Problem 1.1. Deduce from the multiplicativity of the exponential function that for any smooth paths $\gamma_1, \gamma_2, \gamma_3 : [0,1] \to \mathbb{C}^*$ with $\gamma_1(t)\gamma_2(t)\gamma_3(t) = 1$ for all t, one has the identity

$$\int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + \int_{\gamma_3} \frac{dz}{z} \;=\; 0.$$

Problem 1.2. Let $a, b \in \mathbb{R}_{>0}$ with $a \leq b$. Show that for any $\varphi_0 \in [0, \frac{\pi}{2}]$ the arclength of the ellipse segment

$$\mathbb{E}(\varphi_0) = \big\{(a\cos(\varphi), b\sin(\varphi)) \in \mathbb{R}^2 \mid \varphi \in [0, \varphi_0]\big\}$$

can be written as

$$\ell(\varphi_0) \;=\; \frac{b}{2}\int_{x_0}^{x_1} \frac{1-cx}{\sqrt{x(1-x)(1-cx)}}dx \quad \text{with} \quad x_0, x_1 \;\in\; \mathbb{R} \quad \text{and} \quad c \;=\; 1 - \tfrac{a^2}{b^2}.$$

Problem 2.1. Let S be a Riemann surface.

(a) Show that for any branched cover $p : X \to S$ the topological space X has a unique Riemann surface structure making p a morphism of Riemann surfaces.

(b) For $\Sigma \subset S$ discrete, show that any topological covering map $p_0 : X_0 \to S \setminus \Sigma$ extends uniquely to a branched cover $p : X \to S$.

Problem 2.2. Let $f(x) \in \mathbb{C}[x] \setminus \{0\}$, and put $\Sigma = f^{-1}(0) \cup \{\infty\} \subset S = \mathbb{P}^1(\mathbb{C})$.

(a) Check that

$$p_0 : X_0 = \{(x,y) \in \mathbb{C}^2 \mid y^2 = f(x) \neq 0\} \to S \setminus \Sigma$$

is a double cover. Describe its unique extension $p : X \to S$ over each $s \in \Sigma$.

(b) If $f(x) = x(x+1)(x-1)(x-\lambda)$ with $\lambda \in \mathbb{C} \setminus \{0, \pm 1\}$, determine $g(u) \in \mathbb{C}[u]$ such that

$$p^{-1}(S \setminus \{0\}) \;\simeq\; \{(u,v) \in \mathbb{C}^2 \mid v^2 = g(u)\}.$$

Problem 2.3. Let $f(x) = x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{C} \setminus \{0,1\}$, and consider the branched cover

$$p : \quad E \;=\; \{(x,y) \in \mathbb{C}^2 \mid y^2 = f(x)\} \cup \{\infty\} \;\longrightarrow\; \mathbb{P}^1(\mathbb{C}).$$

Show that the differential form

$$\omega \;=\; \frac{dx}{\sqrt{f(x)}},$$

which is a priori only well-defined locally on the complement $E \setminus p^{-1}(\{0,1,\lambda,\infty\})$, extends to a holomorphic differential form on all of E.

All elliptic functions on this sheet are for a given lattice $\Lambda \subset \mathbb{C}$. Put $\wp(z) = \wp_\Lambda(z)$.

Problem 3.1. (a) Show that if f is an elliptic function of degree $\deg(f) = d$, then its derivative is an elliptic function of degree

$$\deg(f') \ \in \ \{d+1, \dots, 2d\},$$

and give examples for the extreme cases $\deg(f') = d+1$ and $\deg(f') = 2d$.

(b) For $n = 1, 2, 3$, find $h_1, h_2 \in \mathbb{C}(x)$ with $(\wp'(z))^{-n} = h_1(\wp(z)) + h_2(\wp(z)) \cdot \wp'(z)$.

Problem 3.2. Show that up to a translation the Weierstrass function is determined uniquely by its differential equation: If F is a meromorphic function on an open domain $\varnothing \neq U \subseteq \mathbb{C}$ satisfying

$$(F'(z))^2 \ = \ 4F(z)^3 - g_2 \cdot F(z) - g_3 \quad \text{for} \quad \begin{cases} g_2 \ = \ 60G_4(\Lambda), \\ g_3 \ = \ 140G_6(\Lambda), \end{cases}$$

then we must have $F(z) = \wp(z+a)$ for some constant $a \in \mathbb{C}$.

Problem 3.3. Show that the following properties are equivalent:

(a) We have $g_2(\Lambda), g_3(\Lambda) \in \mathbb{R}$.

(b) We have $G_{2n}(\Lambda) \in \mathbb{R}$ for all $n \geq 2$.

(c) We have $\wp(\overline{z}) = \overline{\wp(z)}$ for all $z \in \mathbb{C}$.

(d) The lattice $\Lambda \subset \mathbb{C}$ is invariant under complex conjugation.

Put $\mathcal{F} = \{\tau \in \mathbb{H} \mid |\mathrm{Re}(\tau)| \leq 1/2 \text{ and } |\tau| \geq 1\}$.

Problem 4.1. Determine all points $\tau \in \mathcal{F}$ that are equivalent modulo $\Gamma = Sl_2(\mathbb{Z})$ to the point

$$\frac{5i+6}{4i+5} \quad \text{respectively} \quad \frac{2+8i}{17} \ \in \ \mathbb{H}.$$

Problem 4.2. Find natural numbers $n_1, \dots, n_k \in \mathbb{N}$ for which one has the matrix identity

$$\begin{bmatrix} 4 & 9 \\ 11 & 25 \end{bmatrix} \ = \ ST^{n_1} \cdots ST^{n_k} \quad \text{with} \quad S \ = \ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ T \ = \ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Is such a representation unique? More generally, show that the modular group has the presentation

$$Sl_2(\mathbb{Z}) \ \simeq \ \big\langle S, T \mid S^2 = (ST)^3 = 1 \big\rangle.$$

Problem 4.3. (a) Show that $\overline{G_k(\overline{\tau})} = G_k(\tau)$ for all $\tau \in \mathbb{H}$, $k \geq 4$.

(b) Show that the j-function takes real values on the set $\partial\mathcal{F} \cup i \cdot \mathbb{R}_{>0}$.

(c) Conversely, show that any real number arises as $j(\tau_0)$ for some $\tau_0 \in \partial\mathcal{F} \cup i \cdot \mathbb{R}_{>0}$.

Problem 4.4. Verify that the derivative of a meromorphic modular form of weight zero is a meromorphic modular form of weight two. Deduce that if f, g are modular forms of a given weight k, then $fg' - f'g$ is a modular form of weight $2k + 2$.

Let k be a field. For elliptic curves with a flex point at infinity we take this flex point as the neutral element for the group structure.

Problem 5.1. Let $f \in k[x_0, x_1, x_2]$ be a homogenous polynomial and $C_f \subset \mathbb{P}^2$ the corresponding plane curve. Show that for $char(k) = 0$, a smooth point $p \in C_f(k)$ is a flex point of the curve iff

$$\det\Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)(p) \;=\; 0.$$

What happens if the assumption on the characteristic $char(k)$ is dropped?

Problem 5.2. Find the order of the point p on the elliptic curve E when

(a) $p = (3, 12)$ and E is given by $y^2 = x^3 - 14x^2 + 81x$.

(b) $p = (3, 8)$ and E is given by $y^2 = x^3 - 43x + 166$.

Problem 5.3. Put $f(x, y) = y^2 - x^3 + 432c^2$ for fixed $c \in k$.

(a) For which c is the cubic $E = \{(x, y) \mid f(x, y) = 0\} \cup \{\infty\} \subset \mathbb{P}^2$ smooth?

(b) In the smooth case, find $M \in \mathrm{PGl}_3(k)$ that induces on affine coordinates the transformation

$$(x, y) \;\mapsto\; (u, v) \;=\; \Big(\frac{6c}{x} + \frac{y}{6x}, \frac{6c}{x} - \frac{y}{6x}\Big).$$

What is the equation for the cubic in the new affine coordinates (u, v)?

(c) Now let $k = \mathbb{Q}$. Determine the group $E(\mathbb{Q})$ in the case $c = 1$.

Problem 5.4. Let $E \subset \mathbb{P}^2$ be the elliptic curve defined by $y^2 = x^3 - 11$ over $\mathbb{Q}$. Show that

(a) a point $(s, t) \in E(\mathbb{Q})$ is in the image of the map $E(\mathbb{Q}) \to E(\mathbb{Q}), p \mapsto 2p$ iff the polynomial

$$x^4 - 4sx^3 + 88x + 44s \;\in\; \mathbb{Q}[x] \quad \text{has a rational root} \quad x_0 \;\in\; \mathbb{Q}.$$

(b) the images of $p = (3, 4), q = (15, 58)$ in the quotient group $E(\mathbb{Q})/2E(\mathbb{Q})$ are distinct and nonzero, hence linearly independent when the quotient is seen as a vector space over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

(c) the map $(m, n) \mapsto mp + nq$ gives an embedding $\mathbb{Z}^2 \hookrightarrow E(\mathbb{Q})$.

Problem 6.1. Let $E \subset \mathbb{P}^2$ be an irreducible singular cubic over a field k, defined in affine coordinates by a Weierstrass equation.

(a) Show that E has a unique singular point $p = (x_0, y_0) \in E(\overline{k})$.

(b) If p is a cusp with tangent line given by $y = \alpha x + \beta$, check that we have an isomorphism

$$\varphi : \quad E \setminus \{p\} \xrightarrow{\sim} \mathbb{A}^1, \quad (x, y) \mapsto \frac{x - x_0}{y - \alpha x - \beta}$$

such that $a, b, c \in E(\overline{k}) \setminus \{p\} \subset \mathbb{P}^2(\overline{k})$ are collinear iff $\varphi(a) + \varphi(b) + \varphi(c) = 0$.

(c) If p is a node with tangent lines $y = \alpha_i x + \beta_i$ defined over k, check that we have an isomorphism

$$\varphi : \quad E \setminus \{p\} \xrightarrow{\sim} \mathbb{A}^1 \setminus \{0\}, \quad (x, y) \mapsto \frac{y - \alpha_1 x - \beta_1}{y - \alpha_2 x - \beta_2}$$

such that $a, b, c \in E(\overline{k}) \setminus \{p\} \subset \mathbb{P}^2(\overline{k})$ are collinear iff $\varphi(a) \cdot \varphi(b) \cdot \varphi(c) = 1$.

Problem 6.2. Let E be an elliptic curve over the complex numbers. Which of the following four cases can occur for a subgroup $H \subset E(\mathbb{C})$?

(a) H is torsion and $H/2H$ trivial,
(b) H is torsion and $H/2H$ infinite,
(c) H is torsion-free and $H/2H$ trivial,
(d) H is torsion-free and $H/2H$ infinite.

Problem 6.3. Let $a \in \mathbb{Z}$ be an integer which is not divisible by the fourth power of any prime, and consider the elliptic curve E defined by $y^2 = x^3 - ax$.

(a) Show that $|\overline{E}(\mathbb{F}_p)| = p + 1$ for all primes $p \equiv 3 \pmod 4$.

(b) Show that

$$E(\mathbb{Q})_{tors} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } a \text{ is a square,} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } a = -4, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

Problem 6.4. Find the 2-power torsion and sets of representatives for $E(\mathbb{Q})/2E(\mathbb{Q})$ for the elliptic curves defined by the following equations:

(a) $y^2 = x(x - 3)(x + 4)$,

(b) $y^2 = x(x - 1)(x + 3)$.