})$ is the coherent sheaf corresponding to $M \in \operatorname{Mod}_{\text{fingen}}(S)$ (w/ forgotten grading).

<u>Rem 2</u> $\quad \operatorname{Supp} \tilde{M} \subset \mathbb{A}^{2n}$ is stable under the natural action $\mathbb{G}_m \times \mathbb{A}^{2n} \longrightarrow \mathbb{A}^{2n}$, $(\lambda, (x, \xi)) \mapsto (\lambda x, \lambda \xi)$

$(\lambda \in \mathbb{G}_m,\ x, \xi \in \mathbb{A}^n)$

& independent of the chosen $F_\bullet \mathcal{M}$ (see below).

<u>Ex. 3</u> $\quad$ Let $n = 1$. Consider the Dirac module over a fixed point $c \in k$, $\left(\begin{array}{l} \partial \cdot \delta^j := \delta^{j+1} \\ (x-c) \cdot \delta^j := (-1)^j \cdot j \cdot \delta^{j-1} \end{array} \right)$

$$\mathcal{M} := \mathcal{D} / \mathcal{D} \cdot (x-c) \simeq \bigoplus_{j \in \mathbb{N}_0} k \cdot \delta^j$$

w/ the good filtration $F_i \mathcal{M} := F_i \mathcal{D} / (\mathcal{D} \cdot (x-c)) \cap F_i \mathcal{D}$.

$$\simeq \bigoplus_{0 \le j \le i} k \cdot \delta^j$$

$\Longrightarrow \operatorname{gr}^F \mathcal{M} \simeq \bigoplus_{j \in \mathbb{N}_0} k \cdot e_j \quad$ w/ $e_j := [\delta^j] \in \operatorname{gr}^F_j \mathcal{M}$

$\circlearrowright$

$\operatorname{gr}^F \mathcal{D} \simeq k[x, \xi] \quad$ via $\left\{ \begin{array}{l} \xi \cdot e_j = e_{j+1} \\ x \cdot e_j = 0 \end{array} \right.$

$\left(\begin{array}{l} [x-c] = [x] \in \operatorname{gr}^F_1 \mathcal{D} \\ \text{and } [\delta^{j-1}] = [0] \\ \text{in } \operatorname{gr}^F_{j+1} \mathcal{M} \end{array} \right)$

$\Longrightarrow \operatorname{Supp} \tilde{M} = \{0\} \times \mathbb{A}^1$
$= \{(x, \xi) \mid x = 0\}$

Drawback • Not very geometric: Info about c is lost

$\Rightarrow$ don't recover support of the $\mathcal{O}_{\mathbb{A}^n}$-module M from $\operatorname{Supp} \tilde{M}$!

• Bernstein's filtration $F_\bullet \mathcal{D}$ depends on the chosen linear coordinate system on $\mathbb{A}^n$

$\Rightarrow$ doesn't generalize to arbitrary varieties / k !

$\boxed{\text{Can we instead use other filtrations } F_\bullet \mathcal{D} \text{ (eg. the order filtration)?}}$

Back to the axiomatic setup of §4:

Let $(\mathcal{D}, F_\bullet \mathcal{D})$ be a filtered k-algebra, w/ $F_\bullet \mathcal{D}$ increasing filtration by k-subspaces sth

① $\mathcal{D} = \bigcup_i F_i \mathcal{D}$,

② $F_i \mathcal{D} = 0 \quad \forall i < 0 \quad \& \quad 1 \in F_0 \mathcal{D}$,

③ $F_i \mathcal{D} \cdot F_j \mathcal{D} \subseteq F_{i+j} \mathcal{D} \quad \forall i,j \in \mathbb{Z}$.

We further assume

④ $gr_\bullet^F \mathcal{D}$ is a commutative k-algebra of finite type, generated by elements of degree ≤ 1.

(stronger than in §4)

Def Let $M \in \mathrm{Mod}_{\text{fingen}}(\mathcal{D})$.
Pick a good filtration $F_\bullet M$
Put $\mathsf{M} := gr^F M \in \mathrm{Mod}_{\text{fingen}}(S)$, $\quad S := gr^F \mathcal{D}$

$\rightsquigarrow \tilde{\mathsf{M}} \in \mathrm{Coh}(\operatorname{Spec} S)$ associated coherent sheaf

Lemma 4 The support $\operatorname{Supp}(\tilde{\mathsf{M}}) \subseteq \operatorname{Spec}(S)$ only depends on M but not on the chosen good filtration $F_\bullet M$.

Pf. Let $G_\bullet M$ be another good filtration.

a) Case 1: The filtrations are adjacent,

ie $\quad F_i M \subseteq G_i M \subseteq F_{i+1} M \quad \forall i \in \mathbb{Z}$.

Consider $\varphi_i : gr_i^F M \to gr_i^G M$ induced by $F_i M \hookrightarrow G_i M$.

Since $\quad \ker(\varphi_i) \simeq \dfrac{G_{i-1} M}{F_{i-1} M} \simeq \operatorname{cok}(\varphi_{i-1}) \quad \forall i$,

we get an exact sequence

$$0 \to K_\bullet \to gr_\bullet^F M \xrightarrow{\varphi_\bullet} gr_\bullet^G M \to K_{\bullet+1} \to 0$$

w/ $K_\bullet := \ker(\varphi_\bullet)$.

$$\Longrightarrow \quad \mathrm{Supp}(\mathrm{gr}^{F}_{\bullet} M) = \mathrm{Supp}(K_\bullet) \cup \mathrm{Supp}(\mathrm{im}\,\varphi_\bullet)$$

$$\mathrm{Supp}(\mathrm{gr}^{G}_{\bullet} M) = \mathrm{Supp}(\mathrm{im}\,\varphi_\bullet) \cup \mathrm{Supp}(K_{\bullet+1})$$

Note: $\mathrm{Supp}(K_\bullet) = \mathrm{Supp}(K_{\bullet+1})$ since supports don't depend on the grading of the module!

$\Longrightarrow$ claim

b) Case 2: General case.

Put $F_i^{(\nu)} M := F_i M + G_{i+\nu} M \subseteq M$ for $\nu \in \mathbb{Z}$

$$\Longrightarrow \quad F_\bullet^{(\nu)} = \begin{cases} F_\bullet & \text{for } \nu \ll 0 \\ G_\bullet & \text{for } \nu \gg 0 \end{cases}$$

(since any two good filtrations are equivalent, cor. 4.4)

Since each $F_\bullet^{(\nu)}$ is adjacent to $F_\bullet^{(\nu+1)}$ we're then reduced to case 1. $\square$

Rem. 5 The same proof applies to the "cycle-theoretic support"

$$\mathrm{Cycle}(\tilde{M}) := \sum_{\substack{Z \subseteq \mathrm{Supp}\,\tilde{M} \\ \text{irred. cpt}}} m_Z(\tilde{M}) \cdot [Z]$$

$\leftrightarrow$ $\wp \supseteq \mathrm{Ann}_S(\tilde{M})$ minimal prime

$(X = \mathrm{Spec}\, S)$

w/ $m_Z(\tilde{M}) :=$ length of the Artinian $\mathcal{O}_{X,Z}$-module $\tilde{M}_Z$

$=$ $\quad$ $\shortparallel$ $\quad$ $S_\wp$-module $\tilde{M} \otimes_S S_\wp$

Def When $F_\bullet \mathcal{D}$ is the order filtration, we put

$$\mathrm{Char}(M) := \mathrm{Supp}(\widetilde{\mathrm{gr}^F M}) \qquad \text{"characteristic variety"}$$

$$CC(M) := \mathrm{Cycle}(\widetilde{\mathrm{gr}^F M}) \qquad \text{"characteristic cycle"}$$

for $M \in \mathrm{Mod}(\mathcal{D})$ w/ a good filtration $F_\bullet \mathcal{D}$.

Rem $\mathrm{Char}(M) \subset \mathbb{A}^{2n} = \mathrm{Spec}(\underbrace{k[x_1, .., x_n, \xi_1, ..., \xi_n]}_{\mathrm{gr}^F \mathcal{D}})$

is stable under the action

$$\mathbb{G}_m \times \mathbb{A}^{2n} \longrightarrow \mathbb{A}^{2n}$$
$$(\lambda, (x, \xi)) \longmapsto (x, \lambda\xi).$$

Note that the rescaling is only in the "fiber direction" of the cotangent bundle $T^*\mathbb{A}^n = \mathbb{A}^n \times \mathbb{A}^n$ but leaves the "base" untouched!

Ex 6 a) Let $n = 1$. Consider the Dirac module over $c \in k$,
$$M := \mathcal{D} / \mathcal{D} \cdot (x-c) \simeq \bigoplus_{j \in \mathbb{N}_0} k \cdot \delta^j$$

as in example 3. As a good filtration we choose

$$F_i M := \frac{F_i \mathcal{D}}{\mathcal{D} \cdot (x-c) \cap F_i \mathcal{D}} \simeq \underbrace{\bigoplus_{0 \le j \le i} k \cdot \delta^j}_{}$$

same as before, though $F_\bullet \mathcal{D}$ is now the order filtration!

$\Rightarrow \mathrm{gr}^F \mathcal{M} \simeq \bigoplus_{j\in\mathbb{N}_0} \Bbbk \cdot e_j \qquad$ w/ $e_j := [\delta^j] \in \mathrm{gr}^F_j \mathcal{M}$

$\circlearrowleft$

$\mathrm{gr}^F \mathcal{D} \simeq \Bbbk[x,\xi]$ via $\begin{cases} \xi\cdot e_j = e_{j+1} \\ \underline{(x-c)\cdot e_j = 0} \end{cases}$

(unlike for Bernstein's filtration, we now have $[x-c] = [x]-[c] \in \mathrm{gr}^F_0 \mathcal{D}$ where $[c]$ may be nonzero)

$\Rightarrow \mathrm{Char}(\mathcal{M}) = \{c\}\times \mathbb{A}^1$
$= \{(x,\xi) \mid x = c\}$

$\Rightarrow$ Info about c is kept!

b) $\mathcal{M} := \Bbbk[x_1,\dots,x_n]$ w/ $F_i\mathcal{M} := \begin{cases} \Bbbk[x_1,\dots,x_n], & i\geq 0 \\ 0, & i<0 \end{cases}$

good wrt order filtration $F_\bullet\mathcal{D}$ (though not wrt Bernstein filtration, there one could take $F_i\mathcal{M} :=$ polynomials of degree $\leq i$)

$\Rightarrow \mathrm{gr}^F\mathcal{M} \simeq \Bbbk[x_1,\dots,x_n]$

w/ x_i acting as usual
ξ_i acting by zero

$\Rightarrow \mathrm{Char}(\mathcal{M}) = \mathbb{A}^n \times \{0\} = \{(x,\xi) \mid \xi = 0\} \subset \mathbb{A}^n\times\mathbb{A}^n$

c) Can "mix both cases":

Let $n=2$, $\quad \mathcal{M} := \bigoplus_{j\in\mathbb{N}_0} \Bbbk[x_1]\cdot\delta_2^j$ w/
$x_1\cdot f(x_1)\delta_2^j := (x_1 f(x_1))\delta_2^j$
$\partial_1\cdot f(x_1)\delta_2^j := \partial(f)\delta_2^j$
$x_2\cdot f(x_1)\delta_2^j := (-1)^j j\, f(x_1)\delta_2^{j-1}$
$\partial_2\cdot f(x_1)\delta_2^j := f(x_1)\delta_2^{j+1}$.

$\simeq \mathcal{D}/\mathcal{D}\cdot\partial_1 + \mathcal{D}\cdot x_2$

Good filtration induced by $F_\bullet\mathcal{D}$ is $F_i\mathcal{M} := \bigoplus_{j\leq i} \Bbbk[x_1]\cdot\delta_2^j$

$\Rightarrow \mathrm{gr}^F\mathcal{M} \simeq \Bbbk[x_1,\xi_2]$

$\circlearrowleft$

$\mathrm{gr}^F\mathcal{D} \simeq \Bbbk[x_1,x_2,\xi_1,\xi_2]$ w/ x_1,ξ_2 acting in the natural way
x_2,ξ_1 acting by zero.

$\Rightarrow \mathrm{Char}(\mathcal{M}) = \{(x_1,0,0,\xi_2)\}$
$= (\mathbb{A}^1\times 0)\times(0\times\mathbb{A}^1) \subset \mathbb{A}^2\times\mathbb{A}^2$

<u>NB</u> In all the above examples, $\mathrm{Char}(\mathcal{M}) \subset T^*\mathbb{A}^n$ involved the "conormal variety" to $\mathrm{Supp}(\mathcal{M}) \subset \mathbb{A}^n$.

... this is no coincidence!

However, in general $\mathrm{Char}(\mathcal{M})$ needn't be irreducible:

<u>Ex</u> d) Let $n=1$, $\mathcal{M} := \Bbbk[x,x^{-1}]\cdot x^s = \mathcal{D}/\mathcal{D}\cdot(x\partial - s)$, $s\in\Bbbk$.

Put $F_i\mathcal{M} := F_i\mathcal{D}/F_i\mathcal{D}\cap\mathcal{D}\cdot(x\partial-s)$

$\Rightarrow$ this is a good filtration (wrt order filtration $F_\bullet\mathcal{D}$)

where $\mathrm{gr}^F_\bullet\mathcal{M} \cong \Bbbk[x,\xi]/(x\xi)$

$\Rightarrow \mathrm{Char}\,\mathcal{M} = \mathbb{A}^1\times\{0\} \cup \{0\}\times\mathbb{A}^1$

although $\mathcal{M}$ is simple for $s\in\mathbb{Z}$ (exercise)

10. Homological characterization of Hol($\mathcal{D}$)

We've seen to approaches to $\mathcal{D} = \mathcal{D}_{n,k}$:

- Via Bernstein filtration $F_\bullet \mathcal{D}$

  $\rightsquigarrow$ good $F_\bullet \mathcal{M}$ gives Hilbert polynomial $p_{\mathcal{M},F_\bullet}(t)$

  $\rightsquigarrow$ $d(\mathcal{M}) := \deg(p_{\mathcal{M},F_\bullet}) \in \{n, \dots, 2n\}$

  we drop the $\sim$ from now on

  satisfies $d(\mathcal{M}) = \dim \underbrace{\mathrm{Supp}(\mathrm{gr}^F \mathcal{M})}$

  independent of $F_\bullet\mathcal{M}$ but "not very geometric"

- Via order filtration $F_\bullet \mathcal{D}$

  $\rightsquigarrow$ no Hilbert polynomials ($\dim_k F_0 \mathcal{D} = \infty$)

  but for $G_\bullet \mathcal{M}$ good we can still consider

  $\mathrm{Char}(\mathcal{M}) := \mathrm{Supp}(\mathrm{gr}^G \mathcal{M}) \subseteq \mathbb{A}^{2n}$

  ... "sees more geometry" eg. recover $\mathrm{Supp}(\mathcal{M}) \subseteq \mathbb{A}^n$ ...

⚠ The two notions of good filtrations (wrt Bernstein vs. order) are different, and usually $\mathrm{Supp}\, \mathrm{gr}^F \mathcal{M} \neq \mathrm{Supp}\, \mathrm{gr}^G \mathcal{M}$

(they are invariant under two different $\mathbb{G}_m$-actions on $\mathbb{A}^n \times \mathbb{A}^n$).

Goal: Show that $\dim \mathrm{Supp}\, \mathrm{gr}^F \mathcal{M} = \dim \mathrm{Supp}\, \mathrm{gr}^G \mathcal{M}$

$\to$ can define Hol($\mathcal{D}$) using the order filtration ...

We'll again reduce this to commutative algebra

General Setup:

$(\mathcal{D}, F_\bullet \mathcal{D})$ filtered k-algebra

w/ $F_\bullet \mathcal{D}$ increasing filtration by k-subspaces

sth

1. $\mathcal{D} = \bigcup_i F_i \mathcal{D}$
2. $F_i \mathcal{D} = 0 \ \forall i < 0$ & $1 \in F_0 \mathcal{D}$
3. $F_i \mathcal{D} \cdot F_j \mathcal{D} \subseteq F_{i+j} \mathcal{D} \ \forall i,j \in \mathbb{Z}$

We now assume

4. $A := \mathrm{gr}^F \mathcal{D}$ is a commutative regular biequidim k-algebra of finite type, generated by elements of degree ≤ 1.

   (stronger than ④ in §4 & §9)

Notation: For $\mathcal{M} \in \mathrm{Mod}_{fg}(\mathcal{D})$ put

- $j(\mathcal{M}) := \min \{ j \in \mathbb{N}_0 \mid \mathrm{Ext}^j_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \neq 0 \}$
- $d(\mathcal{M}) := \dim \mathrm{Supp}(\mathrm{gr}^F \mathcal{M})$ for $F_\bullet \mathcal{M}$ good wrt $F_\bullet \mathcal{D}$

  independent of $F_\bullet \mathcal{M}$ (Lemma 9.4) but a priori it might depend on $F_\bullet \mathcal{D}$

Thm 1 Put $m = \dim A$. Then

a) $j(\mathcal{M}) + d(\mathcal{M}) = m$

b) for each j we have $\mathrm{Ext}^j_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \in \mathrm{Mod}_{fg}(\mathcal{D}^{op})$

and $d(\mathrm{Ext}^j_{\mathcal{D}}(\mathcal{M}, \mathcal{D})) \leq m - j$

(using analogous notion of "good" for right $\mathcal{D}$-modules)

c) for $j = j(\mathcal{M})$ equality holds in b).

Cor 2 In the above setup $d(\mathcal{M})$ only depends on $\mathcal{M} \in \mathrm{Mod}_{fg}(\mathcal{D})$ (and $m := \dim gr^F \mathcal{D}$) but not on the specific choice of $F_\bullet \mathcal{D}$ & $F_\bullet \mathcal{M}$.

Pf of corollary.

By thm 1a) we have $d(\mathcal{M}) = m - j(\mathcal{M})$ and $j(\mathcal{M})$ is defined in terms of the $\mathrm{Ext}^j_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$, w/ no filtrations involved. $\square$

Pf of thm.

① A strictly filtration-preserving resolution:

Recall a map $f: \mathcal{M} \to \mathcal{N}$ of $\mathcal{D}$-modules w/ given filtrations $F_\bullet \mathcal{M}$, $F_\bullet \mathcal{N}$ is said to be

- filtration-preserving if $f(F_i \mathcal{M}) \subseteq F_i \mathcal{N}$ $\forall i$
- strict if moreover $f(\mathcal{M}) \cap F_i \mathcal{N} = f(F_i \mathcal{M})$, ie if the two filtrations on $f(\mathcal{M})$ induced by $F_\bullet \mathcal{M}$ respectively $F_\bullet \mathcal{N}$ coincide.

Exercise: i) If $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ is an exact sequence of filtered $\mathcal{D}$-modules & strictly f.p. maps, then $gr^F \mathcal{M}' \to gr^F \mathcal{M} \to gr^F \mathcal{M}''$ is exact.

ii) Any well-filtered $(\mathcal{M}, F_\bullet)$ admits a resolution

$$\cdots \to \mathcal{M}_1 \xrightarrow{d_1} \mathcal{M}_0 \xrightarrow{d_0} \mathcal{M} \to 0$$

by "filtered free modules"

$$\mathcal{M}_i := \bigoplus_{j=1}^{n_i} \mathcal{D}(a_{ij})$$

[$\mathcal{D}(a_{ij}) := \mathcal{D}$ w/ the shifted filtration $F_{\bullet + a_{ij}} \mathcal{D}$]

w/ all maps d_i strictly filtration-preserving (proceed inductively).

② Endow $M_j^\vee := \mathrm{Hom}_{\mathcal{D}}(M_j, \mathcal{D}) \in \mathrm{Mod}_{fg}(\mathcal{D}^{op})$ w/ the good filtration

$$F_i M_j^\vee := \{\varphi \mid \varphi(F_\nu M_j) \subseteq F_{\nu+i} M_j \ \forall \nu \in \mathbb{Z}\}$$

so that $$gr^F M_j^\vee \simeq \mathrm{Hom}_A(gr^F M_j, A) \quad \text{for } A := gr^F \mathcal{D}$$ (exercise)

$\Longrightarrow$ filtered cochain cplex $M_\bullet^\vee$

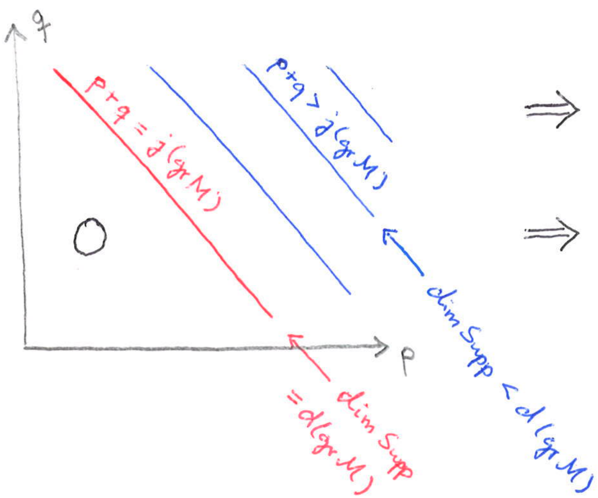
w/
- $H^j(M_\bullet^\vee) = \mathrm{Ext}^j_{\mathcal{D}}(M, \mathcal{D})$
- $H^j(gr^F M_\bullet^\vee) = \mathrm{Ext}^j_A(gr^F M, A)$

③ Spectral sequence of filtered complex:

$$E_1^{pq} := H^{p+q}(gr^F_{-p} M_\bullet^\vee) \Longrightarrow H^{p+q}(M_\bullet^\vee)$$

By thm A.1 (appendix, see below),

$$E_1^j := \bigoplus_{p+q=j} E_1^{pq} = \mathrm{Ext}^j_A(gr^F M, A) \begin{cases} = 0 \text{ for } j < j(gr M), \\ \text{has dim Supp} \leq m - j \text{ else,} \\ \text{w/ equality for } j = j(gr M) \end{cases}$$



$\Longrightarrow$ no cancellation of supports in degree $p+q = j(gr\, M)$

$\Longrightarrow$ $j(M) = j(gr\, M)$

$d(M) = d(gr\, M)$

so thm A.1 gives the result. □

Let's make this explicit for the Weyl algebra.

Cor 3. Let $M \in \mathrm{Mod}_{fg}(\mathcal{D})$ where $\mathcal{D} = \mathcal{D}_{n,k}$.

Then

a) $j(M) := \min\{j \mid \mathrm{Ext}^j_{\mathcal{D}}(M, \mathcal{D}) \neq 0\} \in \{0, 1, \dots, n\}$

b) $d(M) := \deg(p_{M,F_\bullet}(t))$

Hilbert polynomial for $F_\bullet M$ good wrt Bernstein filtrat^n

$= \dim(\mathrm{Char}(M))$

$\mathrm{Supp}(gr^F M)$ for $F_\bullet M$ good wrt order filtrat^n

$= 2n - j(M) \in \{n, \dots, 2n\}$

c) $M \in \mathrm{Hol}(\mathcal{D})$ iff $\mathrm{Ext}^j_{\mathcal{D}}(M, \mathcal{D}) = 0 \ \forall j \neq n$.

Pf. The identifications in b) follow from thm 1 & cor 2.

By Bernstein's inequality we have $d(M) \in \{n, \dots, 2n\}$,

hence $j(M) = 2n - d(M) \in \{0, \dots, n\}$.

Part c) then follows also from thm 1. □

11. Duality

Recall the $\mathrm{Ext}^j_{\mathcal{D}}(\mathcal{M},\mathcal{D})$ are right $\mathcal{D}$-modules.

Exercise $\exists$ equivalence of categories

$$\mathrm{Mod}(\mathcal{D}^{op}) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{D})$$

$$\mathcal{M} \longmapsto \mathcal{M}_{\text{left}} := \text{same underlying vector space as } \mathcal{M} \text{ but w/ the } \mathcal{D}\text{-action}$$

$$P \bullet m := m \cdot P^*$$

for $m \in \mathcal{M}$, $P \in \mathcal{D}$

Here for $P = \sum_I c_I(x)\cdot\partial^I$ we put $P^* := \sum_I (-1)^{|I|}\cdot\partial^I\cdot c_I(x)$

& this gives an iso $\mathcal{D} \xrightarrow{\sim} \mathcal{D}^{op}$, $P \longmapsto P^*$. ("adjoint operator")

Thm 1 On holonomic $\mathcal{D}$-modules we have an exact autoequivalence

$$\mathbb{D}: \mathrm{Hol}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Hol}(\mathcal{D}) \quad \text{w/ } \mathbb{D}\circ\mathbb{D} \simeq \mathrm{id}.$$

$$\mathcal{M} \longmapsto \mathrm{Ext}^n_{\mathcal{D}}(\mathcal{M},\mathcal{D})_{\text{left}}.$$

Pf.

① The functor $\mathbb{D} = \mathrm{Ext}^n_{\mathcal{D}}(-,\mathcal{D})_{\text{left}}$ sends $\mathrm{Hol}(\mathcal{D})$ into itself:

For $\mathcal{M} \in \mathrm{Hol}(\mathcal{D})$ we have $j(\mathcal{M}) = n$ by cor 10.3c)

so thm 10.1c) gives

$$d(\mathrm{Ext}^{n}_{\mathcal{D}}(\mathcal{M},\mathcal{D})) = 2n - n = n$$

$j(\mathcal{M})$

$\Rightarrow \mathrm{Ext}^n_{\mathcal{D}}(\mathcal{M},\mathcal{D}) \in \mathrm{Hol}(\mathcal{D}^{op})$

$\Rightarrow \mathrm{Ext}^n_{\mathcal{D}}(\mathcal{M},\mathcal{D})_{\text{left}} \in \mathrm{Hol}(\mathcal{D})$

② The functor $\mathbb{D}$ is exact on $\mathrm{Hol}(\mathcal{D})$:

For $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ exact in $\mathrm{Hol}(\mathcal{D})$ the long exact Ext sequence reads

$$\cdots\to \underbrace{\mathrm{Ext}^{n-1}_{\mathcal{D}}(\mathcal{M}',\mathcal{D})}_{\substack{=0\\ (\S 10,\ \mathrm{cor}\ 3c)}} \to \underbrace{\mathrm{Ext}^n_{\mathcal{D}}(\mathcal{M}'',\mathcal{D})}_{\mathbb{D}(\mathcal{M}'')} \to \underbrace{\mathrm{Ext}^n_{\mathcal{D}}(\mathcal{M},\mathcal{D})}_{\mathbb{D}(\mathcal{M})} \to \underbrace{\mathrm{Ext}^n_{\mathcal{D}}(\mathcal{M}',\mathcal{D})}_{\mathbb{D}(\mathcal{M}')} \to \underbrace{\cdots}_{\substack{=0\\ (\S 10,\ \mathrm{cor}\ 3c)}}$$

③ To see $\mathbb{D}\circ\mathbb{D}\simeq\mathrm{id}$, take a resolution $0\to P_n\to\cdots\to P_0\to\mathcal{M}\to 0$ w/ $P_i\in\mathrm{Mod}_{fg}(\mathcal{D})$ projective ← see Lemma 2

$\Rightarrow P_i^\vee := \mathrm{Hom}_{\mathcal{D}}(P_i,\mathcal{D}) \in \mathrm{Mod}_{fg}(\mathcal{D}^{op})$ still projective (eg since P_i being a direct summand of a fg free module implies the same for $P_i^\vee$ ...)

$\Rightarrow$ cplex $P_\bullet^\vee$ w/ $H^i(P_\bullet^\vee) \simeq \mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M},\mathcal{D})$ $\forall i\in\mathbb{Z}$

(zero unless $i = n$)

$\Rightarrow \quad 0 \to P^\vee_{0,\text{left}} \to \cdots \to P^\vee_{n,\text{left}} \to \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M},\mathcal{D})_{\text{left}} \to 0$

projective resolution of $\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M},\mathcal{D})_{\text{left}}$

$\Rightarrow$ cplex $P^{\vee\vee}_\bullet = (P^\vee_{\bullet,\text{left}})^\vee_{\text{left}}$

w/ $H^i(P^{\vee\vee}_\bullet) \simeq \mathcal{E}xt^i_{\mathcal{D}}(\mathbb{D}(\mathcal{M}),\mathcal{D})_{\text{left}} \simeq \begin{cases} 0 & \text{if } i \neq n \\ \mathbb{D}^2(\mathcal{M}) & \text{if } i = n \end{cases}$

But $P^{\vee\vee}_\bullet \simeq P_\bullet$ since the P_i are projective (exercise)

$\Rightarrow \quad \mathbb{D}^2\mathcal{M} \simeq H^n(P^{\vee\vee}_\bullet) \simeq H^n(P_\bullet) \simeq \mathcal{M}$. $\square$

In step ③ we've used:

Lemma 2 Any $\mathcal{M} \in \mathrm{Mod}_{fg}(\mathcal{D})$ admits a projective resolution of length n, $\quad 0 \to P_n \to \cdots \to P_0 \to \mathcal{M}$.

Pf. Take any resolution $\cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \to F_0 \xrightarrow{d_0} \mathcal{M} \to 0$ by free modules $F_i \in \mathrm{Mod}_{fg}(\mathcal{D})$.

Put $\mathcal{M}_i := \mathrm{im}(d_i)$. The exact sequences $0 \to \mathcal{M}_{i+1} \to F_i \to \mathcal{M}_i \to 0$ yield

$$\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}_n, -) \simeq \mathcal{E}xt^2_{\mathcal{D}}(\mathcal{M}_{n-1}, -) \simeq \cdots \simeq \underbrace{\mathcal{E}xt^{n+1}(\mathcal{M}, -)}_{=0}$$

(see Lemma 3 below)

$\Rightarrow \quad \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_n, -)$ exact, ie $\mathcal{M}_n$ projective

$\Rightarrow$ can take $\quad P_i := \begin{cases} F_i & \text{for } i < n \\ \mathcal{M}_i & \text{for } i = n. \end{cases}$ $\square$

For completeness we include

Lemma 3 For all $\mathcal{M} \in \mathrm{Mod}_{fg}(\mathcal{D})$, $\mathcal{N} \in \mathrm{Mod}(\mathcal{D})$ one has $\quad \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{N}) = 0 \quad \forall\, i > n$.

Pf.

① For $\mathcal{N} \simeq \mathcal{D}^{\oplus N}$ free this holds by cor 10.3 a)

② For $\mathcal{N} \in \mathrm{Mod}_{fg}(\mathcal{D})$ pick an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{N} \to 0 \quad \text{in } \mathrm{Mod}_{fg}(\mathcal{D})$$

with $\mathcal{F}$ free

$\Rightarrow \quad \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{N}) \simeq \mathcal{E}xt^{i+1}_{\mathcal{D}}(\mathcal{M},\mathcal{K}) \quad \forall \; > n$

by ② & long Ext-sequence

$\Rightarrow$ By induction enough to show $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \text{fingen}) = 0$ for $i \gg 0$.

But already for $i > 2n$ this is OK by passage to $\mathcal{E}xt^i_{gr\mathcal{D}}(gr^F\mathcal{M}, gr^F(\ldots))$ for good F.

(spectral sequence argument as in thm 10.1)

③ $\mathcal{N} \in \mathrm{Mod}_{fg}(\mathcal{D})$ general:

Write $\mathcal{N} = \varinjlim \mathcal{N}_\alpha$ w/ $\mathcal{N}_\alpha \subseteq \mathcal{N}$ fingen submodules

& use that $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \varinjlim \mathcal{N}_\alpha) \simeq \varinjlim \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{N}_\alpha)$. $\square$

II. $\mathcal{D}$-modules on arbitrary varieties

1. The sheaf $\mathcal{D}_X$: Naive viewpoint

Setup: X smooth variety / $k = \bar{k}$ alg closed field w/ char $k = 0$

$\rightsquigarrow$ tangent sheaf ("derivations on X/k w/ values in $\mathcal{O}_X$")

$$\mathcal{T}_X := \mathcal{D}er_{X/k}(\mathcal{O}_X) := \{ \xi \in \mathcal{E}nd_k(\mathcal{O}_X) \mid \forall f, g \in \mathcal{O}_X, \ \xi(fg) = \xi(f)\cdot g + f \cdot \xi(g) \}$$

Def The sheaf of algebraic differential operators on X is the subsheaf of rings

$$\mathcal{D}_X := \langle \mathcal{O}_X, \mathcal{T}_X \rangle \subseteq \mathcal{E}nd_k(\mathcal{O}_X)$$

generated by all functions & derivations.

Ex 1 For $X = \mathbb{A}^n$ we have $\mathcal{T}_X = \bigoplus_{i=1}^n \mathcal{O}_X \cdot \partial_i$

$\Longrightarrow$ global sections of $\mathcal{D}_X$ are the Weyl algebra $H^0(X, \mathcal{D}_X) \simeq k[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$.

This is a "local model" for $\mathcal{D}_X$ in general:

<u>Lemma 2</u> Put $n = \dim X$.

a) $\forall p \in X(k)$ $\exists$ open nbhood $p \in U \subseteq X$

sth $\exists$ functions $x_1, \dots, x_n \in \mathcal{O}_X(U)$

$\exists$ derivations $\partial_1, \dots, \partial_n \in \mathcal{T}_X(U)$

with

- $\mathcal{T}_X|_U = \bigoplus_{i=1}^n \mathcal{O}_U \cdot \partial_i$
- $[\partial_i, \partial_j] = 0 \quad \forall i, j$
- $[\partial_i, x_j] = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

b) In particular then

$$\mathcal{D}_X|_U \simeq \bigoplus_{I \in \mathbb{N}_0^n} \mathcal{O}_U \cdot \partial^I$$

of infinite rank!

is a free $\mathcal{O}_U$-module on generators $\partial^I := \partial_1^{i_1} \cdots \partial_n^{i_n}$.

c) As a sheaf of $\mathcal{O}_U$-algebras

$$\mathcal{D}_X|_U \simeq \mathcal{O}_U\langle \partial_1, \dots, \partial_n \rangle$$

$$:= \frac{\text{free noncommutative } \mathcal{O}_U\text{-algebra on } \partial_1, \dots, \partial_n}{\text{commutator relations from a)}}.$$

Pf. a) $\Rightarrow$ b), c) obvious.

a) $p \in X$ smooth point

$\Rightarrow$ $\mathcal{O}_{X,p}$ regular local ring of dimension $n = \dim X$

$\Rightarrow$ $\exists \underbrace{x_1, \dots, x_n}$ $\in \mathcal{O}_{X,p}$ generating the max. ideal $m_p \trianglelefteq \mathcal{O}_{X,p}$

("regular sequence" since the number of generators is as small as possible)

Recall: The sheaf of Kähler differentials $\Omega^1_X := \Omega^1_{X/k}$ has stalks

$$\Omega^1_{X,p} = \Omega^1_{\mathcal{O}_{X,p}/k}$$

(module of Kähler differentials for the k-algebra $\mathcal{O}_{X,p}$)

$$= \langle df \mid f \in \mathcal{O}_{X,p} \rangle$$

(span as an $\mathcal{O}_{X,p}$-module)

$$= \langle df \mid f \in m_p \rangle$$

(since $df = 0$ for $f \in k$ and $\mathcal{O}_{X,p}/m = k$)

$$= \langle dx_1, \dots, dx_n \rangle$$

(since $m_p = (x_1, \dots, x_n)$)

$$= \bigoplus_{i=1}^n \mathcal{O}_{X,p} \cdot dx_i$$

(since by smoothness $\Omega^1_{X,p}$ is a free $\mathcal{O}_{X,p}$-module of rank n, any set of n generators is a basis)

$\Rightarrow$ for suitable open $p \in U \subseteq X$,

we have $x_1, \dots, x_n \in \mathcal{O}_X(U)$

and $\Omega^1_U = \bigoplus_{i=1}^n \mathcal{O}_U \cdot dx_i$.

Now pass to $\mathcal{T}_X = \mathrm{Der}_{X/k}(\mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$

On $\mathcal{U}$ we denote by $\partial_1, \dots, \partial_n \in \mathcal{T}_{\mathcal{U}} = \mathcal{H}om_{\mathcal{O}_{\mathcal{U}}}(\Omega^1_{\mathcal{U}}, \mathcal{O}_{\mathcal{U}})$ the "dual basis":

$$\partial_i(dx_j) := \delta_{ij} := \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else.} \end{cases}$$

Then:

- $\mathcal{T}_{\mathcal{U}} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{U}} \cdot \partial_i$

- Viewing the ∂_i as derivations via

$$\begin{tikzcd} x_i \in \mathcal{O}_{\mathcal{U}} \arrow[r, "d"] \arrow[dr] & \Omega^1_{\mathcal{U}} \arrow[d, "\partial_j"] \\ & \mathcal{O}_{\mathcal{U}} \end{tikzcd}$$

also denoted ∂_j

we have

$$\begin{aligned} [\partial_i, x_j] &= \partial_i \circ x_j - x_j \circ \partial_i \\ &= \partial_i(x_j) \\ &= \delta_{ij} \end{aligned}$$

and

$[\partial_i, \partial_j] = 0$ (since it is a derivation vanishing on $x_1, \dots, x_n$). $\square$

<u>Rem 3</u> In the above setup we call $(x_1, \dots, x_n)$ "<u>local coordinates</u>" on X.

In the language of algebraic geometry, the morphism $\varphi = (x_1, \dots, x_n) : \mathcal{U} \to \mathbb{A}^n_{k}$ is étale.

Caution: Unlike in differential geometry, the "chart" φ CANNOT be chosen to be an embedding (unless X is a rational variety).

2. The sheaf $\mathcal{D}_X$: Conceptual viewpoint

using a) Lie algebroids,

b) Grothendieck differential operators.

<u>Lie algebroids</u>:

(think of $\mathcal{T} = \mathcal{T}_X$, $\alpha = \mathrm{id} \dots$)

<u>Def</u> A <u>Lie algebroid</u> consists of

- a quasicoherent $\mathcal{O}_X$-module $\mathcal{T}$,
- a k-bilinear map $[\cdot,\cdot] : \mathcal{T} \otimes_{k} \mathcal{T} \to \mathcal{T}$ making $\mathcal{T}$ a sheaf of Lie algebras / k,
- a Lie algebra homomorphism $\alpha : \mathcal{T} \to \mathcal{T}_X$ which is $\mathcal{O}_X$-linear and satisfies

$$[\xi, f \cdot \eta] = f \cdot [\xi, \eta] + \alpha(\xi)(f) \cdot \eta \quad \forall f \in \mathcal{O}_X, \ \xi, \eta \in \mathcal{T}$$

("<u>Leibniz rule</u>")

$\Longrightarrow \mathcal{L} := \mathcal{O}_X \oplus \mathcal{T}$ becomes a sheaf of Lie algebras / k via

$$[f \oplus \xi, g \oplus \eta] := (\alpha(\xi)(g) - \alpha(\eta)(f)) \oplus [\xi, \eta] \quad \forall f, g \in \mathcal{O}_X, \ \xi, \eta \in \mathcal{T}$$

$\Longrightarrow$ Universal enveloping algebra (in the Lie algebra sense)

tensor product over $\Bbbk$ (not over $\mathcal{O}_X$)!

$$\mathcal{U}(\mathcal{L}) := \big(\underbrace{\bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}}_{\substack{\text{sheaf of } \Bbbk\text{-algebras} \\ \text{w/ product given by } \otimes}}\big) / \mathcal{J}$$

w/ $\mathcal{J}$:= twosided ideal generated by

$a\otimes b - b\otimes a - [a,b]$ $\forall a,b \in \mathcal{L}$.

<u>Note:</u> There is no natural structure of $\mathcal{O}_X$-module on $\mathcal{U}(\mathcal{L})$!

<u>Def</u> The <u>universal enveloping algebra of the Lie algebroid $\mathcal{T}$</u> is

$$\mathcal{U}(\mathcal{T}/\mathcal{O}_X) := \mathcal{U}(\mathcal{L})^+ / \mathcal{J}$$

where

$$\mathcal{U}(\mathcal{L})^+ := \Big(\bigoplus_{n>0} \mathcal{L}^{\otimes n}\Big)/\mathcal{J} \subset \mathcal{U}(\mathcal{L}),$$

$\mathcal{J}$:= twosided ideal generated by

$f\otimes a - f\cdot a$ $\forall f \in \mathcal{O}_X$, $a \in \mathcal{L}$.

$\Rightarrow$ $\mathcal{U}(\mathcal{T}/\mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules

<u>Exercise</u>

a) $\mathrm{Mod}(\mathcal{U}(\mathcal{L})) \simeq \mathrm{Mod}(\mathcal{L}) :=$

$$\left\{ \begin{array}{l} \mathcal{M} \in \mathrm{Mod}(\Bbbk_X) \text{ w/ homomorphism} \\ \text{of Lie algebras } \mathcal{L} \to \mathrm{End}_{\Bbbk}(\mathcal{M}) \end{array} \right\}$$

b) $\mathrm{Mod}(\mathcal{U}(\mathcal{T}/\mathcal{O}_X)) \simeq \mathrm{Mod}(\mathcal{T}/\mathcal{O}_X) :=$

$$\left\{ \begin{array}{l} \mathcal{M} \in \mathrm{Mod}(\mathcal{T}) \cap \mathrm{Mod}(\mathcal{O}_X) \\ \text{s.th. } \forall f \in \mathcal{O}_X, \xi \in \mathcal{T}, m \in \mathcal{M}, \\ \quad f\cdot(\xi\cdot m) = (f\xi)\cdot m \\ \quad \xi\cdot(f\cdot m) = \xi(f)\cdot m \\ \qquad\qquad\quad + f\cdot(\xi\cdot m) \end{array} \right\}$$

<u>Cor 1</u>. For any Lie algebroid $(\mathcal{T}, \alpha)$, $\exists$ natural homom. $\mathcal{U}(\mathcal{T}/\mathcal{O}_X) \to \mathrm{End}_{\Bbbk}(\mathcal{O}_X)$.

- For $(\mathcal{T},\alpha) := (\mathcal{T}_X, \mathrm{id})$ we get an epi $\mathcal{U}_X := \mathcal{U}(\mathcal{T}_X/\mathcal{O}_X) \twoheadrightarrow \mathcal{D}_X$.

Pf. Use that

- $\mathcal{O}_X \in \mathrm{Mod}(\mathcal{T}/\mathcal{O}_X)$ via $\alpha : \mathcal{T} \to \mathcal{T}_X = \mathrm{Der}_{\Bbbk}(\mathcal{O}_X)$.
- for $(\mathcal{T},\alpha) = (\mathcal{T}_X, \mathrm{id})$, have

$$\begin{tikzcd} \mathcal{O}_X \oplus \mathcal{T}_X \arrow[r] \arrow[dr] & \mathcal{U}_X \arrow[d] \\ & \mathrm{End}_{\Bbbk}(\mathcal{O}_X) \end{tikzcd}$$

(image generates $\mathcal{D}_X$ by definition) $\square$

We'll see below that in fact $\mathcal{U}_X \xrightarrow{\sim} \mathcal{D}_X$

(for X smooth & char $k = 0$).

But first let's look at

Grothendieck differential operators

Def (Grothendieck) For $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{O}_X)$, $i \in \mathbb{Z}$ we put

$$\mathrm{Diff}_X^i(\mathcal{M},\mathcal{N}) := \begin{cases} 0 & \text{if } i < 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) & \text{if } i = 0 \\ \{P \in \mathcal{H}om_k(\mathcal{M},\mathcal{N}) \mid [P,f] \in \mathrm{Diff}_X^{i-1} \ \forall f \in \mathcal{O}_X\} & \text{else.} \end{cases}$$

$[P,f] := (m \mapsto P(fm) - f \cdot P(m)) \in \mathcal{H}om_k(\mathcal{M},\mathcal{N})$

and

$$\mathrm{Diff}_X(\mathcal{M},\mathcal{N}) := \bigcup_i \mathrm{Diff}_X^i(\mathcal{M},\mathcal{N}) \subseteq \mathcal{H}om_k(\mathcal{M},\mathcal{N}).$$

For $\mathcal{M} = \mathcal{N} = \mathcal{O}_X$ we write

$$\mathrm{Diff}_X := \mathrm{Diff}_X(\mathcal{O}_X, \mathcal{O}_X),$$
$$\mathrm{Diff}_X^i := \mathrm{Diff}_X^i(\mathcal{O}_X, \mathcal{O}_X).$$

Lemma 2. a) $\mathrm{Diff}_X^0 = \mathcal{O}_X$

$\mathrm{Diff}_X^1 = \mathcal{O}_X + \mathcal{T}_X$ inside $\mathcal{E}nd_k(\mathcal{O}_X)$.

b) We have an embedding of sheaves of filtered rings

$$\begin{tikzcd} \mathcal{D}_X \arrow[r, phantom, "\subseteq"] & \mathrm{Diff} \\ F_i \mathcal{D}_X \arrow[u, hook] \arrow[r, phantom, "\subseteq"] & \mathrm{Diff}_X^i \arrow[u, hook] \end{tikzcd}$$

where the <u>order filtration</u> $F_\bullet \mathcal{D}_X$ is defined by

$F_i \mathcal{D}_X :=$ image of $F_i \mathcal{U}_X$ inside $\mathcal{D}_X$

$F_i \mathcal{U}_X := \mathcal{O}_X$-submodule of $\mathcal{U}_X$ generated by the image of $\bigoplus_{m \leq i} \mathcal{T}_X^{\otimes m}$.

Pf.

a) $\mathcal{D}iff_X^0 = \{ P \in End_k(\mathcal{O}_X) \mid [P,f]=0 \ \forall f \in \mathcal{O}_X \}$
$= End_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X.$

For $P \in \mathcal{D}iff_X^1$,
replacing P by $P - \underbrace{P(1)}_{\in \mathcal{O}_X}$ we may assume $P(1) = 0$.

$\Rightarrow \forall f, g \in \mathcal{O}_X$,

$$\begin{aligned} P(fg) - f P(g) &= [P,f](g) \\ &= [P,f](g \cdot 1) \\ &= g \cdot [P,f](1) \\ &= g \cdot (P(f \cdot 1) - f \cdot \underbrace{P(1)}_{=1}) \\ &= g \cdot P(f) \end{aligned}$$

($[P,f] \in \mathcal{O}_X$ for $P \in \mathcal{D}iff_X^1(\mathcal{O}_X)$)

$\Rightarrow P(fg) = f P(g) + g P(f)$

ie. $P \in Der_k(\mathcal{O}_X) = \mathcal{T}_X$

b) By part a) it suffices to show

$$\mathcal{D}iff_X^i \circ \mathcal{D}iff_X^j \subseteq \mathcal{D}iff_X^{i+j} \quad \forall i,j \in \mathbb{N}_0.$$

This can be checked by induction on $i+j$:

- $i+j = 0$ trivial
- induction step: Let $P \in \mathcal{D}iff_X^i$, $Q \in \mathcal{D}iff_X^j$.

$\Rightarrow$ For any $f \in \mathcal{O}_X$,

$$\begin{aligned} [PQ, f] &= PQf - fPQ \\ &= P \circ \underbrace{[Q,f]}_{\in \mathcal{D}iff^{j-1}} + \underbrace{[P,f]}_{\in \mathcal{D}iff^{i-1}} \circ Q \end{aligned}$$

$\in \mathcal{D}iff_X^{i+j-1}(\mathcal{O}_X)$ by induction

$\Rightarrow PQ \in \mathcal{D}iff_X^{i+j}$ $\square$

Conclusion: Have filtered ring homom.

$$\mathcal{U}_X \twoheadrightarrow \mathcal{D}_X \hookrightarrow \mathcal{D}iff_X.$$

Now let's use that X is smooth and $\operatorname{char} k = 0$:

<u>Thm 3</u> We have $\mathcal{U}_X \xrightarrow{\sim} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}iff_X$

inducing isos on each of the filtered pieces,

w/ associated graded $\operatorname{gr}^F \mathcal{D} \simeq \operatorname{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)$.

Pf.

① The map $\mathcal{T}_X^{\otimes d} \to \mathcal{U}_X$ induces an epi

$$\begin{tikzcd} \mathcal{T}_X^{\otimes d} \arrow[r] \arrow[d] & F_d \mathcal{U}_X \arrow[r] & \operatorname{gr}^F_d \mathcal{U}_X \\ \operatorname{Sym}^d_k(\mathcal{T}_X) \arrow[rr, two heads] & & \operatorname{Sym}^d_{\mathcal{O}_X}(\mathcal{T}_X) \arrow[u, "\exists !"'] \end{tikzcd}$$

Indeed:

- Factorization over $\operatorname{Sym}^d_k(\mathcal{T}_X)$ follows from the relations $\underbrace{\xi \otimes \eta - \eta \otimes \xi}_{\in F_2 \mathcal{U}_X} \sim \underbrace{[\xi, \eta]}_{\in F_1 \mathcal{U}_X}$ in $\mathcal{U}_X$ $\forall \xi, \eta \in \mathcal{T}_X$

  $\Rightarrow$ disappears in $\operatorname{gr}^F \mathcal{U}_X$!

- Factorization over $\operatorname{Sym}^d_{\mathcal{O}_X}(\mathcal{T}_X)$ then follows from $\xi \otimes f\eta \sim \xi \otimes f \otimes \eta \sim f \otimes \xi \otimes \eta - \underbrace{\xi(f) \otimes \eta}_{\substack{\text{disappears} \\ \text{in } \operatorname{gr}^F \mathcal{U}_X!}}$

  in $\mathcal{U}$ $\forall \xi, \eta \in \mathcal{T}_X$, $f \in \mathcal{O}_X$.

② Thus we get

$$\begin{tikzcd} \operatorname{Sym}^d_{\mathcal{O}_X}(\mathcal{T}_X) \arrow[r, two heads] \arrow[rrr, bend right, "\varphi_d"'] & \operatorname{gr}^F_d \mathcal{U}_X \arrow[r, two heads] & \operatorname{gr}^F_d \mathcal{D}_X \arrow[r] & \operatorname{gr}_d \mathcal{D}iff_X \end{tikzcd}$$

We'll be done if we can show φ_d is an iso $\forall$

(then also $\mathcal{U}_X \xrightarrow{\sim} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}iff_X$ because the filtrations on all three sheaves start with $F_0 \mathcal{U}_X = F_0 \mathcal{D}_X = F_0 \mathcal{D}iff_X = \mathcal{O}_X$).

Showing φ_d to be an iso is a local problem

$\Rightarrow$ Wlog $\exists$ "local coordinates" $(x_1, \dots, x_n): X \to \mathbb{A}^n$ as in lemma 1.2 (after shrinking X).

$\Rightarrow$ dual coordinates ξ_ν on $T^* X$

sth
$$X \times \mathbb{A}^n_k \xrightarrow{\sim} T^* X$$
$$(p, \xi) \longmapsto (p, \textstyle\sum_\nu \xi_\nu dx_\nu)$$

③ For $f_1, \dots, f_d \in \mathcal{O}_X$ we have

$$[[\cdots[P, f_1], f_2], \dots], f_d] \begin{cases} \in \mathcal{O}_X & \text{for } P \in \mathcal{D}iff^d_X \\ = 0 & \text{for } P \in \mathcal{D}iff^{d-1}_X \end{cases}$$

$\Rightarrow [\cdot[P, f_1], \dots, f_d] \in \mathcal{O}_X$ well defined

for $P \in \mathrm{gr}_d \mathrm{Diff}_X$

Define the <u>symbol map</u>

$$\sigma_d : \mathrm{gr}_d \mathrm{Diff}_X \longrightarrow \mathcal{O}_X[\xi_1, \dots, \xi_n]$$

$$P \longmapsto \frac{1}{d!} [[P, f], \dots], f] \quad \text{(d factors } f\text{)}$$

($d! \in k^*$ since char $k = 0$)

where $f := \sum_{i=1}^{n} \xi_i \cdot x_i$

Note: By expansion in terms of the ξ_i, this is a homogenous polynomial of degree d in the ξ_i w/ coefficients in $\mathcal{O}_X$.

The ξ_i are the coordinate fct^s on our chosen trivialization of $T^*X \simeq X \times \mathbb{A}^n_k$

$\Rightarrow \mathcal{O}_X[\xi_1, \dots, \xi_n] \simeq \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)$

and we may regard σ_d as an $\mathcal{O}_X$-linear homomorphism $\sigma_d : \mathrm{gr}_d \mathrm{Diff}_X \to \mathrm{Sym}^d_{\mathcal{O}_X}(\mathcal{T}_X)$.

④ For $P \in \mathrm{im}(\varphi_d)$ the symbol $\sigma_d(P)$ does what we want: A short computation shows that $\sigma_d(\partial^I) = \xi^I$ for any multiindex I with $|I| = d$

$\Rightarrow \sigma_d \circ \varphi_d = \mathrm{id}$

$\Rightarrow \mathrm{Sym}^d_{\mathcal{O}_X} \mathcal{T}_X \xrightarrow{\sim} \mathrm{gr}^F_d \mathcal{U}_X \xrightarrow{\sim} \mathrm{gr}^F_d \mathcal{D}_X$

and σ_d is surjective

⑤ To show σ_d is injective, let $P \in \mathrm{Diff}^d_X$ w/ $\sigma_d(P) = 0$

$\Rightarrow$ taking coefficients of the monomials ξ^I we get

$[\cdot[P, x_{i_1}], \dots, x_{i_d}] = 0 \quad \forall i_1, \dots, i_d$

(note that the order of $i_1, \dots, i_d$ doesn't matter since $[[-, x_i], x_j] = [[-, x_j], x_i] + [-, \underbrace{[x_i, x_j]}_{=0}]$)

$\Rightarrow [\cdot\cdot[P, x_{i_1}], \dots, x_{i_{d-1}}] \in \mathcal{O}_X \quad \forall i_1, \dots, i_{d-1}$

(exercise, see below)

Put $s_{d-1} := \frac{1}{(d-1)!} [\cdot\cdot[P, f], \dots, f] \in \mathcal{O}_X[\xi_1, \dots, \xi_n]$

(homogenous of degree $d-1$)

By step ④ $\exists\, Q_{d-1} \in F_{d-1}\mathcal{D}_X$ w/ $\sigma_{d-1}(Q_{d-1}) = s_{d-1}$

$\Rightarrow [[P - Q_{d-1}, x_{i_1}], \dots, x_{i_{d-1}}] = 0 \quad \forall\, i_1, \dots, i_{d-1}$

Proceeding inductively we find $Q_j \in F_j \mathcal{D}_X \ \forall j < d$

sth $\quad [[P - \sum_{j=\nu}^{d-1} Q_j, x_{i_1}], \dots x_{i_\nu}] = 0 \quad \forall i_1, \dots, i_\nu$

Taking $\nu = 1$ we get $P - \underbrace{\sum_{j=1}^{d-1} Q_j}_{\in F_{d-1}\mathcal{D}} \in \mathcal{O}_X$

$\Rightarrow P \in F_{d-1}\mathcal{D}_X$ as required. $\square$

Here we've used: (without using that $\mathcal{D}_X = \mathrm{Diff}_X$)

Exercise 4 Show that if $P \in \mathrm{Diff}_X$ and $[P, x_i] = 0 \ \forall i$ (where $x_1, \dots, x_n$ is a system of local coordinates), then we must have $P \in \mathcal{O}_X$.

(Hint: With P also $Q := [P, f] \ \forall f \in \mathcal{O}_X$ satisfies $[Q, x_i] = 0 \ \forall i \dots$ Use this to reduce to the case $P \in \mathrm{Diff}^1(\mathcal{O}_X)$ and apply lemma 2)

Rem 5 The definition of $\mathcal{U}_X, \mathcal{D}_X, \mathrm{Diff}_X$ makes sense also for char $k = p$ or for X singular, and we always have $\quad \mathcal{U}_X \twoheadrightarrow \mathcal{D}_X \hookrightarrow \mathrm{Diff}_X$,

but in general these are not very well-behaved:

Example 6 a) For char $k = p > 0$ and $X = \mathbb{A}^1_k$,

- $0 \neq [\partial^{\otimes p}] \in \ker(\mathcal{U}_X \to \mathcal{D}_X)$

  in fact $F_d \mathcal{D}_X = 0 \ \forall d \geq p$

- Nevertheless $\mathrm{Diff}^n_X \neq 0 \ \forall n \geq 0$,

  eg. look at $\partial^{(n)} := $ "$\frac{1}{n!} \partial^n$" (not welldef. for $n \geq p$)

  defined by $\partial^{(n)}(x^\nu) := \binom{\nu}{n} \cdot x^{\nu - n}$

  (unlike $\frac{1}{n!}$, the binomial coeff $\binom{\nu}{n} \in \mathbb{Z}$ can be read inside k even if $n \geq p$)

$\Rightarrow$ in general the right object to study is Grothendieck's Diff_X.

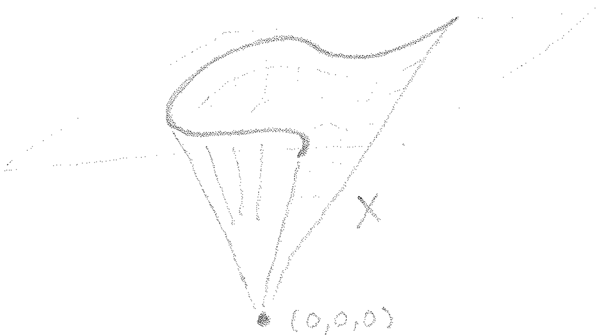
b) Even for char $k = 0$, the ring Diff_X is not very nice if X is singular:

eg for $X = V(x^2+y^2+z^2) \subset \mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$

the affine cone over a smooth plane cubic,

the ring $H^0(X, \mathrm{Diff}_X)$ is NOT fingen $/ k$

and has infinite ascending chains of 2sided ideals!



See Bernstein - Gelfand - Gelfand '72 (Russian Math Surveys 27).

Further reading:

Smith - Stafford '88 (Proc LMS 56)

Musson '91 (Archiv Math. 56)

⋮

So we'll always assume char $k = 0$ & X smooth.

A final remark on the sheaf structure:

<u>Exercise 7</u> Assume $X = \operatorname{Spec} A$ affine and let $U = \operatorname{Spec} A_f \subseteq X$ for $f \in A$.

The ring $D = H^0(X, \mathcal{D}_X)$ has two natural structures of A-module, using left resp right multiplication.

a) Using that $[f, -] : D \to D$ is nilpotent, show that $\forall P \in D$,
- $\exists\, Q \in D,\ m \in \mathbb{N}: \ P \cdot f^m = f \cdot Q$
- $\exists\, R \in D,\ n \in \mathbb{N}: \ f^n \cdot P = R \cdot f$

b) Deduce that $D_f := A_f \otimes_A D \simeq D \otimes_A A_f$ and that this is again a ring (!) with $H^0(U, \mathcal{D}_X) = D_f$.

Thus $\mathcal{D}_X$ is a quasicoherent sheaf both for the left and right $\mathcal{O}_X$-module structures.

3. $\mathcal{D}$-modules: Basic notions

As above X always denotes a smooth var / $k = \bar{k}$ with char$(k) = 0$

$\mathrm{Mod}(\mathcal{D}_X)$ = the category of sheaves of left $\mathcal{D}_X$-modules

$\mathrm{Mod}(\mathcal{D}_X^{op})$ = —— " —— right —— " ——

<u>Ex 1</u> $\mathcal{M} = \mathcal{O}_X \in \mathrm{Mod}(\mathcal{D}_X)$

w/ the natural action via $\mathcal{T}_X = \mathrm{Der}_k(\mathcal{O}_X)$

More generally any vector bdle with a flat connection:

<u>Def</u> For $\nu \in \mathbb{N}_0$ we put

$$\Omega_X^\nu := \mathrm{Alt}^\nu_{\mathcal{O}_X}(\Omega_X^1) := (\Omega_X^1)^{\otimes \nu} \Big/ \begin{matrix}\text{relations generated} \\ \text{by } \alpha\otimes\beta - \beta\otimes\alpha\end{matrix}$$

$$\simeq \left\{ \begin{matrix} \text{alternating } \mathcal{O}_X\text{-multilinear} \\ \text{forms } \mathcal{T}_X^{\otimes\nu} \longrightarrow \mathcal{O}_X \end{matrix} \right\}$$

and put $\alpha_1 \wedge \cdots \wedge \alpha_\nu :=$ image of $\alpha_1 \otimes \cdots \otimes \alpha_\nu$ $\forall \alpha_i \in \Omega_X^1$.

$\Longrightarrow$ $\mathcal{O}_X$-linear epi

$$-\wedge- : \Omega_X^\mu \otimes_{\mathcal{O}_X} \Omega_X^\nu \twoheadrightarrow \Omega_X^{\mu+\nu} \quad \forall \mu, \nu \in \mathbb{N}_0.$$

Recall the "universal derivation" $d : \mathcal{O}_X \twoheadrightarrow \Omega_X^1$ given in local coordinates by $d(f) = \sum_i \partial_i(f)\, dx_i$.

We define the <u>exterior derivative</u> $d : \Omega_X^\nu \to \Omega_X^{\nu+1}$ $\forall \nu \in \mathbb{N}_0$ as the unique k-linear map sth

- $\mathcal{O}_X \xrightarrow{d} \Omega_X^1$ is the universal derivation
- $\mathcal{O}_X \xrightarrow{d\circ d} \Omega_X^2$ is the zero map
- in general

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^\nu \cdot \alpha \wedge d\beta \quad \forall \begin{matrix} \alpha \in \Omega_X^\nu \\ \beta \in \Omega_X^\mu. \end{matrix}$$

<u>Rem</u> It follows that $d\circ d = 0$ in all degrees $\nu \in \mathbb{N}$, and in local coordinates

$$d\Big(\sum_I f_I(x)\, dx_I\Big) = \sum_i \sum_I \partial_i f_I(x) \cdot dx_i \wedge dx_I$$

where $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_\nu}$.

<u>Notation</u> For $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ we put $\Omega_X^\nu(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^\nu$.

Def Let $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$ be a locally free $\mathcal{O}_X$-module.

A <u>connection</u> on $\mathcal{M}$ is a k-linear sheaf hom.

$$\nabla : \mathcal{M} \to \Omega^1_X(\mathcal{M})$$

satisfying <u>Leibniz' rule</u>

$$\nabla(fm) = f \cdot \nabla(m) + m \otimes df \quad \forall m \in \mathcal{M}, \ \forall f \in \mathcal{O}_X.$$

We extend it to
$$\begin{array}{ccc} \nabla : \Omega^\nu_X(\mathcal{M}) & \longrightarrow & \Omega^{\nu+1}_X(\mathcal{M}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ m \otimes \alpha & \longmapsto & \nabla(m) \wedge \alpha + m \otimes d\alpha. \end{array}$$

<u>Exercise 2</u> Show that for $\alpha \in \Omega^1_X(\mathcal{M})$, the form $\nabla\alpha \in \Omega^2_X(\mathcal{M})$ is given by

$$\underbrace{(\nabla\alpha)}_{\in \Omega^2_X(\mathcal{M})}\underbrace{(\xi \otimes \eta)}_{\in \mathcal{T}_X^{\otimes 2}} = \nabla_\xi(\overbrace{\alpha(\eta)}^{\in \mathcal{M}}) - \nabla_\eta(\overbrace{\alpha(\xi)}^{\in \mathcal{M}}) - \alpha(\underbrace{[\xi, \eta]}_{\in \mathcal{T}_X})$$

for all $\xi, \eta \in \mathcal{T}_X$

(we put $\nabla_\xi(\beta) := (\nabla\beta)(\xi)$ for $\beta \in \mathcal{M}$ etc...)

(Hint: First show that in local coordinates for $\alpha = \sum_i m_i \otimes dx_i$ the claim holds when $\xi = \partial_\mu$, $\eta = \partial_\nu$. It then only remains to show that sending $\xi \otimes \eta$ to the RHS gives a well-defined section of $\Omega^2_X(\mathcal{M})$ globally ...)

Def A connection $\nabla : \mathcal{M} \to \Omega^1_X(\mathcal{M})$ is called <u>flat</u> if its <u>curvature</u> $\nabla^2 : \mathcal{M} \to \Omega^2_X(\mathcal{M})$ vanishes.

<u>Note:</u> By exercise 2 the flatness of ∇ means

$$\nabla_\xi(\nabla_\eta(m)) - \nabla_\eta(\nabla_\xi(m)) = \nabla_{[\xi,\eta]}(m) \quad \forall m \in \mathcal{M}, \ \forall \xi, \eta \in \mathcal{T}_X.$$

$\Longrightarrow$ We then get on $\mathcal{M}$ the structure of a left $\mathcal{D}_X$-module via

$$\xi \cdot m := \nabla_\xi(m) \quad \forall m \in \mathcal{M}, \xi \in \mathcal{T}_X.$$

(see lemma 3)

Lemma 3 For $\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$,

a k-linear map $\nabla : \mathcal{T}_X \to \mathrm{End}_k(\mathcal{M})$

$$\xi \longmapsto \nabla_\xi$$

makes $\mathcal{M}$ a left (resp right) $\mathcal{D}_X$-module

via $\xi \cdot m := \nabla_\xi(m)$ (resp $m \cdot \xi := -\nabla_\xi(m)$)

iff

① $\nabla_{f\xi}(m) = f \cdot \nabla_\xi(m)$ ("$\mathcal{O}_X$-linearity for right (resp left) module structure on $\mathrm{End}_k(\mathcal{M})$")

(resp $\nabla_{f\xi}(m) = \nabla_\xi(fm)$)

② $\nabla_\xi(fm) = \xi(f) \cdot m + f \cdot \nabla_\xi(m)$ ("Leibniz rule")

③ $\nabla_{[\xi,\eta]}(m) = [\nabla_\xi, \nabla_\eta](m)$ ("flatness")

$\forall\, m \in \mathcal{M}$, $f \in \mathcal{O}_X$, $\xi, \eta \in \mathcal{T}_X$.

Pf. Since $\mathcal{D}_X = \mathcal{U}(\mathcal{T}_X / \mathcal{O}_X)$ is the universal enveloping algebra of the Lie algebroid $\mathcal{T}_X$, §2 (p.43 exercise) gives $\mathrm{Mod}(\mathcal{D}_X) \simeq \underbrace{\mathrm{Mod}(\mathcal{T}_X / \mathcal{O}_X)}_{\text{"Lie algebroid modules"}}$.

Now ③ makes $\mathcal{M}$ an element of $\mathrm{Mod}(\mathcal{T}_X)$, and ①,② are the compatibilities required for $\mathrm{Mod}(\mathcal{T}_X / \mathcal{O}_X)$. $\square$

Exercise: Check the statements for right modules!

Slogan: Left $\mathcal{D}_X$-modules can be regarded as flat connections on not necessarily locally free $\mathcal{O}_X$-modules.

Q: What are the possible underlying $\mathcal{O}_X$-modules?

Lemma 4 For $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$ the following are equivalent:

a) $\mathcal{M}$ is coherent $/ \mathcal{O}_X$

b) $\mathcal{M}$ is locally free of finite rank $/ \mathcal{O}_X$.

Pf. b) $\Rightarrow$ a) trivial.

a) $\Rightarrow$ b): Assume $\mathcal{M}$ coherent $/ \mathcal{O}_X$.

Want: $\mathcal{M}_p$ free $/ \mathcal{O}_{X,p}$ $\forall\, p \in X(k)$.

Pick a k-basis $\bar{s}_1, \dots, \bar{s}_r$ of $\mathcal{M}_p / m_p \cdot \mathcal{M}_p = \mathcal{M}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/m_p$

(for the maximal ideal $m_p \trianglelefteq \mathcal{O}_{X,p}$)

& lift it to $s_1, \dots, s_r \in \mathcal{M}_p$

with
$$\mathcal{M}_p = \sum_{i=1}^{r} \mathcal{O}_p \cdot s_i \qquad \text{(Nakayama)}$$

Goal: $\exists$ no $\mathcal{O}_{X,p}$-linear relations between the s_i.

For $f \in \mathcal{O}_{X,p}$ put $\operatorname{ord}(f) := \begin{cases} \max\{\nu \mid f \in m_p^\nu\} & \text{if } f \neq 0 \\ \infty & \text{if } f = 0 \end{cases}$

$\Rightarrow$ $\forall f \neq 0$ with $\operatorname{ord}(f) > 0$,

$$\exists\, \partial \in \mathcal{T}_X \text{ with } \operatorname{ord}(\partial(f)) < \operatorname{ord}(f). \qquad (*)$$

(obvious in local coordinates)

If $\exists$ relation $\sum_{i=1}^r f_i s_i = 0$ with $f_i \in \mathcal{O}_{X,p}$ not all zero,

choose one with $\nu := \min_i \{\operatorname{ord}(f_i)\}$ minimal.

Note: $\nu > 0$ because $\bar{s}_1, \dots, \bar{s}_r$ are lin. independent $/k$!

Pick i_0 with $\operatorname{ord}(f_{i_0}) = \nu$ & $\partial \in \mathcal{T}_X$ sth $\operatorname{ord}(\partial(f_{i_0})) < \nu$

($\exists$ by $(*)$)

Write $\partial(s_i) = \sum_{j=1}^r c_{ij} \cdot s_j$ w/ $c_{ij} \in \mathcal{O}_{X,p}$

$$\Rightarrow \quad 0 = \partial\Big(\sum_i f_i s_i\Big) = \sum_i \Big(\underbrace{\partial(f_i)}_{\substack{\operatorname{ord} < \nu \\ \text{for } i = i_0}} + \underbrace{\sum_j f_j c_{ji}}_{\operatorname{ord} \geq \nu} \Big) \cdot s_i$$

$\lightning$ ν not minimal $\lightning$

$\square$

Thus $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are flat connections on vector bundles. Note: There are many more $\mathcal{D}_X$-coherent $\mathcal{D}_X$-modules, e.g.

$$\mathcal{M} := \mathcal{D}_X / \mathcal{J} \quad \text{for any left ideal } \mathcal{J} \trianglelefteq \mathcal{D}_X.$$

Ex. 5 The Dirac module at a point $p \in X(k)$ is given by $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X \cdot \mathcal{J}$ w/ $\mathcal{J} = \{f \in \mathcal{O}_X \mid f(p) = 0\}$

$$\simeq (\mathcal{O}_X/\mathcal{J})[\partial_1, \dots, \partial_n]$$

w/ $\partial_1, \dots, \partial_n$ derivatives wrt local coordinates at the point p.

On the other hand, not every vector bdle admits a flat connection:

Exercise Let X be a smooth projective curve $/k$ ($= \mathbb{C}$ if you like) and $\mathcal{L} \in \operatorname{Pic}(X)$ a line bundle on it.

Show that $\exists$ (automatically flat) connection $\nabla: \mathcal{L} \to \Omega_X^1(\mathcal{L})$ iff $\deg(\mathcal{L}) = 0$.

(Hint: For an elementary argument, write $\mathcal{L} \cong \mathcal{O}(D)$ with a divisor D of degree $d = \deg(\mathcal{L})$. Show that

- $\nabla|_{X \setminus \operatorname{Supp} D} \longleftrightarrow \omega \in \Omega^1_X(X \setminus \operatorname{Supp} D)$

- ∇ extends over $p \in D \longleftrightarrow \operatorname{Res}_p(\omega) = m_p$

 where $D = -\sum_q m_q \cdot q$

and recall:

$\exists$ meromorphic diff'form ω w/ $\operatorname{Res}_p(\omega) = m_p \;\; \forall p$
iff $\sum m_p = 0$.)

<u>Rem 6</u> a) We have $\deg(\mathcal{L}) = c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$ for $k = \mathbb{C}$

$\rightarrow$ topological obstruction to existence of (flat) connection!

For more on this see Atiyah (1956).

b) Similarly, we'll see later that for $\mathcal{L} \in \operatorname{Pic}(X)$

$\exists$ structure of right $\mathcal{D}_X$-module on $\mathcal{L}$ iff $\deg \mathcal{L} = 2g - 2$.

smooth proj curve / k of genus g

This gives a mnemonic for the following result:

<u>Lemma 7.</u> Let $\mathcal{M}, \mathcal{N} \in \operatorname{Mod}(\mathcal{D}_X)$, $\mathcal{M}', \mathcal{N}' \in \operatorname{Mod}(\mathcal{D}_X^{op})$, then:

(Note: By remark 6 b) these are all combinations where a $\mathcal{D}$-module structure can exist. For instance, $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}'$ doesn't work...)

a) $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \operatorname{Mod}(\mathcal{D}_X)$ via $\xi(m \otimes n) := \xi m \otimes n + m \otimes \xi n$

b) $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N} \in \operatorname{Mod}(\mathcal{D}_X^{op})$ via $(m' \otimes n)\xi := m'\xi \otimes n - m' \otimes \xi n$

c) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \operatorname{Mod}(\mathcal{D}_X)$ via $(\xi f)(m) := \xi(f(m)) - f(\xi m)$

d) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') \in \operatorname{Mod}(\mathcal{D}_X)$ via $(\xi f)(m') := -f(m')\xi + f(m'\xi)$

e) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') \in \operatorname{Mod}(\mathcal{D}_X^{op})$ via $(f\xi)(m) := f(m)\xi + f(\xi m)$

Pf. Exercise, using lemma 3. For instance let's check b):

① $(m' \otimes n) f\xi = (m'(f\xi)) \otimes n - m' \otimes (f\xi n)$

$= ((fm')\xi) \otimes n - fm' \otimes \xi n$

$= (f \cdot (m' \otimes n))\xi \quad \Rightarrow$ right $\mathcal{O}_X$ linear

② $(f \cdot (m' \otimes n))\xi = -\xi(f) \cdot m' \otimes n + f((m' \otimes n)\xi)$

(same computation plus $(fm')\xi = f \cdot (m'\xi) - \xi(f) m'$.

$\Rightarrow$ Leibniz

③ $(m'\otimes n)\cdot[\xi,\eta] = m'[\xi,\eta]\otimes n - m'\otimes[\xi,\eta]n$

$$= ((m'\xi)\eta - (m'\eta)\xi)\otimes n$$

$$- m'\otimes(\xi(\eta n) - \eta(\xi n))$$

$$= ((m'\otimes n)\xi)\eta - ((m'\otimes n)\eta)\xi$$

$\Rightarrow$ flatness $\square$

So, what about left $\leftrightarrow$ right?

Recall: On $X = \mathbb{A}^n_k$ we had $\mathrm{Mod}(\mathcal{D}_X) \simeq \mathrm{Mod}(\mathcal{D}_X^{op})$

$$M_{left} \longleftarrow\!\shortmid M$$

where $M_{left} := M$ as a k vector space

but with $\underset{\underset{M_{left}}{\cap}}{P\cdot m} := \underset{\underset{M}{\cap}}{m\cdot P^*}$

for $P^* := \sum_I (-1)^I \partial^I \cdot c_I(x)$

if $P = \sum_I c_I(x)\partial^I \in \mathcal{D}_X$.

Can we "globalize" this?

Def Let $\xi \in \mathcal{T}_X$. The Lie derivative $L_\xi : \Omega^\bullet_X \to \Omega^\bullet_X$ is the unique degree-preserving endomorphism such that

a) $L_\xi(\alpha\wedge\beta) = L_\xi(\alpha)\wedge\beta + \alpha\wedge L_\xi(\beta)$

b) $L_\xi \circ d = d\circ L_\xi$

c) $L_\xi(f) = \xi(f)$ $\quad \forall \alpha,\beta\in\Omega^\bullet_X$, $\forall f\in\mathcal{O}_X$.

Explicitly:

Assume we have local coordinates $x_1,\dots,x_n$ on X.

Put $\xi = f\cdot\partial_i$ with $f\in\mathcal{O}_X$ and $i\in\{1,\dots,n\}$.

Then for any section $\alpha = g\cdot dx_1\wedge\dots\wedge dx_n \in \Omega^n_X$ w/ $g\in\mathcal{O}_X$ one computes

$$L_\xi(\alpha) = L_\xi(g)\,dx_1\wedge\dots\wedge dx_n + \sum_{j=1}^n g\cdot dx_1\wedge\dots\wedge L_\xi(dx_j)\wedge\dots\wedge dx_n$$

$$= \xi(g)\,\underset{dx_1\wedge\dots\wedge dx_n}{} + \sum_{j=1}^n g\cdot dx_1\wedge\dots\wedge d(\underbrace{\xi(x_j)}_{=f\cdot\delta_{ij}})\wedge\dots\wedge dx_n$$

$$= \underbrace{(f\partial_i(g) + \partial_i(f)g)}_{=(\partial_i f)(g)}\,dx_1\wedge\dots\wedge dx_n$$

$\Longrightarrow$ On $\Omega^n_X = \mathcal{O}_X \cdot dx_1 \wedge \dots \wedge dx_n$

we have $\underset{\substack{\uparrow \\ f\cdot\partial_i}}{-L_\xi} = \underset{\substack{\uparrow \\ -\partial_i f}}{\xi^*}$ (the adjoint of ξ)

Notation: $\omega_X := \Omega^n_X$.

Cor. 8 a) We have $\omega_X \in \mathrm{Mod}(\mathcal{D}_X^{op})$

via $\alpha \cdot \xi := -L_\xi(\alpha)$ $\forall\, \alpha \in \omega_X$, $\xi \in \mathcal{T}_X$

b) Hence $\exists$ equivalence of categories

$$\mathrm{Mod}(\mathcal{D}_X) \simeq \mathrm{Mod}(\mathcal{D}_X^{op})$$

$$M \longmapsto M \otimes_{\mathcal{O}_X} \omega_X =: M_{\text{right}}$$

$$N_{\text{left}} := N \otimes_{\mathcal{O}_X} \omega_X^{-1} \longleftarrow\!\shortmid\; N$$

Pf. a) Exercise $\square$

b) follows from a) via lemma 7.

4. Direct and inverse images I

$\pi: X \longrightarrow Y$ morphism of smooth varieties / k

Inverse images

Recall: $\pi^*: \mathrm{Mod}(\mathcal{O}_Y) \longrightarrow \mathrm{Mod}(\mathcal{O}_X)$

$$M \longmapsto \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1} M$$

via the natural homom. $\pi^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

Analog for $\mathcal{D}$-modules?

Don't have a natural homom. $\pi^{-1}\mathcal{D}_Y \not\rightarrow \mathcal{D}_X$

but: $\mathcal{T}_X \xrightarrow{d\pi} \pi^* \mathcal{T}_Y = \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{T}_Y$.

(in local coordinates: $\frac{\partial}{\partial x_i} \mapsto \sum_j \frac{\partial \pi_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j}$ where $\pi_j := y_j \circ \pi$ "chain rule")

Lemma 1 The inverse image functor π^* for $\mathcal{O}$-modules has a natural lift to $\mathcal{D}$-modules:

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{D}_Y) \arrow[r, dashed, "\exists \pi^*"] \arrow[d, "\text{forget}"'] & \mathrm{Mod}(\mathcal{D}_X) \arrow[d, "\text{forget}"] \\ \mathrm{Mod}(\mathcal{O}_Y) \arrow[r, "\pi^*"] & \mathrm{Mod}(\mathcal{O}_X) \end{tikzcd}$$

Pf. Let $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Y)$.

- action of $\mathcal{O}_X$ on $\pi^*\mathcal{M} \in \mathrm{Mod}(\mathcal{O}_X)$: the usual one
- action of $\mathcal{T}_X$:

For $\xi \in \mathcal{T}_X$ write $d\pi(\xi) = \sum_i f_i \otimes \eta_i$ w/ $f_i \in \mathcal{O}_X$, $\eta_i \in \pi^{-1}\mathcal{T}_Y$

and put

$$\xi(f \otimes m) := \xi(f) \otimes m + f \cdot \sum_i f_i \otimes \eta_i(m) \qquad \text{"chain rule"}$$

for $f \in \mathcal{O}_X$, $m \in \pi^{-1}\mathcal{M}$. Exercise: This makes $\pi^*\mathcal{M}$ a $\mathcal{D}_X$-module! $\square$

Rem $\mathcal{E} \in \mathrm{Mod}(\mathcal{O}_Y)$ locally free w/ flat conn. $\nabla: \mathcal{E} \to \Omega^1_Y(\mathcal{E})$

$$\Rightarrow \pi^*(\mathcal{E}, \nabla) = (\pi^*(\mathcal{E}), \nabla: \pi^*\mathcal{E} \xrightarrow{\nabla} \pi^*(\Omega^1_Y(\mathcal{E})) \xrightarrow{d\pi} \Omega^1_X(\pi^*\mathcal{E}))$$

Alternative description:

Put $\mathcal{D}_{X\to Y} := \pi^*\mathcal{D}_Y := \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{D}_Y$

This is a left $\mathcal{D}_X$-module (special case of lemma 1)

but also a right $\pi^{-1}\mathcal{D}_Y$-module (by right multiplicat^n)

& the two actions commute

$\Rightarrow \mathcal{D}_{X\to Y} \in \mathrm{Mod}(\mathcal{D}_X \times \pi^{-1}\mathcal{D}_Y^{op})$ is a bimodule

and
$$\boxed{\pi^*\mathcal{M} = \mathcal{D}_{X\to Y} \otimes_{\pi^{-1}\mathcal{D}_Y} \pi^{-1}\mathcal{M}.}$$

Example 2

a) We have $\mathcal{D}_{X\to pt} = \mathcal{O}_X$ as a $\mathcal{D}_X \times k$-bimodule,

for $\pi: X \to \mathrm{Spec}\, k = pt$.

$$\Rightarrow \pi^*: \mathrm{Vect}(k) \longrightarrow \mathrm{Mod}(\mathcal{D}_X)$$
$$V \longmapsto V \otimes_k \mathcal{O}_X$$

b) For $j: U \hookrightarrow Y$ open we have $\mathcal{D}_{U\to Y} = \mathcal{D}_U$

as a bimodule for $\mathcal{D}_U \times j^{-1}\mathcal{D}_Y^{op}$

$$\Rightarrow j^*: \mathrm{Mod}(\mathcal{D}_Y) \longrightarrow \mathrm{Mod}(\mathcal{D}_U)$$
$$\mathcal{M} \longmapsto \mathcal{M}|_U.$$

c) In general, if $\exists$ local coordinates $y_1, \ldots, y_n$ on Y

sth $\mathcal{D}_Y = \bigoplus_I \mathcal{O}_Y \cdot \partial_y^I$ w/ $\partial_y^I := \partial_{y_1}^{i_1} \cdots \partial_{y_n}^{i_n}$,

then for any $\pi: X \to Y$ one has

$$\mathcal{D}_{X\to Y} = \bigoplus_I \mathcal{O}_X \cdot \partial_y^I$$

as a left module for $\mathcal{D}_X$ (via "chain rule": $\xi \cdot (f\partial_y^I) := \xi(f)\partial_y^I + f \cdot d\pi(\xi)\cdot\partial_y^I$ for $\xi \in \mathcal{T}_X, f \in \mathcal{O}_X$)

and right module for $\pi^{-1}(\mathcal{D}_Y) = \pi^{-1}(\mathcal{O}_Y)\langle \partial_{y_1}, \ldots, \partial_{y_n}\rangle$.

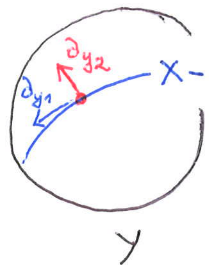
(in the "natural" way...)

Important special cases:

(c1) $i: X \hookrightarrow Y$ closed embedding w/ $X = \{y_{m+1} = \cdots = y_n = 0\}$

($m = \dim X \in \{0, 1, .., n-1\}$)

$$\Rightarrow \quad \mathcal{D}_{X\to Y} = \mathcal{D}_X \otimes_k k[\underbrace{\partial_{y_{m+1}}, \dots, \partial_{y_n}}_{\text{derivations in "normal direction"}}]$$



$\Rightarrow \mathcal{D}_{X\to Y}$ is

- flat over $\mathcal{D}_X$
- generated by $\underset{i^*(1)}{1} \in \underset{i^*\mathcal{D}_Y}{\mathcal{D}_{X\to Y}}$ as an $i^{-1}\mathcal{D}_Y^{op}$-module

⚠ $i^*: \mathrm{Mod}(\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_X)$

does NOT preserve finite generation of $\mathcal{D}$-modules,

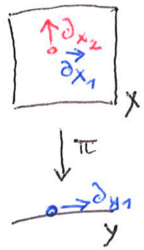
eg $i^*(\mathcal{D}_X) = \mathcal{D}_{X\to Y}$ is NOT fin.gen. as $\mathcal{D}_X$-module!

(c2) $\pi: X \to Y$ smooth

$\Rightarrow$ can take local coordinates x_i on X, y_j on Y sth $d\pi(\partial_{x_i}) = \begin{cases} \partial_{y_i} & \text{for } i \le \dim Y \\ 0 & \text{else} \end{cases}$

$\Rightarrow \mathcal{D}_{X\to Y}$ is

- flat over $i^{-1}\mathcal{D}_Y^{op}$
- generated by $1 \in \mathcal{D}_{X\to Y}$ as a $\mathcal{D}_X$-module.



Globally we get:

Cor 3 a) $i: X \hookrightarrow Y$ closed immersion ← as right $i^{-1}\mathcal{D}_Y$-module

$$\Rightarrow \quad \mathcal{D}_{X\to Y} = i^{-1}(\mathcal{D}_Y / \mathcal{J}_X \cdot \mathcal{D}_Y)$$

where $\mathcal{J}_X := \{f \in \mathcal{O}_Y \mid f|_X \equiv 0\} \trianglelefteq \mathcal{O}_Y$

b) $\pi: X \twoheadrightarrow Y$ smooth morphism ← as left $\mathcal{D}_X$-module

$$\Rightarrow \quad \mathcal{D}_{X\to Y} = \mathcal{D}_X / \mathcal{D}_X \cdot \mathcal{T}_{X/Y}$$

where $\mathcal{T}_{X/Y} := \ker(d\pi: \mathcal{T}_X \to \pi^*\mathcal{T}_Y)$.

Rem 1) The functor $\pi^*: \mathrm{Mod}(\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_X)$

is right exact for any $\pi: X \to Y$

(and even exact if π is a smooth morphism).

2) For $X \xrightarrow{\pi} Y \xrightarrow{\varrho} Z$ we have $(\varrho \circ \pi)^* \simeq \pi^* \circ \varrho^*$,

hence the above corollary is enough to compute π^*

for any $\pi: X \to Z$ by writing

$$\begin{tikzcd} X \arrow[r, hook, "(id{,}\pi)"] \arrow[rr, bend right, "\pi"'] & X \times Z \arrow[r, "pr_2"] & Z \end{tikzcd}.$$

Rem Since π^* extends the pullback for $\mathcal{O}$-modules,

is preserves quasicoherence:

$$\pi^*: \mathrm{Mod}_{qc}(\mathcal{D}_Y) := \{M \in \mathrm{Mod}(\mathcal{D}_X) \mid \text{quasicoherent as } \mathcal{O}_Y\text{-module}\}$$
$$\to \mathrm{Mod}_{qc}(\mathcal{D}_X) \subset \mathrm{Mod}(\mathcal{D}_X).$$

Direct images (naively)

These are easier for right-$\mathcal{D}$-modules:

("you integrate distributions, not functions")

Recall $\mathcal{D}_{X\to Y} \in \mathrm{Mod}(\mathcal{D}_X \times \pi^{-1}\mathcal{D}_Y^{op})$ for $\pi: X \to Y$

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Mod}(\mathcal{D}_X^{op}) \arrow[r, "(-)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}"] \arrow[dr] & \mathrm{Mod}(\pi^{-1}\mathcal{D}_Y^{op}) \arrow[d, "\pi_*"] \\ & \mathrm{Mod}(\mathcal{D}_Y^{op}) \end{tikzcd}$$

For left $\mathcal{D}$-modules we define pushforward via left $\leftrightarrow$ right side change:

Def Put $\mathcal{D}_{Y\leftarrow X} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\omega_Y$

$$\in \mathrm{Mod}(\mathcal{D}_X^{op} \times \pi^{-1}\mathcal{D}_Y)$$

(a left-$\pi^{-1}\mathcal{D}_Y$ and right $\mathcal{D}_X$-module)

We get $\pi_{naive}: \mathrm{Mod}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}(\mathcal{D}_Y)$,

$$\pi_{naive}(\mathcal{M}) := \pi_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M})$$
$$\simeq \pi_*(\mathcal{M}_{right} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})_{left}$$

Caution:

a) In general $\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} (-)$ is only RIGHT exact

and $\pi_*(-)$ is only LEFT exact

$\Rightarrow$ π_{naive} is usually not well-behaved,

eg it can happen that $(\varrho \circ \pi)_{naive} \not\simeq \varrho_{naive} \circ \pi_{naive}$

for $X \xrightarrow{\pi} Y \xrightarrow{\varrho} Z \ldots$

b) With our definition using $\pi_*: \mathrm{Mod}(\pi^{-1}\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_Y)$,

it's not obvious whether π_{naive} preserves quasicoherence.

We'll later resolve issue a) by replacing π_{naive} by its derived category version $\pi_*^{\mathcal{D}} := R\pi_*(\mathcal{D}_{Y\leftarrow X} \otimes^{L}_{\mathcal{D}_X} (-))$ and check b) there (eg writing $\pi: X \xhookrightarrow{(id,\pi)} X\times Y \xrightarrow{pr} Y$)

But let's first take a look at a case where we don't need derived categories: Closed immersions $X \hookrightarrow Y$.

5. Kashiwara's thm

Let $i : X \hookrightarrow Y$ be a closed immersion of smooth var^s / k

<u>Note</u>: In this case $\mathcal{D}_{X\to Y}$ is flat over $\mathcal{D}_X$ (example 4.2 (c1))

and furthermore $i_* : \mathrm{Mod}(i^{-1}\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_Y)$ is exact (pushforward under closed immersion of any topological spaces is exact).

$\Longrightarrow$ $i_*^{\mathcal{D}} := i_{naive} : \mathrm{Mod}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}(\mathcal{D}_Y)$ is an exact (not so naive) functor

<u>Rem</u> We use the notation $i_*^{\mathcal{D}}$ to distinguish since on the underlying $\mathcal{O}$-modules the functor is NOT given by $i_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$.

<u>Ex 1</u> Let $i : X = \{0\} \hookrightarrow Y = \mathbb{A}^1_k = \mathrm{Spec}\, k[y]$

$$\Longrightarrow \mathcal{D}_{Y\leftarrow X} \simeq k[\partial]$$

w/ left $\mathcal{D}_Y$-action by
$$\begin{cases} \partial \cdot \partial^\nu := -\partial^{\nu+1} \\ x\cdot \partial^\nu := [\partial^\nu, x] \\ \qquad = \nu\cdot\partial^{\nu-1} \end{cases}$$

(use example 4.2 (c1) and left/right trafo $P \mapsto P^*$)

$\Longrightarrow$ For $\mathcal{M} := \mathcal{O}_X = k$ we get

$$i_*^{\mathcal{D}}(\mathcal{M}) = i_*(k[\partial] \otimes_k k)$$
$$= \bigoplus_{\nu\geq 0} k\cdot\delta^\nu \quad (\text{with } \delta^\nu := \partial^\nu \otimes 1)$$

... the Dirac module!

The underlying $\mathcal{O}_Y$-module is much larger than $i_*(\mathcal{O}_X) = k\cdot\delta^0$...

<u>Note</u>: For $\nu > 0$ the summands $k\cdot\delta^\nu \subset i_*^{\mathcal{D}}(\mathcal{M})$ are NOT $\mathcal{O}_Y$-submodules, indeed $x\cdot\delta^\nu = \nu\cdot\delta^{\nu-1}$!

Back to the general case: What about quasicoherence?

<u>Lemma 2</u> For any $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$, we have

$$\mathcal{N} := i_*^{\mathcal{D}}(\mathcal{M}) \in \mathrm{Mod}_{qc}(\mathcal{D}_Y)$$

and $\mathrm{Supp}(\mathcal{N}) \subseteq X$

(support as an $\mathcal{O}_Y$-module,
ie $\forall s \in \mathcal{N}\ \exists N \gg 0 : \mathcal{J}_X^N \cdot s = 0$
for the ideal $\mathcal{J}_X := \{ f \in \mathcal{O}_Y \mid f|_X = 0 \}$)

Pf. Claim is local on Y & X is a local complete intersect^n

$\Rightarrow$ wlog $\exists$ coordinates $y_1, \ldots, y_n$ on Y w/ $X = \{ y_{m+1} = \cdots = y_n = 0 \}$

$\Rightarrow$ $\mathcal{D}_{Y \leftarrow X} \simeq k[\partial_{m+1}, \ldots, \partial_n] \otimes_k \mathcal{D}_X$ (example 4.2(c1))

$\Rightarrow$ $\mathcal{N} \simeq k[\partial_{m+1}, \ldots, \partial_n] \otimes_k i_*(\mathcal{M})$

iso of sheaves
but the $\mathcal{O}_Y$-module structure on the RHS
is not just the one on $i_*(\mathcal{M})$,
it involves commutators with $\partial_{m+1}, \ldots, \partial_n$!

Put $F_\bullet \mathcal{N}$:= filtration by the order in $\partial_{m+1}, \ldots, \partial_n$

$\Rightarrow$ each $F_d \mathcal{N} \subset \mathcal{N}$ is an $\mathcal{O}_Y$-submodule
and on $\mathrm{gr}_d^F \mathcal{N} \simeq \mathrm{Sym}^d(k^{n-m}) \otimes_k i_*(\mathcal{M})$
the $\mathcal{O}_Y$-module structure is the one from $i_*(\mathcal{M})$

$\Rightarrow$ each $F_d \mathcal{N}$ is quasicoh / $\mathcal{O}_Y$ with $\mathrm{Supp}(F_d \mathcal{N}) \subseteq X$

(by induction, since extensions of quasicoherent sheaves are quasicoherent [Hartshorne, prop II.5.7])

$\Rightarrow$ $\mathcal{N} = \varinjlim F_d \mathcal{N}$ is quasicoh / $\mathcal{O}_Y$ w/ $\mathrm{Supp}\, \mathcal{N} \subseteq X$. $\square$

<u>Conclusion:</u> Get functor

$$i_*^{\mathcal{D}} : \mathrm{Mod}_{qc}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}_{qc}^X(\mathcal{D}_Y) := \{ \mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_Y) \mid \mathrm{Supp}\, \mathcal{N} \subseteq X \}$$

<u>Q:</u> Is this an equivalence?

Ex 3 Let $i : X = \{0\} \hookrightarrow Y = \mathbb{A}^1_k$ & $M = \mathcal{O}_X = k$.

By ex 1,

$$i^D_* M = \bigoplus_{\nu \geq 0} k \cdot \delta^\nu \quad w/ \quad y \cdot \delta^\nu = \nu \cdot \delta^{\nu-1}$$

$\Rightarrow$ multiplication by y is surjective on $i^D_* M$

$$\Rightarrow \quad i^*(i^D_* M) = \mathcal{O}_Y/(y) \otimes_{\mathcal{O}_Y} i^D_* M \overset{!}{=} 0$$

Better work in the derived sense:

Take $Li^*(\dots) = [\mathcal{O}_Y \xrightarrow{y \cdot} \mathcal{O}_Y] \otimes_{\mathcal{O}_Y} (\dots)$ ...

Indeed:

$$M = k \cdot \delta^0 \overset{!}{=} \ker(i^D_* M \xrightarrow{y \cdot} i^D_* M).$$

Back to the general case:

Def For $\mathcal{N} \in \mathrm{Mod}(\mathcal{O}_Y)$ and $\mathcal{J} \trianglelefteq \mathcal{O}_Y$ put

$$\mathcal{N}^{\mathcal{J}} := \{ s \in \mathcal{N} \mid \mathcal{J} \cdot s = 0 \} \subseteq \mathcal{N}$$

By construction $\mathcal{J} \cdot \mathcal{N}^{\mathcal{J}} = 0$

$\Rightarrow$ For $\mathcal{J} = \mathcal{J}_X$ we have $\mathcal{N}^{\mathcal{J}} \in \mathrm{Mod}(\mathcal{O}_X)$

Lemma 4 For $\mathcal{J} = \mathcal{J}_X$ the functor $\mathcal{N} \mapsto \mathcal{N}^{\mathcal{J}}$ has a natural lift to $\mathcal{D}$-modules:

$$\begin{tikzcd}
\mathrm{Mod}_{qc}(\mathcal{D}_Y) \arrow[r, dashed, "\exists"] \arrow[d, "\text{forget}"'] & \mathrm{Mod}_{qc}(\mathcal{D}_X) \arrow[d, "\text{forget}"] \\
\mathrm{Mod}_{qc}(\mathcal{O}_Y) \arrow[r] & \mathrm{Mod}_{qc}(\mathcal{O}_X) \\
\mathcal{N} \arrow[r, mapsto] & \mathcal{N}^{\mathcal{J}}
\end{tikzcd}$$

Pf. For $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_Y)$, want natural $\mathcal{D}_X$-module structure on the $\mathcal{O}_X$-module $\mathcal{N}^{\mathcal{J}}$.

For this consider $\quad di : \mathcal{T}_X \hookrightarrow i^* \mathcal{T}_Y = \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1} \mathcal{T}_Y$:

Given $\xi \in \mathcal{T}_X$, $\exists$ locally an extension to $\tilde{\xi} \in \mathcal{T}_Y$ s.th. $\tilde{\xi}|_X = di(\xi)$. (*)

$$\Rightarrow \quad \begin{tikzcd}
\mathcal{J}_X \arrow[r, dashed, "\exists"] \arrow[d, hook] & \mathcal{J}_X \arrow[d, hook] \\
\mathcal{O}_Y \arrow[r, "\tilde{\xi}"] \arrow[d, two heads] \arrow[dr, phantom, "(*)"] & \mathcal{O}_Y \arrow[d, two heads] \\
\mathcal{O}_X \arrow[r, "\xi"'] & \mathcal{O}_X
\end{tikzcd}$$

diagram commutes

$\Rightarrow \xi(\mathcal{J}_X) \subseteq \mathcal{J}_X$

For $s \in N^J$ define $\xi \cdot s := \tilde{\xi} \cdot s \in N$

(using $\mathcal{D}_Y$-module structure on N)

Claim:

a) $\tilde{\xi} \cdot s \in N^J$

b) only depends on ξ but not on the chosen $\tilde{\xi}$

c) this $\mathcal{T}_X \to \mathrm{End}_k(N^J)$ makes N^J a $\mathcal{D}_X$-module.

Indeed,

a) $\forall f \in J$ we have $f \cdot (\tilde{\xi} \cdot s) = \tilde{\xi} \cdot \underbrace{(f \cdot s)}_{=0} - \underbrace{\tilde{\xi}(f)}_{\in J} \cdot \underbrace{s}_{\in N^J} \overset{!}{=} 0$

b) for any other $\hat{\xi}$ with $\hat{\xi}|_X = \tilde{\xi}|_X$,

have $(\hat{\xi} - \tilde{\xi})|_X = 0$

$\Longrightarrow \hat{\xi} - \tilde{\xi} \in J \cdot \tilde{\mathcal{T}}_X$ w/ $\tilde{\mathcal{T}}_X := \{ \eta \in \mathcal{T}_Y \mid \exists \sigma \in \mathcal{T}_X$ w/ $\eta|_X = di(\sigma) \}$

$\Longrightarrow (\hat{\xi} - \tilde{\xi}) \cdot s \in \underbrace{J \cdot \underbrace{\tilde{\mathcal{T}}_X \cdot N^J}_{\subseteq N^J \text{ as shown above}}}_{\subseteq J \cdot N^J = 0}$

$\Longrightarrow \hat{\xi} \cdot s = \tilde{\xi} \cdot s$

c) follows from $N \in \mathrm{Mod}(\mathcal{D}_Y)$. $\square$

Thm 5 (Kashiwara) Put $J = J_X$.

a) The functor $(-)^J$ is right adjoint to $i_*^{\mathcal{D}}$,

ie $\exists$ natural iso

$$\mathrm{Hom}_{\mathcal{D}_Y}(i_*^{\mathcal{D}} M, N) \cong \mathrm{Hom}_{\mathcal{D}_X}(M, N^J)$$

$\forall M \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$

$\forall N \in \mathrm{Mod}_{qc}(\mathcal{D}_Y)$.

b) We get mutally inverse equivalences of categories

$$\mathrm{Mod}_{qc}(\mathcal{D}_X) \underset{(-)^J}{\overset{i_*^{\mathcal{D}}}{\rightleftarrows}} \mathrm{Mod}_{qc}^X(\mathcal{D}_Y).$$

Pf. We prove the corresponding statement for right modules.

a) Let $M \in \mathrm{Mod}_{qc}(\mathcal{D}_X^{op})$,

$N \in \mathrm{Mod}_{qc}(\mathcal{D}_Y^{op})$.

- Given $\varphi \in \mathrm{Hom}_{\mathcal{D}_Y^{op}}(i_*^{\mathcal{D}} M, N)$,

precompose with

$$\imath : M \hookrightarrow i_*^{\mathcal{D}} M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$$
$$m \longmapsto m \otimes 1$$

(we drop $i_* : \mathrm{Mod}_{qc}(\mathcal{O}_X) \hookrightarrow \mathrm{Mod}_{qc}(\mathcal{O}_Y)$ from the notations because i is an affine morphism)

$\Rightarrow \varphi^{\#} := \varphi \circ \imath : \mathcal{M} \xrightarrow{\imath} \mathcal{M} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \xrightarrow{\varphi} \mathcal{N}$

$\cap$

$\mathrm{Hom}_{\mathcal{D}_X^{op}}(\mathcal{M}, \mathcal{N}^{\mathcal{J}})$ since $\mathcal{J} \cdot \mathrm{im}(\varphi \circ \imath) = 0$.

($\mathcal{D}_X^{op}$-linearity: Exercise using the construction in lemma 4)

- Conversely,

given $\psi \in \mathrm{Hom}_{\mathcal{D}_X^{op}}(\mathcal{M}, \mathcal{N}^{\mathcal{J}})$,

consider

$$\begin{aligned} i_*^{\mathcal{D}} \mathcal{M} &= \mathcal{M} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \\ &= \mathcal{M} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_Y / \mathcal{J}_Y \cdot \mathcal{D}_Y \xrightarrow{\psi^{\natural}} \mathcal{N} \\ m \otimes P &\longmapsto \underbrace{\psi(m) \cdot P}_{\substack{\text{well-defined} \\ \text{since } \psi(m) \in \mathcal{N}^{\mathcal{J}} \\ \text{is killed by } \mathcal{J}_Y \cdot \mathcal{D}_Y}} \end{aligned}$$

$\Rightarrow \psi^{\natural} \in \mathrm{Hom}_{\mathcal{D}_Y^{op}}(i_*^{\mathcal{D}} \mathcal{M}, \mathcal{N})$

- Exercise: $(\varphi^{\#})^{\natural} = \varphi$

$(\psi^{\natural})^{\#} = \psi$ hence a) holds.

b) By part a) we have adjunction maps

$\mathrm{id}^{\natural} : i_*^{\mathcal{D}}(\mathcal{N}^{\mathcal{J}}) \longrightarrow \mathcal{N}$

$\mathrm{id}^{\#} : \mathcal{M} \longrightarrow (i_*^{\mathcal{D}} \mathcal{M})^{\mathcal{J}}$

Want: These are isomorphisms $\forall\, \mathcal{N} \in \mathrm{Mod}_{qc}^X(\mathcal{D}_Y)$
$\forall\, \mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$!

This is a local problem

$\Rightarrow$ wlog $\exists$ coordinate system $y_1, \dots, y_m$ on Y

sth $X = \{ y_1 = \dots = y_n = 0 \}$.

For compositions $i : X = X_1 \overset{i_1}{\hookrightarrow} X_2 \overset{i_2}{\hookrightarrow} Y$ of closed embeddings

one has: $((-)^{\mathcal{J}_2})^{\mathcal{J}_1} = (-)^{\mathcal{J}}$ w/ $\mathcal{J}_1 = \mathcal{J}_{X_1} \trianglelefteq \mathcal{O}_{X_2}$
$\mathcal{J}_2 = \mathcal{J}_{X_2} \trianglelefteq \mathcal{O}_Y$,

hence by a): $i_*^{\mathcal{D}} = i_{2*}^{\mathcal{D}} \circ i_{1*}^{\mathcal{D}}$

$\Rightarrow$ wlog $n = m-1$,

ie $X \hookrightarrow Y$ hypersurface.

Put $y := y_m$ (generator of $\mathcal{J}_X$)

$\partial := \partial_m := \partial_{y_m}$.

$(\#)$: Recall $\mathcal{D}_{X\to Y} = \bigoplus_{\nu\geq 0} \mathcal{D}_X \cdot \partial^\nu$

$\Rightarrow \quad i_*^D M = M \otimes_k k[\partial] = \bigoplus_{\nu \geq 0} M\partial^\nu$

w/ $\quad m\partial^\nu \circ y := \nu \cdot m\partial^{\nu-1} \qquad \forall m \in M, \nu \geq 0$

$\Rightarrow \quad M \overset{!}{=} \ker(i_*^D M \xrightarrow{\cdot y} i_*^D M) = (i_*^D M)^J$

$\Rightarrow \quad id^\#$ iso

(4): Put $\tilde{N} := \sum_{\nu \geq 0} N^J \cdot \partial^\nu \subset N \quad$ (a $\mathcal{D}_Y^{op}$-submodule!)

Using $[\partial^\nu, y] = \nu\partial^{\nu-1}$, have $n\partial^\nu \circ y = \nu \cdot n\partial^{\nu-1} \quad \forall n \in N^J$

$\Rightarrow \quad \tilde{N} = \bigoplus_{\nu\geq 0} N^J \cdot \partial^\nu = i_*^D(N^J)$ (<u>direct</u> sum)

and $\quad \tilde{N} \xrightarrow{\cdot y} \tilde{N}$ epi.

Want: $\tilde{N} = N$.

So let $s \in N$. Locally $\exists N$ s.th. $s \cdot y^N = 0 \in \tilde{N}$

(since $\operatorname{Supp} N \subset X$)

$\Rightarrow$ By ind^n on N

enough to show: $\quad s \cdot y \in \tilde{N}$ implies $s \in \tilde{N}$.

Assume $s \in N$ w/ $s \cdot y \in \tilde{N}$

$\Rightarrow \quad \exists \tilde{s} \in \tilde{N}$ w/ $s \cdot y = \tilde{s} \cdot y \quad$ (since $\tilde{N} \xrightarrow{\cdot y} \tilde{N}$ epi)

$\Rightarrow \quad s - \tilde{s} \in N^J \subset \tilde{N}$

$\Rightarrow \quad s = (s - \tilde{s}) + \tilde{s} \in \tilde{N}$ as required. $\square$

<u>Rem</u> The naive (ie using the reduced support, not the scheme-theoretic) analogue for $\mathcal{O}$-modules obviously fails:

e.g. take $X = \{0\} \overset{i}{\hookrightarrow} Y = \mathbb{A}^1_k = \operatorname{Spec} k[y]$

then $\quad N := \mathcal{O}_Y/(y^2)$ has $\operatorname{Supp} N = \{0\}$

but $\quad N \notin \operatorname{im}(i_* : \operatorname{Mod}_{qc}(\mathcal{O}_X) \longrightarrow \operatorname{Mod}_{qc}(\mathcal{O}_Y))$

$= \{(\mathcal{O}_Y/(y))^{\oplus n} \mid n \in \mathbb{N}_0\}$

$\Rightarrow$ Again "the $\mathcal{D}$-action removes nilpotents"

(similar to what we saw for showing in §3.4 that $\mathcal{O}$-coherent $\mathcal{D}$-modules are vector bdles)

Intuitively: $\mathcal{D}$-modules are closer to topology than the theory of qcoh $\mathcal{O}$-modules...

Rem / Def. By construction the functor $\mathcal{N} \mapsto \mathcal{N}^{\mathcal{J}}$ for $\mathcal{J} = \mathcal{J}_X$ factors as

$$\begin{tikzcd} \mathrm{Mod}_{qc}(\mathcal{D}_Y) \arrow[rr, "(-)^{\mathcal{J}}"] \arrow[dr, dashed, "\Gamma_{[X]}(-)"'] & & \mathrm{Mod}_{qc}(\mathcal{D}_X) \\ & \mathrm{Mod}^X_{qc}(\mathcal{D}_Y) \arrow[ur, "(-)^{\mathcal{J}}"'] & \end{tikzcd}$$

where
$$\Gamma_{[X]}(\mathcal{M}) := \{ s \in \mathcal{M} \mid \mathcal{J}_X^N \cdot s = 0 \ \forall N \gg 0 \}$$

is the functor of "sections supported on X", which restricts to the identity on $\mathrm{Mod}^X_{qc}(\mathcal{D}_Y) \subset \mathrm{Mod}_{qc}(\mathcal{D}_Y)$.

6. An application: $\mathcal{D}$-affine varieties

Def The variety X is called $\mathcal{D}$-affine if the global sections functor
$$\Gamma(X, -) : \mathrm{Mod}_{qc}(\mathcal{D}_X) \to \mathrm{Mod}(R)$$
$(R := \mathcal{D}_X(X))$

is an equivalence of categories & exact.

Exercise 1 This happens iff

a) $\Gamma(X,-) : \mathrm{Mod}_{qc}(\mathcal{D}_X) \to \mathrm{Mod}(R)$ is exact,

b) $\Gamma(X, \mathcal{M}) \simeq 0$ only for $\mathcal{M} \simeq 0$.

(Hint: Show first that assuming a) & b), any $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$ is generated by its global sections: $\mathcal{D}_X \otimes_R \Gamma(X, \mathcal{M}) \twoheadrightarrow \mathcal{M}$ is epi...)

$\Longrightarrow$ Any affine variety is $\mathcal{D}$-affine, but there are more:

Thm 2 $X = \mathbb{P}^n_k$ is $\mathcal{D}$-affine.

Pf.

a) Consider the natural action of $\mathbb{G}_m$ on $\Gamma(\tilde{X}, p^*\mathcal{M})$

where $p : \tilde{X} := \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow X = \mathbb{P}^n$

$$\Longrightarrow \Gamma(\tilde{X}, p^*\mathcal{M}) = \bigoplus_{e \in \mathbb{Z}} \underbrace{\Gamma(\mathcal{M}(e))}_{:= \text{ eigenspace where } \mathbb{G}_m \text{ acts via } z \mapsto z^e}$$
w/ $\Gamma(X, \mathcal{M}) = \Gamma(\mathcal{M}(0))$

Note:

The Euler vector field $\xi := \sum_{\alpha=0}^{n} x_\alpha \cdot \partial_\alpha \in \mathcal{T}_{\tilde{X}/X}$

acts on
$$p^*\mathcal{M} = \mathcal{O}_{\tilde{X}} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{M}$$
via $\xi \otimes \mathrm{id}$.

$$\Longrightarrow \xi|_{\Gamma(\mathcal{M}(e))} = e \cdot \mathrm{id}$$

If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence in $\mathrm{Mod}_{qc}(\mathcal{D}_X)$, applying the left exact functor $j_* = j_{naiv}$ for $j : \tilde{X} \hookrightarrow V := \mathbb{A}^{n+1}$ after the exact p^* (note: p is smooth!) we get

$$(*) \quad 0 \to j_* p^* \mathcal{M}' \to j_* p^* \mathcal{M} \to j_* p^* \mathcal{M}'' \to \underset{\substack{\| \\ \mathcal{N} \in \mathrm{Mod}_{qc}^{\{0\}}(V)}}{\mathrm{coker}} \to 0$$

ie $\mathrm{Supp}\, \mathcal{N} \subseteq \{0\}$

Kashiwara's thm
$$\Longrightarrow \quad \mathcal{N} \simeq i_*^{\mathcal{D}}(k^r)$$
$$= k[\partial_0, \dots, \partial_n] \otimes_k k^r \quad \text{for } i : \{0\} \hookrightarrow V \ \& \text{ some } r \in \mathbb{N}_0$$

But
$$x_\alpha \cdot (\partial^I \otimes v) = -i_\alpha \cdot \partial^{I - e_\alpha} \otimes v$$
$$\partial_\alpha \cdot (\partial^I \otimes v) = \partial^{I + e_\alpha} \otimes v \qquad \forall v \in k^r$$

$\Longrightarrow$ $\xi = \sum_{\alpha=0}^n x_\alpha \partial_\alpha$ acts on $\mathcal{N}$ via
$$\xi \cdot (\partial^I \otimes v) = -(n+1+|I|) \cdot \partial^I \otimes v$$

$\Longrightarrow$ All eigenvalues of ξ on $\Gamma(V, \mathcal{N})$ are integers < 0

But V is affine, so $(*)$ remains exact after taking $\Gamma(V, -)$.
Looking at the zero eigenspaces of ξ we get that

$$0 \to \underset{\substack{\| \\ \Gamma(X, \mathcal{M}')}}{\Gamma(p^* \mathcal{M}')^\xi} \to \underset{\substack{\| \\ \Gamma(X, \mathcal{M})}}{\Gamma(p^* \mathcal{M})^\xi} \to \underset{\substack{\| \\ \Gamma(X, \mathcal{M}'')}}{\Gamma(p^* \mathcal{M}'')} \to \underbrace{\Gamma(V, \mathcal{N})^\xi}_{=0}$$

is exact

$\Longrightarrow$ $\Gamma(X, -)$ exact as required for a)

b) $\mathcal{M} \neq 0 \implies j_* p^* \mathcal{M} \neq 0$

$\implies \exists \ell : \Gamma(\mathcal{M}(\ell)) \neq 0$

- If $\ell = 0$: Done.
- If $\ell < 0$: $\exists \alpha$ sth $\Gamma(\mathcal{M}(\ell)) \xrightarrow{x_\alpha \cdot} \Gamma(\mathcal{M}(\ell+1))$ is not zero
  (indeed $j_* p^* \mathcal{M}$ has no sections w/ supp $= \{0\}$)
  $\implies$ Reduce to $\ell = 0$
- If $\ell > 0$: $\exists \alpha$ sth $\Gamma(\mathcal{M}(\ell)) \xrightarrow{\partial_\alpha \cdot} \Gamma(\mathcal{M}(\ell-1))$ is not zero
  (because $\xi = \sum x_\alpha \partial_\alpha$ acts by $\ell \cdot \mathrm{id} \neq 0$ on LHS) $\square$
  $\implies$ Again reduce to $\ell = 0$.

Rem
- This works because for $X = \mathbb{P}^n$ the ring $R = \mathcal{D}_X(X)$ is "big enough", e.g. $x_\alpha \cdot \partial_\beta \in R \ \forall \alpha, \beta \in \{0, 1, \dots, n\}$.
- More generally any "flag variety" $\leftarrow$ G/P w/ G reductive gp, P parabolic subgp
  is $\mathcal{D}$-affine (Beilinson-Bernstein, 1981)

  [e.g. $G = Gl_{n+1}$, $P = \mathbb{G}_m \cdot \mathrm{Stab}(v)$ for any $v \in \mathbb{A}^{n+1} \setminus \{0\}$ gives $G/P \simeq \mathbb{P}^n$]
- Are these the only smooth projective $\mathcal{D}$-affine varieties?
  $\to$ Still an open conjecture!
  (see also the recent preprint by A. Langer, "On smooth projective $\mathcal{D}$-affine var^{s}" (June 2019))

7. Coherent $\mathcal{D}$-modules & good filtrations

Recall: • The Weyl algebra $\mathcal{D} = \mathcal{D}_{n,k} = k[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ is both left + right Noetherian (since for the order filtration $gr^F_\cdot \mathcal{D} \simeq k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ is Noetherian)

• $M \in Mod(\mathcal{D})$ is fingen $\Longleftrightarrow$ $\exists$ good $F_\cdot M$ wrt order filtration (ie $gr^F_\cdot M$ fingen / $gr^F_\cdot \mathcal{D}$)

Goal: Generalize these ideas to arbitrary smooth var's X.

Lemma 1 $\forall$ open affine $U \subseteq X$ the ring $\mathcal{D} = \mathcal{D}_X(U)$ is both left + right Noetherian.

Pf. Put $F_\cdot \mathcal{D} := (F_\cdot \mathcal{D}_X)(U)$.

$\Rightarrow$ filtration w/

$$gr^F_\cdot \mathcal{D} = \bigoplus_{i \in \mathbb{N}_0} \frac{(F_i \mathcal{D}_X)(U)}{(F_{i-1}\mathcal{D}_X)(U)} \overset{U \text{ affine}}{\cong} \bigoplus_i \left(\frac{F_i \mathcal{D}_X}{F_{i-1}\mathcal{D}_X}\right)(U)$$

$$\cong (gr^F_\cdot \mathcal{D}_X)(U)$$

$$\cong (Sym^\cdot_{\mathcal{O}_X}(\mathcal{T}_X))(U)$$

$$\cong \Gamma(T^*_U, \mathcal{O}_{T^*_U})$$ ← fin gen. comm. k-algebra $\Rightarrow$ Noetherian!

$\Rightarrow$ $\mathcal{D}$ Noetherian (left + right) $\square$

Recall Over a Noetherian ring R, every submodule of a fin gen left R-module is fin presented.

Sheaf-theoretic analog:

Def Let $\mathcal{R}$ be a sheaf of rings on X.

a) $\mathcal{M} \in Mod(\mathcal{R})$ is called coherent if
- $\mathcal{M}$ is locally fin.gen. / $\mathcal{R}$
- $\forall U \subseteq X$ open, any fingen submodule $\mathcal{N} \subset \mathcal{M}|_U$ is locally fin.presented.

b) $\mathcal{R}$ is called coherent if it is so as a left $\mathcal{R}$-module.

Lemma 2 For $\mathcal{M} \in Mod(\mathcal{D}_X)$, TFAE:

a) $\mathcal{M}$ is coherent / $\mathcal{D}_X$

b) $\mathcal{M}$ is locally fingen / $\mathcal{D}_X$ & quasicoh / $\mathcal{O}_X$

Pf. a) $\Rightarrow$ b) obvious

b) $\Rightarrow$ a) : Consider $\mathcal{D}_U^{\oplus N} \twoheadrightarrow \mathcal{N} \subset \mathcal{M}|_U$ for $U \subseteq X$ open

Wlog U affine $\Longrightarrow$ $\mathcal{D}(U)^{\oplus N} \twoheadrightarrow \mathcal{N}(U)$ epi

$\overset{\text{Lemma 1}}{\Longrightarrow}$ $\exists\ \mathcal{D}(U)^{\oplus M} \to \mathcal{D}(U)^{\oplus N} \to \mathcal{N}(U) \to 0$ exact

$\overset{U \text{ affine}}{\Longrightarrow}$ $\mathcal{D}_U^{\oplus M} \to \mathcal{D}_U^{\oplus N} \to \mathcal{N} \to 0$ exact $\square$

Cor • $\mathcal{D}_X$ is a coherent sheaf of rings.
• Coherent $\mathcal{D}_X$-modules are a Serre subcat of $\mathrm{Mod}_{qcoh}(\mathcal{D}_X)$.

We put
$$\begin{aligned} \mathrm{Coh}(\mathcal{D}_X) &:= \mathrm{Mod}_{coh}(\mathcal{D}_X) \\ &:= \{\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X) \text{ coherent} / \mathcal{D}_X\} \\ &\subset \mathrm{Mod}_{qcoh}(\mathcal{D}_X). \end{aligned}$$

Lemma 3 a) Any $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$ is generated $/ \mathcal{D}_X$ by a (global) coherent $\mathcal{O}_X$-submodule $\mathcal{N}$.

b) Any $\mathcal{M} \in \mathrm{Mod}_{qcoh}(\mathcal{D}_X)$ is a union of coherent $\mathcal{D}_X$-submodules.

Pf. a) Write $X = \bigcup_{i=1}^N U_i$ affine open cover w/ each $\mathcal{M}|_{U_i}$ fin gen $/ \mathcal{D}_{U_i}$, say $\mathcal{M}|_{U_i} = \mathcal{D}_{U_i} \cdot \mathcal{N}_{U_i}$ for some coherent $\mathcal{O}_X$-submodule $\mathcal{N}_{U_i}$.

[Hartshorne, Ex. II.5.15]

Pick any coherent $\mathcal{O}_X$-submodules $\mathcal{N}_i \subseteq \mathcal{M}$ w/ $\mathcal{N}_i|_{U_i} = \mathcal{N}_{U_i}$.
$\Longrightarrow$ $\mathcal{N} := \sum_{i=1}^N \mathcal{N}_i \subseteq \mathcal{M}$ works.

b) Similar (exercise). $\square$

Like for the Weyl algebra, we now have:

Def For $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$, a filtration $F_\bullet\mathcal{M}$ by quasicoh. $\mathcal{O}_X$-submodules is called compatible if

- $\bigcup_i F_i\mathcal{M} = \mathcal{M}$ ("exhaustive")
- $\bigcap_i F_i\mathcal{M} = \{0\}$ ("separated")
- $\underbrace{F_i\mathcal{D}_X}_{\text{(order filtration on } \mathcal{D}_X)} \cdot F_j\mathcal{M} \subseteq F_{i+j}\mathcal{M}$ $\forall i,j \in \mathbb{Z}$. ("compatible")

It is called a good filtration if moreover $\mathrm{gr}^F_\bullet \mathcal{M}$ is a coherent sheaf of modules $/ \mathrm{gr}^F_\bullet \mathcal{D}_X$.

Lemma 4 For a compatible filtration $F_\bullet\mathcal{M}$, TFAE:

a) $F_\bullet\mathcal{M}$ is good

b) Each $F_i\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, we have $F_i\mathcal{M} = \{0\}$ $\forall i \ll 0$, and $\exists j_0 \in \mathbb{Z}$ s.th. $\forall j \geq j_0$, $F_i\mathcal{D} \cdot F_j\mathcal{M} = F_{i+j}\mathcal{M}$ $\forall i > 0$.

Pf. Same as for the Weyl algebra in § I, prop. 4.1, after taking sections on an affine open $U \subset X$. $\square$

Cor. 5 For $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$, TFAE:

a) $\mathcal{M}$ admits a good filtration

b) $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$.

Pf. "a) $\Rightarrow$ b)"

Pick j_0 as in lemma 4 b), then $F_{j_0}\mathcal{M}$ is a coherent $\mathcal{O}_X$-module
$\hookrightarrow$ hence locally fingen $/ \mathcal{O}_X$

generating $\mathcal{M}$ as a $\mathcal{D}_X$-module.

$\Rightarrow$ $\mathcal{M}$ locally fingen $/ \mathcal{D}_X$

$\Rightarrow$ coherent $/ \mathcal{D}_X$ by lemma 2

"b) $\Rightarrow$ a)"

By lemma 3a), $\exists$ $\mathcal{O}_X$-coherent $\mathcal{N} \subseteq \mathcal{M}$ w/ $\mathcal{M} = \mathcal{D}_X \cdot \mathcal{N}$.

$\Rightarrow$ $F_\bullet \mathcal{M} := F_\bullet \mathcal{D}_X \cdot \mathcal{N}$ gives a good filtration by lemma 4. $\square$

Exercise 6 a) A good $F_\bullet\mathcal{M}$ refines any compatible $G_\bullet\mathcal{M}$:

$$\exists\, s \in \mathbb{Z}: \quad F_i\mathcal{M} \subseteq G_{i+s}\mathcal{M} \ \forall i \in \mathbb{Z}.$$

b) Any two good $F_\bullet\mathcal{M}, G_\bullet\mathcal{M}$ are equivalent:

$$\exists\, \varepsilon, s \in \mathbb{Z}: \quad G_{i-\varepsilon}\mathcal{M} \subseteq F_i\mathcal{M} \subseteq G_{i+s}\mathcal{M} \quad \forall i \in \mathbb{Z}.$$

Def For $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$,

pick a good filtration $F_\bullet\mathcal{M}$

and consider $\quad \tilde{\mathcal{M}} := gr^F_\bullet \mathcal{M} \in \mathrm{Mod}^{gr}_{coh}(gr^F_\bullet \mathcal{D}_X)$,

a coherent graded sheaf of modules over the graded sheaf of rings $\quad gr^F_\bullet \mathcal{D}_X \simeq \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)$.

$\Rightarrow$ $\tilde{\mathcal{M}}$ is a coherent $\mathcal{O}_S$-module for the relative spectrum

$$S := \mathrm{Spec}_{\mathcal{O}_X}(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{T}_X)$$
$$= T^*X$$

(the total space of the cotangent bdle!)

Def $\quad \mathrm{Char}(\mathcal{M}) := \mathrm{Supp}(\tilde{\mathcal{M}}) \subseteq T^*X \quad$ "char. variety"

$$CC(\mathcal{M}) := \sum_{\substack{\Lambda \subset \mathrm{Char}\,\mathcal{M} \\ \text{irred cpt}}} m_\Lambda(\tilde{\mathcal{M}}) \cdot [\Lambda] \quad \text{"char. cycle"}$$

w/ $m_\Lambda(\tilde{\mathcal{M}}) :=$ length of the Artinian $\mathcal{O}_{T^*X,\Lambda}$-module $\tilde{\mathcal{M}}_\Lambda$.

Rem a) While $\tilde{\mathcal{M}}$ depends on the chosen $F_\bullet\mathcal{M}$, the same argument as in §I, lemma 9.4 shows that $\mathrm{Char}\,\mathcal{M}$ & $CC\,\mathcal{M}$ only depend on $\mathcal{M}$.

b) $\mathrm{Char}\,\mathcal{M} \subseteq T^*X$ is a conic subset (ie stable under the action of $\mathbb{G}_m$ on the fibers of $T^*X \xrightarrow{\pi} X$) and $\pi\,\mathrm{Char}\,\mathcal{M} = \mathrm{Supp}\,\mathcal{M}$.

Simplest example:

Prop. 7 For $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$, TFAE:

a) $\mathcal{M}$ is coherent $/\mathcal{O}_X$

b) $\mathcal{M} \simeq (\mathcal{E}, \nabla)$ is a vbdle $\mathcal{E}$ w/ a flat connection $\nabla: \mathcal{E} \to \Omega^1_X(\mathcal{E})$

c) $\mathrm{Char}\,\mathcal{M} = $ Zero section $\subset T^*X$.

Pf. a) $\Leftrightarrow$ b): see lemma 3.4.

a) $\Rightarrow$ c): If $\mathcal{M}$ is coherent $/\mathcal{O}_X$, then

$$F_i\mathcal{M} := \begin{cases} \mathcal{M} & \text{if } i \geq 0 \\ 0 & \text{else} \end{cases}$$ is a good filtrat^n

But in local coordinates on $U \subset X$, we have $\mathrm{gr}^F_\bullet \mathcal{D}_U \simeq \mathcal{O}_U[\xi_1, \dots, \xi_n]$ w/ $\xi_1, \dots, \xi_n \in \mathrm{Ann}(\tilde{\mathcal{M}})$

$\Rightarrow \mathrm{Supp}\,\tilde{\mathcal{M}}|_U \subseteq$ Zero section $= V(\xi_1, \dots, \xi_n) \subset T^*X$.

c) $\Rightarrow$ a): In local coordinates as above,

$\mathrm{Supp}(\tilde{\mathcal{M}}) \subseteq V(\xi_1, \dots, \xi_n)$

$\Rightarrow (\xi_1, \dots, \xi_n) \subseteq \mathrm{Rad}(\mathrm{Ann}_{\mathcal{O}_X[\xi_1, \dots, \xi_n]}(\tilde{\mathcal{M}}))$

"nilradical"
(use the Nullstellensatz)

$\Rightarrow \exists\, i \in \mathbb{N}: \ (\xi_1, \dots, \xi_n)^i \trianglelefteq \mathcal{O}_X[\xi_1, \dots, \xi_n]$ acts trivially on $\tilde{\mathcal{M}} = \mathrm{gr}^F_\bullet \mathcal{M}$

$\Rightarrow \partial^I F_\bullet\mathcal{M} \subseteq F_{\bullet + i - 1}\mathcal{M} \quad \forall\, I = (i_1, \dots, i_n)$ w/ $|I| \leq i$ (*)

But $F_{i+j}\mathcal{M} \overset{\substack{\text{for } j \gg 0 \\ \text{since } F_\bullet\mathcal{M} \text{ is} \\ \text{a good filtrat}^n}}{=} F_i\mathcal{D}_X \cdot F_j\mathcal{M} = \sum_{|I| \leq i} \mathcal{O}_X \cdot \partial^I F_j\mathcal{M}$

$\overset{(*)}{\subseteq} F_{i+j-1}\mathcal{M}$

$\Longrightarrow \mathcal{F}_e\mathcal{M} = \mathcal{F}_{e+1}\mathcal{M} \quad \forall\, e \gg 0$ big enough, say $\geq e_0$

$\Longrightarrow \mathcal{M} = \bigcup_e \mathcal{F}_e\mathcal{M} \overset{!}{=} \mathcal{F}_{e_0}\mathcal{M}$ $\square$

... which is coherent $/\mathcal{O}_X$.

Opposite extreme:

For $p \in X$ the Dirac module $\delta_p := i^{\mathcal{D}}_*(k) \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$
$(i : \{p\} \hookrightarrow X)$

has $\quad \mathrm{Char}(\delta_p) \subseteq T_p^*X \subset T^*X$

... a single fiber of the cotangent bdle.

More generally:

Any morphism $f: X \to Y$ of smooth var $/k$ induces a correspondence

$$\begin{tikzcd} & X \times_Y T^*Y \arrow[dl, "\varrho_f := df"'] \arrow[dr, "pr =: \varpi_f"] & \\ T^*X & & T^*Y \end{tikzcd}$$

Prop 8 a) If $i: X \hookrightarrow Y$ is a closed immersion,

$$\mathrm{Char}(i^{\mathcal{D}}_*\mathcal{M}) = \varpi_i(\varrho_i^{-1}\,\mathrm{Char}\,\mathcal{M}) \quad \forall\, \mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X).$$

b) If $\pi: X \to Y$ is smooth,

$$\mathrm{Char}(\pi^*\mathcal{N}) = \varrho_\pi(\varpi_\pi^{-1}\,\mathrm{Char}\,\mathcal{N}) \quad \forall\, \mathcal{N} \in \mathrm{Coh}(\mathcal{D}_Y).$$

Pf. a) Claim is local on Y

$\Longrightarrow$ Wlog $\exists$ local coordinates $y_1, \dots, y_n$ on Y

w/ $X = V(y_1, \dots, y_m) \subset Y$.

$\Longrightarrow i^{\mathcal{D}}_*\mathcal{M} \simeq k[\partial_{m+1}, \dots, \partial_n] \otimes_k i_*\mathcal{M}$

(for $s \in \mathcal{M}$, $I = (i_{m+1}, \dots, i_n) \in \mathbb{N}_0^{n-m}$, $\alpha \in \{1, \dots, n\}$ and $f \in \mathcal{O}_Y$)

w/ the left $\mathcal{D}_Y$-module structure given by

- $\partial_\alpha \cdot (\partial^I \otimes s) := \begin{cases} \partial^I \otimes \partial_\alpha s & \text{if } \alpha \leq m \\ \partial^{I+e_\alpha} \otimes s & \text{if } \alpha > m \end{cases}$

- $f \cdot (\partial^I \otimes s) := \sum_{J \leq I} \partial^J \otimes f_J \cdot s$

for $f \cdot \partial^I = \sum_{J \leq I} \partial^J \cdot f_J$ in $\mathcal{D}_Y$.

$\Rightarrow$ For any good filtration $F_\bullet \mathcal{M}$, the filtration

$$F_\bullet(i^D_* \mathcal{M}) := \bigoplus_{|I|\le \bullet} \partial^I \otimes i_*(F_{\bullet-|I|}\mathcal{M})$$

= "tensor product of the order filtration on $k[\partial_{m+1},\dots,\partial_n]$ with the given filtration on $\mathcal{M}$"

is compatible.

It is even good:

$$gr^F_\bullet(i^D_*\mathcal{M}) \simeq \bigoplus_{|I|\le\bullet} \xi^I \otimes i_*(gr^F_{\bullet-|I|}\mathcal{M})$$

$$= \left(k[\xi_{m+1},\dots,\xi_n] \otimes_k i_*(gr^F\mathcal{M})\right)_\bullet$$

(tensor product in the sense of graded modules)

($gr^F\mathcal{M}$ coherent / $\mathcal{O}_X[\xi_1,\dots,\xi_m]$ by goodness of $F_\bullet\mathcal{M}$

$\Rightarrow$ $k[\xi_{m+1},\dots,\xi_n] \otimes_k i_* gr^F\mathcal{M}$ coherent / $i_*\mathcal{O}_X[\xi_1,\dots,\xi_n]$ hence over $\mathcal{O}_Y[\xi_1,\dots,\xi_n]$

$\Rightarrow$ $F_\bullet(i^D_*\mathcal{M})$ good)

This also shows $gr^F_\bullet(i^D_*\mathcal{M}) \simeq \varpi_{i*}\rho_i^*(gr^F\mathcal{M})$ via the diagram

$$\begin{tikzcd}
\mathrm{Coh}(\mathcal{O}_{T^*X}) \arrow[r, phantom, "\simeq"] \arrow[d, "\rho_i^*"] & \mathrm{Mod}_{fg}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)) \arrow[r, phantom, "\simeq"] \arrow[d, "{\mathrm{Sym}(i^*\mathcal{T}_Y)\otimes_{\mathrm{Sym}(\mathcal{T}_X)}(-)\ (\text{via } \mathcal{T}_X\to i^*\mathcal{T}_Y)}"] & \mathrm{Mod}_{fg}(\mathcal{O}_X[\xi_1,\dots,\xi_m]) \arrow[d, "{k[\xi_{m+1},\dots,\xi_n]\otimes_k(-)}"] \\
\mathrm{Coh}(\mathcal{O}_{X\times_Y T^*Y}) \arrow[r, phantom, "\simeq"] \arrow[d, "\varpi_{i*}"] & \mathrm{Mod}_{fg}(\mathrm{Sym}_{\mathcal{O}_X}(i^*\mathcal{T}_Y)) \arrow[r, phantom, "\simeq"] \arrow[d, "i_*"] & \mathrm{Mod}_{fg}(\mathcal{O}_X[\xi_1,\dots,\xi_n]) \arrow[d, "i_*"] \\
\mathrm{Coh}(\mathcal{O}_{T^*Y}) \arrow[r, phantom, "\simeq"] & \mathrm{Mod}_{fg}(\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{T}_Y)) \arrow[r, phantom, "\simeq"] & \mathrm{Mod}_{fg}(\mathcal{O}_Y[\xi_1,\dots,\xi_n])
\end{tikzcd}$$

b) Fix a good filtration $F_\bullet\mathcal{N}$

$\Rightarrow$ the filtration $F_\bullet(\pi^*\mathcal{N}) := \pi^*(F_\bullet\mathcal{N})$

$$:= \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}F_\bullet\mathcal{N}$$

is compatible, even good:

This is a local problem on X and Y

$\Rightarrow$ wlog $\exists$ local coordinates $x_1, \dots, x_m$ on X
$y_1, \dots, y_n$ on Y

with $d\pi(\partial_{x_i}) = \begin{cases} \partial_{y_i} & \text{for } i \le m \\ 0 & \text{for } i > m. \end{cases}$

(flatness of $\mathcal{O}_X$ over $\mathcal{O}_Y$ for π smooth)

$\Rightarrow$ $gr^F(\pi^* \mathcal{N}) \simeq \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}(gr^F \mathcal{N})$

has $\xi_i := \text{image}(\partial_{x_i}) \in gr^F_1(\mathcal{D}_X)$

the term $\xi_i(f) \otimes s$ vanishes in gr^F !

acting via $\xi_i \cdot (f \otimes s) = \begin{cases} f \otimes \eta_i s & \text{for } i \le m \\ 0 & \text{for } i > m \end{cases}$

(for $f \in \mathcal{O}_X$, $s \in gr^F \mathcal{N}$ and $\eta_i := \text{image}(\partial_{y_i}) \in gr^F_1 \mathcal{D}_Y$)

$\Rightarrow$ $gr^F(\pi^* \mathcal{N}) \simeq \rho_{\pi *} \varpi_\pi^* (gr^F \mathcal{N})$ via

$$\begin{array}{ccccc}
Coh(\mathcal{O}_{T^*Y}) & \xrightarrow{\varpi_\pi^*} & Coh(\mathcal{O}_{X \times_Y T^*Y}) & \xrightarrow{\rho_{\pi *}} & Coh(\mathcal{O}_{T^*X}) \\
\cup\!\!| & & \cup\!\!| & & \cup\!\!| \\
Mod_{fg}(\mathcal{O}_Y[\eta_1, \dots, \eta_n]) & \xrightarrow{\pi^*} & Mod_{fg}(\mathcal{O}_X[\eta_1, \dots, \eta_n]) & \longrightarrow & Mod_{fg}(\mathcal{O}_X[\xi_1, \dots, \xi_m])
\end{array}$$

(induced by the epi $\mathcal{O}_X[\xi_1, \dots, \xi_m] \twoheadrightarrow \mathcal{O}_X[\eta_1, \dots, \eta_n]$)

$\Rightarrow$ Claim follows. $\square$

Thm 9 ("Bernstein's inequality")

$$\dim \mathrm{Char}(\mathcal{M}) \ge \dim X \quad \forall \mathcal{M} \in Coh(\mathcal{D}_X).$$

Pf. Since $\mathrm{Char}(\mathcal{M}|_U) = \mathrm{Char}(\mathcal{M}) \cap pr^{-1}(U)$ for $U \subset X$ open,

restrict to $U \subset X$ open w/ $\mathrm{Supp}\,\mathcal{M} \cap U \neq \emptyset$ smooth

$\Rightarrow$ Wlog $Z := \mathrm{Supp}\,\mathcal{M} \xhookrightarrow{i} X$ smooth subvariety

By Kashiwara's equivalence then

$$\mathcal{M} \simeq i_*^{\mathcal{D}} \mathcal{N} \quad \text{for some } \mathcal{N} \in Coh(\mathcal{D}_Z)$$

$\Rightarrow$ $\mathrm{Char}\,\mathcal{M} = \varpi_i(\rho_i^{-1}\, \mathrm{Char}\,\mathcal{N})$

for $T^*Z \xleftarrow{\rho_i} Z \times_X T^*X \xrightarrow{\varpi_i} T^*X$

smooth with fiber dimension $= \dim X - \dim Z$

closed immersion

$$\Rightarrow \dim \operatorname{Char} M = \underbrace{\dim \operatorname{Char} N}_{\geq \dim Z} + \dim X - \dim Z$$

(because $\operatorname{Char} N \twoheadrightarrow \operatorname{Supp} N = Z$)

$\square$

Rem • The above proof shows that $\operatorname{Char} M$ always contains the conormal variety

$$\Lambda_Z := (\text{Zariski closure of } N^*_{Z^{sm}/X}) \subseteq T^*X$$

conormal bundle to smooth locus of Z inside X

(though maybe not as an irreducible cpt).

• It does NOT show that all irreducible components $\Lambda \subseteq \operatorname{Char} M$ satisfy $\dim \Lambda \geq \dim X$.

However, this is true and in fact much more:

Recall T^*X is a symplectic variety,

ie it comes w/ a natural $\omega \in H^0(T^*X, \Omega^2_{T^*X})$ that induces a nondegenerate alternating bilinear form on each tangent space $T_{(p,\xi)}(T^*X)$; in local coordinates (x_i, ξ_i) on T^*X this ω is given by $\omega = \sum_i dx_i \wedge d\xi_i$.

$\Rightarrow$ each $V := T_{(p,\xi)}(T^*X)$ is a symplectic vector space, ie comes w/ an alternating nondegenerate bilinear form ω. A subspace $W \subseteq V$ is called

• involutive if $W \supseteq W^\perp := \{v \in V \mid \omega(v,w) = 0 \ \forall w \in W\}$ $\left(\Rightarrow \dim W \geq \frac{\dim V}{2}\right)$

• isotropic if $W \subseteq W^\perp$ $\left(\Rightarrow \dim W \leq \frac{\dim V}{2}\right)$

• Lagrangian if $W = W^\perp$ $\left(\Rightarrow \dim W = \frac{\dim V}{2}\right)$.

Def A subvariety $\Lambda \subset T^*X$ is called involutive / isotropic / Lagrangian if $T_{(p,\xi)}(\Lambda) \subseteq T_{(p,\xi)}(T^*X)$ is so for every smooth point $(p,\xi) \in Sm(\Lambda)$.

Ex For any closed subvariety $Z \subseteq X$ the conormal variety $\Lambda_Z \subseteq T^*X$ is Lagrangian.

(in fact every conic Lagrangian subvariety of T^*X arises like this)

A deep thm of Gabber says:

Thm For $\mathcal{M} \in Coh(\mathcal{D}_X)$, every irred. cpt $\Lambda \subseteq Char\,\mathcal{M}$ is involutive

($\Rightarrow$ in particular $\dim \Lambda \geq \dim X$)

(We won't prove this here, but it will not be used in these notes)

8. Holonomic $\mathcal{D}$-modules

Recall $\mathcal{M} \in Coh(\mathcal{D}_X) \Rightarrow \dim Char\,\mathcal{M} \geq \dim X$

(Bernstein inequality)

Def $\mathcal{M}$ is holonomic if $\dim Char\,\mathcal{M} = \dim X$.

Ex
- $\mathcal{M} = (\mathcal{E}, \nabla)$ locally free $/\mathcal{O}_X$
  $\Rightarrow$ holonomic
- The Dirac module $\mathcal{M} = i_*^{\mathcal{D}}(\mathcal{O}_{pt})$ in a point $\{pt\} \xhookrightarrow{i} X$ is holonomic, more generally:

Lemma 1 a) If $i: X \hookrightarrow Y$ is a closed immersion, then

$$i_*^{\mathcal{D}}: \underbrace{Hol(\mathcal{D}_X)}_{\{\text{holonomic } \mathcal{D}_X\text{-modules}\}} \hookrightarrow Hol(\mathcal{D}_Y).$$

b) If $\pi: X \to Y$ is a smooth morphism, then

$$\pi^*: Hol(\mathcal{D}_Y) \to Hol(\mathcal{D}_X).$$

Pf. a) For $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_X)$,

$$\mathrm{Char}(i_*^{\mathcal{D}}\mathcal{M}) = \varpi_i(\varrho_i^{-1}\,\mathrm{Char}\,\mathcal{M})$$

where

$$\begin{array}{ccccc} T^*X & \xleftarrow{\varrho_i} & X \times_Y T^*Y & \xrightarrow{\varpi_i} & T^*Y \\ & \text{smooth, with fibers of dim } \dim Y - \dim X & & \text{closed immersion} & \end{array}$$

(prop. 7.8)

$$\Rightarrow \dim \mathrm{Char}(i_*^{\mathcal{D}}\mathcal{M}) = \dim(\mathrm{Char}\,\mathcal{M}) + \dim Y - \dim X$$
$$= \dim Y$$

b) Similar: For $\mathcal{N} \in \mathrm{Hol}(\mathcal{D}_Y)$,

$$\mathrm{Char}(\pi^*\mathcal{N}) = \varrho_\pi(\varpi_\pi^{-1}\,\mathrm{Char}\,\mathcal{N})$$

where ϱ_π is a closed immersion

ϖ_π is smooth w/ fiber dim $= \dim X - \dim Y$.

$\square$

$\Rightarrow$ many interesting examples by starting with flat v'bdles $\mathcal{M} = (\mathcal{E}, \nabla)$ & applying $i_*^{\mathcal{D}}, \pi^*, \ldots$

Subquotients and extensions also work:

Lemma 2 a) $\mathrm{Hol}(\mathcal{D}_X) \subset \mathrm{Coh}(\mathcal{D}_X)$ is a Serre subcat.

b) Every holonomic $\mathcal{D}_X$-module has finite length.

Pf.

a) Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be exact in $\mathrm{Coh}(\mathcal{D}_X)$.

Pick a good $F_\bullet\mathcal{M}$

$\Rightarrow$ induced filtrations $F_\bullet\mathcal{M}' := \mathcal{M}' \cap F_\bullet\mathcal{M} \subset \mathcal{M}'$

$F_\bullet\mathcal{M}'' := \mathrm{image}(F_\bullet\mathcal{M}) \subset \mathcal{M}''$

are again good and

$$0 \to \mathrm{gr}^F\mathcal{M}' \to \mathrm{gr}^F\mathcal{M} \to \mathrm{gr}^F\mathcal{M}'' \to 0$$

is exact (exercise, as in §I, sect. 7)!

$$\Rightarrow \mathrm{Char}\,\mathcal{M} = \mathrm{Char}\,\mathcal{M}' \cup \mathrm{Char}\,\mathcal{M}''$$

each has $\dim \geq \dim X$

$\Rightarrow$ $\mathcal{M}$ holonomic iff both $\mathcal{M}', \mathcal{M}''$ are holonomic.

b) For $d \in \mathbb{N}$ put $CC(\mathcal{M})_d := \sum\limits_{\substack{\Lambda \subset \mathrm{Char}\,\mathcal{M} \\ \text{irred. cpt} \\ \text{of dim} = d}} m_\Lambda(\tilde{\mathcal{M}}) \cdot \Lambda$

where $m_\Lambda(\tilde{\mathcal{M}}) \in \mathbb{N}_0$ is the multiplicity of $\tilde{\mathcal{M}} = gr^F \mathcal{M}$ along Λ.

$\Longrightarrow$ If $d = \dim \mathrm{Char}\,\mathcal{M}$, then

$$CC(\mathcal{M})_d = CC(\mathcal{M}')_d + CC(\mathcal{M}'')_d$$

(Caution: For $d < \dim \mathrm{Char}\,\mathcal{M}$ this could fail if irred. cpt^s of $\mathrm{Char}\,\mathcal{M}'$ are properly contained in irred. cpt^s of $\mathrm{Char}\,\mathcal{M}$ ...)

$\Longrightarrow$ If $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_X)$ & $\mathcal{M}', \mathcal{M}'' \neq 0$ then

$$0 < CC(\mathcal{M}')_d < CC(\mathcal{M})_d$$
$$0 < CC(\mathcal{M}'')_d < CC(\mathcal{M})_d$$

(inequality "$\leq$" on all cpt^s, and strict on at least one)

$\Longrightarrow$ $\mathcal{M}$ has finite length. $\square$

Homological characterization?

<u>Def</u> For $\mathcal{M}, \mathcal{N} \in \mathrm{Coh}(\mathcal{D}_X)$, $j \in \mathbb{N}_0$, let

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$$

be the sheaf associated with the presheaf

$$\left(\begin{smallmatrix} U \subset X \\ \text{open} \end{smallmatrix}\right) \mapsto \mathrm{Ext}^j_{\mathcal{D}_X(U)}(\mathcal{M}(U), \mathcal{N}(U)).$$

<u>Exercise</u> a) For $U \subset X$ affine open we have

$$(\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}))(U) \simeq \mathrm{Ext}^j_{\mathcal{D}_X(U)}(\mathcal{M}(U), \mathcal{N}(U)).$$

b) If $\mathcal{F}_\bullet \to \mathcal{M}$ is a resolution by locally free $\mathcal{D}_X$-modules, then

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}) \simeq \mathcal{H}^j(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}_\bullet, \mathcal{N})).$$

[<u>Aside</u> Although in general $\mathrm{Coh}(\mathcal{D}_X)$ does NOT have enough projectives, locally free resolutions exist if X is quasiproj (thus locally free $\not\Rightarrow$ projective in $\mathrm{Coh}(\mathcal{D}_X)$)

Alternatively you can use injective resolutions $\mathcal{N} \to \mathcal{J}^\bullet$, since $\mathrm{Mod}(\mathcal{R})$ has enough injectives for ANY sheaf of rings $\mathcal{R}$.]

Note: Taking $\mathcal{N} := \mathcal{D}_X \in \mathrm{Mod}(\mathcal{D}_X \times \mathcal{D}_X^{op})$ (a bimodule!)
we get

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \in \mathrm{Mod}(\mathcal{D}_X^{op}) \quad \forall j \in \mathbb{N}_0.$$

Def For $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$ put

- $j(\mathcal{M}) := \min \{ j \in \mathbb{N}_0 \mid \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \neq 0 \}$
- $d(\mathcal{M}) := \dim \mathrm{Char}\, \mathcal{M}$

Thm 3 a) $j(\mathcal{M}) + d(\mathcal{M}) = 2n \qquad (n := \dim X)$

b) For each j we have $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \in \mathrm{Coh}(\mathcal{D}_X^{op})$
and

$$d(\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)_{\text{left}}) \leq 2n - j$$

pass back to left module via $(-) \otimes_{\mathcal{O}_X} \omega_X^{-1}$

c) For $j = j(\mathcal{M})$,
equality holds in b).

Pf. For $X = \mathrm{Spec}\, B$ affine w/ coordinates $x_1, \dots, x_n$
this holds by thm I.10.1 with $\mathcal{D} = \mathcal{D}_X(X)$

$$A = gr^F \mathcal{D} \simeq B[\xi_1, \dots, \xi_n].$$

In general take a cover $X = \bigcup_i U_i$ by such open affine $U_i \subset X$
then

$$j(\mathcal{M}) = \min_i j(\mathcal{M}|_{U_i}) =: j(\mathcal{M}|_{U_{i_0}})$$

$$d(\mathcal{M}) = \max_i d(\mathcal{M}|_{U_i}) =: d(\mathcal{M}|_{U_{i_1}})$$

Since $j(\mathcal{M}|_{U_i}) + d(\mathcal{M}|_{U_i}) = 2n$ for all i,
we can take $i_0 = i_1$ $\Rightarrow$ claim. $\square$

Cor 4 a) $j(\mathcal{M}) \in \{0, 1, \dots, n\}$.

b) $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_X)$ iff $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \simeq 0 \quad \forall j \neq n.$

c) $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \in \mathrm{Hol}(\mathcal{D}_X)$

for all $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$.

Pf. Thm 3 + Bernstein's inequality. $\square$

Like for the Weyl algebra we get a duality on $\mathrm{Hol}(\mathcal{D}_X)$:

Cor 5 $\exists$ exact autoequivalence

$$\mathbb{D} : \mathrm{Hol}(\mathcal{D}_X) \longrightarrow \mathrm{Hol}(\mathcal{D}_X)$$

$$\mathcal{M} \mapsto \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)_{\text{left}}$$

with $\mathbb{D} \circ \mathbb{D} \simeq \mathrm{id}$.

Pf. Same as in §1, thm 11.1. $\square$

Note: The map $\mathrm{id} \xrightarrow{\sim} \mathbb{D} \circ \mathbb{D}$ exists globally!

[put examples here!]

We can now also discuss direct images under open embeddings $j : U \hookrightarrow X$.

Note: Here $j_* := j_{\text{naive}} : \mathrm{Mod}_{qc}(\mathcal{D}_U) \to \mathrm{Mod}_{qc}(\mathcal{D}_X)$ is the usual sheaf-theoretic pushforward on the level of the underlying quasicoherent sheaves.

If U is affine, then $j : U \hookrightarrow X$ is an affine morphism (because X is separated) and then j_* is exact (in general it is left exact).

Caution: In general j_* does NOT preserve coherence!

E.g. $\mathcal{M} := \mathcal{D}_U \in \mathrm{Coh}(\mathcal{D}_U)$ on $U = X \setminus \{0\} \overset{j}{\hookrightarrow} X = \mathbb{A}^1_k$

$$\rightsquigarrow j_*(\mathcal{M}) = \mathcal{O}_X[\tfrac{1}{z}] \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$$

$$\notin \mathrm{Coh}(\mathcal{D}_X)!$$

For holonomic $\mathcal{D}$-modules life is much nicer:

Thm 6 For $j : U \hookrightarrow X$ open

$$j_* : \mathrm{Hol}(\mathcal{D}_U) \longrightarrow \mathrm{Hol}(\mathcal{D}_X).$$

Pf. Let $\mathcal{N} \in \mathrm{Hol}(\mathcal{D}_U)$.

① $\exists\, \mathcal{M} \in \mathrm{Hol}(\mathcal{D}_X)$ with $\mathcal{M}|_U \simeq \mathcal{N}$:

Indeed:

Start with $j_*\mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$.

By lemma 7.3 it is a union of $\mathcal{D}_X$-coherent submodules $\mathcal{M}_i$.

$$\Rightarrow \quad \mathcal{N} = (j_*\mathcal{N})|_U = \bigcup_i \mathcal{M}_i|_U$$

But $\mathcal{N}$ is holonomic, hence of finite length (lemma 2)

$$\Rightarrow \exists\, i_0 : \quad \mathcal{N} = \mathcal{M}_{i_0}|_U$$

Now $\mathcal{M}_{i_0} \in Coh(\mathcal{D}_X)$.

$\mathcal{M}' := \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}_{i_0}, \mathcal{D}_X)_{left} \in Hol(\mathcal{D}_X)$ by cor 4

and $\mathcal{M}'|_U \simeq \mathbb{D}(\mathcal{M}_{i_0}|_U) \simeq \mathbb{D}(\mathcal{N})$

(note: $\mathcal{M}_{i_0}|_U$ IS holonomic!)

$\Rightarrow$ $\mathcal{M} := \mathbb{D}(\mathcal{M}') \in Hol(\mathcal{D}_X)$

and $\mathcal{M}|_U \simeq \mathbb{D}(\mathbb{D}(\mathcal{N})) \simeq \mathcal{N}$ by cor 5.

We need two boring reduction steps before we'll apply this:

② Reduction to the case $X = \mathbb{A}^n_k$:

Claim local on X $\Rightarrow$ Wlog $X \underset{\text{closed}}{\overset{i}{\hookrightarrow}} \mathbb{A}^n_k$ affine

Pick $V \underset{\text{open}}{\overset{j'}{\hookrightarrow}} \mathbb{A}^n_k$ with $U = V \cap X \underset{\text{closed}}{\overset{i'}{\hookrightarrow}} V$.

Then
$$\begin{tikzcd} V \arrow[r, hook, "j'"] & \mathbb{A}^n_k \\ U \arrow[u, hook, "i'"] \arrow[r, hook, "j"] & X \arrow[u, hook, "i"'] \end{tikzcd}$$
commutes $\Rightarrow$ $i^{\mathcal{D}}_* \circ j_* = j'_* \circ i'^{\mathcal{D}}_*$

(obvious in this simple case)

Thus: $j_* \mathcal{N} \in Hol(\mathcal{D}_X) \Leftrightarrow i^{\mathcal{D}}_* j_* \mathcal{N} \in Hol(\mathcal{D}_{\mathbb{A}^n})$

$\Leftrightarrow j'_* \mathcal{N}' \in Hol(\mathcal{D}_{\mathbb{A}^n})$

where $\mathcal{N}' := i'^{\mathcal{D}}_*(\mathcal{N})$

is holonomic if $\mathcal{N}$ is so

(lemma 1)

③ Reduction to the case $U = X \setminus V(f) \subset X = \operatorname{Spec} A$

basic open subset corresponding to $f \in A$:

(in our case $A = k[x_1, \dots, x_n]$)

Pick a finite open cover $U = \bigcup_\alpha U_\alpha$

by such $U_\alpha = X \setminus V(f_\alpha)$, $f_\alpha \in A$.

Let $j_\alpha : U_\alpha \hookrightarrow X$

$j_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \hookrightarrow X$ be the inclusions.

$\Rightarrow$ $j_*(\mathcal{N}) \simeq \ker\left(\bigoplus_\alpha j_{\alpha *}(\mathcal{N}|_{U_\alpha}) \rightrightarrows \bigoplus_{\alpha<\beta} j_{\alpha\beta *}(\mathcal{N}|_{U_{\alpha\beta}})\right)$

$\Rightarrow$ $j_*(\mathcal{N})$ holonomic

if all the $j_{\alpha *}(\mathcal{N}|_{U_\alpha})$ are holonomic.

④ Left to show:

Given $\mathcal{M} \in Hol(\mathcal{D}_{\mathbb{A}^n})$ & $U = \mathbb{A}^n \setminus V(f) \overset{j}{\hookrightarrow} \mathbb{A}^n$

we have
$$j_*(\mathcal{M}|_U) \in Hol(\mathcal{D}_{\mathbb{A}^n}).$$

View $\mathcal{M}$ as a module under the Weyl algebra $\mathcal{D} = \mathcal{D}_{n,k}$
$= k[x_1,..x_n]\langle \partial_1,..,\partial_n\rangle$

then
$$j_*(\mathcal{M}|_U) = \mathcal{M}[\tfrac{1}{f}]$$
$$:= k[x_1,..,x_n,{}^1/_f] \otimes_{k[x_1,..,x_n]} \mathcal{M}.$$

Now mimic the proof of Bernstein's thm I.8.2:

Pick a good $F_\bullet^B\mathcal{M}$ wrt Bernstein filtration $F_\bullet^B\mathcal{D}$

and put
$$F_i^B(\mathcal{M}[\tfrac{1}{f}]) := \tfrac{1}{f^i} \otimes F^B_{i\cdot(\deg f+1)}\mathcal{M}, \quad i \in \mathbb{N}_0$$

This is a compatible filtration w/ $\dim_k F_i^B(\mathcal{M}[\frac{1}{f}]) \leq c \cdot \frac{i^n}{n!} + \dots$
(some $c > 0$)

by the same arguments as in I.8.2.

$\Rightarrow \mathcal{M}[\frac{1}{f}] \in Hol(\mathcal{D})$ by prop I.8.1. $\square$

<u>Exercise</u> Let $f \in H^0(X, \mathcal{O}_X)$

$\Rightarrow \exists\, b(s) \in k[s]\setminus\{0\}$
$\exists\, P(s) \in \mathcal{D}_X[s] := \mathcal{D}_X \otimes_k k[s]$

sth
$$P(s)(f^{s+1}) = b(s) \cdot f^s.$$

(put $U := X \setminus V(f)$ & apply thm 6 after passing to the larger field $K := k(s)$, proceeding as in chapter I).

In fact this can be deduced directly from ① (with a bit more work) and then allows to give an alternative proof of thm 6 avoiding ②, ④.

9. Minimal extensions

Let $U \hookrightarrow X$ be open affine ← only needed since I want notations $j_*^{\mathcal{D}}$, $j_!^{\mathcal{D}}$ etc. compatible w/ later sections

The exact fctor $j_*^{\mathcal{D}} := j_* : Hol(\mathcal{D}_U) \to Hol(\mathcal{D}_X)$

doesn't preserve simplicity:

Ex 1 For $U = X \setminus V(f)$ w/ non-constant $f \in H^0(X, \mathcal{O}_X)$ we have

$$j_*^{\mathcal{D}} \mathcal{O}_U = \mathcal{O}_X[\tfrac{1}{f}] \overset{\neq}{\hookleftarrow} \mathcal{O}_X$$

(proper $\mathcal{D}_X$-submodule w/ quotient supported on $V(f) = X \setminus U$!)

However:

Lemma 2 For $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_U)$, the pushforward $j_*^{\mathcal{D}}\mathcal{M}$ has no submodule supported inside $X \setminus U$.

In fact $j_*^{\mathcal{D}}$ is right adjoint to $(-)|_U$:

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{N}, j_*^{\mathcal{D}}\mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{D}_U}(\mathcal{N}|_U, \mathcal{M})$$

$\forall \mathcal{N} \in \mathrm{Hol}(\mathcal{D}_X)$. $\square$

Pf. Obvious from the definitions.

For the dual statement, put

$$j_!^{\mathcal{D}} := \mathbb{D} \circ j_*^{\mathcal{D}} \circ \mathbb{D} : \mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_X),$$

again an exact functor.

Ex 3 Let $X = \mathbb{A}^1 \supset U = \mathbb{A}^1 \setminus \{0\}$

By ex 1 we have an exact sequence

$$0 \to \mathcal{O}_X \to j_*^{\mathcal{D}} \mathcal{O}_U \to \delta_0 \to 0.$$

Dualizing & using
$$\mathbb{D}\mathcal{O}_X \simeq \mathcal{O}_X$$
$$\mathbb{D}\mathcal{O}_U \simeq \mathcal{O}_U$$
$$\mathbb{D}\delta_0 \simeq \delta_0$$
(exercise sheet 6)

we get:

$$0 \to \delta_0 \to j_!^{\mathcal{D}} \mathcal{O}_U \to \mathcal{O}_X \to 0.$$

Exercise More generally: For any $X \supset U = X \setminus V(f)$ w/ $f \in H^0(X, \mathcal{O}_X)$ non-constant, one has

$$j_!^{\mathcal{D}} \mathcal{O}_U = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X[\tfrac{1}{f}], \mathcal{O}_X)$$

$$\Downarrow$$

$$\mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

as $\mathcal{O}_X$-modules, with the $\mathcal{D}_X$-module structure given by $(\partial\varphi)(g) := \partial(\varphi(g)) - \varphi(\partial(g))$ $\forall \partial \in \mathcal{T}_X$,
$\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\ldots, \mathcal{O}_X)$.

Back to the general case:

Lemma 4 For $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_X)$, the "proper pushforward" $j_!^{\mathcal{D}} \mathcal{M}$ has no quotient supported inside $X \setminus U$.

In fact $j_!^{\mathcal{D}}$ is left adjoint to $(-)|_U$:

$$\mathrm{Hom}_{\mathcal{D}_X}(j_!^{\mathcal{D}} \mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{D}_U}(\mathcal{M}, \mathcal{N}|_U)$$

$$\forall \mathcal{N} \in \mathrm{Hol}(\mathcal{D}_X).$$

Pf. Apply duality to lemma 2:

$$\begin{aligned}
\mathrm{Hom}(j_! \mathcal{M}, \mathcal{N}) &\simeq \mathrm{Hom}(\mathbb{D}\mathcal{N}, \mathbb{D} j_! \mathcal{M}) && \text{by duality} \\
&\simeq \mathrm{Hom}(\mathbb{D}\mathcal{N}, j_* \mathbb{D}\mathcal{M}) && \text{by def}^{\text{n}} \text{ of } j_! \\
&\simeq \mathrm{Hom}((\mathbb{D}\mathcal{N})|_U, \mathbb{D}\mathcal{M}) && \text{by lemma 2} \\
&\simeq \mathrm{Hom}(\mathbb{D}(\mathcal{N}|_U), \mathbb{D}\mathcal{M}) && \text{since } (\mathbb{D}\mathcal{N})|_U \simeq \mathbb{D}(\mathcal{N}|_U) \\
&\simeq \mathrm{Hom}(\mathcal{M}, \mathcal{N}|_U) && \text{by duality.} \quad \square
\end{aligned}$$

Cor 5 $\exists$ natural map $j_!^{\mathcal{D}} \mathcal{M} \longrightarrow j_*^{\mathcal{D}} \mathcal{M}$ $\quad \forall \mathcal{M} \in \mathrm{Hol}(\mathcal{D}_U)$.

Pf. Apply lemma 4 to $\mathrm{id}: \mathcal{M} \to \mathcal{N}|_U$ with $\mathcal{N} := j_*^{\mathcal{D}} \mathcal{M}$. $\square$

Def We call

$$j_{!*}^{\mathcal{D}}(\mathcal{M}) := \mathrm{im}(j_!^{\mathcal{D}} \mathcal{M} \longrightarrow j_*^{\mathcal{D}} \mathcal{M}) \in \mathrm{Hol}(\mathcal{D}_X)$$

the minimal extension of $\mathcal{M}$.

(also: "middle"
"intermediate"
"Deligne-Goresky-MacPherson" etc...)

$\Rightarrow$ get a functor $j_{!*}^{\mathcal{D}} : \mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_X)$.

Prop. 6 For $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_U)$, the minimal extension $\mathcal{N} := j_{!*}^{\mathcal{D}} \mathcal{M}$ is the unique $\mathcal{D}_X$-module (up to iso) with

(i) $\mathcal{N}|_U \simeq \mathcal{M}$

(ii) $\mathcal{N}$ has neither subobjects nor quotients supported in $X \setminus U$.

Pf. "$\Rightarrow$": The maps $j_! \mathcal{M} \twoheadrightarrow j_{!*} \mathcal{M} \hookrightarrow j_* \mathcal{M}$ induce injections

$$\mathrm{Hom}(j_{!*}\mathcal{M}, \mathcal{N}') \hookrightarrow \mathrm{Hom}(j_! \mathcal{M}, \mathcal{N}')$$

$$\mathrm{Hom}(\mathcal{N}', j_{!*}\mathcal{M}) \hookrightarrow \mathrm{Hom}(\mathcal{N}', j_* \mathcal{M})$$

$$\forall \mathcal{N}' \in \mathrm{Hol}(\mathcal{D}_X).$$

But if $\operatorname{Supp} \mathcal{N}' \subseteq X \setminus U$,

then $\operatorname{Hom}(j_! \mathcal{M}, \mathcal{N}') = \operatorname{Hom}(\mathcal{N}', j_* \mathcal{M}) = 0$

by lemma 2 & 4.

$\Rightarrow$ (ii) holds

(and (i) is obvious from the definition
because the map $j_! \mathcal{M} \to j_* \mathcal{M}$ is the identity on U)

"$\Leftarrow$" Given any other $\mathcal{N} \in \operatorname{Hol}(\mathcal{D}_X)$ w/ (i), (ii),

the iso $\mathcal{N}|_U \simeq \mathcal{M}$ from (i) gives maps

$$j_! \mathcal{M} \xrightarrow{\alpha} \mathcal{N} \xrightarrow{\beta} j_* \mathcal{M}$$

by adjunction (lemma 2 & 4).

But $\alpha|_U$ and $\beta|_U$ are isomorphisms

$\Rightarrow$ By (ii) we must have $\operatorname{cok}(\alpha) = 0$
$\ker(\beta) = 0$

$$\Rightarrow \quad \begin{array}{ccccc} j_! \mathcal{M} & \overset{\alpha}{\twoheadrightarrow} & \mathcal{N} & \overset{\beta}{\hookrightarrow} & j_* \mathcal{M} \\ \| & & \downarrow {\scriptstyle s!\,\exists \text{ iso!}} & & \| \\ j_! \mathcal{M} & \twoheadrightarrow & j_{!*} \mathcal{M} & \hookrightarrow & j_* \mathcal{M} \end{array} \quad \Rightarrow \mathcal{N} \simeq j_{!*} \mathcal{M}. \quad \square$$

Cor 7 a) $\mathbb{D} \circ j_{!*}^{\mathcal{D}} = j_{!*}^{\mathcal{D}} \circ \mathbb{D}$

b) If $\mathcal{M} \in \operatorname{Hol}(\mathcal{D}_U)$ is simple, so is $j_{!*}^{\mathcal{D}} \mathcal{M}$.

c) Conversely, $\mathcal{N} \simeq j_{!*}^{\mathcal{D}}(\mathcal{N}|_U)$ $\forall$ simple $\mathcal{N} \in \operatorname{Hol}(\mathcal{D}_X)$
w/ $\operatorname{Supp} \mathcal{N} \not\subseteq X \setminus U$.

Pf. a) (i) $\mathbb{D}(j_{!*}^{\mathcal{D}} \mathcal{M})|_U \simeq \mathbb{D}(\mathcal{M})|_U$

(ii) Quotients of $\mathbb{D}(j_{!*}^{\mathcal{D}} \mathcal{M})$ are $\mathbb{D}$(submodules of $j_{!*}^{\mathcal{D}} \mathcal{M}$)
Submodules —— " —— $\mathbb{D}$(quotients —— " ——)

& the duality functor $\mathbb{D}(\dots)$ preserves supports

$\Rightarrow$ $\exists$ NO quotients or submodules w/ $\operatorname{Supp} \subseteq X \setminus U$

Thus $\mathbb{D}(j_{!*}^{\mathcal{D}} \mathcal{M}) \simeq j_{!*}^{\mathcal{D}} \mathcal{M}$ by prop 6.

b) $j_{!*}^{\mathcal{D}} \mathcal{M} \overset{p}{\twoheadrightarrow} \mathcal{N}$ nontrivial quotient & $\mathcal{M}$ simple

$\Rightarrow$ $\mathcal{M} \overset{p|_U}{\twoheadrightarrow} \mathcal{N}|_U$ is an iso or the zero map

$\Rightarrow$ either $\ker(p)$ has $\operatorname{Supp}(\dots) \subseteq X \setminus U \underset{\text{prop 6}}{\Rightarrow} \ker(p) = 0$ $\lightning$

or $\mathcal{N}|_U = 0 \underset{\text{prop 6}}{\Rightarrow} \mathcal{N} = 0$ $\lightning$

c) By adjunction we have maps $j_!^{\mathcal{D}}(\mathcal{N}|_U) \xrightarrow{\alpha} \mathcal{N} \xrightarrow{\beta} j_*^{\mathcal{D}}(\mathcal{N}|_U)$

For $\operatorname{Supp} \mathcal{N} \not\subseteq X \setminus U$ these are non-zero $\Rightarrow$ for $\mathcal{N}$ simple,
α is epi & β mono. $\square$

We can now classify all simple holonomic modules:

Recall:

- W smooth var$/k$

  $\mathcal{E}$ coherent vbdle $/\mathcal{O}_W$ w/ flat conn. $\nabla : \mathcal{E} \to \Omega^1_W(\mathcal{E})$

  $\Leftrightarrow \quad \mathcal{N} := (\mathcal{E}, \nabla) \in \mathrm{Hol}(\mathcal{D}_W)$ w/ $\mathrm{Char}\, \mathcal{N} \subset$ Zero Section.

  (see §7, prop. 7)

- Such an $\mathcal{N}$ is simple as a $\mathcal{D}_W$-module iff $\nexists$ no subbundles $0 \neq \mathcal{F} \overset{\neq}{\hookrightarrow} \mathcal{E}$ w/ $\nabla(\mathcal{F}) \subseteq \Omega^1_W(\mathcal{F})$.

- In that case, given embeddings $W \underset{\text{closed}}{\overset{i}{\hookrightarrow}} U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X$, the $\mathcal{D}_X$-module

$$\mathcal{M} := j^{\mathcal{D}}_{!*}(i^{\mathcal{D}}_{*}(\mathcal{E}, \nabla)) \in \mathrm{Hol}(\mathcal{D}_X)$$

is again simple by Kashiwara's thm & cor 7 b).

<u>Thm 8</u> Every simple $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_X)$ arises like this, in an "essentially unique" way.

Pf. Put $Z := \mathrm{Supp}\, \mathcal{M} \subseteq X$.

Pick $U \hookrightarrow X$ open dense with $W := U \cap Z$ smooth.

Kashiwara's thm: $\exists\, \mathcal{N} \in \mathrm{Hol}(\mathcal{D}\ \ )$ sth $\mathcal{M}|_U \simeq i^{\mathcal{D}}_*(\mathcal{N})$ for $i : W \underset{\text{closed}}{\hookrightarrow} U$.

Now: $\mathcal{N}$ holonomic $\Rightarrow \dim \mathrm{Char}\, \mathcal{N} = \dim W$

Since $\mathrm{Char}\, \mathcal{N} \twoheadrightarrow \mathrm{Supp}\, \mathcal{N} = W$ & $\mathrm{Char}\, \mathcal{N} \subset T^*W$ is conic (ie stable under $\mathbb{G}_m$-action),

$\exists\, W_0 \subseteq W$ open dense sth $\mathrm{Char}\, \mathcal{N}|_{W_0} =$ zero section.

Shrinking U we may assume $W_0 = W$

$\Rightarrow \mathcal{N} = (\mathcal{E}, \nabla)$ coherent $/\mathcal{O}_W$

and $\mathcal{M} = j^{\mathcal{D}}_{!*}(i^{\mathcal{D}}_*\mathcal{N})$ by simplicity of $\mathcal{M}$ (use cor 7c).

Same argument shows:

The Zariski closure $\overline{W} = Z = \mathrm{Supp}\, \mathcal{M}$ is determined by $\mathcal{M}$, and so is the Zariski germ of $(\mathcal{E}, \nabla)$ at the generic point of Z, ...

... ie given $(\mathcal{E}_\alpha,\nabla_\alpha)$ on $W_\alpha \xhookrightarrow{i_\alpha} U_\alpha \xhookrightarrow{j_\alpha} X$, $\alpha = 1,2$,

we have

$$j_{1!*}\, i_{1*}(\mathcal{E}_1,\nabla_1) \simeq j_{2!*}\, i_{2*}(\mathcal{E}_2,\nabla_2)$$

iff $\exists\, W \hookrightarrow W_1 \cap W_2$ open dense in both W_1 & W_2

sth $\quad (\mathcal{E}_1,\nabla_1)|_W \simeq (\mathcal{E}_2,\nabla_2)|_W$. $\square$

Notation: $\mathcal{M} = IC(\mathcal{E},\nabla)$ "intersection complex of $(\mathcal{E},\nabla)$"

Special case: $Z \subset X$ any irred. closed subvariety

$\rightsquigarrow$ take $W := Sm(Z)$ smooth locus of Z

$\rightsquigarrow$ $IC_Z := IC(\mathcal{O}_W)$ "intersection cplex of Z"

Outlook: For $k = \mathbb{C}$, its solution complex

$$\mathcal{S}ol(IC_Z) := R\mathcal{H}om_{\mathcal{D}_X^{an}}(IC_Z, \mathcal{O}_X^{an}) \in Perv(\mathbb{C}_X)$$

is the perverse IC sheaf whose hypercohomology computes the intersection cohomology of Z ...

Appendix A. Some commutative algebra

Let A be a commutative regular Noetherian biequidim ring,

↳ ie $\forall \mathfrak{p} \in \operatorname{Spec} A$, the localization $A_{\mathfrak{p}}$ is a regular local ring

↳ ie all maximal chains of primes have the same length

e.g. the coordinate ring of a smooth affine equidim. var / k

(think of $A = k[y_1, \ldots, y_m]$).

Def

For $M \in \operatorname{Mod}_{fg}(A)$ let $\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\}$

$$= \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \operatorname{Ann}_A M\}$$

We put

- $d(M) := \dim \operatorname{Supp}(M) = \dim(A/\operatorname{Ann} M)$

  Krull dimension

  $= \max\{\dim(A/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp} M\}$

- $j(M) := \min\{j \in \mathbb{N}_0 \mid \operatorname{Ext}^j_A(M, A) \neq 0\}$

  ($j(M)$ is called "grade number")

<u>Thm 1</u> Put $m = \dim A$. Then

a) $j(M) + d(M) = m$

b) for each j, we have $\mathrm{Ext}^j_A(M, A) \in \mathrm{Mod}_{fg}(A)$

and $d(\mathrm{Ext}^j_A(M,A)) \leq m - j$.

c) for $j = j(M)$ equality holds in b).

<u>Ex 2.</u> For $M = A/_{(x)}$ w/ x a non-zero-divisor,

have the free resolution $0 \to A \xrightarrow{x} A \to M \to 0$

hence $$\mathrm{Ext}^j_A(M, A) \cong \begin{cases} A/(x), & j = 1 \\ 0 & j \neq 1 \end{cases}$$

- More generally,

for $x_1, \dots, x_m \in A$ put ("Koszul complex")

$$K(\underline{x}) := \text{Total complex of } \bigotimes_{i=1}^{m} [\underset{\wr}{A} \xrightarrow{x_i} \underbrace{A}].$$

(notation: basis vector e_i in degree -1 $\mapsto$ basis vector 1 in degree 0)

By defn

$K(\underline{x})_\nu \simeq$ free A-module generated by the tensors

$$e_{i_1} \wedge \dots \wedge e_{i_\nu} := 1 \otimes \dots \otimes e_{i_1} \otimes \dots \otimes e_{i_\nu} \otimes \dots \otimes 1$$

$$(\text{w/ } 1 \leq i_1 < \dots < i_\nu \leq m),$$

w/ differential

$$d(e_{i_1} \wedge \dots \wedge e_{i_\nu}) := \sum_{\mu=1}^{\nu} (-1)^{\mu+1} x_{i_\mu} \cdot e_{i_1} \wedge \dots \wedge \widehat{e_{i_\mu}} \wedge \dots \wedge e_{i_\nu}$$

μ^{th} factor omitted.

e.g. for $m = 2$:

$$K(\underline{x}) = [\underset{\substack{\wr \\ \text{basis} \\ e_1 \wedge e_2}}{A} \xrightarrow{(x_1, -x_2)} \underset{\substack{\wr \\ \text{basis} \\ e_1, e_2}}{A^2} \xrightarrow{\binom{x_1}{x_2}} \underset{\substack{\wr \\ \text{basis} \\ 1}}{A}]$$

<u>Exercise</u> Show that if $x_1, \dots, x_m \in A$ form a <u>regular sequence</u>, ie each x_i is a non-zero-divisor in $A/(x_1, \dots, x_{i-1})$, then $K(\underline{x})$ is a resolution of $M = A/(x_1, \dots, x_n)$.

Deduce that in this case

$$\mathrm{Ext}^j_A(M, A) \simeq \begin{cases} 0 & \text{if } j \neq n \\ A/_{(x_1, \dots, x_n)} & \text{if } j = n \end{cases}$$

$\hookrightarrow$ $\dim \mathrm{Supp}(\dots) = m - n$!

Pf of thm 1.

a) Have $j(M) = \min \{ j(M_m) \mid m \in \mathrm{Spm}\, A \}$ (maximal ideals of A)

$d(M) = \max \{ d(M_m) \mid m \in \mathrm{Spm}\, A \}$

[here $d(M) := d_{A_m}(M_m)$ refers to M_m as a module over A_m (not over A), ditto for $j(M_m)$.]

and $\dim A_m = \dim A$ for $m \in \mathrm{Spm}\, A$ (equidim!)

$\Longrightarrow$ May replace $A \rightsquigarrow A_m$, $M \rightsquigarrow M_m$

(if we can show $j(M_m) + d(M_m) = \dim A \ \forall m$, then the m w/ $j(M_m)$ min. are also those w/ $d(M_m)$ max.)

$\Longrightarrow$ Wlog A regular local ring w/ max ideal $\mathfrak{m}$.

Pick $\mathfrak{p} \in \operatorname{Supp}(M)$ minimal,

ie $\quad d(M) = \dim(A/\mathfrak{p})$. ①

Then $M_\mathfrak{p}$ has length zero over $A_\mathfrak{p}$ $\Longrightarrow$ $j(M_\mathfrak{p}) = \dim(A_\mathfrak{p})$ ②

(reduce by dévissage to the case where $M_\mathfrak{p}$ is replaced by $A_\mathfrak{p}/\mathfrak{p}$ & apply Koszul for the regular local ring $A_\mathfrak{p}$)

Since $(\operatorname{Ext}^j_A(M,A))_\mathfrak{p} \simeq \operatorname{Ext}^j_{A_\mathfrak{p}}(M_\mathfrak{p}, A_\mathfrak{p})$, we also know

$$\underset{(\text{wrt } A)}{j(M)} \leq \underset{(\text{wrt } A_\mathfrak{p})}{j(M_\mathfrak{p})}. \quad ③$$

$$\Longrightarrow j(M) + d(M) \underset{①+③}{\leq} j(M_\mathfrak{p}) + \dim(A/\mathfrak{p}) \underset{②}{=} \dim(A_\mathfrak{p}) + \dim(A/\mathfrak{p})$$

$$\underset{(\text{equidim.})}{=} \dim(A)$$

NB: A priori we don't know whether a minimal $\mathfrak{p} \in \operatorname{Supp} M$ will have $j(M_\mathfrak{p})$ minimal, so we only get "$\leq$" in the above!

It remains to show $j(M) + d(M) \geq \dim(A)$.

Use induction on $d(M)$:

For $d(M) = 0$ one has $j(M) = \dim(A)$

(again by dévissage reduce to $M \simeq A/\mathfrak{m}$)

Assume now the claim holds for all modules with $d(-) < d$, $(d>0)$

and let $d(M) = d$. We want: $j(M) \geq m - d$.

Write $0 = M_0 \subset M_1 \subset \cdots \subset M_e = M$

sth $\quad M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} A \quad \forall i$.

$\Longrightarrow j(M) \geq \min\{ j(A/\mathfrak{p}_i) \mid i = 1, \dots, e\}$

and $d(A/\mathfrak{p}_i) \leq d \quad \forall i$

NB: We don't know whether $j(A/\mathfrak{p}_i)$ is minimal for $i = e$, so only get inequalities

$\Longrightarrow$ By induction we may assume $M = A/\mathfrak{p}$ w/ $d(M) = d$.

$d > 0 \Longrightarrow \exists x \in \mathfrak{m} \setminus \mathfrak{p}$

$\Longrightarrow$ exact sequence $0 \to A/\mathfrak{p} \xrightarrow{x\cdot} A/\mathfrak{p} \to \underbrace{A/(\mathfrak{p},x)}_{d(\dots) = d-1} \to 0$

$\Rightarrow j(\dots) \geq m - d + 1$ by induction

$\Rightarrow \operatorname{Ext}^i_A(A/\wp) \xrightarrow{x\cdot} \operatorname{Ext}^i_A(A/\wp)$ Iso $\forall i < m-d$

(by long exact Ext-sequence)

But $x \in \mathfrak{m}$ then implies

by Nakayama's lemma: $\operatorname{Ext}^i_A(A/\wp) = 0 \ \forall i < m-d$

$\Rightarrow j(A/\wp) \geq m-d$ as required.

b) Take $\wp \in \operatorname{Supp}(\operatorname{Ext}^i_A(M,A))$ minimal

$\Rightarrow d(\operatorname{Ext}^i_A(M,A)) = \dim(A/\wp) = m - \dim(A_\wp)$ (4)

But

$$\operatorname{Ext}^i_{A_\wp}(M_\wp, A_\wp) = (\operatorname{Ext}^i_A(M,A))_\wp \neq 0$$

by definition of Supp (...)

$$\Rightarrow i \underset{\substack{\uparrow \\ \text{homological dim of regular local ring is = Krull dim (proof similar to a))}}}{\leq} \dim(A_\wp) \underset{\substack{\uparrow \\ (4)}}{=} m - d(\operatorname{Ext}^i_A(M,A))$$

$\Rightarrow$ claim.

c) Exercise: $\operatorname{Supp} M = \bigcup_j \operatorname{Supp} \operatorname{Ext}^j_A(M,A)$

(by part a) we know that $M \neq 0$ implies $\operatorname{Ext}^j_A(M,A) \neq 0$ for some $j \in \mathbb{N}_0$; apply this to the localizations $M \rightsquigarrow M_\wp$, $A \rightsquigarrow A_\wp$)

$$\Rightarrow m - j(M) \underset{\substack{\uparrow \\ \text{by a)}}}{=} d(M) \underset{\substack{\uparrow \\ \text{by the exercise}}}{=} \max_{j \geq j(M)} \underbrace{d(\operatorname{Ext}^j_A(M,A))}_{\substack{\leq m-j \\ \text{by b)}}}$$

$\Rightarrow$ equality for $j = j(M)$.

$\square$

Appendix B. Spectral sequences

$\mathcal{A} = \mathrm{Mod}(R)$ for a ring R

[or more generally:
$\mathcal{A}$ abelian category w/ arbitrary limits & colimits
w/ Grothendieck's axiom (Ab5): filtered $\varinjlim$ are exact
(a priori only right exact!)]

Motivation: Let $C = [\cdots \to C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} \cdots]$ be a complex in $\mathcal{A}$ w/ a descending filtration by subcomplexes $F^pC \subseteq C$, ie $\forall n \in \mathbb{Z}$ we have

$$C^n \supseteq \cdots \supseteq F^pC^n \supseteq F^{p+1}C^n \supseteq \cdots$$

w/ $d(F^pC^n) \subseteq F^pC^{n+1}$.

Get associated graded complex

$$\mathrm{gr}^F C := [\cdots \to \mathrm{gr}^F C^n \xrightarrow{d} \mathrm{gr}^F C^{n+1} \to \cdots]$$

Q: Can we compute $H^\bullet(C)$ via $H^\bullet(\mathrm{gr}^F C)$?

<u>Heuristic:</u> In $H^n(gr_p^F C) \subseteq H^n(gr^F C)$,

we have classes represented by any

$\alpha \in d^{-1}(\underbrace{F_{p+1} C^{n+1}})$, more than $\ker(C^n \xrightarrow{d} C^{n+1})$!

(becomes zero in $gr_p^F C^{n+1}$ but needn't be zero in C^{n+1})

But for any such α, the class of $d(\alpha) \in F_{p+1} C^{n+1}$ gives a "first order error term"

$$[d(\alpha)] \in H^{n+1}(gr^F_{p+1} C)$$

$\Rightarrow$ restrict attention to those $[\alpha] \in H^n(gr_p^F C)$ w/ $[d(\alpha)] = 0 \in H^{n+1}(gr^F_{p+1} C)$.

Idea: Iterate this to remove "higher error terms"!

<u>Rem 1</u> The exact sequences $0 \to F^{p+1} C \to F^p C \to gr_p^F C \to 0$ of complexes give a long exact sequence

$$\cdots \to \bigoplus_p H^n(F^{p+1}C) \xrightarrow{i} \bigoplus_p H^n(F^p C) \xrightarrow{j} \bigoplus_p H^n(gr_p^F C) \xrightarrow{k} \bigoplus_p H^{n+1}(F^{p+1} C) \xrightarrow{i} \cdots \xrightarrow{j} \bigoplus_p H^{n+1}(gr_p^F C)$$

with $[d(\alpha)] = j\,k([\alpha])$

(read in the summand for $p+1$)

Putting $D := \bigoplus_{n,p} H^n(F^p C)$

$E := \bigoplus_{n,p} H^n(gr_p^F C)$

we arrive at the following abstract setup:

<u>Def</u> An <u>exact couple</u> in $\mathcal{A}$ is a tuple (D, E, i, j, k) w/ objects $D, E \in \mathcal{A}$ and morphisms

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E & \end{array} \qquad \text{sth} \begin{cases} \ker(j) = \operatorname{im}(i) \\ \ker(k) = \operatorname{im}(j) \\ \ker(i) = \operatorname{im}(k) \end{cases}$$

<u>Note:</u> $jk: E \to E$ satisfies $(jk)^2 = j(kj)j = 0$.

<u>Lemma 2</u> Putting $D' := \operatorname{im}(i)$

$E' := \dfrac{\ker(jk)}{\operatorname{im}(jk)}$ $\leftarrow$ Heuristic: passing to $\ker(jk)$ kills the "first order error"

we get an exact couple $(D', E', \underbrace{i', j', k'}_{\text{induced by } i, j, k})$.

Pf. Define

- $i' := i|_{D'}: D' \to D'$
- $j': D' \to E'$ by $j'(i(\alpha)) := [j(\alpha)]$ $\leftarrow$ (NB: $j(\alpha) \in \ker(jk)$)
- $k': E' = \dfrac{\ker(jk)}{\operatorname{im}(jk)} \to D'$ by $k'([\alpha]) := k(\alpha)$

(NB: k vanishes on $\operatorname{im}(jk)$)

Exactness: Exercise! $\square$

Let's iterate this:

Given an exact couple $\mathcal{C} := (D, E, i, j, k)$, consider the couples

$$(D_r, E_r, i_r, j_r, k_r) := \underset{\substack{\uparrow \\ (r-1) \text{ fold iterate} \\ \text{of lemma 2}}}{\mathcal{C}^{(r-1)}} \quad \forall r \in \mathbb{N}.$$

(heuristic: kill successively "higher errors"...)

$\Rightarrow$ arrive at a spectral sequence $(E_r, d_r := j_r k_r)_{r \in \mathbb{N}}$:

Def
- A differential object in $\mathcal{A}$ is a pair (E, d) where $E \in \mathcal{A}$ and $d \in \mathrm{End}_{\mathcal{A}}(E)$ w/ $d \circ d = 0$. We then put $H(E,d) := \dfrac{\ker(d)}{\mathrm{im}(d)}$.
- A spectral sequence in $\mathcal{A}$ is a family $(E_r, d_r)_{r \in \mathbb{N}}$ of differential objects w/ $E_{r+1} \cong H(E_r, d_r)$ $\forall r$.

Lemma 3 a) For any SS

$$\exists\ 0 = B_1 \subseteq \cdots \subseteq B_r \subseteq \cdots \subseteq Z_r \subseteq \cdots \subseteq Z_1 = E_1$$

sth $\quad E_r \simeq Z_r / B_r \quad \forall r \in \mathbb{N}$.

(heuristic: We approximate by successively finer subquotients...)

b) For the SS of a couple $\mathcal{C} = (D, E, i, j, k)$ one has

$$Z_r = k^{-1}(\mathrm{im}(i^r))$$
$$B_r = j(\ker(i^r))$$
$$d_{r+1} = j \circ i^{-r} \circ k$$

$\uparrow$ any preimage under $i^r := \underbrace{i \circ \cdots \circ i}_{r \text{ times}}$

Pf. a) Since $E_r = H(E_{r-1}, d_{r-1})$ $\forall r$, we have

$$\begin{array}{ccccccccc}
E_r & \twoheadleftarrow & \ker(d_{r-1}) & \hookrightarrow & E_{r-1} & \twoheadleftarrow & \cdots & \hookrightarrow & E_1 \\
\cup\!\!| & & & & & & & & \cup\!\!| \\
\ker(d_r) & \twoheadleftarrow & \bullet & \twoheadleftarrow & & & \cdots & \twoheadleftarrow & Z_r \\
\cup\!\!| & & & & & & & & \cup\!\!| \\
\mathrm{im}(d_r) & \twoheadleftarrow & \bullet & \twoheadleftarrow & & & \cdots & \twoheadleftarrow & B_r
\end{array}$$

where $Z_r :=$ full preimage of $\ker(d_r)$
$B_r :=$ —— " —— $\mathrm{im}(d_r)$
$\Rightarrow \dfrac{Z_r}{B_r} \simeq \dfrac{\ker(d_r)}{\mathrm{im}(d_r)}$

b) Note: i_r is induced by i
k_r is induced by k
j_r is induced by $j \circ i^{-r+1}$ (see proof of lemma 2)

$$\Rightarrow Z_r = \ker(j \circ i^{-r+1} \circ k)$$
$$= k^{-1}(\ker(j \circ i^{-r+1}))$$
$$= k^{-1}(i^{r-1}(\underbrace{\ker j}_{= \mathrm{im}\, i})) = k^{-1}(\mathrm{im}(i^r))$$

and $B_r = \operatorname{im}(j \circ i^{-r+1} \circ k)$

$$= j(i^{-r+1}(\underbrace{\operatorname{im} k}_{=\ker i})) = j(i^{-r}(0)) = j(\ker i^r). \qquad \square$$

<u>Notation:</u>

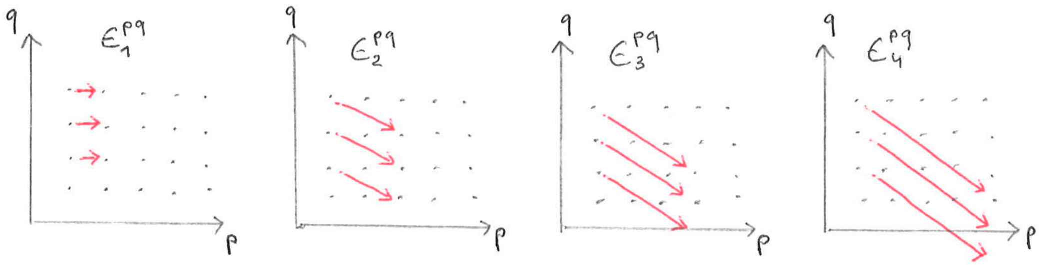
$$\begin{array}{rcl} Z_\infty & := & \varprojlim Z_r \\ \rotatebox{90}{$\subseteq$} & & \\ B_\infty & := & \varinjlim B_r \end{array} \qquad \text{(using (Ab5))}$$

and $E_\infty := Z_\infty / B_\infty$ (still a subquot. of E_1).

In real life the objects are usually bigraded:

- exact couples as $D = \bigoplus_{p,q} D^{pq}$, $E = \bigoplus_{p,q} E^{pq}$ w/ degree$(i) = (-1,1)$, degree$(j) = (0,0)$, degree$(k) = (1,0)$
- spectral sequences as

$$E_r = \bigoplus_{p,q} E_r^{pq} \quad \text{w/ degree}(d_r) = (r, 1-r)$$



<u>Def</u> A bigraded SS $(E_r^{pq}, d_r)_{r\in\mathbb{N}}$ <u>converges</u> to a graded object $H = \bigoplus_n H^n \in \mathcal{A}$ w/ an exhaustive, separated filtration $F^\bullet H = \bigoplus_n F^\bullet H^n$ if

$$E_\infty^{p,q} := \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} \simeq gr_p^F H^{p+q} \quad \forall p,q \in \mathbb{Z}.$$

We then write $E_r^{pq} \Rightarrow H^{p+q}$ (by abuse of notation the gr_p^F is dropped)

<u>Rem</u> Often the convergence holds <u>at finite level</u> already, ie $\forall p,q\ \exists r = r(p,q)$ sth

$$Z_r^{pq} = Z_{r+1}^{pq} = \dots = Z_\infty^{pq}$$
$$B_r^{pq} = B_{r+1}^{pq} = \dots = B_\infty^{pq},$$

but the above definition does not require this.

<u>Ex 4</u> For a filtered complex $F^\bullet C = [\dots \to F^\bullet C^n \to F^\bullet C^{n+1} \to \dots]$ we have the bigraded exact couple with

$$D^{pq} := H^{p+q}(F^pC)$$
$$E^{pq} := E_1^{pq} := H^{p+q}(gr^F C).$$

The long exact sequence reads

$$\begin{array}{ccccccccc} \dots \to & D^{p+1,q-1} & \xrightarrow{i} & D^{pq} & \xrightarrow{j} & E^{pq} & \xrightarrow{k} & D^{p+1,q} & \xrightarrow{i} \dots \\ & \| & & \| & & \| & & \| & \\ & H^n(F^{p+1}C) & & H^n(F^pC) & & H^n(gr_p^F C) & & H^{n+1}(F^{p+1}C) & \end{array}$$

Thm 5 Let (D, E, i, j, k) be a bigraded exact couple.

Put $H^n := \varinjlim D^{p,n-p}$ w/ $F^pH^n := \mathrm{im}(D^{p,n-p})$,

where the $\varinjlim$ uses the transition maps $i : D^{p,n-p} \to D^{p-1,n-p+1}$

Then for the corresponding SS $(E_r^{pq}, d_r)_{r\in\mathbb{N}}$ we have

$$E_r^{pq} \Rightarrow H^{p+q}$$

iff $\quad k^{-1}\left(\bigcap_r \mathrm{im}(i^r)\right) = \ker(k).$

Pf. Fix $n \in \mathbb{N}$ and put $D^p := D^{p,n-p}$,

We have $K^p := \ker(D^p \to F^pH^n) \underset{\substack{\uparrow \\ \text{(using (Ab5))}}}{=} \bigcup_r \ker(i^r : D^p \to D^{p-r})$

$\Rightarrow j(K^p) = B_\infty^{p,n-p}$

by lemma 3b)

$\Rightarrow$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K^{p+1} \arrow[r] \arrow[d, "i"] & D^{p+1} \arrow[r] \arrow[d, "i"] & F^{p+1}H^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^p \arrow[r] \arrow[d] & D^p \arrow[r] \arrow[d, "j"] & F^pH^n \arrow[r] \arrow[d] & 0 \\
 & B_\infty^{p,q} \arrow[r] & j(D^p) \arrow[r] & gr_p^F H^n \arrow[r] & 0
\end{tikzcd}$$

snake lemma $\Rightarrow \quad gr_p^F H^n \simeq j(D^p)/B_\infty^{p,q} = \ker(k) \cap E_1^{p,n-p} / B_\infty^{p,n-p}$

Note: $\ker(k) \subseteq Z_\infty = k^{-1}\left(\bigcap_r \mathrm{im}(i^r)\right)$ by lemma 3b,

hence the desired convergence holds iff we have "$=$". $\square$

Cor 6 Let $F^\bullet C = [\cdots \to F^\bullet C^n \to F^\bullet C^{n+1} \to \cdots]$ be a filtered complex where $F^\bullet$ is exhaustive, separated & bounded below, ie $\forall n \in \mathbb{Z}\ \exists p = p(n)$ with $F^{p(n)}C^n = 0$.

Then we have a convergent SS

$$E_1^{pq} := H^{p+q}(gr_p^F C) \Rightarrow H^{p+q}(C).$$

Pf. For any p,q we have

$$Z_\infty = k^{-1}\Big(\bigcap_r \mathrm{im}\big(i^r : \underbrace{H^{p+q}(gr^F_{p+r} C)}_{\substack{=0 \text{ for } r \gg 0 \\ \text{when } p,q \text{ are fixed,} \\ \text{since } F^\bullet \text{ is bounded below!}}} \to H^{p+q}(gr_p^F C)\big)\Big)$$

$$= k^{-1}(0)$$

$$= \ker(k),$$ so thm 5 applies. $\square$