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Intersection Theory

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Introduction

Intersection theory is one of the most powerful tools in algebraic geometry, and its applications are ubiquitous. Its general paradigm is that in the study of algebraic varieties, we can extract discrete invariants such as Euler characteristics of coherent sheaves via a simple calculus of intersection products. More specifically, the basic goal is to understand intersections of subvarieties in a given ambient variety. Usually the ambient variety will be smooth, but even when the subvarieties are smooth as well, their intersection can be very singular, reducible and nonreduced. What makes intersection theory work is the possibility to deform subvarieties, as one does in topology by means of singular cohomology. Before discussing the analogy with topology that will be the blueprint for our development of intersection theory in the rest of this lecture, let us look at a few simple examples.

Prehistory: Bézout's theorem

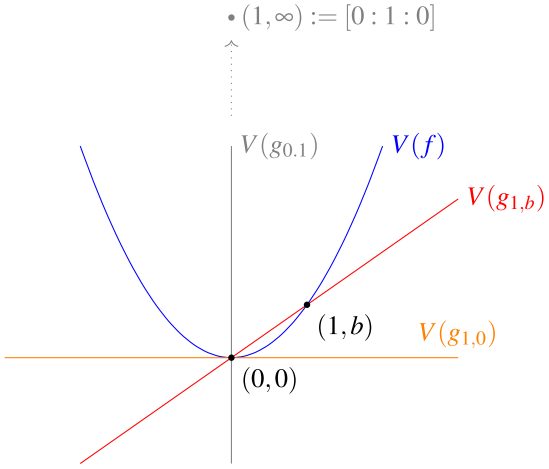
Let k be an algebraically closed field. How many points are there in the intersection of two plane curves $V(f), V(g) \subset \mathbb{A}^2_k$ cut out by polynomials $f, g \in k[x,y]$ of positive degree? It is easy to decide whether the intersection is finite:

$$\begin{aligned} |V(f) \cap V(g)| < \infty &\iff \dim V(f) \cap V(g) = 0 \\ &\iff V(f) \text{ and } V(g) \text{ have no common component} \\ &\iff f \text{ and } g \text{ have no common factor} \end{aligned}$$

When the intersection is finite, we expect that it consists of precisely $\deg(f) \cdot \deg(g)$ many points, but this statement has to be understood properly: For instance, consider the zero loci of

$$f = y - x^2 \quad \text{and} \quad g_{a,b} = ay - bx \quad \text{for fixed} \quad (a,b) \in k^2 \setminus \{(0,0)\}.$$

For $ab \neq 0$ we have $|V(f) \cap V(g_{a,b})| = 2$ as expected. For $b = 0$ the intersection consists only of a single point, but this point should be counted with multiplicity two because the two curves are tangent at this point; for $a = 0$ the two curves again meet only in a single point, but their closures in the projective plane will meet in the point $[0:1:0] \in \mathbb{P}^2(k)$ at infinity:



Hence if we want intersection numbers to stay constant in families, then

- we should work with projective varieties, and
- we should count points with appropriate multiplicities.

Finding the right definition of intersection multiplicities was one of the key points in the development of intersection theory. Here we can write down things by hand:

Definition. Let $f, g \in k[x,y]$ be polynomials of positive degree. Let $p \in V(f) \cap V(g)$ be a point which does not lie on any common irreducible component of the two curves $V(f)$ and $V(g)$. Then we define the *intersection multiplicity* of the curves at p to be

$$i_p(f,g) := \dim_k \mathscr{O}_{\mathbb{A}^2,p}/(f,g) \quad \text{where} \quad \mathscr{O}_{\mathbb{A}^2,p} := \{\tfrac{a}{b} \mid a,b \in k[x,y], b(p) \neq 0\}$$

One easily checks that the above definition is invariant under affine-linear coordinate changes, which allows to define $i_p(C,D) \in \mathbb{N}$ for projective curves $C, D \subset \mathbb{P}^2_k$ and any point $p \in (C \cap D)(k)$ by using affine charts. We then have:

Theorem (Bézout). *For any curves $C, D \subset \mathbb{P}^2_k$ without common components, we have*

$$\sum_{p \in (C \cap D)(k)} i_p(C,D) = \deg(C)\deg(D).$$

More generally, let $H_1, \dots, H_n \subset \mathbb{P}^n_k$ *be hypersurfaces,* $n \in \mathbb{N}$. *If* $Z = H_1 \cap \cdots \cap H_n$ *is finite, then*

$$\sum_{p \in Z(k)} i_p(H_1, \dots, H_n) \;=\; \deg(H_1) \cdots \deg(H_n)$$

for the intersection multiplicities

$$i_p(H_1, \dots, H_n) \;=\; \dim_k \mathscr{O}_{\mathbb{P}^n,p}/(f_1, \dots, f_n),$$

where $f_i \in \mathscr{O}_{\mathbb{P}^n,p}$ *is the germ of a local equation that cuts out* $H_i \subset \mathbb{P}^n_k$.

The proof is elementary, using Hilbert polynomials. We will recover it later as a special case of general results about the intersection product on smooth varieties.

An example from enumerative geometry

We can also count more complicated objects in terms of intersection numbers. For instance, it is a classical problem to determine the number of all conics tangent to a suitable number of other conics in general position in the projective plane. Any conic can be written as

$$C_s \;=\; V_+(f_s) \;\subset\; \mathbb{P}^2_k \quad \text{with} \quad f_s = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

for a unique point $s = [a : b : c : d : e : f] \in S = \mathbb{P}^5_k$. Here the parameter space S includes also all singular conics, but using discriminants one may check that inside it the smooth conics form an open dense subset.

Lemma. *Assume that* $char(k) \notin \{2, 3\}$. *Then for every smooth conic* $Q \subset \mathbb{P}^2_k$ *the subset*

$$Z_Q \;:=\; \{s \in S \mid C_s \textit{ is tangent to } Q \textit{ at some point } p \in Q(k)\} \;\subset\; S \;=\; \mathbb{P}^5_k$$

is an irreducible hypersurface of degree $\deg(Z_Q) = 6$.

Proof. Since $char(k) \neq 2$, any two smooth conics are isomorphic via a linear change of coordinates in the projective plane. Hence we may assume that Q is given by

$$Q \;=\; V_+(yz - x^2) \;\subset\; \mathbb{P}^2_k.$$

Thus Q is the image of the closed immersion $i\colon \mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k, [u : v] \mapsto [uv : u^2 : v^2]$, and its scheme-theoretic intersection with C_s is obtained by pulling back the defining equation $f_s(x, y, z) = 0$ under this closed immersion. As a closed subscheme of $\mathbb{P}^1_k$ we have

$$C_s \cap Q = V_+(i^*(f_s)) \subset \mathbb{P}^1_k \quad \text{with} \quad i^*(f_s) = f_s(uv, u^2, v^2)$$
$$= (a + f)u^2v^2 + bu^4 + cv^4 + du^3v + euv^3$$

Thus $C_s \cap Q$ is cut out by a homogenous polynomial of degree $d = 4$ in u, v. By definition the two curves C_s and Q are tangent at some point of the intersection iff this degree four equation has a double root. In the chart $v = 1$, this condition is equivalent to the condition that the quartic polynomial

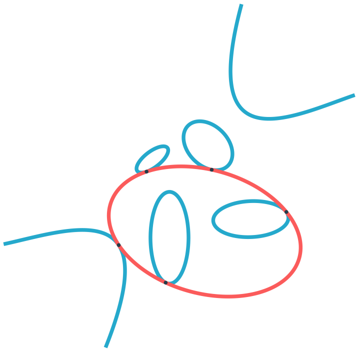
$$g_s(u) = f_s(u, 1) \in k[u]$$

has a double root. If we factor the polynomial as $g_s(u) = c \cdot \prod_{i=1}^{d}(u - \alpha_{s,i})$, then the existence of a double root of the polynomial is tantamount to the vanishing of the discriminant

$$\Delta_s \;=\; c^{2d-2} \cdot \prod_{i<j}(\alpha_{s,i} - \alpha_{s,j})^2$$

By Galois theory this discriminant is a polynomial in the coefficients $a + f, b, c, d, e$ of g_s. If all coefficients of g_s are multiplied by a scalar λ, then the roots of g_s do not change but the leading coefficient c is rescaled by c^{2d-2}. Hence Δ_s is a homogenous polynomial of degree $2d - 2 = 6$ in the coefficients of g_s, and therefore its vanishing locus is a hypersurface of degree six in S. □

For dimension reasons, to arrive at a finite number of points in $\mathbb{P}^5_k$ we need to intersect *five* hyperplanes. Hence there should exist a finite number of smooth conics that are tangent to *five* other conics in general position as illustrated in the picture below (taken from `www.juliahomotopycontinuation.org/3264/`):



We want to know how many solutions there are:

Problem (Steiner 1848). *Given five general conics in the plane, how many smooth conics are there which are tangent to all five given conics?*

By the higher-dimensional Bézout theorem the expected answer is $6^5 = 7776$, as claimed by Steiner. However, there was a mistake in his argument; the problem is that for *any* smooth conic Q, the hypersurface $Z_Q \subset \mathbb{P}^5_k$ contains the Veronese surface

$$\mathbb{P}^2_k \;=\; \{[u^2 : v^2 : w^2 : uv : uw : vw] \mid [x : y : z] \in \mathbb{P}^2_k\} \;\subset\; \mathbb{P}^5_k$$

because this surface parametrizes double lines $V_+((ux+vy+wz)^2) \subset \mathbb{P}^2_k$ and any such double line intersects our smooth conic in a point with multiplicity two. Hence for any five smooth conics $Q_1, \dots, Q_5 \subset \mathbb{P}^2_k$, the intersection $Z_{Q_1} \cap \dots \cap Z_{Q_5}$ will never be finite, it always contains the Veronese surface. To find the correct count of conics, one has to deal with the *excess intersection* along the Veronese surface in a suitable way, which leads to the following famous result:

Theorem (de Jonquiéres 1859, Chasles 1864, Fulton-MacPherson 1978). *The number of smooth conics that are tangent to five given smooth conics in general position in the plane is*

$$N \;=\; 3264.$$

This explains the title of the intersection theory book by Eisenbud and Harris. We will see later in the lecture where the number 3264 comes from, and learn techniques to deal with many similar counting problems. In the above result the specific value of N does not seem to have a big theoretical significance, the interesting thing is that the number is well-defined and computable. However, sometimes the numbers obtained from intersection theory do have a theoretical meaning: For instance, it is a classical result that any smooth cubic surface $S \subset \mathbb{P}^3_k$ contains precisely 27 lines, and the configuration of these lines is related to the root system E_6 from Lie theory. While the above results are all very old, enumerative geometry continues to be an important topic of current research for instance in Gromov-Witten theory, quantum cohomology and mirror symmetry.

Singular (co)homology as a blueprint

Before we develop the basic notions of intersection theory, let take a brief look at some analogous notions from topology as a blueprint. For any complex projective variety X its singular cohomology $H^*(X) = \bigoplus_{i\in\mathbb{Z}} H_i(X)$ is a graded commutative ring for the cup product

$$\cup\colon \quad H^i(X) \otimes H^j(X) \;\longrightarrow\; H^{i+j}(X).$$

The singular homology $H_*(X) = \bigoplus_{i\in\mathbb{Z}} H^i(X)$ is a graded module for this ring via the cap product

$$\cap\colon \quad H^i(X) \otimes H_j(X) \;\longrightarrow\; H_{j-i}(X).$$

Cohomology is a contravariant functor while homology is a covariant functor; the functorialities are compatible in the sense that for any morphism $f\colon Y \to X$ we have the *projection formula*

$$f_*(f^*(\alpha) \cap \beta) \;=\; \alpha \cap f_*(\beta) \quad \text{for} \quad \alpha \,\in\, H^*(Y), \beta \,\in\, H_*(X),\; f\colon Y \to X.$$

If X is smooth of complex dimension n, we have the Poincaré duality isomorphism

$$\mathrm{PD}\colon \quad H^i(X) \xrightarrow{\sim} H_{2n-i}(X), \quad \alpha \;\mapsto\; \alpha \cap [X]$$

taking the cup product with the fundamental class $[X] \in H_{2n}(X)$. In this case, we define the *intersection product* $\cdot$ on homology to be Poincaré dual to the cup product on cohomology:

$$\begin{tikzcd}
H_i(X) \otimes H_j(X) \arrow[r, "\cdot"] \arrow[d, "\mathrm{PD}^{-1}"'] & H_{i+j-2n}(X) \\
H^{2n-i}(X) \otimes H^{2n-j}(X) \arrow[r, "\cup"] & H^{4n-i-j}(X) \arrow[u, "\mathrm{PD}"']
\end{tikzcd}$$

Generalizing the fundamental class that appears in Poincaré duality, we can define for any closed subvariety $Z \subset X$ of dimension d a *cycle class* $[Z] \in H_{2d}(X)$ by taking a triangulation of the subvariety. From the viewpoint of algebraic geometry, the span of these cycle classes is the really interesting part of the homology.

Remark. For simplicity, we have formulated the above only in the case when X is projective and hence compact. However, everything in this section works in the same way also for smooth non-compact varieties if we replace singular homology by *Borel-Moore* homology, defined like singular homology but using *locally* finite rather than finite chains of simplices. In particular, we always have an isomorphism between cohomology and Borel-Moore homology in the complementary degree, and every closed subvariety has a cycle class in Borel-Moore homology.

Example (pushforward of cycles). Let $Z \subset X$ be a closed subvariety, $f \colon X \to Y$ a morphism to another smooth variety, and put $W = f(Z) \subset Y$. Then there are two cases:

- Either $f \colon Z \to W$ restricts over some open subset of W to a topological covering map. We then denote the degree of this covering map by $\deg(Z/W) \in \mathbb{N}$.
- Or $\dim(W) < \dim(Z)$. In this case we formally put $\deg(Z/W) = 0$.

In both cases, it follows from the definition via triangulations that the pushforward of the cycle class $[Z] \in H_{2d}(X)$ (where $d = \dim Z$) is

$$f_*[Z] \;=\; \deg(Z/W) \cdot [W] \;\in\; H_{2d}(Y).$$

Example (intersection numbers). For closed subvarieties $Z_1, Z_2 \subset X$, we may view their cycle classes not only in the homology of the ambient smooth variety X but also as classes $[Z_i] \in H_{2d_i}(Z_i)$ where $d_i = \dim(Z_i)$. If in the above diagram for the definition of the intersection product, we replace the Poincaré duality isomorphism by its relative version

$$\mathrm{PD}\colon \quad H^{2(n-d_i)}(X, X \setminus Z_i) \xrightarrow{\sim} H_{2d_i}(Z_i)$$

and the cup product by

$$\cup\colon \quad H^{2(n-d_1)}(X, X \setminus Z_1) \otimes H^{2(n-d_2)}(X, X \setminus Z_2) \longrightarrow H^{4n-2d_1-2d_2}(X, X \setminus (Z_1 \cap Z_2)),$$

we can regard the intersection product between the two given cycle classes as an element

$$[Z_1]\cdot[Z_2] \;\in\; H_{2(d_1+d_2-n)}(Z_1\cap Z_2).$$

This is most useful if the subvarieties intersect properly in the sense that $Z_1\cap Z_2$ is of pure dimension d_1+d_2-n: In this case the top homology group of the intersection is the free group

$$H_{2(d_1+d_2-n)}(Z_1\cap Z_2) \;\simeq\; \bigoplus_W \mathbb{Z}\cdot[W]$$

where the direct sum runs over the irreducible components W of $Z_1\cap Z_2$, and hence we obtain a decomposition

$$[Z_1]\cdot[Z_2] \;=\; \sum_W i_W(Z_1,Z_2)\cdot[W] \quad \text{with} \quad i_W(Z_1,Z_2) \;\in\; \mathbb{N},$$

so we have found a topological definition of intersection multiplicities.

Example (Chern classes). A *Weil divisor* on X is by definition an element of the free abelian group

$$\operatorname{Div}(X) \;:=\; \bigoplus_{Z\subset X} \mathbb{Z}$$

where the sum runs over all codimension one subvarieties $Z\subset X$. We write Weil divisors as finite formal sums

$$D \;=\; \sum_{Z\subset X} n_Z(D)\cdot[Z] \quad \text{with} \quad n_Z(D) \;\in\; \mathbb{Z}.$$

If X is smooth, then every line bundle $\mathscr{L}\in\operatorname{Pic}(X)$ can be written as $\mathscr{L}\simeq\mathscr{O}_X(D)$ for some Weil divisor $D\in\operatorname{Div}(X)$ which is unique up to linear equivalence. One can show that the cycle class

$$[D] \;:=\; \sum_{Z\subset X} n_Z(D)\cdot[Z] \;\in\; H^2(X)$$

only depends on the linear equivalence class of the divisor $D\in\operatorname{Div}(X)$. So there is a unique homomorphism $c_1\colon\ \operatorname{Pic}(X)\to H^2(X)$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Div}(X) \arrow[r] \arrow[d] & H^2(X) \\
\operatorname{Div}(X)/\sim \arrow[r, "\sim"] & \operatorname{Pic}(X) \arrow[u, "\exists ! c_1"']
\end{tikzcd}$$

We call $c_1(\mathscr{L})\in H^2(X)$ the *first Chern class* of the line bundle $\mathscr{L}\in\operatorname{Pic}(X)$. More generally, one can show there is a unique way to attach to every vector bundle $\mathscr{E}$ on a smooth complex variety X a *total Chern class*

$$c(\mathscr{E}) \;=\; \sum_{i\geq 0} c_i(\mathscr{E}) \;\in\; H^*(X) \quad \text{with} \quad c_i \;\in\; H^{2i}(X)$$

such that the following axioms hold:

a) Naturality: For any morphism $f\colon Y \to X$ we have $c_i(f^*(\mathscr{E})) = f^*(c_i(\mathscr{E}))$.

b) Whitney formula: For direct sums of vector bundles $c(\mathscr{E} \oplus \mathscr{F}) = c(\mathscr{E}) \cup c(\mathscr{F})$.

c) Normalization: For line bundles $\mathscr{L} \in \mathrm{Pic}(X)$ we have $c(\mathscr{L}) = 1 + c_1(\mathscr{L})$ for the first Chern class $c_1(\mathscr{L}) \in H^2(X)$ constructed above.

Chern classes measure how far a vector bundle is from being trivial. To get a feeling of their geometric meaning, we mention the following interpretation that could be used for an alternative definition:

Theorem. *Let $\mathscr{E}$ be a vector bundle of rank r which is generated by its global sections. Then for generically chosen sections $s_0, s_1, \dots, s_r \in H^0(X, \mathscr{E})$ and $0 \leq i \leq r$ the loci*

$$Z(s_0, \dots, s_i) \ := \ \left\{ p \in X \mid s_0(p), \dots, s_i(p) \textit{ are linearly dependent in } \mathscr{E}/\mathfrak{m}_p\mathscr{E} \right\}$$

are closed in X and we have

$$c_{r-i}(\mathscr{E}) \ = \ [Z(s_0, \dots, s_i)].$$

The goal of this lecture is to develop an algebraic analog of the above results in singular homology that works for algebraic varieties X over any field k. Concretely, we will

- define Chow groups $A_*(X)$, the algebraic analog of Borel-Moore homology,
- develop the notion of cycle classes and Chern classes in these Chow groups,
- show that if X is smooth, then $A_*(X)$ comes with a natural intersection product,
- study the basic properties and sample applications of this intersection product.

As we will see, the notion of Chern classes is a crucial ingredient in the development of the theory. To conclude this introduction, let us illustrate its use in applications by a brief outlook on the Riemann-Roch theorem.

Outlook: Riemann-Roch

For smooth projective curves the Euler characteristic of a line bundle $\mathscr{L} \in Pic(C)$ is given in terms of its degree and the genus of the curve by the Riemann-Roch theorem

$$\dim_k H^0(C, \mathscr{L}) - \dim_k H^1(C, \mathscr{L}) \ = \ \deg(\mathscr{L}) + 1 - g.$$

The message here is that it is much easier to compute the Euler characteristic of a coherent sheaf than to compute its individual sheaf cohomology groups. The right hand side of the Riemann-Roch formula is an expression in Chern classes, since the degree of a line bundle only depends on its first Chern class. Motivated by the integration of differential forms in de Rham cohomology, it is common to use the notation

$$\int_C : \quad H^2(C) \longrightarrow \mathbb{Z}$$

for the homomorphism which is the composite of $H^2(X) \to H_0(X), \alpha \mapsto \alpha \cap [X]$ with the natural homomorphism $H_0(X) \to \mathbb{Z}$ counting the number of points; with this notation the right hand side of the Riemann-Roch formula can be rewritten as the 'integral'

$$\deg(\mathscr{L}) + 1 - g = \int_X (c_1(\mathscr{L}) + \tfrac{1}{2} c_1(\mathscr{T}_C))$$

where $\mathscr{T}_C$ denotes the tangent bundle of the curve. More generally, for any smooth projective variety X of dimension d over an arbitrary field there is a natural degree homomorphism

$$\int_X : \quad A_d(X) \longrightarrow \mathbb{Z}$$

and the Riemann-Roch theorem has the following far-reaching generalization:

Theorem (Hirzebruch-Riemann-Roch). *Let $\mathscr{E}$ be a vector bundle on a smooth projective variety X. Then*

$$\sum_i (-1)^i \dim_k H^i(X, \mathscr{E}) \; = \; \int_X ch(\mathscr{E}) \cdot td(X).$$

Here the *Chern character $ch(\mathscr{E})$* and the *Todd class $td(X)$* and certain expressions in Chern classes that will be studied in more detail later; their first terms are given explicitly by

$$\begin{aligned} ch(\mathscr{E}) &= rk(\mathscr{E}) + c_1(\mathscr{E}) + \tfrac{1}{2} c_1(\mathscr{E})^2 - \cdots \\ td(X) &= 1 + \tfrac{1}{2} c_1(X) + \tfrac{1}{12}(c_1(X)^2 + c_2(X)) + \cdots \quad \text{where} \quad c_i(X) := c_i(T_X). \end{aligned}$$

Conventions

For the rest of the lecture we fix an arbitrary base field k. By a *scheme* we mean a separated scheme of finite type over a field. By a *variety* we mean an integral scheme. Subvarieties and subschemes will always be assumed to be closed unless we explicitly say otherwise.

Chapter I
Chow groups

1 The order of zeroes and poles

Let X be a variety, i.e. an integral scheme of finite type over a field k, and consider its function field

$$\begin{aligned} k(X) &= \mathscr{O}_{X,\eta} && \text{for the generic point } \eta \in X \\ &= \operatorname{colim}_U \mathscr{O}_X(U) && \text{with the colimit over all open } U \subset X \\ &= \operatorname{Quot}(\mathscr{O}_X(U)) && \text{for any \textit{affine} open } U \subset X. \end{aligned}$$

The elements of $k(X)$ are called *rational functions*. Explicitly, rational functions are given by eLet $W_1, \dots, W_r \subset W$ be its irreducible components,quivalence classes of pairs (f, U) where $U \subset X$ is a nonempty open subset and $f \in \mathscr{O}_X(U)$, and where we declare two such pairs (f_1, U_1) and (f_2, U_2) to be equivalent if they satisfy $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$. We will often drop the domain of definition of rational functions and simply denote them by $f \in k(X)$.

Example 1.1. We have

$$\begin{aligned} k(\mathbb{P}^n_k) &= k(\tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}) \qquad \text{by restriction to the affine open } U = \mathbb{A}^n_k \subset \mathbb{P}^n_k \\ &= \{ \tfrac{f}{g} \mid f, g \in k[x_0, \dots, x_n] \text{ homogenous of the same degree and } g \neq 0 \} \end{aligned}$$

Similarly one obtains that

$$\begin{aligned} k(\mathbb{P}^m_k \times \mathbb{P}^n_k) = \{ \tfrac{f}{g} \mid f, g \in k[x_0, \dots, x_m][y_0, \dots, y_n] \\ \text{bihomogenous of the same bidegree and } g \neq 0 \} \end{aligned}$$

For any square-free polynomial $f \in k[x] \setminus \{0\}$, the curve $X = V(y^2 - f(x)) \subset \mathbb{A}^3_k$ has the function field

$$k(X) = k(x)[y]/(y^2 - f(x))$$

To any rational function we will attach a divisor of zeroes and poles consisting of certain subvarieties of codimension one. Recall that the *dimension* of a variety X is given by

$$\begin{aligned} \dim X &= \sup\{n \in \mathbb{N}_0 \mid \exists \text{closed subvarieties} \varnothing \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X\} \\ &= \dim \mathscr{O}_X(U) \qquad \text{Krull dimension of } \mathscr{O}_X(U) \text{ for any affine open } U \subset X \\ &= \operatorname{trdeg}(k(X)/k) \qquad \text{transcendence degree of the field extension } k(X) \supset k \end{aligned}$$

The *codimension* of a subvariety $Z \subset X$ is given by

$$\begin{aligned} \operatorname{codim}_X Z &= \dim X - \dim Z \\ &= \sup\{d \in \mathbb{N}_0 \mid \exists \text{closed subvarieties} Z = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d = X\} \\ &= \dim \mathscr{O}_{X,Z} \qquad \text{Krull dimension of the local ring of } X \text{ along } Z \end{aligned}$$

where as usual we put $\mathscr{O}_{X,Z} := \mathscr{O}_{X,\xi}$ for the generic point $\xi \in Z$.

Example 1.2. Let $Z \subset X$ be a subvariety with $\operatorname{codim}_X Z = 1$. If $Z \not\subset \operatorname{Sing}(X)$, then the generic point of the subvariety Z must lie the smooth locus of X. Hence the local ring $\mathscr{O}_{X,Z}$ is a regular ring. Since this local ring is also a domain of Krull dimension one, it follows that $\mathscr{O}_{X,Z}$ is a discrete valuation ring. We then denote the corresponding valuation by

$$\operatorname{ord}_Z\colon \quad k(X)^\times \;=\; \operatorname{Quot}(\mathscr{O}_{X,Z})^\times \;\longrightarrow\; \mathbb{Z}$$

and call $\operatorname{ord}_Z(f)$ the *order of pole or vanishing* of the rational function $f \in k(X)$.

If $Z \subset \operatorname{Sing}(X)$, then the local ring $\mathscr{O}_{X,Z}$ is no longer a discrete valuation ring, but we can still define the order pole or vanishing using the notion of length. Recall that any finitely generated module M over a Noetherian ring A has a composition series

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_r = 0$$

with $M_{i-1}/M_i \simeq A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i \trianglelefteq A$. We say M has *finite length* if all the $\mathfrak{p}_i$ are maximal ideals. In this case the length r of the chain is independent of the chosen chain and is called the *length* $\ell_A(M)$ of the module.

Example 1.3. If A contains a subfield $k \subset A$ that maps isomorphically to the residue field modulo every maximal ideal, then for every finite length A-module M its length is the dimension $\ell_A(M) = \dim_k M$.

The following example illustrates why in general we want to use the length rather than the dimension over the base field:

Example 1.4. Let $A = \mathscr{O}_{X,Z}$ for $Z = V(y) \subset X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. For $M := A/(y)$ we have

$$M \;\simeq\; k(x), \quad \text{hence} \quad \dim_k(A/(y)) \;=\; \infty \quad \text{but} \quad \ell_A(M) \;=\; 1.$$

The finiteness of the length in the above example carries over to local rings of any variety along a subvariety of codimension one:

Lemma 1.5. *Let A be a Noetherian local domain of Krull dimension one. Then for any $a \in A \setminus \{0\}$ the quotient $A/(a)$ has finite length as an A-module.*

Proof. Let $M_0 = A/(a) \supsetneq M_1 \supsetneq \cdots \supsetneq M_r = 0$ with $M_i/M_{i+1} \simeq A/\mathfrak{p}_i$ for suitable prime ideals $\mathfrak{p}_i \trianglelefteq A$. Since A has Krull dimension one, every nonzero prime ideal is maximal, so we only need to show $\mathfrak{p}_i \neq 0$ for all i. But this is clear: If $\mathfrak{p}_i = 0$, then we have a surjective homomorphism

$$M_i \twoheadrightarrow M_i/M_{i+1} \simeq A/(0) = A.$$

But a acts by zero on M_i while it acts nontrivially on A, a contradiction. □

Corollary 1.6. *For any subvariety $Z \subset X$ with $\operatorname{codim}_X Z = 1$, there exists a unique group homomorphism*

$$\operatorname{ord}_Z\colon \quad k(X)^\times \longrightarrow \mathbb{Z}$$

such that on nonzero elements of the local ring $A = \mathscr{O}_{X,Z} \subset k(X)$ it is given by the length

$$\operatorname{ord}_Z(f) = \ell_A(A/(f)) \quad \textit{for all} \in A \setminus \{0\}.$$

Proof. Lemma 1.5 shows that for any $f \in A \setminus \{0\}$ the A-module $M = A/(f)$ has finite length, hence the map

$$\operatorname{ord}_Z\colon \quad A \setminus \{0\} \longrightarrow \mathbb{N}_0, \quad f \mapsto \ell_A(A/(f)).$$

is well-defined. The map is additive in the sense that $\operatorname{ord}_Z(fg) = \operatorname{ord}_Z(f) + \operatorname{ord}_Z(g)$ for all $f, g \in A \setminus \{0\}$, since the length is additive in short exact sequences and we have an exact sequence

$$0 \longrightarrow A/(f) \xrightarrow{g} A/(fg) \longrightarrow A/(g) \longrightarrow 0.$$

Since $k(X) = \operatorname{Quot}(A)$, the claim now follows with

$$\operatorname{ord}_Z\left(\frac{f}{g}\right) = \operatorname{ord}_Z(f) - \operatorname{ord}_Z(g) \quad \text{for all} \quad f, g \in A \setminus \{0\};$$

the additivity ensures that this is well-defined and gives a homomorphism. □

Example 1.7. If $\dim X = 1$, any codimension one subvariety has the form $Z = \{p\}$ for some closed point $p \in X$. The local ring along the subvariety is then the local ring $\mathscr{O}_{X,p}$ at that point, with residue field $\kappa(p) = \mathscr{O}_{X,p}/\mathfrak{m}_p$. Hence if $\kappa(p) = k$, then example 1.3 implies

$$\operatorname{ord}_Z(f) = \dim_k(\mathscr{O}_{X,p}/(f)) \quad \text{for all} \quad f \in \mathscr{O}_{X,p}.$$

To illustrate the behaviour of the vanishing order at a singular point, consider the cuspidal cubic $X = V(y^2 - x^3) \subset \mathbb{A}_k^2 = \operatorname{Spec} k[x,y]$. Let $Z = \{(0,0)\}$ be its singular point. Then

$$\begin{aligned}\operatorname{ord}_Z(x) &= \dim_k k[x,y]/(y^2-x^3,y) = \dim_k k[x]/x^3 = 3,\\ \operatorname{ord}_Z(y) &= \dim_k k[x,y]/(y^2-x^3,x) = \dim_k k[y]/(y^2) = 2.\end{aligned}$$

Hence the rational function $f = y/x \in k(X)^\times$ has

$$\operatorname{ord}_Z(f) = \operatorname{ord}_Z(y) - \operatorname{ord}_Z(x) = 1 > 0$$

even though clearly $f \notin \mathscr{O}_{X,p}$. Note that $\mathscr{O}_{X,p}$ is not a discrete valuation ring.

2 Cycles and rational equivalence

We can now define the algebraic analog of Borel-Moore homology. It generalizes the Weil divisor class group to subvarieties of higher codimension:

Definition 2.1. Let X be a scheme (not necessarily integral) and $d \in \mathbb{N}_0$.

a) A *d-cycle* on X is an element of the free abelian group

$$Z_d(X) := \bigoplus_{Z \subset X} \mathbb{Z} \cdot [Z]$$

over all subvarieties $Z \subset X$ of dimension $\dim Z = d$. We write d-cycles as finite formal sums

$$\alpha = \sum_{Z \subset X} n_Z(\alpha) \cdot [Z] \quad \text{with multiplicities} \quad n_Z(\alpha) \in \mathbb{N}_0.$$

b) For subvarieties $W \subset X$ of dimension $d+1$ and rational functions $f \in k(W)^\times$ we consider the cycle

$$\operatorname{cyc}(f) := \sum_{Z \subset W} \operatorname{ord}_Z(f) \cdot [Z] \in Z_d(W) \subset Z_d(X)$$

where the sum runs over all subvarieties $Z \subset W$ with $\operatorname{codim}_W Z = 1$. We say that a cycle $\alpha \in Z_d(X)$ is *rationally equivalent to zero* if there exists a finite collection of subvarieties $W_1, \dots, W_r$ of dimension $d+1$ and rational functions $f_i \in k(W_i)^\times$ such that

$$\alpha = \sum_{i=1}^r \operatorname{cyc}(f_i).$$

c) The cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}_d(X) \subset Z_d(X)$ since we have $\operatorname{cyc}(1/f) = -\operatorname{cyc}(f)$. The quotient $A_d(X) := Z_d(X)/\operatorname{Rat}_d(X)$ is called the *d-th Chow group of* X. We put

$$Z_*(X) := \bigoplus_{d \geq 0} Z_d(X) \quad \text{and} \quad A_*(X) := \bigoplus_{d \geq 0} A_d(X).$$

d) Two cycles $\alpha, \beta \in Z_d(X)$ are called *rationally equivalent* if they have the same image in the Chow group, i.e. if $\alpha - \beta \in \mathrm{Rat}_d(X)$. We then write $\alpha \sim \beta$.

Note that since $Z_*(X)$ and $A_*(X)$ are defined in terms of integral subschemes, the only depend on the underlying reduced closed subscheme $X^{\mathrm{red}} \subset X$. For a more visual explanation of rational equivalence, we need a notion of fundamental cycles for arbitrary (not necessarily integral) subschemes:

Definition 2.2. For any subscheme $Z \subset X$ with irreducible components $Z_1, \dots, Z_r$ its *fundamental cycle* is defined by

$$[Z] \;:=\; \sum_{i=1}^{r} m_i \cdot [Z_i] \;\in\; Z_d(X). \quad \text{where} \quad m_i \;:=\; \ell_{\mathscr{O}_{Z,Z_i}}(\mathscr{O}_{Z,Z_i}) \;\in\; \mathbb{N}.$$

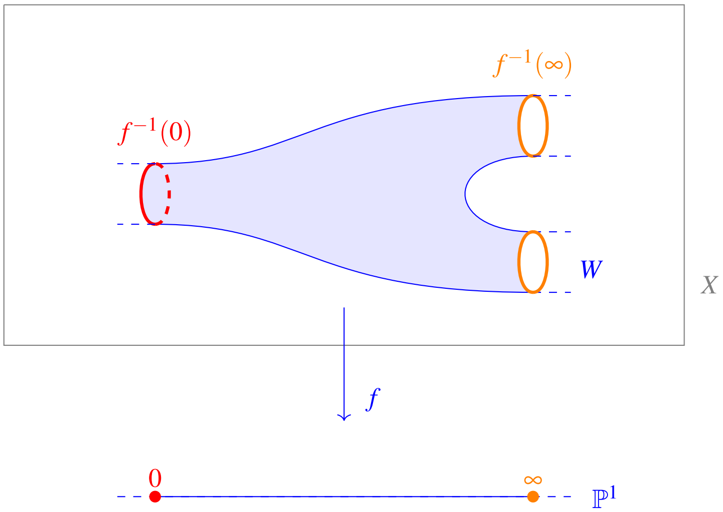
Its image in the Chow group $A_\bullet(X)$ is called the *fundamental class* of Z in X.

This leads to the following interpretation of the notion of rational equivalence via morphisms to the projective line:

Example 2.3. For any subvariety $W \subset X$ and any rational function $f \in k(W)^\times$ the definitions imply

$$\mathrm{cyc}(f) \;=\; [f^{-1}(0)] - [f^{-1}(\infty)]$$

where $f^{-1}(0), f^{-1}(\infty) \subset W$ denote the scheme-theoretic fibers of $f \colon W \to \mathbb{P}^1$. We can therefore regard rational equivalence as a way of saying that two cycles are related by an algebraic deformation over $\mathbb{P}^1$ as indicated in the following picture:



By construction $A_d(X) = 0$ for $d \notin \{0, 1, \dots, \dim(X)\}$, and the Chow group in the top degree is freely generated by irreducible components:

Example 2.4. Let $n = \dim X$, and let $X_1, \dots, X_r$ be the irreducible components of X of dimension n. Then

$$A_n(X) \;=\; Z_n(X) \;=\; \bigoplus_{i=1}^{r} \mathbb{Z} \cdot [X_i] \quad \text{since} \quad \mathrm{Rat}_n(X) \;=\; 0.$$

So the top degree Chow group of a variety looks like the top degree Borel-Moore homology (= degree zero cohomology) of a complex manifold. At the opposite end, the degree zero Chow groups of affine and projective spaces also look like degree zero Borel-Moore homology (= top degree cohomology):

Lemma 2.5. *For any $n > 0$ we have $A_0(\mathbb{A}^n) \simeq 0$ and $A_0(\mathbb{P}^n) \simeq \mathbb{Z}$.*

Proof. Let $p \in \mathbb{A}^n$ be a closed point, and let $q = \pi(p) \in \mathbb{A}^{n-1}$ be its image under the projection

$$\pi\colon \quad \mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}).$$

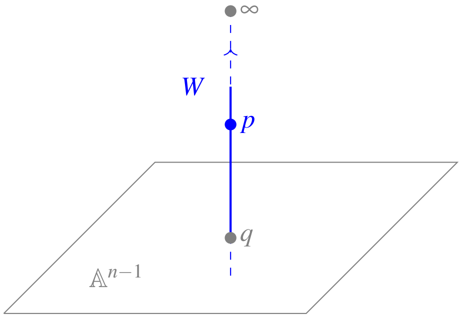
The subvariety $W = \pi^{-1}(q) \subset \mathbb{A}^n$ is a line over the residue field $\kappa = \kappa(q)$, indeed we have

$$W \;\simeq\; \mathrm{Spec}\, \kappa[x_n].$$

Now p is a closed point on this line, hence we have $\{p\} = V(f) \subset \mathrm{Spec}\, \kappa[x_n]$ for some $f \in \kappa[x_n] = \mathscr{O}_W(W) \subset k(W)$. Then

$$[p] \;=\; \mathrm{cyc}(f) \;\in\; \mathrm{Rat}_0(\mathbb{A}^n) \;\subset\; Z_0(\mathbb{A}^n).$$

Hence the fundamental cycle of any point in affine space is rationally equivalent to zero. Since $A_0(\mathbb{A}^n)$ is generated by such cycles, we get $A_0(\mathbb{A}^n) = 0$. Note that this is an algebraic version of a homotopy argument: We are "deforming" a point in $\mathbb{A}^n$ along a line W to a point at infinity as one would do in Borel-Moore homology:



Essentially the same argument works in the projective case: Let $p \in \mathbb{P}^n$ be a closed point different from $\infty := [0 : \cdots : 0 : 1]$. Up to a permutation of the first n coordinates, we may assume the given point lies in the affine chart $\mathbb{A}^n \subset \mathbb{P}^n$ of points with coordinates $[1 : x_1 : \cdots : x_n]$. Let $W \subset \mathbb{A}^n$ be the affine line constructed previously. Its closure is a projective line given by $\overline{W} = W \cup \{\infty\}$. The function f from above is still a rational function on this projective line, but now it has a pole at the point ∞, indeed $\mathrm{cyc}(f) = [p] - d \cdot [\infty]$ where $d = [\kappa : k]$ is the degree of the residue field extension. This shows that the homomorphism

$$\mathbb{Z} \longrightarrow A_0(\mathbb{P}^n), \quad m \mapsto m \cdot [\infty]$$

is surjective. It is then an isomorphism: The cycle $m \cdot [\infty]$ is rationally equivalent to zero only for $m = 0$. Indeed, otherwise there would exist a curve $W \subset \mathbb{P}^n$ and principal divisors on each irreducible component of the curve such that the sum of the principal divisors (taken over all components of W) is $m \cdot [\infty]$. But the degree of principal divisors is zero on each component, so this can happen only if $m = 0$. □

The simplicity of the above example is misleading, in general the Chow groups of complex varieties capture much finer geometric information than Borel-Moore homology, even in degree zero! This becomes visible already for smooth projective curves X of genus $g > 0$, where $A_0(X) \simeq \mathrm{Pic}(X)$. More generally we have:

Example 2.6. Let X be a variety of dimension n. Then $Z_{n-1}(X)$ is the group of Weil divisors on the variety, and a Weil divisor is rationally equivalent to zero iff it is a principal divisor. Hence $A_{n-1}(X)$ is the Weil divisor class group. In particular, we get a homomorphism

$$c_1 : \quad \mathrm{Pic}(X) \longrightarrow A_{n-1}(X)$$

by sending a Cartier divisor to the associated Weil divisor. From the comparison between Cartier divisors and Weil divisors in algebraic geometry, we know that this homomorphism $c_1 : \mathrm{Pic}(X) \to A_{n-1}(X)$ is

- injective if X is normal,
- an isomorphism if X locally factorial.

The assumptions of normality resp. local factoriality cannot be dropped:

Exercise 2.7. Show that for the cuspidal cubic $X = V_+(y^2z - x^3) \subset \mathbb{P}^2$ over any algebraically closed field k with $\mathrm{char}\, k \neq 2, 3$, the Picard group fits into an exact sequence

$$0 \to k^\times \to \mathrm{Pic}(X) \to A_0(X) \to 0 \quad \text{with} \quad A_0(X) \simeq \mathbb{Z}.$$

Exercise 2.8. Show that on the cone $X = V(z^2 - xy) \subset \mathbb{A}^3$ the line $Z = V(x, z)$ is a Weil divisor that does not come from a Cartier divisor, and deduce that for this cone we have

$$\mathrm{Pic}(X) \simeq 0 \quad \text{but} \quad A_1(X) \simeq \mathbb{Z}/2\mathbb{Z}.$$

3 Localization and Mayer-Vietoris

In algebraic topology one often computes homology groups by restriction to open subsets, for instance using the long exact sequence of a pair or the Mayer-Vietoris sequence. We now develop similar techniques for Chow groups of a scheme X.

Proposition 3.1 (Localization sequence). *Let $i\colon Y \hookrightarrow X$ be a closed subscheme, and denote the open embedding of its complement by $j\colon U = X \setminus Y \hookrightarrow X$. Then for any $d \geq 0$ we have natural homomorphisms i_*, j^* between Chow groups that fit into the exact sequence*

$$A_d(Y) \xrightarrow{i_*} A_d(X) \xrightarrow{j^*} A_d(U) \longrightarrow 0.$$

Proof. We have a natural inclusion $Z_d(Y) \subset Z_d(X)$ since any subvariety $Z \subset Y$ is also a subvariety of X. In the same way the definitions imply that for the subgroups of cycles rationally equivalent to zero we have $\mathrm{Rat}_d(Z) \subset \mathrm{Rat}_d(Y)$. So i induces a group homomorphism

$$i_*\colon \quad A_d(Z) \longrightarrow A_d(X) \quad \text{given on generators by} \quad [Z] \mapsto [i(Z)].$$

To define the pullback under the open immersion $j\colon U \hookrightarrow X$, consider the group homomorphism

$$j^*\colon \quad Z_d(X) \longrightarrow Z_d(U) \quad \text{given on generators by} \quad [Z] \mapsto [Z \cap U].$$

We need to verify that this homomorphism preserves rational equivalence in the sense that $j^*(\mathrm{Rat}_d(X)) \subset \mathrm{Rat}_d(U)$. We check this on generators: Let $Z_1, Z_2 \subset X$ be two subvarieties which define rationally equivalent cycles. By definition there exists a subvariety $W \subset X$ of dimension $d+1$ and a rational function $f \in k(W)^\times$ with $\mathrm{cyc}(f) = [Z_1] - [Z_2]$. If $W \cap U = \varnothing$, then also $Z_1 \cap U = Z_2 \cap U = \varnothing$ and there is nothing to show. Hence we may assume that $W \cap U \neq \varnothing$. In this case $W \cap U$ is an open dense subset of W and by restriction of rational functions to this open dense subset we get a commutative diagram

$$\begin{tikzcd}
k(W)^\times \arrow[r, "\sim"] \arrow[d, "\mathrm{cyc}"'] & k(W \cap U)^\times \arrow[d, "\mathrm{cyc}"] \\
Z_d(W) \arrow[r, "j^*"] & Z_d(W \cap U)
\end{tikzcd}$$

Thus $j^*(\mathrm{Rat}_d(X)) \subset \mathrm{Rat}_d(U)$. So we get a homomorphism $j^*\colon A_d(X) \to A_d(U)$ on Chow groups induced by the above one on cycles.

It remains to show exactness of the localization sequence. Surjectivity of j^* is clear since for any subvariety $Z \subset U$ its closure $\overline{Z} \subset X$ is a subvariety of X such that $j^*[Z] = [U]$. For the exactness in the middle of the localization sequence, we clearly have the inclusion $\mathrm{im}(i_*) \subset \ker(j^*)$. Conversely, let $\alpha \in Z_d(X)$ be a cycle with $j^*(\alpha) \sim 0$. By definition there are subvarieties $W_\nu \subset U$ of dimension $d+1$ and rational functions $f_\nu \in k(W_\nu)^\times$ such that

$$j^*(\alpha) \; = \; \sum_\nu \mathrm{cyc}(f_\nu).$$

The closure of W_ν is a subvariety $\overline{W}_\nu \subset X$ such that $k(\overline{W}_\nu) \simeq k(W_\nu)$. Let $\overline{f}_\nu \in k(\overline{W}_\nu)$ be the unique rational function extending f_ν. Then as in the previous commutative square $j^*(\mathrm{cyc}(\overline{f}_\nu)) = \mathrm{cyc}(f_\nu)$, so

$$\beta \; := \; \alpha - \sum_\nu \mathrm{cyc}(\overline{f}_\nu) \; \in \; Z_d(X)$$

is rationally equivalent to α and $j^*(\beta) = 0$. The latter means that β is a cycle whose intersection with $U \subset X$ is zero, hence β is a linear combination of subvarieties of $Y = X \setminus U$. So $\beta \in i_*(Z_d(Y))$, i.e. in Chow groups $\alpha \in i_*(A_d(Y))$. □

Example 3.2. In general $i_* \colon A_d(Y) \longrightarrow A_d(X)$ need not be injective:

a) For $i \colon Y = \{0\} \hookrightarrow X = \mathbb{A}^1$ we have $i_* \colon A_0(Y) = \mathbb{Z} \longrightarrow A_0(X) = 0$.

b) For smooth curves $i \colon Y \hookrightarrow X = \mathbb{P}^2$ we have $i_* \colon A_0(Y) = \mathrm{Pic}(Y) \to A_0(X) = \mathbb{Z}$.

c) More generally, let $i \colon Y \hookrightarrow X = \mathbb{P}^n$ be a reduced hypersurface and $Y_1, \dots, Y_r \subset \mathbb{P}^n$ its irreducible components. We have natural identifications

$$A_{n-1}(Y) \; = \; \bigoplus_i \mathbb{Z} \cdot [Y_i] \; \simeq \; \mathbb{Z}^r \quad \text{and} \quad A_{n-1}(X) \; = \; \mathrm{Pic}(\mathbb{P}^n) \; \simeq \; \mathbb{Z}$$

such that the following diagram commutes:

$$\begin{tikzcd} A_{n-1}(Y) \arrow[r, "i_*"] \arrow[d, equal, "\wr"] & A_{n-1}(X) \arrow[d, equal, "\wr"] \\ \mathbb{Z}^r \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Thus i_* cannot be injective for $r > 1$. Explicitly, the map at the bottom of the diagram is given by $(a_1, \dots, a_r) \mapsto a_1 d_1 + \dots + a_r d_r$ where $d_i = \deg(Y_i)$. By the localization sequence the top Chow group of the complement of the reduced hypersurface $Y \subset \mathbb{P}^n$ is then given by

$$A_n(\mathbb{P}^n \setminus Y) \; \simeq \; \mathbb{Z}/(d_1, \dots, d_r).$$

Remark 3.3. In Borel-Moore homology for varieties over $k = \mathbb{C}$, one has a long exact sequence

$$\cdots \; \to \; H_i^{BM}(Z) \; \to \; H_i^{BM}(X) \; \to \; H_i^{BM}(U) \; \to \; H_{i-1}^{BM}(Z) \; \to \; \cdots$$

As the above example shows, the Chow groups do *not* fit into such a long exact sequence. One can extend the localization sequence on Chow groups to a long exact sequence via the higher Chow groups introduced by Bloch.

Similarly to the localization sequence, we obtain the following analog of the Mayer-Vietoris sequence in topology:

Proposition 3.4 (Mayer-Vietoris). *Let X_1, X_2 be closed subschemes of X. Then we have an exact sequence*

$$A_d(X_1 \cap X_2) \longrightarrow A_d(X_1) \oplus A_d(X_2) \longrightarrow A_d(X_1 \cup X_2) \longrightarrow 0 \quad \textit{for each} \quad d \geq 0.$$

Proof. Consider the subgroups $Z_d(X_\nu) \subset Z_d(X_1 \cup X_2)$ and $\mathrm{Rat}_d(X_\nu) \subset \mathrm{Rat}_d(X_1 \cup X_2)$ for $\nu = 1, 2$. Since subvarieties are by definition irreducible, every subvariety of the union $X_1 \cup X_2$ is contained in X_1 or X_2 (or both). So the definition of cycles and rational equivalence imply

- $Z_d(X_1 \cup X_2) = Z_d(X_1) + Z_d(X_2)$,
- $\mathrm{Rat}_d(X_1 \cup X_2) = \mathrm{Rat}_d(X_1) + \mathrm{Rat}_d(X_2)$,

where the right hand side denotes the sum of subgroups inside the given ambient group. Hence the difference map

$$\Delta\colon \quad Z_d(X_1) \oplus Z_d(X_2) \twoheadrightarrow Z_d(X_1 \cup X_2), \quad (\alpha, \beta) \mapsto \alpha - \beta$$

is surjective and restricts to a surjection Δ_{Rat} between subgroups of cycles rationally equivalent to zero as shown in the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \ker(\Delta_{\mathrm{Rat}}) \arrow[r] \arrow[d, "\exists\varphi"] & \mathrm{Rat}_d(X_1) \oplus \mathrm{Rat}_d(X_2) \arrow[r, "\Delta_{\mathrm{Rat}}"] \arrow[d, hook] & \mathrm{Rat}_d(X_1 \cup X_2) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & Z_d(X_1 \cap X_2) \arrow[r] & Z_d(X_1) \oplus Z_d(X_2) \arrow[r, "\Delta"] & Z_d(X_1 \cup X_2) \arrow[r] & 0
\end{tikzcd}$$

The snake lemma then gives an exact sequence

$$0 \longrightarrow \operatorname{coker}(\varphi) \longrightarrow A_d(X_1) \oplus A_d(X_2) \longrightarrow A_d(X_1 \cup X_2) \longrightarrow 0$$

The claim now follows from the fact that $\operatorname{coker}(\varphi)$ is a quotient of $A_d(X_1 \cap X_2)$, which is clear since $R_d(X_1 \cap X_2) \subset \ker(\Delta_{\mathrm{Rat}})$. □

4 Example: Affine bundles

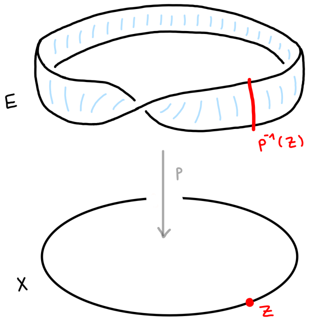
As an application of the localization sequence, let us take a look at the Chow groups of affine bundles. By an *affine bundle of rank n* we mean a morphism $p\colon E \to X$ of schemes such that there is a cover $X = \bigcup_i U_i$ by open subsets $U_i \subset X$ over which we have isomorphisms

$$p^{-1}(U_i) \simeq \mathbb{A}^n \times U_i \quad \text{for all} \quad i.$$

If the transition morphisms between different charts are linear in the sense that they are given by morphisms $U_i \cap U_j \to \mathrm{GL}_n$, then we call $p\colon E \to X$ a *vector bundle* or GL_n-bundle. More generally, one could impose that the transition maps only lie in the group of affine-linear transformations $\mathrm{Aff}_n = (\mathbb{A}^n, +) \rtimes \mathrm{GL}_n$ and then call p an Aff_n-bundle. In what follows we need neither of these assumptions, we use the most general notion of affine bundles where the transition functions can be arbitrary automorphisms. The following is a special case of the flat pullback that we will discuss in more detail later:

Definition 4.1. Let $p\colon E \to X$ be an affine bundle. For any subvariety $Z \subset X$ of dimension d the preimage $p^{-1}(Z) \subset E$ is a subvariety of dimension $d+n$, so we get a homomorphism

$$p^*\colon \quad Z_d(X) \longrightarrow Z_{d+n}(E), \quad [Z] \mapsto [p^{-1}(Z)].$$



If $W \subset X$ is a subvariety of dimension $d+1$, then the preimage $V = p^{-1}(W)$ is a subvariety of dimension $d+n+1$, and one easily checks that we have a commutative diagram

$$\begin{tikzcd}
k(W)^\times \arrow[r, "p^*"] \arrow[d, "\mathrm{cyc}"'] & k(V)^\times \arrow[d, "\mathrm{cyc}"] \\
Z_d(X) \arrow[r, "p^*"] & Z_{d+n}(Y)
\end{tikzcd}$$

Hence $p^*(\mathrm{Rat}_d(X)) \subset \mathrm{Rat}_{d+n}(E)$ and the pullback on cycles descends to a natural homomorphism

$$p^*\colon \quad A_d(X) \longrightarrow A_{d+n}(E).$$

If $n > 1$, then the homomorphism $Z_d(X) \to Z_{d+n}(E)$ is clearly not surjective: Not every cycle on an affine bundle comes by pullback from the base space. But every cycle can be *deformed* into one that comes by pullback from the base:

Theorem 4.2. *Let $p\colon E \to X$ be an affine bundle of rank n. Then for all $d \in \mathbb{Z}$ the homomorphism*

$$p^*\colon \quad A_d(X) \twoheadrightarrow A_{d+n}(E) \quad \textit{is surjective.}$$

Proof. We first reduce to the case of a trivial bundle: By definition of affine bundles we may pick a closed subscheme $Y \subset X$ such that $p^{-1}(U) \simeq \mathbb{A}^n \times U$ over the open subset $U = X \setminus Y \subset X$. The localization sequences for the open embedding $U \hookrightarrow X$ and for the embedding of its preimage $E_U = p^{-1}(U) \hookrightarrow E$ fit into a commutative diagram

$$\begin{tikzcd}
A_d(Y) \arrow[r] \arrow[d, "p_Y^*"'] & A_d(X) \arrow[r] \arrow[d, "p^*"'] & A_d(U) \arrow[r] \arrow[d, "p_U^*"'] & 0 \\
A_d(E_Y) \arrow[r] & A_d(E) \arrow[r] & A_d(E_U) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where $p_Y \colon E_Y = p^{-1}(Y) \to Y$ and $p_U \colon E_U = p^{-1}(U) \to U$ are again affine bundles. By induction on the dimension we may assume that p_Y^* is surjective, and by the diagram it then only remains to show p_U^* is surjective.

Replacing X be U we may hence assume $E \simeq \mathbb{A}^n \times X$. In this case, the projection factors as

$$E \simeq \mathbb{A}^n \times X \longrightarrow \mathbb{A}^{n-1} \times X \longrightarrow \cdots \longrightarrow \mathbb{A}^1 \times X \longrightarrow ;X$$

where each step is a trivial bundle of rank one. So we may assume $n = 1$ and use a similar argument as in lemma 2.5: We want to show that for any subvariety $Z \subset E$ of dimension $d+1$, its class is rationally equivalent to a class that comes by pullback from X. Replacing X by the closure of $f(Z)$, we may assume X is a variety and the morphism $p \colon Z \to X$ is dominant. There are then two cases:

Either $\dim X = \dim Z - 1$. In this case $Z = E$, so $[Z] = p^*[X]$ and we are done. Or $\dim X = \dim Z$. In this case $\dim E > \dim Z$, so Z cannot contain the generic fiber of the projection $p \colon E = \mathbb{A}^1 \times X \to X$. By base change to the function field $K = k(X)$ we then see that

$$Z_K \;=\; Z \times_X \operatorname{Spec} K \;\subsetneq\; E_K \;=\; \operatorname{Spec} K[t]$$

is a proper closed subscheme, hence of the form $Z_K = V(f) \subset \operatorname{Spec} K[t]$ for some polynomial $f \in K[t]$ with $\deg f > 0$. View f as a rational function on the variety E, then we obtain for the corresponding cycle that $\operatorname{cyc}(f) = Z + Z'$ where $Z' \in Z_{d+1}(E)$ is a linear combination of subvarieties which do not meet the generic fiber. Thus every component of Z' maps to a subvariety of dimension $\leq d$ in X and therefore has positive-dimensional fibers over its image. Since $p \colon E \to X$ has relative dimension one, it follows that Z' is the pullback of a cycle on X, and again we are done. □

If $p \colon E \to X$ is a vector bundle, we will later deduce from the theory of Chern classes that the epimorphism p^* in the above theorem is an isomorphism. For the trivial bundle $E = X \times \mathbb{A}^n$ this can be seen as a *homotopy invariance* for Chow groups, analogous to the homotopy invariance for Borel-Moore homology. Before proceeding further, let us note the following trivial consequence:

Corollary 4.3. *We have*

$$A_d(\mathbb{A}^n) \simeq \begin{cases} \mathbb{Z} & \textit{for } d = n, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Clearly $A_i(\operatorname{Spec} k)$ is isomorphic to $\mathbb{Z}$ for $i = 0$ and zero otherwise. So the result follows from the previous theorem for the affine bundle $p\colon \mathbb{A}^n \to \operatorname{Spec} k$. $\square$

5 Proper pushforward

In the localization sequence we have seen that any closed immersion gives rise to a pushforward functor on Chow groups. The following example shows that there is no way to make Chow groups into a covariant functor for arbitrary morphisms:

Example 5.1. Let $i\colon \operatorname{Spec} k \hookrightarrow \mathbb{A}^1$ be the embedding of a point and $p\colon \mathbb{A}^1 \to \operatorname{Spec} k$ the structure morphism. Then $p \circ i = id$. If Chow groups can be made a covariant functor for arbitrary morphisms, it follows that the identity map on $A_0(\operatorname{Spec} k) = \mathbb{Z}$ factors as

$$\begin{tikzcd} A_0(\operatorname{Spec} k) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & A_0(\mathbb{A}^1) \arrow[r, "p_*"] & A_0(\operatorname{Spec} k) \end{tikzcd}$$

which is absurd since $A_0(\mathbb{A}^1) \simeq 0$ by lemma 2.5, whereas $A_0(\operatorname{Spec} k) \simeq \mathbb{Z} \neq 0$.

The problem is that the obvious pushforward $p_*\colon Z_0(\mathbb{A}^1) \to Z_0(\operatorname{Spec} k)$ does not respect rational equivalence: On the affine line we can move points to infinity as in lemma 2.5. This is analogous to the case of Borel-Moore homology which also is not a covariant functor for all continuous maps. But Borel-Moore homology is a covariant functor for all *proper* maps: Continuous maps such that the preimage of any compact set is compact. We will see the same holds for Chow groups.

Recall that a morphism of schemes is *proper* if it is separated, of finite type and universally closed. Closed immersions and projective morphisms are proper, while affine morphisms are not proper unless they are finite.

Definition 5.2. Let $f\colon X \to Y$ be a proper morphism. For any subvariety $Z \subset X$, the properness implies that the image $f(Z) \subset Y$ is again closed, and we endow it with the reduced subscheme structure. Then for the dominant morphism $f\colon Z \twoheadrightarrow W = f(Z)$ of varieties we have:

- Either $\dim W < \dim X$, in which case we put $\deg(Z/W) = 0$.
- Or $\dim W = \dim X$, in which case $f\colon Z \to W$ is a generically finite morphism and we denote its generic degree by $\deg(Z/W) = [k(Z) : k(W)] \in \mathbb{N}$.

We define $f_*[Z] := \deg(Z/W) \cdot [W]$ and extend this definition linearly to a group homomorphism

$$f_*\colon \quad Z_d(X) \longrightarrow Z_d(Y).$$

Remark 5.3. The pushforward on cycles is a functor: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two proper morphisms, then we have

$$(g \circ f)_* = g_* \circ f_*\colon \quad A_d(X) \longrightarrow A_d(Z)$$

as one sees immediately from the transitivity of degrees in field extensions.

Since properness is stable under base change, a proper morphism $f\colon X \to Y$ has all fibers $f^{-1}(y) \to \operatorname{Spec} \kappa(y)$ proper. Intuitively, this means that the fibers do not allow for deformations of points to "extra points at infinity" as in example 5.1, and indeed the proper pushforward descends to Chow groups:

Theorem 5.4. *Let $f\colon X \to Y$ be a proper morphism. Then $f_*(\mathrm{Rat}_d(X)) \subset \mathrm{Rat}_d(Y)$, hence we obtain an induced homomorphism*

$$f_*\colon \quad A_d(X) \longrightarrow A_d(Y) \quad \textit{for each} \quad d \geq 0.$$

Proof. By definition, any cycle rationally equivalent to zero is a sum of cycles of the form $\mathrm{cyc}(r)$ where $r \in k(W)^\times$ is a rational function on a subvariety $W \subset X$ of dimension $d+1$. We must show that each cycle of the form $f_*(\mathrm{cyc}(r))$ is rationally equivalent to zero. Replacing X by W and replacing Y by the underlying reduced closed subscheme of $f(W)$, we are left with the following

Claim. *Let $f\colon X \to Y$ be a proper surjective morphism of varieties. Then for any rational function $r \in k(X)^\times$ we have*

$$f_*(\mathrm{cyc}(r)) = \begin{cases} 0 & \textit{if} \dim Y < \dim X, \\ \mathrm{cyc}(N_{L/K}(r)) & \textit{if} \dim Y = \dim X, \end{cases}$$

where $N_{L/K}: L \to K$ denotes the norm of the field extension $K = k(Y) \subset L = k(X)$.

We will divide the proof of this claim in two parts, dealing first with the case where f is generically finite and then with the case where $\dim Y < \dim X$.

Case 1: $\dim Y = \dim X$. Here f is generically finite. We want to compare the multiplicity with which a given subvariety $Z \subset Y$ of codimension one enters in the cycles $f_*(\mathrm{cyc}(r))$ and $\mathrm{cyc}(N_{L/K}(f))$. This can be done locally near the generic point of Z. Note that for dimension reasons the morphism f has finite fibers over the generic point of Z, because $\mathrm{codim}_Y(Z) = 1$ and $\dim Y = \dim X$. So after shrinking Y to a neighborhood of the generic point of Z we may assume that $f\colon X \to Y$ has finite fibers. From algebraic geometry we know that any proper morphism with finite fibers is a finite morphism, so we may assume $f\colon X \to Y$ is a finite morphism.

Shrinking Y further, we may assume $Y = \operatorname{Spec} A_0$ is affine. Then $X = \operatorname{Spec} B_0$ for a finite A_0-algebra. Let $A = \mathscr{O}_{Y,Z}$ be the local ring along the given subvariety, then the base change

$$B \;=\; B_0 \otimes_{A_0} A$$

is a domain with quotient field $B \otimes_A K = L$. The ring extension $A \subset B$ is finite and we have a bijection between

- maximal ideals $\mathfrak{m}_i \trianglelefteq B$, and
- subvarieties $V_i \subset X$ dominating Z,

such that the corresponding local rings are given by $B_{\mathfrak{m}_i} \simeq \mathscr{O}_{X,V_i}$. What we need to show is

$$\sum_i \operatorname{ord}_{V_i}(r) \cdot [k(V_i) : k(Z)] \;=\; \operatorname{ord}_Z(N_{L/K}(r)),$$

indeed the left hand side is by definition the multiplicity of Z in the cycle $f_*(\operatorname{cyc}(r))$ and the right hand side is the one in $\operatorname{cyc}(N_{L/K}(r))$. To check the above equality, notice that both sides are additive with respect to factorizations of r as a product of rational functions. Since $k(Y) = \operatorname{Quot}(B)$, it will therefore suffice to show the above equality when $r \in B$. In this case, the left hand side of the equality can be rewritten as

$$\sum_i \operatorname{ord}_{V_i}(r) \cdot [k(V_i) : k(Z)] \;=\; \sum_i \ell_{B_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}/(r)) \cdot [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \;=\; \ell_A(B/(r))$$

for the maximal ideal $\mathfrak{m} \trianglelefteq A$ as one easily sees using the definition of length via composition series. On the other hand, the right hand side of the desired equality can be rewritten as

$$\operatorname{ord}_Z(N_{L/K}(r)) \;=\; \operatorname{ord}_Z(\det(\varphi)) \quad \text{for} \quad \varphi \;=\; (x \mapsto rx) \;\in\; \operatorname{End}_K(L).$$

by definition of the norm. The claim then boils down to $\ell_A(B/(r)) = \operatorname{ord}(\det(\varphi))$, and this is shown in the appendix in corollary 8.6.

Case 2: $\dim X > \dim Y$. In this case we want to show that $f_*(\operatorname{cyc}(r)) = 0$ for any given rational function $r \in k(X)^\times$. This is trivial if $\dim X > \dim Y + 1$, since $\operatorname{cyc}(r)$ is a cycle of codimension one on X. Hence in what follows we may assume that $\dim X = \dim Y + 1$. By definition of the pushforward on cycles then

$$f_*(\operatorname{cyc}(r)) \;=\; \sum_V \operatorname{ord}_V(r) \cdot \deg(V/Y) \cdot [Y]$$

where the sum runs over all subvarieties $V \subset X$ dominating Y. Since the sum on the right hand side is a multiple of the fundamental class $[Y]$, we only need to show the scalar identity

$$\sum_V \operatorname{ord}_V(r) \cdot \deg(V/Y) \;=\; 0.$$

For this we may replace $X \to Y$ by its generic fiber $X_K \to \operatorname{Spec} K$. Thus we may assume that $Y = \operatorname{Spec} K$ is the spectrum of a field. By our dimension assumption then X is a curve over K. Consider now the normalization $n\colon \tilde{X} \to X$. Since $\tilde{X}$ is a smooth curve, we may find a finite morphism $g\colon \tilde{X} \to \mathbb{P}^1_K$. We get a commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "n"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
\mathbb{P}^1_K \arrow[r, "p"] & Y = \operatorname{Spec} K
\end{tikzcd}$$

Let $\tilde{r} \in k(\tilde{X})^\times$ be the rational function corresponding to r via $k(\tilde{X}) \simeq k(X)$. Then by functoriality of the pushforward on cycles

$$f_*(\mathrm{cyc}(r)) \;=\; f_* n_*(\mathrm{cyc}(\tilde{r})) \;=\; p_* g_*(\mathrm{cyc}(\tilde{r})) \;=\; p_*(N(\tilde{r}))$$

where in the last step we have used the norm $N : k(\tilde{X})^\times \to k(\mathbb{P}^1_K)^\times$ and the result for generically finite morphisms from step 1. It then only remains to observe that for any $s \in k(\mathbb{P}^1_K)^\times$ we have $p_*(\mathrm{cyc}(s)) = 0$, which is clear by writing s as a quotient of two homogenous polynomials of the same degree. □

In the last step of the above argument, we have reproven the well-known fact that any principal divisor on a smooth projective curve has degree zero. Similarly we can count points on any proper scheme:

Definition 5.5. Let $f\colon X \to \operatorname{Spec} k$ be proper. Then the *degree homomorphism* is defined as the composite of the pushforward f_* and the identification $A_0(\operatorname{Spec} k) = \mathbb{Z}$ given by the fundamental class of a point:

$$\deg\colon \quad A_0(X) \xrightarrow{f_*} A_0(\operatorname{Spec} k) \;\simeq\; \mathbb{Z}, \qquad \sum_i n_i \cdot [p_i] \;\mapsto\; \sum_i n_i \cdot [\kappa(p_i) : k].$$

We extend this to $A_*(X)$ by precomposing with the projection $A_*(X) \twoheadrightarrow A_0(X)$ and also denote it by

$$\textstyle\int_X\colon \quad A_0(X) \longrightarrow \mathbb{Z}$$

to evoke the analogy with the integral of top forms in de Rham cohomology.

Corollary 5.6 (Bézout). *Let k be algebraically closed, and let $C, D \subset \mathbb{P}^2$ be reduced curves without common components. Then*

$$\sum_{p \in \mathbb{P}^2(k)} i_p(C,D) \;=\; \deg(C)\deg(D).$$

Proof. Both sides are additive with respect to the union of irreducible components, so we may assume C is irreducible and we can talk about its function field. Pick homogenous polynomials f, g of degree $\deg(f) = \deg(C)$ and $\deg(g) = \deg(D)$ such that

$$C = V_+(f),\; D = V_+(g) \;\subset\; \mathbb{P}^2$$

as closed subschemes. We first reduce to the case where D is a line:

Let h be a linear form such that the line $L = V_+(h) \subset \mathbb{P}^2$ intersects the curve C in finitely many points. Let $d = \deg(g)$. The rational function $g/h^d \in k(\mathbb{P}^2)^\times$ restricts to a well-defined rational function

$$r \;=\; (g/h^d)|_C \;\in\; k(C)^\times$$

because $L \cap C$ is finite, and

$$\begin{aligned}\operatorname{cyc}(r) \;&=\; \sum_{p \in C(k)} \operatorname{ord}_p(r) \cdot [p] \\ &=\; \sum_{p \in C(k)} (\ell(\mathscr{O}_{C,p}/(g)) - d \cdot \ell(\mathscr{O}_{C,p}/(h)) \cdot [p] \\ &=\; \sum_{p \in C(k)} (i_p(C,D) - d \cdot i_p(C,L)) \cdot [p]\end{aligned}$$

in $Z_0(C)$. Since $\operatorname{cyc}(r) \in Z_0(C)$ is rationally equivalent to zero and the degree map preserves rational equivalence, it follows that

$$0 \;=\; \sum_{p \in C(k)} (i_p(C,D) - d \cdot i_p(C,L)),$$

in other words

$$\sum_{p \in C(k)} i_p(C,D) \;=\; d \cdot \sum_{p \in C(k)} i_p(C,L) \quad \text{where} \quad d \;=\; \deg(D).$$

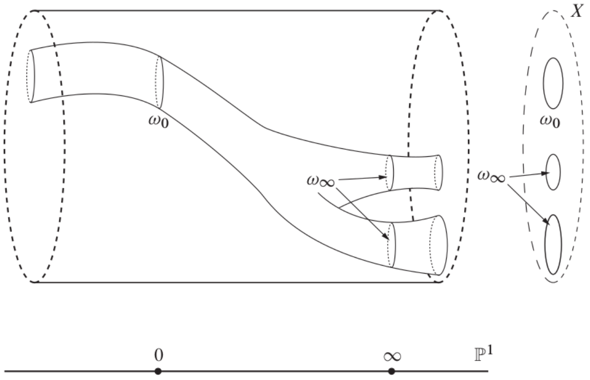
Thus it suffices to prove the claim when $D = L$ is a line. Repeating the argument with the roles of the two curves interchanged, we are left with the case where C and D are two distinct lines, in which case the result is trivial. □

Remark 5.7. In theorem 5.4, it is essential that proper morphisms are by definition separated. For instance, let X be the projective line with a doubled origin, i.e. the scheme obtained by glueing two copies of $\mathbb{P}^1$ along the open subset $\mathbb{P}^1 \setminus \{0\}$, and let $f\colon X \to \operatorname{Spec} k$ be the structure morphism. The function $r = x_1/x_0 \in k(X)^\times$ has a simple zero at each of the two copies of the origin $0_1, 0_2 \in X$ and a simple pole at infinity, so $\operatorname{cyc}(r) = [0_1] + [0_2] - [\infty]$ and hence $f_*[\operatorname{cyc}(r)] \not\sim 0$.

Another application of theorem 5.4 is the translation between two different views of rational equivalence: We have defined the notion of rational equivalence in terms of principal divisors on subvarieties of dimension $d+1$. We could also define it via families of subschemes of dimension d parametrized by the projective line: Given a subvariety $V \subset X \times \mathbb{P}^1$ such that the projection $p\colon V \to \mathbb{P}^1$ is dominant, we can view V as a family of subschemes

$$i\colon \quad V_t \;:=\; p^{-1}(t) \;\hookrightarrow\; X \times \{t\} \;=\; X \quad \text{for} \quad t \in \mathbb{P}^1(k)$$

as illustrated in the following picture (stolen from the book by Eisenbud-Harris):



This leads to the following alternative interpretation of rational equivalence:

Proposition 5.8. *A cycle $\alpha \in Z_d(X)$ is rationally equivalent to zero iff there exists a finite collection of subvarieties $V \subset X \times \mathbb{P}^1$ of dimension $d+1$ dominating $\mathbb{P}^1$ such that*

$$\alpha \;=\; \sum_V i_*\left([V_0] - [V_\infty]\right).$$

Proof. If $V \subset X \times \mathbb{P}^1$ is a subvariety of dimension $d+1$ dominating $\mathbb{P}^1$ via the second projection, then viewing that projection as a rational function $p \in k(V)^\times$ we have $[V_0] - [V_\infty] = \mathrm{cyc}(p) \in \mathrm{Rat}_d(V)$. So $i_*[V_0] - i_*[V_\infty] \in \mathrm{Rat}_d(X)$ by theorem 5.4, hence any cycle α of the form given above is rationally equivalent to zero.

Conversely, starting from our definition of rational equivalence, suppose we are given a rational function $r \in k(W)^\times$ on a subvariety $W \subset X$ of dimension $d+1$. We also write $r\colon W \to \mathbb{P}^1$ for the morphism defined by the rational function and denote by

$$V \;:=\; \overline{\mathrm{graph}(r)} \;\subset\; W \times \mathbb{P}^1 \;\subset\; X \times \mathbb{P}^1$$

the closure of its graph. The projection $q\colon V \to W$ is a birational proper morphism, hence we have

$$\mathrm{cyc}(r) \;=\; p_*[\mathrm{cyc}(r \circ q)] \;=\; [V_0] - [V_\infty].$$

Hence any cycle rationally equivalent to zero has the form in the proposition. □

6 Flat pullback

We have already seen two examples of a pullback between Chow groups: In the localization sequence we have taken the pullback under an open embedding, and for affine bundles we have defined the pullback under the projection to the base. Both are special cases of the flat pullback to be defined below.

Recall that a ring homomorphism $A \to B$ is said to be *flat* the functor $(-) \otimes_A B$ from A-modules to B-modules is exact. A morphism $f\colon X \to Y$ of schemes is *flat* if it has the following equivalent properties:

a) for every affine open subset $V \subset Y$ and every affine open $U \subset f^{-1}(V)$ the ring homomorphism
$$f^\sharp\colon \quad \mathscr{O}_Y(V) \longrightarrow \mathscr{O}_X(U) \quad \text{is flat.}$$

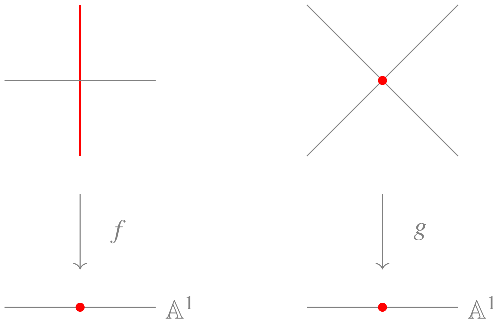
b) for every $x \in X$ and $y = f(x)$ the ring homomorphism $f^\sharp\colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat.

By the *local flatness criterion* from algebraic geometry, the flatness condition in b) is equivalent to
$$\operatorname{Tor}_1(\kappa(y), \mathscr{O}_{X,x}) \;=\; 0$$
where $\kappa(y) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$ and the Tor functor is taken over the local ring $\mathscr{O}_{Y,y}$.

Example 6.1. Let $X = V(xy) \subset \mathbb{A}^2$ be the union of the coordinate axes in the affine plane. Then

- $f\colon X \longrightarrow \mathbb{A}^1$ given by $g(x,y) = x$ is not flat,
- $g\colon X \longrightarrow \mathbb{A}^1$ given by $f(x,y) = x + y$ is flat.

Note that $g^{-1}(0) \simeq \operatorname{Spec} k[x]/(x^2)$, so flat morphisms may have some non-reduced fibers. The failure of flatness in the second example is due to the jump of the fiber dimension over the origin:



In algebraic geometry one shows that a morphism from a reduced scheme to a smooth curve is flat iff every irreducible component of the source dominates that curve. Moreover, for any flat morphism $f\colon X \to Y$ of schemes and any $x \in X$ with image $y = f(x)$ we have
$$\dim_x X \;=\; \dim_x f^{-1}(y) + \dim_y Y$$
where the *local dimension* of X at x is defined by $\dim_x X = \dim \mathscr{O}_{X,x}$. For reducible schemes the fibers may still have irreducible components of different dimension even if the source of the morphism is connected:

Example 6.2. The scheme $X = V(xy, xz) \subset \mathbb{A}^3 = \operatorname{Spec} k[x,y,z]$ is connected and the morphism

$$f\colon X \to \mathbb{A}^1, (x,y,z) \mapsto x+y$$

is easily seen to be flat. Its fiber over $c \in \mathbb{A}^1(k)$ is given as a set by

$$f^{-1}(p) \;=\; \{(c,0,0)\} \cup \{(0,c,t) \mid t \in \mathbb{A}^1\}.$$

Definition 6.3. A scheme X is *equidimensional* (also called *pure dimensional*) if all its irreducible components have the same dimension. A morphism $f : X \to Y$ is called *equidimensional of relative dimension n* if for any subvariety $V \subset Y$ its preimage

$$W \;=\; f^{-1}(V) \quad \text{is equidimensional with} \quad \dim W \;=\; \dim V + n.$$

Flat morphisms between varieties are automatically equidimensional:

Proposition 6.4. *Let $f\colon X \to Y$ be a flat morphism, where Y is irreducible and X is an equidimensional scheme. Then any base change of f is equidimensional of relative dimension*

$$n \;=\; \dim X - \dim Y.$$

Proof. See Hartshorne, cor. III.9.6. □

In what follows, we make the convention that the term flat always means flat and equidimensional. For any flat morphism $f\colon X \to Y$ of relative dimension n, proposition 6.4 allows to define a homomorphism

$$f^*\colon \quad Z_d(Y) \longrightarrow Z_{d+n}(X), \quad [Z] \mapsto [f^{-1}(Z)]$$

by additive extension of the map sending a subvariety $Z \subset Y$ of dimension d to the fundamental cycle of the subscheme $f^{-1}(Z) \subset X$. Note that in the definition we only use subvarieties $Z \subset Y$, but a simple bookkeeping of lengths shows that the same formula then holds for fundamental cycles of arbitrary subschemes:

Lemma 6.5. *For any subscheme $Z \subset Y$ we have*

$$f^*[Z] = [f^{-1}(Z)].$$

Proof. Let $V \subset f^{-1}(Z)$ be an irreducible component of the preimage of Z, seen as a subvariety of the scheme $f^{-1}(Z)$. Let $W = \overline{f(V)} \subset Y$ be its closure, endowed with the reduced subscheme structure so that it becomes a subvariety of Z. The flatness of f implies that W is an irreducible component of Z (else we could find a local section $s \in \mathscr{O}_Z$ which is not a zero divisor in $\mathscr{O}_Z$ but satisfies $s|_W = 0$. The latter condition would mean that the local section pulls back to a zero divisor in $\mathscr{O}_{f^{-1}(Z)}$, which is impossible because flat ring homomorphisms preserve the property of not being a zero divisor). Algebraically, the fact that $W \subset Z$ is an irreducible component means that the local ring $\mathscr{O}_{Z,W}$ is an Artin ring, i.e. has Krull dimension zero.

Consider now the homomorphism $A = \mathscr{O}_{Z,W} \to B \longrightarrow \mathscr{O}_{f^{-1}(Z),V}$. By definition of fundamental cycles

- $[f^{-1}(Z)]$ contains $[V]$ with multiplicity $\ell_B(B)$,
- $[f^{-1}(W)]$ contains $[V]$ with multiplicity $\ell_{B/\mathfrak{m}_A B}(B/\mathfrak{m}_A B) = \ell_A(B/\mathfrak{m}_A B)$,
- $[Z]$ contains $[W]$ with multiplicity $\ell_A(A)$.

Hence the claim boils down to the identity $\ell_B(B) = \ell_A(A) \cdot \ell_A(B/\mathfrak{m}_A B)$, which is easily seen to hold for any local homomorphism between Artinian local rings. □

Corollary 6.6. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are flat morphisms, then $f \circ g$ is flat and we have*

$$(g \circ f)^* \;=\; f^* \circ g^*\colon \quad Z_d(X) \;\longrightarrow\; Z_{d+n}(Z) \quad \textit{for} \quad n \;=\; \dim X - \dim Z.$$

Proof. For any subvariety $V \subset Z$ we have

$$(g \circ f)^*[V] = [(g \circ f)^{-1}(V)] = [f^{-1}(g^{-1}(V))] = f^*[g^{-1}(V)] = f^* g^*[V]$$

by repeated application of lemma 6.5. □

We next want to show that the flat pullback preserves rational equivalence of cycles. For this we will need the following compatibility of flat pullback and proper pushforward. Suppose that we are given a Cartesian square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Recall that if f is proper, then f' is proper. Similarly, if f is proper, then f' is proper.

Lemma 6.7. *For any Cartesian square as above where f is proper and g is flat, we have*

$$f'_* g'^*(\alpha) \;=\; g^* f_*(\alpha) \quad \textit{for all} \quad \alpha \;\in\; Z_d(X).$$

Proof. If f is a closed immersion, this follows directly from the definitions. To deal with the general case, note that for any subvariety $Z \subset X$ the cycle $[Z] \in Z_d(X)$ is the pushforward of $[Z] \in Z_d(Z)$ under the closed immersion of the subvariety. Hence by the case of closed immersions we may replace X by Z. By the same argument we can replace Y by the closure of $f(X)$ with its reduced subscheme structure. So we may assume that $f\colon X \to Y$ is a dominant morphism between varieties and $\alpha = [X]$ is the fundamental class of the variety X. Now

$$\begin{aligned} g^*[Y] \;&=\; [Y'], & f_*[X] \;&=\; \deg(X/Y) \cdot [Y], \\ g'^*[X] \;&=\; [X'], & f'_*[X'] \;&=\; \deg(X'/Y') \cdot [Y'], \end{aligned}$$

so everything boils down to showing the equality $\deg(X/Y)=\deg(X'/Y')$. Note that by flatness $g\colon Y'\to Y$ is dominant. So if f is not generically finite, then the claim is trivial since both sides in the equality are zero. But f is generically finite. then replacing Y by an open dense subset we can assume f is finite and g is surjective, in which case clearly $\deg(X/Y)=\deg(X'/Y')$. □

Theorem 6.8. *Let $f\colon X\to Y$ be a flat morphism of relative dimension n. Then we have*

$$f^*(\mathrm{Rat}_d(Y)) \;\subset\; \mathrm{Rat}_{d+n}(X),$$

hence we obtain an induced homomorphism of Chow groups $f^*\colon A_d(Y)\to A_{d+n}(X)$.

Proof. By proposition 5.8 the subgroup $\mathrm{Rat}_d(Y)\subset Z_d(Y)$ is generated by cycles of the form $\alpha=[V_0]-[V_\infty]$ where $V\subset Y\times\mathbb{P}^1$ is a subvariety which dominates $\mathbb{P}^1$ via the projection on the second factor. For any such cycle we consider the diagram below, where

$$W \;=\; (f\times id)^{-1}(V) \;\subset\; X\times\mathbb{P}^1$$

and where p,q,g,h are the restriction of the projections to the two factors:

$$\begin{tikzcd}
X \arrow[d, "f"'] & W \arrow[l, "p"'] \arrow[r, "g"] \arrow[d, "f\times id"'] & \mathbb{P}^1 \arrow[d, equal] \\
Y & V \arrow[l, "q"'] \arrow[r, "h"] & \mathbb{P}^1
\end{tikzcd}$$

We have

$$\begin{aligned}
f^*\alpha \;&=\; f^*q_*([h^{-1}(0)]-[h^{-1}(\infty)]) && \text{since } \alpha=[V_0]-[V_\infty]\\
&=\; p_*(f\times id)^*([h^{-1}(0)]-[h^{-1}(\infty)]) && \text{by lemma 6.7}\\
&=\; p_*([g^{-1}(0)]-[g^{-1}(\infty)]) && \text{by lemma 6.5}\\
&=\; [W_0]-[W_\infty].
\end{aligned}$$

At this point we would like to say that $[W_0]-[W_\infty]\in\mathrm{Rat}_{d+n}(X)$ by proposition 5.8, but we have to be slightly more careful because in general the subscheme W may be reducible and non-reduced. We do the bookkeeping of multiplicities as follows:

Since by assumption the morphism $f\times id\colon X\times\mathbb{P}^1\to Y\times\mathbb{P}^1$ is equidimensional, the scheme $W=(f\times id)^{-1}(V)$ is equidimensional as well. Let $W_1,\dots,W_r\subset W$ be its irreducible components, endowed with the reduced subscheme structure, and denote by $g_i=g|_{W_i}\colon W_i\to\mathbb{P}^1$ the restriction of g. The fundamental cycle of the scheme W has the form

$$[W] \;=\; \sum_{i=1}^r m_i\cdot[W_i] \quad \text{with certain} \quad m_i \;\in\; \mathbb{N}_0,$$

and we will be done if we can show that

$$[W_0]-[W_\infty] \;=\; \sum_{i=1}^r m_i\cdot\mathrm{cyc}(g_i) \quad \text{for the rational functions} \quad g_i \;\in\; k(W_i)^\times$$

Since by definition $\mathrm{cyc}(g_i) = [g_i^{-1}(0)] - [g_i^{-1}(\infty)]$, everything then boils down to the identity

$$[h^{-1}(p)] \;=\; \sum_{i=1}^{r} m_i \cdot [h_i^{-1}(p)] \quad \text{for all} \quad p \;\in\; \mathbb{P}^1$$

which is a special case of lemma 6.9 below. □

To explain the last step in the above proof, recall that an *effective Cartier divisor* on a scheme W is a closed subscheme $D \subset W$ of codimension one defined locally by a single function that is not a zero divisor. By this we mean that each $p \in W$ has an affine open neighborhood $U = \operatorname{Spec} A \subset W$ such that

$$D \cap U \;=\; \operatorname{Spec}(A/(f)) \quad \text{for some } f \in A \text{ which is not a zero divisor.}$$

If $W_i \subset W$ is an irreducible component meeting U, then $U_i := U \cap W_i = \operatorname{Spec}(A/\mathfrak{p}_i)$ for some minimal prime ideal $\mathfrak{p}_i \trianglelefteq A$. Now the set of zero divisors of any Noetherian ring is the union of its associated prime ideals, and for reduced rings these are just the minimal prime ideals. Hence $f \notin \mathfrak{p}_i$, so the image $f_i := (f \bmod \mathfrak{p}_i) \in A_i := A/\mathfrak{p}_i$ is not zero and thus not a zero divisor. So we get on the chosen chart an effective Cartier divisor $D \cap U_i = \operatorname{Spec}(A_i/(f_i)) \subset U_i$, and these glue to an effective Cartier divisor

$$D \cap W_i \;\subset\; W_i.$$

Suppose $\dim W_i = d$ for all i. There are two sources for multiplicities in $[D]$:

- Vanishing orders on the irreducible components: The cycle $[D \cap W_i] \in Z_{d-1}(W_i)$ on the integral scheme W_i has multiplicities given by the order of zeroes of a local defining function for the Cartier divisor $D \cap W_i$,
- Multiplicities of the components: If W is non-reduced, the cycle $[W] \in Z_d(W)$ contains the component W_i by definition with multiplicity given by the length of the local ring along that component.

Lemma 6.9. *Let W be an equidimensional scheme of dimension n. Let $W_1, \dots, W_r$ be its irreducible components and $[W] = \sum_{i=1}^{r} m_i \cdot [W_i]$ its fundamental cycle, with multiplicities $m_i \in \mathbb{N}$. Then for any effective Cartier divisor $D \subset W$ we have*

$$[D] \;=\; \sum_{i=1}^{r} m_i \cdot [D \cap W_i] \quad \textit{in} \quad Z_{n-1}(X).$$

Proof. Let $V \subset W$ be a subvariety of codimension one. We must show that it enters with the same multiplicity in the cycles $[D]$ and $\sum_i m_i [D \cap W_i]$. Working locally, we may assume that

$$D \;=\; V(f) \;\subset\; W \quad \text{for some non-zero-divisor} \quad f \;\in\; \Gamma(W, \mathscr{O}_W).$$

The irreducible components $W_i \subset W$ which contain V correspond bijectively to the minimal prime ideals

$$\mathfrak{p}_i \;\trianglelefteq\; A \;=\; \mathscr{O}_{W,V},$$

and by definition the multiplicities of the components in the fundamental cycle $[W]$ are given by $m_i = \ell_{A_{\mathfrak{p}_i}}(A_{\mathfrak{p}_i})$. By definition

- $[V]$ enters in $[D]$ with multiplicity $\ell_A(A/(f))$,
- $[V]$ enters in $[D \cap W_i]$ with multiplicity $\ell_{A/\mathfrak{p}_i}(A/(\mathfrak{p}_i + fA))$.

The claim now follows from the identity

$$\ell_A(A/(f)) \;=\; \sum_i \ell_{A_{\mathfrak{p}_i}}(A_{\mathfrak{p}_i}) \cdot \ell_{A/\mathfrak{p}_i}(A/(\mathfrak{p}_i + fA))$$

which is shown in corollary 8.9. □

Remark 6.10. The equidimensionality of W is important: Let $W = V(xz, yz) \subset \mathbb{A}^3$ be the scheme with irreducible components $W_1 = V(z)$, $W_2 = V(x,y)$. Clearly the function $f = z - x \in \Gamma(W, \mathscr{O}_W)$ is not a zero divisor. The subscheme cut out by f is a line with a fat point

$$D \;=\; \operatorname{Spec} k[x,y,z]/(xz, yz, z-x) \;\simeq\; \operatorname{Spec} k[x,y]/(x^2, xy).$$

Hence one computes

$$[D] \;=\; [V(x,z)] \quad \text{but} \quad \sum_{i=1}^{2} [D \cap W_i] \;=\; [V(x,z)] + [V(x,y,z)]$$

The problem arises from the cycle $[D \cap W_2] = [V(x,y,z)] \in Z_0(W)$ which has too high codimension in W. In fact one can show in general that for any effective Cartier divisor D on a not necessarily equidimensional scheme W with $[W] = \sum_{i=1}^r m_i [W_i]$ one has

$$[D] \;=\; \sum_{i \in I} m_i \cdot [D \cap W_i]$$

where $I = \{1 \leq i \leq r \mid \dim W_i = \dim W\}$, i.e. we sum only over top-dimensional irreducible components. The proof is similar to the above.

7 More examples: Cellular varieties

In general, computing the Chow groups of a variety is not an easy task, but there are some cases where it can be reduced to our earlier computation for affine space via the localization sequence. The simplest example is projective space:

Proposition 7.1. *Let $n \in \mathbb{N}$. Then*

$$A_d(\mathbb{P}^n) \;\simeq\; \begin{cases} \mathbb{Z} & \text{for } 0 \leq d \leq n, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We first claim that $A_*(\mathbb{P}^n) = \mathbb{Z}\cdot[L_0]+\cdots+\mathbb{Z}\cdot[L_n]$ where $L_d \subset \mathbb{P}^n$ denotes a linear subspace of dimension d. This follows by induction on n, since looking at the hyperplane $H = L_{n-1}$ with complement $U = \mathbb{P}^n \setminus H$ we have the localization sequence

$$A_*(H) \xrightarrow{i_*} A_*(\mathbb{P}^n) \longrightarrow A_*(U) \xrightarrow{j^*} 0.$$

We may assume $L_0 \subset L_1 \subset \cdots \subset L_{n-1} = H \simeq \mathbb{P}^{n-1}$. By induction on n then

$$A_*(H) \simeq A_*(\mathbb{P}^{n-1}) \simeq \mathbb{Z}\cdot[L_0]+\cdots+\mathbb{Z}\cdot[L_{n-1}],$$

where by abuse of notation we apply the same notation for classes in $A_*(H)$ and their images in $A_*(\mathbb{P}^n)$. Moreover $A_*(U) = \mathbb{Z}\cdot[U]$ is generated by $[U] = j^*[L_n]$, being the Chow group of an affine space. Hence the claim follows.

It remains to be shown that there exist no nontrivial relations between the given generators. Since the generators are cycles of different dimension, we only need to show that for each d the map

$$\mathbb{Z} \longrightarrow A_d(\mathbb{P}^n), \quad m \mapsto m\cdot[L_d] \quad \text{is injective.}$$

For $d = n$ this is clear, and for $d = n-1$ it holds since $A_{n-1}(\mathbb{P}^n) = \mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$. So in what follows we may assume $d < n-1$. Suppose that

$$m\cdot[L_d] = \sum_{i=1}^{r} \mathrm{cyc}(f_i)$$

for suitable $f_i \in k(W_i)^\times$ and subvarieties $W_i \subset \mathbb{P}^n$ of dimension $d+1$. The union

$$W = W_1 \cup \cdots \cup W_r \subset \mathbb{P}^n$$

is a closed subscheme of dimension $d+1$. By taking an intersection of $d+2$ general hyperplanes, we find a linear subspace

$$L \subset \mathbb{P}^n \quad \text{of dimension} \quad \dim L = n-d-2 \quad \text{with} \quad L \cap W = \varnothing.$$

The projection from L is a rational map $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ that restricts on W to a proper morphism

$$\pi\colon \quad W \longrightarrow \mathbb{P}^{d+1}$$

with finite fibers, and $L_d' := \pi(L_d) \subset \mathbb{P}^{d+1}$ is a hyperplane. Since by theorem 5.4 the proper pushforward preserves rational equivalence, we have

$$m\cdot[L_d'] = \pi_*(m\cdot[L_d]) \sim 0 \quad \text{in} \quad Z_d(\mathbb{P}^{d+1}),$$

and hence $m = 0$ because $A_d(\mathbb{P}^{d+1}) = \mathbb{Z}\cdot[L_d']$ is freely generated by $[L_d']$. □

A similar description exists more generally for varieties which can be stratified by affine spaces in the following sense:

Example 7.2. A scheme X is called *cellular* if it admits a finite filtration by closed subschemes $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that the complement of each in the next is a disjoint union

$$X_i \setminus X_{i-1} \;=\; U_{i1} \sqcup \cdots \sqcup U_{ir_i} \quad \text{with} \quad U_{ij} \;\simeq\; \mathbb{A}^{n_{ij}} \quad \text{for suitable} \quad n_{ij} \;\in\; \mathbb{N}_0.$$

The same localization argument as in the first step of the above proof then shows that the total Chow group is generated by the classes of the closure $Z_{ij} = \overline{U}_{ij} \subset X$, i.e. that

$$\bigoplus_{i=1}^{n} \mathbb{Z}^{r_i} \;\twoheadrightarrow\; A_*(X), \quad (a_{ij}) \;\mapsto\; \sum_{i,j} a_{ij} \cdot [Z_{ij}]$$

is surjective. In fact one can again show that this map is an isomorphism, but this is harder — it is a special case of a result by Totaro (2014) that relies on higher Chow groups. Challenge: Can you come up with an elementary proof?

The notion of a cellular scheme is motivated by CW complexes in algebraic topology, but it is much more restrictive: In particular, any cellular variety is rational, i.e. birational to an affine space. However, the class of cellular varieties includes very important examples such as Grassmann varieties:

Example 7.3. The projective space $\mathbb{P}(V) = \operatorname{Proj}(\operatorname{Sym}^*(V^\vee))$ on a vector space V parametrizes lines in the vector space. We will see later that more generally, for any $d \in \{1, \dots, \dim V - 1\}$ the d-dimensional subspaces of V are parametrized by a smooth projective variety called the *Grassmann variety* $\operatorname{Gr}(d,V)$. Let us look at the case $d = 2$, $\dim V = 4$: To describe it set-theoretically on the level of points, consider the map

$$\iota\colon\; \operatorname{Gr}(2,V) \;:=\; \{\text{subspaces } W \subset V \text{ with } \dim W = 2\} \;\hookrightarrow\; \mathbb{P}(\wedge^2 V),\; W \;\mapsto\; [\wedge^2 W].$$

This map is injective since

$$W \;=\; \{w \in V \mid u \wedge w = 0\} \quad \text{for any} \quad u \in \wedge^2 W \setminus \{0\}.$$

We call ι the *Plücker embedding*. Its image consists of the points $[u] \in \mathbb{P}(\wedge^2 V)$ represented by decomposable vectors, i.e. vectors that can be written as a wedge product $u = v_1 \wedge v_2$ of two vectors $v_1, v_2 \in V$. It is a simple exercise in linear algebra that

$$u \;\in\; \wedge^2(V) \quad \text{is decomposable iff} \quad u \wedge u \;=\; 0 \;\in\; \wedge^4 V$$

Hence set-theoretically the Plücker embedding gives a bijection from $\operatorname{Gr}(2,V)$ onto the subset

$$G \;:=\; \{[u] \in \mathbb{P}(\wedge^2 V) \mid u \wedge u = 0\} \;\subset\; \mathbb{P}(\wedge^2 V).$$

One easily sees that this subset is Zariski closed. To give explicit equations, pick a basis $e_1, \dots, e_4$ of V. The vectors $e_{ij} = e_i \wedge e_j$ with $i < j$ form a basis of $\wedge^2(V)$. Take its dual basis consisting of the vectors

$$p_{ij} \in \wedge^2(V^\vee) = H^0(\mathbb{P}(\wedge^2 V), \mathscr{O}(1)) \quad \text{with} \quad p_{ij}(v_{i'j'}) = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j', \\ 0 & \text{otherwise.} \end{cases}$$

The p_{ij} are called *Plücker coordinates*. By construction any vector $u \in \wedge^2 V$ has the decomposition

$$u = \sum_{i<j} p_{ij}(u) \cdot e_{ij}$$

and in these terms we have:

$$u \wedge u = 0 \quad \Longleftrightarrow \quad (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})(u) = 0.$$

Thus we can define the Grassmannian by a quadratic equation in the homogenous coordinates:

$$G = V_+(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subset \mathbb{P}(\wedge^2 V) = \operatorname{Proj} k[p_{12}, \dots, p_{34}].$$

This Grassmannian is a cellular variety. To see this, consider the following chain of closed subsets:

- Let $X_4 = G$ be the full Grassmannian.
- Let $X_3 = G \cap V_+(p_{12})$ be the set of planes $W \subset V$ with $W \cap \langle e_3, e_4 \rangle \neq 0$.
- Let $X_2 = G \cap V_+(p_{12}, p_{13}) = X_2' \cup X_2''$ be the union of the following two sets:

$$\begin{aligned} X_2' &= G \cap V_+(p_{12}, p_{13}, p_{23}) = \{\text{planes } W \subset V \text{ with } W \ni e_4\}, \\ X_2'' &= G \cap V_+(p_{12}, p_{13}, p_{14}) = \{\text{planes } W \subset \langle e_2, e_3, e_4 \rangle\}. \end{aligned}$$

- Let $X_1 = X_2' \cap X_2''$ be the set of planes $W \subset \langle e_2, e_3, e_4 \rangle$ with $W \ni e_4$.
- Let $X_0 = \{W\}$ be the subvariety consisting of the single point $W = \langle e_3, e_4 \rangle$.

Then we have isomorphisms

- $X_4 \setminus X_3 \simeq \operatorname{Spec} k[s_{13}, s_{14}, s_{23}, s_{24}] \simeq \mathbb{A}^4$ where $s_{ij} = p_{ij}/p_{12}$.
- $X_3 \setminus X_2 \simeq \operatorname{Spec} k[t_{14}, t_{23}, t_{34}] \simeq \mathbb{A}^3$ where $t_{ij} = p_{ij}/p_{13}$.
- $X_2' = \mathbb{P}\langle e_{14}, e_{24}, e_{34} \rangle \supset X_1 = \mathbb{P}\langle e_{24}, e_{34} \rangle$, hence $X_2' \setminus X_1 \simeq \mathbb{A}^2$.
- $X_2'' = \mathbb{P}\langle e_{23}, e_{24}, e_{34} \rangle \supset X_1 = \mathbb{P}\langle e_{24}, e_{34} \rangle$, hence $X_2'' \setminus X_1 \simeq \mathbb{A}^2$.
- $X_1 = \mathbb{P}\langle e_{24}, e_{34} \rangle \setminus X_0 = \mathbb{P}\langle e_{34} \rangle$, hence $X_1 \setminus X_0 \simeq \mathbb{A}^1$ and $X_0 \simeq \mathbb{A}^0$.

Thus the Grassmann variety $G = \operatorname{Gr}(2, V)$ is cellular, and its Chow groups are given by

$$A_d(G) = \begin{cases} \mathbb{Z} \cdot [X_d] & \text{for } d = 0, 1, 3, 4, \\ \mathbb{Z} \cdot [X_2'] + \mathbb{Z} \cdot [X_2''] & \text{for } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

With more work, one may check that there are no relations between the given classes, so we have

$$A_d(G) \simeq \begin{cases} \mathbb{Z} & \text{for } d = 0,1,3,4, \\ \mathbb{Z}^2 & \text{for } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

However, the $\mathbb{Z}$-linear independence of classes in Chow groups is much easier to check using the intersection product; we will later give a complete description of the intersection product on the Chow ring of arbitrary Grassmann varieties.

The Chow groups of cellular varieties behave very nicely, for instance they satisfy the following analog of the Künneth decomposition in cohomology:

Exercise 7.4. Let X,Y be schemes.

a) Show that for any $d,e \in \mathbb{N}_0$ we have a well-defined product

$$\times: \quad A_d(X) \otimes A_e(Y) \longrightarrow A_{d+e}(X \times Y), \quad [Z] \times [W] := [Z \times W].$$

b) Show that if X and Y are cellular varieties, then for any $m \in \mathbb{N}_0$ the resulting homomorphism

$$\times: \quad \bigoplus_{d+e=m} A_d(X) \otimes A_e(Y) \longrightarrow A_m(X \times Y)$$

is surjective. Is it also injective? What happens if X and Y are not cellular?

8 Appendix: Length and determinant

In this appendix we recall the relation between the length and determinant that we have used in the proof of theorem 5.4. We fix a Noetherian local domain A with $\dim A = 1$ and a finite-dimensional vector space V over $K = \mathrm{Quot}(A)$.

Definition 8.1. A *lattice* in V is a finitely generated A-submodule $M \subset V$ with the property that

$$V = M \otimes_A K.$$

Remark 8.2. Lattices always exist: For any basis of V over K, the A-submodule generated by it is a lattice. From this observation one easily verifies that for any given lattice $M_0 \subset V$ and any A-submodule $M \subset M_0$ the following conditions are equivalent:

a) $M \subset V$ is a lattice.

b) M_0/M has finite length as an A-module.

Moreover, for any two lattices $M_1, M_2 \subset V$ their intersection $M_1 \cap M_2 \subset V$ is again a lattice. We can therefore define a measure for how different two lattices are by comparing both to their intersection:

Definition 8.3. The *distance* between two lattices $M_1, M_2 \subset V$ is defined to be the integer

$$\begin{aligned} d(M_1,M_2) \;&=\; \ell_A(M_1/M_{12}) - \ell_A(M_2/M_{12}) && \text{for } M_{12} = M_1 \cap M_2 \\ &=\; \ell_A(M_1/N) - \ell_A(M_2/N) && \text{for any sublattice } N \subset M_{12} \end{aligned}$$

The last formula implies that the distance between lattices is additive:

Lemma 8.4. *We have* $d(M_1,M_3) \;=\; d(M_1,M_2) + d(M_2,M_3)$ *for lattices* M_1, M_2, M_3.

Proof. Put $N = M_1 \cap M_2 \cap M_3$. Then

$$\begin{aligned} d(M_1,M_3) \;&=\; \ell_A(M_1/N) - \ell_A(M_3/N) \\ &=\; \ell_A(M_1/N) - \ell_A(M_2/N) + \ell_A(M_2/N) - \ell_A(M_3/N) \\ &=\; d(M_1,M_2) + d(M_2,M_3) \end{aligned}$$

because N is a sublattice of $M_i \cap M_j$ for all $i, j \in \{1,2,3\}$. □

After these preliminaries we can formulate the main result of this section. For any automorphism $\varphi \in \mathrm{Aut}_K(V)$, consider its determinant $\det(\varphi) \in K^\times$ and consider its order

$$\mathrm{ord}(\det(\varphi)) \;:=\; \ell_A(A/(a)) - \ell_A(A/(b)) \quad \text{for } \det(\varphi) = \frac{a}{b} \text{ with } a, b \in A,$$

i.e. the order of zeroes or poles at the closed point of $\operatorname{Spec} A$.

Proposition 8.5. *Let* $M \subset V$ *be a lattice. For* $\varphi \in \mathrm{Aut}_K(V)$ *the image* $\varphi(M) \subset V$ *is again a lattice and*

$$d(M, \varphi(M)) \;=\; \mathrm{ord}(\det(\varphi)).$$

Proof. We first claim that both sides of the equation are additive with respect to the composition of automorphisms: For all $\varphi, \eta \in \mathrm{Aut}_K(V)$ the order of the determinant satisfies

$$\begin{aligned} \mathrm{ord}(\det(\varphi \circ \eta)) \;&=\; \mathrm{ord}(\det(\varphi) \cdot \det(\eta)) && \text{by multiplicativity of det} \\ &=\; \mathrm{ord}(\det(\varphi)) + \mathrm{ord}(\det(\eta)) && \text{by additivity of ord} \end{aligned}$$

and the distance of lattices satisfies

$$\begin{aligned} d(M, \varphi(\eta(M)) \;&=\; d(M, \varphi(M)) + d(\varphi(M), \varphi(\eta(M))) && \text{by lemma 8.4} \\ &=\; d(M, \varphi(M)) + d(M, \eta(M)) && \text{since } d \circ (\varphi, \varphi) = d. \end{aligned}$$

It therefore suffices to verify the claim of the proposition when φ runs over a set of generators of the group $\mathrm{Aut}_K(V)$. By choosing a basis we may assume $V = K^n$ and hence $\mathrm{Aut}_K(V) = \mathrm{GL}_n(K)$. Then our set of generators can be chosen to be the elementary matrices

$$\begin{aligned} E_{ij}(\lambda) &= id + \lambda \cdot \delta_{ij} && \text{with } i \neq j \text{ and } \lambda \in K^{\times} \\ E_i(\lambda) &= id + (\lambda - 1) \cdot \delta_{ii} && \text{with any } i \text{ and } \lambda \in K. \end{aligned}$$

For the lattice $M = R^n \subset V = K^n$ the claim now follows by a direct computation. It then only remains to observe that the function $d(M, \varphi(M))$ is independent of the chosen lattice $M \subset V$ (as it should be if the proposition is supposed to be true), since we have

$$\begin{aligned} d(M, \varphi(M)) &= d(M,M') + d(M', \varphi(M')) + d(\varphi(M'), \varphi(M)) \\ &= d(M,M') + d(M', \varphi(M')) + d(M',M) \\ &= d(M', \varphi(M') \end{aligned}$$

by repeated application of lemma 8.4 and because $d \circ (\varphi, \varphi) = d$. □

Corollary 8.6. *Let $A \subset B$ be a finite extension of domains, where A is a Noetherian local ring of dimension one as above. Then for any nonzero element $r \in B \setminus \{0\}$ we have*

$$\ell_A(B/rB) = \mathrm{ord}(N_{L/K}(r)),$$

where $N_{L/K} \colon L = \mathrm{Quot}(B) \to K = \mathrm{Quot}(A)$ is the norm of the field extension.

Proof. Take $V = L$ and $\varphi = (x \mapsto bx) \in \mathrm{Aut}_K(V)$ in proposition 8.5. □

In the above result we have only been dealing with domains. In particular, for the A-module $M = B$ the multiplication $M \to M, m \mapsto rm$ was injective. This is no longer the case for more general A-modules M, and as a consequence the length $\ell(M/rM)$ is less well-behaved; for instance, the function $M \mapsto \ell(M/rM)$ is in general not additive in short exact sequences of modules. This leads to the following notion:

Definition 8.7. Let A be a commutative Noetherian ring and $a \in A$. Let M be a finitely generated module over A. We say $e(a,M)$ is defined if

$${}_aM := \ker(M \xrightarrow{a\cdot} M) \quad \text{and} \quad M_a := \mathrm{cok}(M \xrightarrow{a\cdot} M) = M/aM$$

both have finite length over A, and we then put $e(a,M) := \ell_A(M_a) - \ell_A({}_aM)$.

Lemma 8.8. *Let A be a commutative Noetherian ring and $a \in A$,*

a) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated modules over A. If two of the three numbers $e(a,M), e(a,M'), e(a,'')$ are defined, then so is the third, and then

$$e(a,M) = e(a,M') + e(a,M'').$$

b) For any finite length A-module M we have $e(a,M) = 0$.

Proof. For $a)$, the snake lemma for the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "a"] & M \arrow[r] \arrow[d, "a"] & M'' \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

gives an exact sequence $0 \to_a M' \to_a M \to_a M'' \to M'_a \to M_a \to M''_a \to 0$. For $b)$ use the exact sequence

$$0 \to_a M \to M \to M \to M_a \to 0$$

and the additivity of the length in short exact sequences. □

Corollary 8.9. *Let A be a one-dimensional Noetherian local ring,* $\mathfrak{p}_1,\dots\mathfrak{p}_r \trianglelefteq A$ *its minimal prime ideals, and* $a \in A \setminus \bigcup_{i=1}^r \mathfrak{p}_i$. *Then any finitely generated A-module M satisfies*

$$e(a,M) \;=\; \sum_{i=1}^{r} \ell_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}) \cdot e(a,A/\mathfrak{p}_i) \quad \textit{where} \quad e(a,A/\mathfrak{p}_i) \;=\; \ell_A(A/(\mathfrak{p}_i + aA)).$$

Proof. The second equality follows from our assumption that $a \notin \bigcup_{i=1}^r \mathfrak{p}_i$. For the first equality, note that both sides of the equality are additive with respect to short exact sequences, hence we may assume $M = A/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \trianglelefteq A$. There are then only two cases:

a) If $\mathfrak{p}$ is a maximal ideal, then $M/\mathfrak{p}$ has finite length over A, hence by the previous lemma we get $e(a,M) = 0$. So the left hand side of the desired equality is zero. But the right hand side is also zero, since $\mathfrak{p} \not\subset \mathfrak{p}_i$ implies that $M_{\mathfrak{p}_i} = 0$ for all i.

b) If $\mathfrak{p} = \mathfrak{p}_i$ for some i, then $\ker(M \xrightarrow{a} M) = 0$ by our assumption $a \notin \mathfrak{p}_i$. Hence we get

$$e(a,M) = \ell_A(M/(a)) = \ell_A(A/(\mathfrak{p}_i + aA))$$

and the claim follows because $\ell_{A_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) = \ell_{A_{\mathfrak{p}_j}}((A/\mathfrak{p}_i)_{\mathfrak{p}_j}) = \delta_{ij}$. □

Chapter II
Vector bundles and Chern classes

1 Intersection with Cartier divisors

In this section we study the intersection of arbitrary cycles with codimension one cycles attached to Cartier divisors. Recall that for integral schemes X of dimension n we have defined the first Chern class $c_1\colon \operatorname{Pic}(X) \to A_{n-1}(X)$ by the commutative diagram

$$\begin{tikzcd} \operatorname{Div}(X) \arrow[r] \arrow[d] & Z_{n-1}(X) \arrow[d] \\ \operatorname{Pic}(X) \arrow[r, "\exists ! c_1"] & A_{n-1}(X) \end{tikzcd}$$

where $\operatorname{Div}(X)$ is the group of Cartier divisors on X and the top row maps a Cartier divisor to the associated Weil divisor. We will define the intersection of a subvariety and a Cartier divisor as the pullback of the divisor to the subvariety, but this has to be understood in the correct sense since the pullback of Cartier divisors is not always well-defined as a Cartier divisor:

Definition 1.1. Let $D \in \operatorname{Div}(X)$ be a Cartier divisor, represented by a family $(f_\alpha)_{\alpha \in I}$ of functions

$$f_\alpha \;\in\; k(U_\alpha)^\times \;=\; k(X)^\times \quad \text{for an open cover} \quad X \;=\; \bigcup_{\alpha \in I} U_\alpha$$

such that $f_\alpha / f_\beta \in \mathscr{O}_X(U_\alpha \cap U_\beta)^\times$ for all α, β. The cycles $\operatorname{cyc}(D_\alpha) \in Z_{n-1}(U_\alpha)$ then agree on the overlap of charts, and by gluing them we obtain the associated Weil divisor

$$[D] \;\in\; Z_{n-1}(X)$$

The *support* $|D| := \operatorname{Supp}(D) \subset X$ is defined as the union of all subvarieties $Z \subset X$ of codimension one that enter in the divisor with a nonzero coefficient. The *pullback* of the divisor under a morphism $f\colon Y \to X$ of varieties is defined as follows:

a) If $f(Y) \not\subset |D|$, we define $f^*(D) \in \mathrm{Div}(Y)$ to be the Cartier divisor given by the functions

$$g_\alpha = f_\alpha \circ f \in k(V_\alpha)^\times \quad \text{for the open cover} \quad Y = \bigcup_{\alpha \in I} V_\alpha \quad \text{with} \quad V_\alpha := f^{-1}(U_\alpha).$$

b) If $f(Y) \subset |D|$, we cannot define the pullback as a Cartier divisor like this, but we can still consider the line bundle $\mathscr{O}_X(D) \in \mathrm{Pic}(X)$ and define the pullback of D as the cycle class

$$f^*(D) \;:=\; c_1(f^*\mathscr{O}_X(D)) \;\in\; A_{d-1}(Y) \quad \text{where} \quad d \;=\; \dim Y.$$

In both cases we get a cycle class

$$f^*(D) \;\in\; A_{d-1}(f^{-1}(|D|)).$$

In case a) this class determines the underlying Weil divisor on the variety Y and hence also the corresponding Cartier divisor, while in case b) it only remembers its rational equivalence class. Even worse, by abuse of notation we will sometimes also write

$$f^*(\alpha) \;:=\; c_1(f^*\mathscr{O}_X(D)) \;\in\; A_{d-1}(Y) \quad \text{for the class} \quad \alpha \;=\; [D] \;\in\; A_{n-1}(X)$$

of a Cartier divisor $D \in \mathrm{Div}(X)$ even when $f(Y) \not\subset |D|$. All three cases can be unified formally via the notion of a *pseudodivisor* as in Fulton's book, but we here take a more casual view, assuming that it will be clear from the context whether a notation refers to a Cartier divisor or to its rational equivalence class.

Definition 1.2. Let D be a Cartier divisor on a scheme X, and let $i\colon Z \hookrightarrow X$ be a subvariety of dimension $\dim Z = d$. We define the *intersection product* of the divisor with the subvariety by

$$D \cdot [Z] \;:=\; [i^*(D)] \;\in\; A_{d-1}(Z \cap |D|).$$

More generally, for a cycle

$$\alpha \;=\; \sum_{Z \subset X} n_Z \cdot [Z] \;\in\; Z_d(X)$$

we define its *support* $|\alpha| := \mathrm{Supp}(\alpha) \subset X$ to be the union of all Z with $n_Z \neq 0$ and put

$$D \cdot \alpha \;:=\; \sum_{Z \subset X} n_Z \cdot i_{Z*}(D \cdot [Z]) \;\in\; A_{d-1}(|\alpha| \cap |D|)$$

for the inclusion $i_Z\colon Z \cap |D| \hookrightarrow |\alpha| \cap |D|$ (usually omitted from the notation).

We emphasize that in the above definition α is a cycle, not a cycle class, and D is a Cartier divisor, not a linear equivalence class of Cartier divisors. The following proposition summarizes some basic properties of the intersection product:

Proposition 1.3. *The intersection product has the following properties:*

a) Additivity in the cycle: Let $D \in \operatorname{Div}(X)$, $\alpha_1, \alpha_2 \in Z_d(X)$, *then*

$$D \cdot (\alpha_1 + \alpha_2) \;=\; D \cdot \alpha_1 + D \cdot \alpha_2 \quad \textit{in} \quad A_{d-1}\big(\cup_{i=1,2}|D| \cap |\alpha_i|\big).$$

b) Additivity in the divisor: Let $D_1, D_2 \in \operatorname{Div}(X)$, $\alpha \in Z_d(X)$, *then*

$$(D_1 + D_2) \cdot \alpha \;=\; D_1 \cdot \alpha + D_2 \cdot \alpha \quad \textit{in} \quad A_{d-1}\big(\cup_{i=1,2}|D_i| \cap |\alpha|\big).$$

c) Projection formula: Let $f\colon X \to Y$ *be proper and* $D \in \operatorname{Div}(Y)$, $\alpha \in Z_d(X)$, *then*

$$f_*(f^*(D) \cdot \alpha) \;=\; D \cdot f_*(\alpha) \quad \textit{in} \quad A_{d-1}\big(|D| \cap f(|\alpha|)\big).$$

d) Pullback: Let $f\colon Y \to X$ *be flat with* $\dim(Y/X) = n$ *and* $D \in \operatorname{Div}(X)$, $\alpha \in Z_d(X)$, *then*

$$f^*(D) \cdot f^*(\alpha) \;=\; f^*(D \cdot \alpha) \quad \textit{in} \quad A_{d+n-1}(f^{-1}\big(|D| \cap |\alpha|\big)).$$

e) Linear equivalence: If $D \in \operatorname{Div}(X)$ *is a principal Cartier divisor, then*

$$D \cdot \alpha \;=\; 0 \quad \textit{in} \quad A_{d-1}\big(|\alpha|\big) \quad \textit{for all} \quad \alpha \;\in\; Z_d(X).$$

f) Rational equivalence: If $\alpha \in Z_d(X)$ *is rationally equivalent to zero, then*

$$D \cdot \alpha \;=\; 0 \quad \textit{in} \quad A_{d-1}\big(|D|\big) \quad \textit{for all} \quad D \;\in\; \operatorname{Div}(X).$$

Proof. The parts *a)*, *b)* and *e)* follow directly from our definition of the intersection product between cycles and Cartier divisors. For *c)* and *d)* we can assume $\alpha = [Z]$ for a subvariety $Z \subset X$. For the projection formula we can then by functoriality of pushforward and pullback even assume $Z = X$ and $Y = f(Z)$, in which case the claim boils down to the identity

$$f_*([f^*D]) \;=\; \deg(X/Y) \cdot [D] \quad \text{in} \quad Z_{d-1}(Y).$$

This can be checked locally, so we may assume $D = \operatorname{cyc}(r)$ for some $r \in k(Y)^\times$ and in this case we know

$$f_*([f^*\operatorname{cyc}(r)]) \;=\; f_*(\operatorname{cyc}(f^*(r))) \;=\; \operatorname{cyc}(N(f^*(r))) \;=\; \operatorname{cyc}(f^n) \;=\; n \cdot \operatorname{cyc}(r)$$

for $n = \deg(Y/X)$ and the norm $N\colon k(X) \to k(Y)$. For *d)* we can assume $Z = X$ and must show

$$[f^*D] \;=\; f^*([D]) \quad \text{in} \quad A_{d+n-1}(Y).$$

This can be checked locally, so we may assume D is the difference of two effective Cartier divisors. As both sides are additive, we may assume D is an effective Cartier divisor. Then the required identity is a special case of the fact that the pullback of cycles is compatible with fundamental cycles of schemes, see lemma 6.5. For *f)* we may assume $\alpha = \operatorname{cyc}(r)$ for a rational function $r \in k(W)^\times$ on a subvariety $W \subset X$ of dimension $d + 1$. Using the pushforward under closed immersions we may assume that $W = X$. Then $\alpha = [E] \in A_d(X)$ is the Weil divisor attached to a principal Cartier divisor $E \in \mathrm{Div}(X)$. Now the intersection product is commutative in the sense that for any two Cartier divisors $D, E \in \mathrm{Div}(X)$ we have

$$D \cdot [E] \;=\; E \cdot [D] \quad \text{in} \quad A_{d-1}(|D| \cap |E|).$$

We will verify this property in the next section. Assuming it for the moment, it follows from *e)* that if E is principal, then $D \cdot [E] = 0$. □

Corollary 1.4. *For any $d \in \mathbb{N}_0$ the intersection product $\cdot$ descends to a bilinear pairing*

$$\mathrm{Pic}(X) \times A_d(X) \;\longrightarrow\; A_{d-1}(X).$$

Proof. By pushforward to the ambient variety, we may view the intersection product as a pairing with values in $A_*(X)$. By proposition 1.3 we then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Div}(X) \times Z_d(X) \arrow[rr, "(D{,}\alpha) \mapsto D\cdot\alpha"] \arrow[dr] & & A_{d-1}(X) \\
& \mathrm{Pic}(X) \times A_d(X) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

where the top row is bilinear by *a), b)* and the dashed arrow exists by *e), f)*. □

Definition 1.5. For line bundles $\mathscr{L} = \mathscr{O}_X(D) \in \mathrm{Pic}(X)$ and a cycle $\alpha \in A_d(X)$ we write

$$c_1(\mathscr{L}) \cap \alpha \;:=\; D \cdot \alpha \;\in\; A_{d-1}(X).$$

to emphasize the analogy with the cup product in topology. Thus we obtain a group homomorphism

$$c_1(\mathscr{L}) \cap - \colon \quad A_d(X) \;\longrightarrow\; A_{d-1}(X).$$

From proposition 1.3 we immediately obtain:

a) Additivity: For $\mathscr{L}, \mathscr{L}' \in \mathrm{Pic}(X)$, the dual $\mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$ and $\alpha \in A_d(X)$ we have

$$\begin{aligned} c_1(\mathscr{L} \otimes \mathscr{L}') \cap \alpha \;&=\; c_1(\mathscr{L}) \cap \alpha + c_1(\mathscr{L}') \cap \alpha, \\ c_1(\mathscr{L}^\vee) \cap \alpha \;&=\; -c_1(\mathscr{L}) \cap \alpha. \end{aligned}$$

b) Projection formula: Let $f\colon X \to Y$ be proper and $\mathscr{L} \in \mathrm{Pic}(Y), \alpha \in Z_d(X)$, then

$$f_*(c_1(f^*\mathscr{L}) \cap \alpha) \;=\; c_1(\mathscr{L}) \cap f_*(\alpha).$$

c) Pullback: Let $f\colon Y \to X$ be flat with $\dim(Y/X) = n$ and $\mathscr{L} \in \mathrm{Pic}(X), \alpha \in Z_d(X)$, then

$$c_1(f^*\mathscr{L}) \cap f^*(\alpha) \;=\; f^*(c_1(\mathscr{L}) \cap \alpha).$$

Example 1.6. For each d, let $L_d \subset \mathbb{P}^n$ be a linear subspace of dimension d. Then for the cup product with the tautological line bundle we have

$$c_1(\mathscr{O}(1)) \cap [L_d] \;=\; [L_{d-1}].$$

This gives a much easier proof that $A_d(\mathbb{P}^n) \simeq \mathbb{Z}$ (without linear projections). More generally, we can define the degree homomorphism on cycles of arbitrary dimension by

$$\deg\colon \quad A_d(\mathbb{P}^n) \longrightarrow \mathbb{Z}, \quad \alpha \mapsto \int_\alpha c_1(\mathscr{L})^n \cap \alpha$$

where $c_1(\mathscr{L})^n \cap - \colon A_d(\mathbb{P}^n) \to A_0(\mathbb{P}^n)$ denotes the n-fold iterate of $c_1(\mathscr{L}) \cap (-)$.

Instead of only fixing a line bundle $\mathscr{L} \in \operatorname{Pic}(X)$, we can also fix an effective Cartier divisor $D \subset X$ and take the intersection product with values in $A_*(D)$. In this case there is again another notation:

Definition 1.7. Let $i\colon D \hookrightarrow X$ be the embedding of an effective Cartier divisor. Then the homomorphism

$$i^*\colon \quad A_d(X) \longrightarrow A_{d-1}(D), \quad i^*(\alpha) := D \cdot \alpha$$

is called the *Gysin homomorphism* attached to the Cartier divisor.

Remark 1.8. The definitions easily imply:

a) For $\alpha \in A_d(X)$ we have $i_* i^*(\alpha) = c_1(\mathscr{O}_X(D) \cap \alpha)$.
b) For $\alpha \in A_d(D)$ we have $i^* i_*(\alpha) = c_1(N) \cap \alpha$ where $N = i^*(\mathscr{O}_X(D))$.
c) If X is equidimensional of dimension d, then $i^*[X] = [D]$ in $A_{d-1}(D)$.

The equidimensionality in part *c)* is needed: For instance, let $X = V(xz, yz) \subset \mathbb{A}^3$ be the union of the xy-plane and the z-axis, and consider the effective Cartier divisor given by $D = V(z - x)$. Let $0 \in \mathbb{A}^3$ be the origin. Then one has

$$i^*[X] \;=\; D \cdot [X] \;=\; [D] + [\{0\}] \;\neq\; [D] \;\in\; A_*(D).$$

2 Commutativity of the intersection product

We defined the intersection $D \cdot \alpha \in A_{d-1}(|D| \cup |\alpha|)$ of a Cartier divisor $D \in \operatorname{Div}(X)$ and a cycle $\alpha \in Z_d(X)$ by restricting the divisor in a suitable sense to the support of the cycle. Here the divisor and the cycle take a completely different role. The goal of this section is to see that if $\alpha = [E]$ also underlies a Cartier divisor $E \in \operatorname{Div}(X)$, then the intersection product does not change if the roles of the two divisors are interchanged. Let us start with the simplest case:

Proposition 2.1. *Let X be a variety of dimension n. Let $D,E \subset X$ two effective Cartier divisors that intersect properly, i.e. do not have any common irreducible component. Then*

$$D\cdot[E] \;=\; E\cdot[D] \quad \textit{in} \quad A_{n-2}(|D|\cap|E|).$$

Proof. Let $W \subset X$ be any subvariety with $\operatorname{codim}_X(W)=2$. We want to compare the multiplicity with which the subvariety enters in both sides of the equation. To do this, let $f,g\in A=\mathscr{O}_{X,W}$ be local equations that cut the Cartier divisors D,E on some affine chart. Let us now look at the irreducible components of the two divisors. The irreducible components $Z\subset|E|$ with $Z\supset W$ correspond bijectively to the minimal primes $\mathfrak{p}\trianglelefteq B=A/(g)$. For any such component we have:

- $[Z]$ enters in $[E]$ with multiplicity $\ell_{B_\mathfrak{p}}(B_\mathfrak{p})$,
- $[W]$ enters in $D\cdot[Z]$ with multiplicity $\ell_{B/\mathfrak{p}}(B/(\mathfrak{p}+fB))$,

Taking the sum over all such components $Z\subset|E|$, we see that $[W]$ enters in $D\cdot[E]$ with multiplicity

$$\sum_{\mathfrak{p}}\ell_{B_\mathfrak{p}}(B_\mathfrak{p}/(g))\cdot\ell_{B/\mathfrak{p}}(B/(\mathfrak{p}+fB)) \;=\; e_B(f,B) \;=\; e_A(f,A/(g))$$

where the first equality holds by corollary 8.9 applied to the one-dimensional local Noetherian ring B. Similarly $[W]$ enters in $E\cdot[D]$ with multiplicity $e_A(g,A/(f))$, so the claim boils down to the identity $e_A(f,A/(g)) \;=\; e_A(g,A/(f))$ which is easily checked from the definitions. □

One way to generalize the above to divisors that do not intersect properly would be to deform divisors in their linear equivalence class. But this would only give an identity in $A_{n-2}(X)$; we want to get a class in $A_{n-2}(|D|\cap|E|)$ to keep as much information about supports as possible. One situation where this is easy is if D and E are linear combinations of effective Cartier divisors whose supports are irreducible and contained in $|D|\cup|E|$: Then the identity $D\cdot[E]=E\cdot[D]$ reduces by bilinearity to the two basic cases where

- either D and E intersect properly as discussed above,
- or $D=E$, in which case the desired identity is trivial.

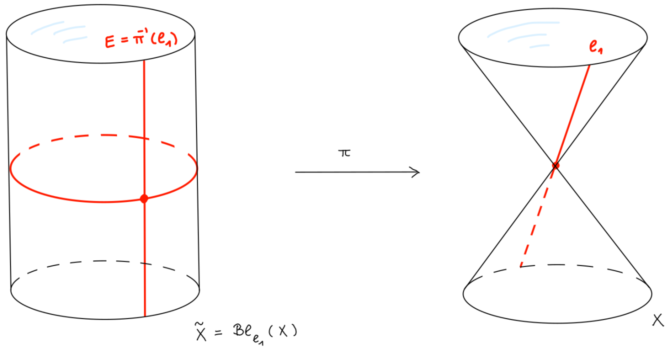
On singular varieties we cannot always reduce to these two basic cases directly:

Example 2.2. Take a quadric cone $X=V(z^2-x^2-y^2)\subset\mathbb{A}^3$ and let $D,D'\in\operatorname{Div}(X)$ be the principal Cartier divisors cut out by linear forms such that the underlying Weil divisors are

$$[D] \;=\; [\ell_1]+[\ell_2] \quad \text{and} \quad [D'] \;=\; 2[\ell_1] \quad \text{for two distinct lines} \quad \ell_1,\ell_2\subset X.$$

The only Cartier divisors with irreducible support contained in $|D|\cup|D'|$ are *even* multiples of the two lines, and D cannot be written as a linear combination of such even multiples. The problem is that the scheme theoretic intersection $D\cap D'\subset X$

has the fundamental cycle $[D \cap D'] = [\ell_1]$ which is not a Cartier divisor. Now there is a universal way to turn a closed subscheme into an effective Cartier divisor, the blowup along that subscheme:



We will see that in general such blowups reduce us to the two basic cases above.

To describe this strategy in more detail, let $D, D' \subset X$ be any effective Cartier divisors on a variety X. As a measure for the failure of proper intersection, define the *excess intersection* by

$$\varepsilon(D,D') \ := \ \max\{\mathrm{ord}_Z(D) \cdot \mathrm{ord}_Z(D') \mid Z \subset X \text{ subvariety with } \mathrm{codim}_X Z = 1\}.$$

Then $\varepsilon(D,D') \geq 0$, with equality iff D and D' intersect properly. We want to reduce the excess intersection by blowing up the scheme-theoretic intersection of the two divisors, i.e. the closed subscheme

$$D \cap D' \ := \ V(\mathscr{I}) \ \hookrightarrow \ X$$

which is cut out by the sum $\mathscr{I} := \mathscr{I}_D + \mathscr{I}_{D'}$ of the ideal sheaves $\mathscr{I}_D, \mathscr{I}_{D'} \trianglelefteq \mathscr{O}_X$ of the subschemes $D, D' \subset X$. Let

$$\pi\colon \quad \tilde{X} \ = \ \mathrm{Bl}_{D \cap D'}(X) \ \longrightarrow \ X$$

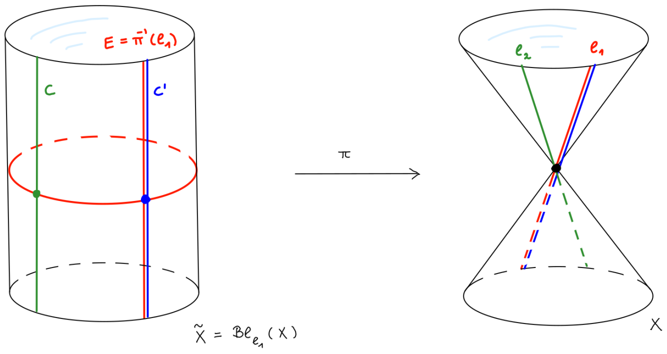
be the blowup and $E = \pi^{-1}(D \cap D') \subset \tilde{X}$ the exceptional divisor.

Lemma 2.3. *We have $\pi^*(D) = C + E$ and $\pi^*(D') = C' + E$, where $C, C' \subset \tilde{X}$ are effective Cartier divisors such that*

a) $|C| \cap |C'| = \varnothing$,

b) If $\varepsilon(D,D') > 0$, then $\max\{\varepsilon(C,E), \varepsilon(C',E)\} < \varepsilon(D,D')$.

For instance, in the above example $\varepsilon(D,D') = 2$ but $\varepsilon(C',E) = 1, \varepsilon(C,E) = 0$ as illustrated in the following picture:



Proof. Working locally we may assume that $X = \mathrm{Spec}(A)$ is an affine scheme and that $D, D' \subset X$ are principal divisors cut out by functions $f, f' \in A$. The blowup is defined by

$$\tilde{X} \; = \; \mathrm{Proj}_A \bigoplus_{n \geq 0} I^n \quad \text{for the ideal} \quad I \; = \; (f, f') \trianglelefteq A.$$

It is the closed subscheme $\tilde{X} = V_+(\ker \varphi) \subset \mathbb{P}^1_A = \mathrm{Proj}_A A[s,t]$ cut out by the kernel of the graded A-algebra epimorphism

$$\varphi: \quad A[s,t] \twoheadrightarrow \bigoplus_{n \geq 0} I^n \quad \text{with} \quad \begin{cases} \varphi(s) = f \\ \varphi(t) = f' \end{cases}$$

By the universal property of the blowup, the preimage $E = \pi^{-1}(D \cap D') \subset \tilde{X}$ is an effective Cartier divisor. Since its underlying closed subscheme is contained in the scheme-theoretic preimages $\pi^{-1}(D), \pi^{-1}(D') \subset \tilde{X}$, the local defining equation for the Cartier divisor E divides those for the Cartier divisors $\pi^*(D), \pi^*(D') \in \mathrm{Div}(\tilde{X})$, hence

$$\pi^*(D) \; = \; E + C \quad \text{and} \quad \pi^*(D') \; = \; E + C'$$

for certain effective Cartier divisors $C, C' \in \mathrm{Div}(\tilde{X})$. To control the latter, note that we have inclusions[1]

$$\tilde{X} \; \subset \; V_+(f's - ft) \; \subset \; \mathbb{P}^1_A.$$

It follows that

- on $U_0 = \{s \neq 0\} \subset \tilde{X}$ the pullback of f divides f' and hence $E \cap U_0 = V(f_{|U_0})$,
- on $U_\infty = \{t \neq 0\} \subset \tilde{X}$ the pullback of f' divides f and hence $E \cap U_\infty = V(f'_{|U_\infty})$.

---

[1] The inclusion $\tilde{X} \subset V_+(f's - ft)$ may be strict: For example, consider the affine cone $X = \mathrm{Spec}\, A$ for $A = k[x,y,z,w]/(xw - yz)$, and take $f = x$, $f' = y$. Then the subscheme $V_+(f's - ft) \subset X \times \mathbb{P}^1_A$ is reducible, hence it must be different from the blowup $\tilde{X}$ as the latter is irreducible.

On the other hand, the Cartier divisors $\pi^*(D), \pi^*(D') \in \mathrm{Div}(\tilde{X})$ are given in the charts by

- $\pi^*(D) \cap U_0 = V(f_{|U_0})$ and $\pi^*(D') \cap U_\infty = V(f'|_{|U_\infty})$,
- $\pi^*(D) \cap U_\infty = V(f_{|U_\infty}) = V(f'_{|U_\infty}) + V(s/t)$ since $f_{|U_\infty} = f'_{|U_\infty} \cdot s/t$,
- $\pi^*(D') \cap U_0 = V(f'_{|U_0}) = V(f_{|U_0}) + V(t/s)$ since $f'_{|U_0} = f_{|U_0} \cdot t/s$.

A direct comparison therefore shows

$$C \;=\; V_+(s) \quad \text{and} \quad C' \;=\; V_+(t) \quad \text{for the two sections} \quad s,t \;\in\; H^0(\tilde{X}, \mathscr{O}(1))$$

where $\mathscr{O}(1) \in \mathrm{Pic}(\tilde{X})$ is the restriction of the tautological bundle to $\tilde{X} \subset \mathbb{P}^1_A$. Hence we have

$$C \;\subset\; X \times \{0\} \quad \text{and} \quad C' \;\subset\; X \times \{\infty\},$$

which implies that C and C' are disjoint and map isomorphically onto D and D' via the projection. This last statement shows in particular that for any codimension one subvariety $\tilde{Z} \subset \tilde{X}$ with $\tilde{Z} \subset E \cap C$, its image under $\pi\colon \tilde{X} \to X$ will be a codimension one subvariety

$$Z \;=\; \pi(\tilde{Z}) \quad \text{with} \quad \tilde{Z} \;\subset\; D \cap D'.$$

The corresponding multiplicities then satisfy

$$\mathrm{ord}_Z(D) \;\geq\; \mathrm{ord}_{\tilde{Z}}(E) + \mathrm{ord}_{\tilde{Z}}(C),$$

because we have $[D] = \pi_*[E+C]$ by the projection formula.

To prove claim *b)*, we now argue by contradiction. If the claim is not true, then up to interchanging the two divisors we may assume $\varepsilon(C,E) \geq \varepsilon(D,D') > 0$. Pick a subvariety $\tilde{Z} \subset \tilde{X}$ of codimension one with

$$\mathrm{ord}_{\tilde{Z}}(C) \cdot \mathrm{ord}_{\tilde{Z}}(E) \;=\; \varepsilon(C,E),$$

and let $Z = \pi(\tilde{Z}) \subset X$ be its image. From the above we then obtain

$$\begin{aligned} \mathrm{ord}_{\tilde{Z}}(C) \cdot \mathrm{ord}_{\tilde{Z}}(E) \;\geq\; \varepsilon(D,D') \;&\geq\; \mathrm{ord}_Z(D) \cdot \mathrm{ord}_Z(D') \\ &\geq\; (\mathrm{ord}_{\tilde{Z}}(E) + \mathrm{ord}_{\tilde{Z}}(C)) \cdot (\mathrm{ord}_{\tilde{Z}}(E) + \mathrm{ord}_{\tilde{Z}}(C')) \\ &\geq\; (\mathrm{ord}_{\tilde{Z}}(E))^2 + \mathrm{ord}_{\tilde{Z}}(C) \cdot \mathrm{ord}_{\tilde{Z}}(E) \end{aligned}$$

which is a contradiction since $\mathrm{ord}_{\tilde{Z}}(E) > 0$. $\square$

We can now finally prove the main result of this section, the commutativity of the intersection product for arbitrary Cartier divisors:

Theorem 2.4. *Let X be a variety of dimension n, and let $D, D' \in \mathrm{Div}(X)$ be arbitrary Cartier divisors. Then*

$$D \cdot [D'] \;=\; D' \cdot [D] \quad \textit{in} \quad A_{n-2}(|D| \cap |D'|).$$

Proof. Let us first assume that $D, D' \in \mathrm{Div}(X)$ are both effective. In this case we will prove the theorem by induction on the excess intersection $\varepsilon(D, D')$. For $\varepsilon(D, D') = 0$ the two divisors intersect properly, and in this case the claim has been shown in proposition 2.1. For $\varepsilon(D, D') > 0$, we consider the blowup

$$\pi\colon \quad \tilde{X} \;=\; \mathrm{Bl}_{D\cap D'}(X) \;\longrightarrow\; X$$

with exceptional divisor $E = \pi^{-1}(D \cap D') \subset \tilde{X}$ as above. By lemma 2.3 we know that

$$\begin{aligned} \pi^*(D) \;&=\; E + C \qquad\qquad & \varepsilon(E, C) < \varepsilon(D, D'),\\ \pi^*(D') \;&=\; E + C' \qquad\qquad & \varepsilon(E, C') < \varepsilon(D, D'), \end{aligned}$$

where $C, C' \in \mathrm{Div}(\tilde{X})$ are effective Cartier divisors with $C \cap C' = \varnothing$. Now

- $E \cdot [C] = C \cdot [E]$ by induction since $\varepsilon(E, C) < \varepsilon(D, D')$,
- $E \cdot [C'] = C' \cdot [E]$ by induction since $\varepsilon(E, C') < \varepsilon(D, D')$,
- $C \cdot [C'] = C' \cdot [C]$ by proposition 2.1 since C and C' intersect properly,

Using the projection formula and the compatibility with pullback in proposition 1.3 we therefore obtain that

$$\begin{aligned} D \cdot [D'] \;&=\; \pi_*(\pi^*(D \cdot [D']))\\ &=\; \pi_*(\pi^*(D) \cdot \pi^*[D'])\\ &=\; \pi_*((E + C) \cdot [E + C'])\\ &=\; \pi_*(E \cdot [E] + E \cdot [C'] + C \cdot [E] + C \cdot [C'])\\ &=\; \cdots\\ &=\; D' \cdot [D]. \end{aligned}$$

This proves the claim if $D, D' \in \mathrm{Div}(X)$ are both effective. By bilinearity one then immediately deduces the claim also for differences $D = A - B$ and $D' = A' - B'$ of effective Cartier divisors $A, B, A', B' \in \mathrm{Div}(X)$. At this point we are not completely finished yet, since there are examples of varieties with Cartier divisors $D \in \mathrm{Div}(X)$ that cannot be written as a difference of two effective divisors; the reason is that the ideal sheaf

$$\mathscr{I} \;:=\; \{ f \in \mathscr{O}_X \mid f \cdot \mathscr{O}_X(D) \subset \mathscr{O}_X \}$$

of denominators in the local equations for the divisor need not be locally free. But passing to the blowup $p\colon \mathrm{Bl}_Z(X) \to X$ of $Z = V(\mathscr{I}) \subset X$, the Cartier divisor $p^*(D)$ will become a difference of effective Cartier divisors, so a similar book-keeping as above gives the theorem also when D is arbitrary and only D' is assumed to be effective. Repeating the same blowup argument also for the ideal of denominators of D' we finally obtain the result when D, D' are both arbitrary Cartier divisors. □

3 More about intersection products

The commutativity of the intersection product between two Cartier divisors leads to the following result:

Corollary 3.1. *For any $D, D' \in \mathrm{Div}(X)$ and $\alpha \in Z_d(X)$, we have*

$$D \cdot (D' \cdot \alpha) \;=\; D' \cdot (D \cdot \alpha) \quad \textit{in} \quad A_{d-2}(|D| \cap |D'| \cap |\alpha|).$$

Proof. By definition both sides are additive in α, so we may assume $\alpha = i_*[Z]$ for a subvariety $i\colon Z \hookrightarrow X$. By definition

$$D' \cdot \alpha \;=\; [D'_Z] \;\in\; A_{d-1}(|D'| \cap Z).$$

where by our conventions $D'_Z = i^*(D)$ is an actual Cartier divisor on Z if $Z \not\subset |D'|$, and a linear equivalence class of such divisors otherwise. In both cases we get from the definitions

$$D \cdot (D' \cdot \alpha) \;=\; D_Z \cdot [D'_Z] \;\in\; A_{d-1}(|D| \cap |D'| \cap Z),$$

where $D_Z = i^*(D)$ again denotes an actual Cartier divisor on Z if $Z \not\subset |D|$ and a linear equivalence class of such divisors otherwise. The claim now follows from the fact that $D_Z \cdot [D'_Z] = D'_Z \cdot [D_Z]$ by theorem 2.4. □

Definition 3.2. For $D_1, \dots, D_r \in \mathrm{Div}(X)$ and $\alpha \in Z_d(X)$, the corollary says that the interated intersection product

$$D_1 \cdot D_2 \cdots D_r \cdot \alpha \;:=\; D_1 \cdot (D_2 \cdots D_r \cdot \alpha) \;\in\; A_{d-r}(|D_1| \cap \cdots \cap |D_r| \cap |\alpha|)$$

is invariant under permutation of the divisors $D_1, \dots, D_r$. Hence for any homogenous polynomial $P \in \mathbb{Z}[x_1, \dots, x_r]$ of degree e, inserting the divisors we get a well-defined class

$$P(D_1, \dots, D_r) \cdot \alpha \;\in\; A_{d-e}(Z) \quad \text{on} \quad Z \;:=\; \bigcup_{1 \leq i \leq r} |D_i| \cap |\alpha|.$$

In the special case where Z is proper and $d = e$, we may in particular consider the intersection number

$$(P(D_1, \dots, D_r) \cdot \alpha)_X \;:=\; \int_Z P(D_1, \dots, D_r) \cdot \alpha \;\in\; \mathbb{Z}.$$

Depending on the context, the above notation will often be shortened: For instance, if X is an equidimensional scheme of dimension d, then for $D_1, \dots, D_r \in \mathrm{Div}(X)$ and any homogenous polynomial $P \in \mathbb{Z}[x_1, \dots, x_r]$ of degree e, we will use the shorthand notation

$$P(D_1, \dots, D_r) \;:=\; P(D_1, \dots, D_r) \cdot [X] \;\in\; Z_{d-e}(|D_1| \cup \cdots \cup |D_r|).$$

If moreover X is proper and $d = e$, we denote by

$$(P(D_1,\dots,D_r))_X \;:=\; (P(D_1,\dots,D_r)\cdot[X])_X \;\in\; \mathbb{Z}$$

the corresponding intersection number. For divisors that intersect in finitely many points, these intersection numbers count the points with certain multiplicities; but for divisors whose intersection is not finite one has to be more careful:

Example 3.3. Let X be a smooth surface.

a) If $D_1, D_2 \subset X$ are effective Cartier divisors that intersect properly, let $Z = D_1 \cap D_2$ be their scheme-theoretic intersection. Then the counting argument in the proof of proposition 2.1 shows that

$$(D_1 \cdot D_2)_X \;=\; \sum_{p\in Z} \ell_{\mathscr{O}_{Z,p}}(\mathscr{O}_{Z,p}) \;\geq\; 0.$$

b) This may fail if the divisors do not intersect properly: Let $\pi\colon X = \mathrm{Bl}_p(S) \to S$ be the blowup of a smooth surface S in a point p. Then $E = \pi^{-1}(p)$ is an effective Cartier divisor on X. By definition of the intersection product its self-intersection is the class

$$E\cdot[E] \;=\; [\mathscr{O}_X(E)|_E] \;\in\; A_0(E).$$

Now $E \simeq \mathbb{P}^1$ and via this isomorphism we have $\mathscr{O}_X(E)|_E \simeq \mathscr{O}_{\mathbb{P}^1}(-1)$, hence we find that

$$(E\cdot E)_X \;=\; \deg(\mathscr{O}_{\mathbb{P}^1}(-1)) \;=\; -1.$$

In fact the Chow groups of a blowup of a smooth surface can be described as follows:

Lemma 3.4. *Let S be a smooth projective surface. Let $\pi\colon X = \mathrm{Bl}_p(S) \to S$ be its blowup in a point $p \in S(k)$, and $E = \pi^{-1}(p) \subset X$ the exceptional divisor. Then we have isomorphisms*

$$\begin{aligned} A_d(S) &\xrightarrow{\sim} A_d(X) \quad \textit{for } d \neq 1,\\ A_1(S)\oplus\mathbb{Z} &\xrightarrow{\sim} A_1(X) \quad \textit{with } ([Z],m) \;\mapsto\; [\pi^{-1}(Z)] + m\cdot[E], \end{aligned}$$

and via this the intersection product on the middle Chow group $A_1(X) = \mathrm{Pic}(X)$ is given by

$$((\alpha,m)\cdot(\beta,n))_X \;=\; (\alpha,\beta)_S - mn \quad \textit{for} \quad \alpha,\beta \;\in\; A_1(S),\; m,n \;\in\; \mathbb{Z}.$$

Proof. The localization sequence for $V = X\setminus E \subset X$ and $U = S\setminus\{p\} \subset S$ gives a commutative diagram

$$\begin{tikzcd} A_d(E) \arrow[r, "i_*"] \arrow[d, "\pi_{E*}"] & A_d(X) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & A_d(V) \arrow[r] \arrow[d, "\pi_{V*}"] & 0 \\ A_d(\{p\}) \arrow[r] & A_d(S) \arrow[r] & A_d(U) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Here π_{V*} is an isomorphism since $\pi_V : V \xrightarrow{\sim} U$ is so. Moreover, since $E \simeq \mathbb{P}^1$ we know that for $d \neq 1$ the homomorphism $\pi_{E*} : A_d(E) \to A_d(\{p\})$ is an isomorphism, which gives

$$A_d(S) \xrightarrow{\sim} A_d(X) \quad \text{for} \quad d \neq 1.$$

For the middle Chow group, it suffices by the same diagram to observe that the proper pushforward $i_* : \mathbb{Z} = A_1(E) \to A_1(X)$ is split by $A_1(X) \to A_1(E), \gamma \mapsto \gamma \cdot [E]$ since $(E \cdot E)_X = -1$. This splitting of the top row of the diagram gives rise to a decomposition

$$A_1(X) \simeq A_1(V) \oplus \mathbb{Z} \simeq A_1(U) \oplus \mathbb{Z} \simeq A_1(S) \oplus \mathbb{Z}.$$

Let $\pi^* : A_1(S) \to A_1(X)$ be the inclusion obtained from this decomposition. Then by construction of the splitting we have $(\pi^*(\alpha) \cdot \pi^*(\beta))_X = (\alpha, \beta)_S$ and it follows that

$$((\pi^*(\alpha) + m \cdot [E]) \cdot (\pi^*(\beta) + n \cdot [E]))_X = (\alpha, \beta)_S - mn$$

as desired. Note that by construction $\pi^*([Z]) = [\pi^{-1}(Z)]$ for any curve $Z \subset S$. □

Example 3.5. We have only defined an intersection product with Cartier divisors, not with Weil divisors. In fact, on singular varieties X there is no good way to define an intersection product of Weil divisors:

a) Let $H \subset \mathbb{P}^3$ be a hyperplane and $C \subset H$ a smooth conic. For $p \in \mathbb{P}^3 \setminus H$ consider the cone

$$X = \overline{pC} \subset \mathbb{P}^3$$

with vertex p. Let $\ell_1, \ell_2 \subset X$ be two lines, and let $D \in \mathrm{Div}(X)$ be the effective Cartier divisor cut out by a hyperplane so that $[D] = 2[\ell_1]$. By restricting D to ℓ_2 one finds that

$$D \cdot [\ell_2] = [p].$$

But $[D] = 2[\ell_1]$, hence an intersection product for Weil divisors compatible with the one for Cartier divisors would necessarily take values in $A_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ since it would satisfy

$$[\ell_1] \cdot [\ell_2] = \tfrac{1}{2} \cdot [p].$$

On normal surfaces such an intersection product of Weil divisors can indeed by defined [Fulton, ex. 8.3.11]. But in higher dimension things are worse:

b) Let $H \subset \mathbb{P}^4$ be a hyperplane and $Q \subset H$ a smooth quadric. For $p \in \mathbb{P}^4 \setminus H$ consider the cone

$$X = \overline{pQ} \subset \mathbb{P}^4$$

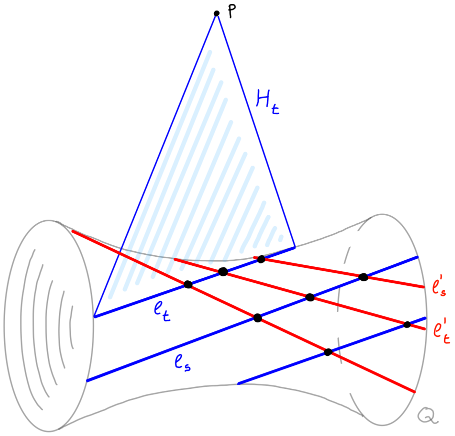
with vertex p. Now Q has two different rulings given by 1-parameter families of lines

$$\{\ell_t \subset Q\}_{t \in \mathbb{P}^1} \quad \text{and} \quad \{\ell_t' \subset Q\}_{t \in \mathbb{P}^1}$$

with $\ell_s \cap \ell_t = \ell'_s \cap \ell'_t = \varnothing$ for all $s \neq t$, and each ℓ_s intersects each ℓ'_t transversely in a single point. Taking the cone through the lines in the first ruling we obtain a family of planes

$$H_t \;:=\; \overline{p\,\ell_t} \;\subset\; X$$

that sweep out X as illustrated in the following picture (though the true picture would have to be drawn in a four-dimensional ambient space):



If there is an intersection product of Weil divisors on X that for finite transverse intersections counts the number of intersection points, then for general $s, t \in \mathbb{P}^1$ we have

$$([\ell_s]\cdot[H_t])_X \;=\; 0 \quad \text{and} \quad ([\ell'_s]\cdot[H_t])_X \;=\; 1.$$

But any two lines in a plane are rationally equivalent. Applying this first to the plane H_s from the first ruling and then to the plane $H'_t = \overline{p\,\ell'_t}$ from the second ruling, we see that

$$[\ell_s] \;\sim\; [H_s \cap H'_t] \quad \text{in } A_1(H_s),$$
$$[\ell'_t] \;\sim\; [H_s \cap H'_t] \quad \text{in } A_1(H'_t).$$

By pushforward to X we obtain

$$[\ell_s] \;\sim\; [H_s \cap H'_t] \;\sim\; [\ell'_t] \quad \text{in} \quad A_1(X),$$

so the above hypothetical intersection numbers between Weil divisors cannot be invariant under rational equivalence. The moral is that intersections with Cartier divisors behave much better than those with Weil divisors; this is one of the reasons why Chern classes play an important role in intersection theory.

4 Segre classes of vector bundles

We will define Chern classes of vector bundles as the inverse of Segre classes. To motivate the latter, let us begin with a simple geometric example:

Example 4.1. Let A be a smooth variety with trivial tangent bundle $\mathscr{T}_A$, for instance an affine space or an abelian variety. Let $i\colon X \hookrightarrow A$ be the embedding of a smooth hypersurface. Its normal bundle is a quotient of the trivial bundle $i^*(\mathscr{T}_A)$ sits in the exact sequence

$$0 \longrightarrow \mathscr{T}_X \longrightarrow i^*(\mathscr{T}_A) \longrightarrow \mathscr{N}_{X/A} \longrightarrow 0.$$

By dualizing this sequence, we see that the conormal bundle to the hypersurface is a line subbundle

$$\mathscr{E} := \mathscr{N}_{X/A}^\vee \subset i^*(\mathscr{T}_A^\vee) \simeq \mathscr{O}_X^{n+1} \quad \text{where} \quad \dim(A) = n+1.$$

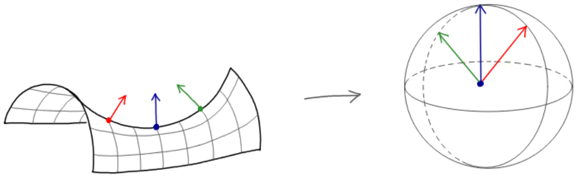
Fixing a trivialization $i^*(\mathscr{T}_A^\vee) \simeq \mathscr{O}_X^{n+1}$ as above, we obtain for each point $p \in X$ a line

$$\mathscr{E} \otimes \kappa(p) \subset i^*(\mathscr{T}_A^\vee) \otimes \kappa(p) \simeq \mathbb{A}^{n+1}$$

in a fixed affine space. As in differential geometry we define the *Gauss map* to be the morphism

$$\gamma\colon \quad X \longrightarrow \mathbb{P}^n, \quad p \mapsto [\mathscr{E} \otimes \kappa(p)]$$

that sends a point of the hypersurface to the conormal direction at that point:



Such Gauss maps have been studied in algebraic geometry for instance in relation with projective duality and with Torelli's theorem. In particular, we can consider the preimages

$$\gamma^{-1}(H) \subset X \quad \text{of linear subspaces} \quad H \subset \mathbb{P}^n$$

Their fundamental classes are simple examples of more general Segre classes to be defined soon. Conceptually one may view the Gauss map in terms of projective bundles as the composite

$$X \xleftarrow{\sim} \mathbb{P}(\mathscr{E}) \subset \mathbb{P}(i^*(\mathscr{T}_A^\vee)) \simeq X \times \mathbb{P}^n \twoheadrightarrow \mathbb{P}^n$$

so before proceeding we need to recall some general notions for projective bundles.

Let X be a scheme. For a locally free sheaf $\mathscr{E} \in \mathrm{Coh}(X)$, consider the associated vector bundle

$$E \;=\; \mathrm{Spec}_X(\mathrm{Sym}_X^* \mathscr{E}^\vee) \;\longrightarrow\; X \quad \text{where} \quad \mathscr{E}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X).$$

Here the symmetric algebra and the spectrum are taken in the relative sense. We denote by

$$p\colon \quad \mathbb{P}(E) \;=\; \mathrm{Proj}_X(\mathrm{Sym}_X^* \mathscr{E}^\vee) \;\longrightarrow\; X$$

the projective bundle of lines in the vector bundle E and by $\mathscr{O}_{\mathbb{P}(E)}(1) \in \mathrm{Pic}(\mathbb{P}(E))$ the natural relatively ample line bundle on it. The latter is the dual of the tautological subbundle

$$\mathscr{O}_{\mathbb{P}(E)}(-1) \;\subset\; p^*(\mathscr{E})$$

whose fiber over any point is the line corresponding to that point; it fits in the relative Euler sequence

$$0 \;\longrightarrow\; \Omega^1_{\mathbb{P}(E)/E}(1) \;\longrightarrow\; p^*(\mathscr{E}^\vee) \;\longrightarrow\; \mathscr{O}_{\mathbb{P}(E)}(1) \;\longrightarrow\; 0.$$

Remark 4.2. The tautological bundle from above depends on the vector bundle E, not just on the associated projective bundle: Indeed, let $\mathscr{L} \in \mathrm{Pic}(X)$ be a line bundle and $L \to X$ the associated vector bundle of rank one. Then we have a canonical isomorphism

$$\varphi\colon \quad \mathbb{P}(E) \;\xrightarrow{\sim}\; \mathbb{P}(E \otimes L)$$

of schemes over X, but in these terms $\varphi^*(\mathscr{O}_{\mathbb{P}(E\otimes L)}(1)) \simeq \mathscr{O}_{\mathbb{P}(E)}(1) \otimes p^*(\mathscr{L}^\vee)$.

In the last section we have seen that the first Chern class of a line bundle can be viewed as an endomorphism of Chow groups via the cap product. We now take a similar approach for the definition of Segre classes:

Definition 4.3. With notations as above, the projection $p\colon \mathbb{P}(E) \to X$ is a proper flat morphism of relative dimension $r = \mathrm{rk}(E) - 1$, so we have pullback and pushforward functors

$$A_*(X) \;\xrightarrow{p^*}\; A_{*+r}(\mathbb{P}(E)) \quad \text{and} \quad A_*(\mathbb{P}(E)) \;\xrightarrow{p_*}\; A_*(X).$$

For $i \in \mathbb{N}$ we define the cap product with the *i-th Segre class* of the vector bundle E to be the homomorphism

$$\begin{aligned} A_*(X) \;&\longrightarrow\; A_{*-i}(X), \\ \alpha \;&\mapsto\; s_i(E) \cap \alpha \;:=\; p_*\big(c_1(\mathscr{O}_{\mathbb{P}(E)}(1))^{r+i} \cap p^*(\alpha)\big). \end{aligned}$$

The geometric meaning of this is illustrated by the following example, generalizing the case of Gauss maps that we considered above:

Example 4.4. Suppose we have an embedding $\mathscr{E} \subset \mathscr{O}_X^{n+1}$ in a trivial bundle, and consider the diagram

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[r, hook, "\iota"] \arrow[dr, "p"'] & X \times \mathbb{P}^n \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}^n \\
 & X &
\end{tikzcd}$$

Then $\mathscr{O}_{\mathbb{P}(E)}(1) \simeq f^*(\mathscr{O}_{\mathbb{P}^n}(1))$ for the morphism $f := p_2 \circ \iota$. If $H_1, \dots, H_{r+i} \subset \mathbb{P}^n$ are hyperplanes whose preimages intersect the projective bundle properly in the sense that

$$\gamma^{-1}(H_1 \cap \cdots \cap H_{j+1}) \quad \text{is an effective Cartier divisor on} \quad \gamma^{-1}(H_1 \cap \cdots \cap H_j)$$

for all $j < r+i$, then by definition of the intersection product with Cartier divisors we have

$$c_1(\mathscr{O}_{\mathbb{P}(E)}(1))^{r+i} \cap p^*[X] \;=\; [\gamma^{-1}(H_1 \cap \cdots \cap H_{r+i})] \quad \text{in} \quad Z_{d-i}(\mathbb{P}(E))$$

where $d = \dim X$. In this case

$$s_i(E) \cap [X] \;=\; p_*[\gamma^{-1}(H_1 \cap \cdots \cap H_{r+i})] \quad \text{in} \quad Z_{d-i}(X).$$

For $i < 0$ the pushforward cycle on the right is zero for dimension reasons. For $i \geq 0$ its support is

$$|s_i(E) \cap [X]| \;=\; \{x \in X \mid p^{-1}(x) \cap H_1 \cap \cdots \cap H_{r+i} \neq \varnothing\},$$

where the intersection on the right hand side uses the embedding $p^{-1}(x) \subset \mathbb{P}^n$ given by ι. Thus Segre classes are classes of loci defined by incidence conditions which generalize the preimages of linear spaces under Gauss maps of hypersurfaces.

In fact the above incidence locus can be endowed with a natural scheme structure as follows: Let $\mathscr{F}$ be the quotient vector bundle defined by the exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X^{n+1} \longrightarrow \mathscr{F} \longrightarrow 0.$$

Pick a generic basis $\sigma_0, \dots, \sigma_n$ of $W := H^0(X, \mathscr{O}_X^{n+1})$. Then $\mathscr{F}$ is generated by the global sections

$$\tau_\nu \;=\; \text{image}(\sigma_\nu) \;\in\; H^0(X, \mathscr{F}).$$

Suppose that the above hyperplanes $H_1, \dots, H_{r+i} \subset \mathbb{P}^n$ have been chosen in such a way that

$$H_1 \cap \cdots \cap H_{r+i} \;=\; \mathbb{P}(U) \quad \text{for the subspace} \quad U_i \;:=\; \langle \sigma_0, \dots, \sigma_{n-r-i} \rangle \;\subset\; W.$$

Consider the fibers $\mathscr{E}(x) := \mathscr{E} \otimes \kappa(x)$ and $\mathscr{F}(x) := \mathscr{F} \otimes \kappa(x)$ at a point $x \in X$. Then set-theoretically the condition that describes the support of the Segre class can be reformulated as follows:

$$\begin{aligned} p^{-1}(x)\cap H_1\cap\cdots\cap H_{r+i}\neq\varnothing &\iff \mathscr{E}(x)\cap\langle\sigma_0,\dots,\sigma_{n-r-i}\rangle\neq 0\\ &\iff \tau_0(x),\dots,\tau_{n-r-i}(x)\in\mathscr{F}(x)\\ &\qquad\text{are linearly dependent over } k\\ &\iff \tau(x) \;=\; 0 \text{ for the section}\\ &\qquad \tau:=\tau_0\wedge\cdots\wedge\tau_{n-r-i}\in H^0(X,\mathrm{Alt}^{n-r-i+1}\mathscr{F}), \end{aligned}$$

and the locus $V(\tau)=\{x\in X\mid \tau(x)=0\}\subset X$ has a natural scheme structure as the vanishing locus of a section of a vector bundle. If the base field k is algebraically closed of characteristic zero, then a transversality result by Kleiman says that for generically chosen hyperplanes and sections as above, the above loci are generically reduced of the expected dimension and

$$s_i(\mathscr{E})\cap[X] \;=\; [V(\tau)].$$

Thus Segre classes for subbundles of trivial bundles are represented as degeneraci loci where sections of the quotient bundle become linearly dependent.

After this somewhat heuristic discussion where transversality issues have been glossed over, let us now develop the general properties of Segre classes starting from the definition given above. We begin with the trivial case of Segre classes for line bundles (as in the above example of Gauss maps for smooth hypersurfaces):

Lemma 4.5. *Let L be a line bundle on a scheme X. Then*

$$s_i(L)\cap(-) \;=\; (-1)^i\cdot c_1(L)^i\cap(-) \quad \textit{for all } i\in\mathbb{N}_0.$$

Proof. For line bundles L one has $\mathbb{P}(L)\simeq X$ and $\mathscr{O}_{\mathbb{P}(E)}(1)\simeq\mathscr{L}^\vee$ by remark 4.2. Hence we get

$$s_i(L)\cap\alpha \;=\; c_1(\mathscr{O}_{\mathbb{P}(L)}(1))^i\cap\alpha \;=\; c_1(L^\vee)^i\cap\alpha \;=\; (-1)^i\cdot c_1(L)^i\cap\alpha$$

by taking $r=0$ in the definition of Segre classes and using $c_1(\mathscr{L}^\vee)=-c_1(\mathscr{L})$. □

In particular, this shows that the Segre classes of a line bundle can be nonzero in all degrees $i\in\{0,1,\dots,\dim X\}$. For vector bundles of arbitrary rank we have the following properties:

Proposition 4.6. *For any vector bundle E on X we have:*

a) Pullback: If $f\colon Y\to X$ is flat, then

$$s_i(f^*E)\cap f^*(-) \;=\; f^*(s_i(E)\cap(-)).$$

b) Projection formula: If $f\colon Y\to X$ is proper, then

$$f_*(s_i(f^*E)\cap(-)) \;=\; s_i(E)\cap f_*(-).$$

c) Commutativity: If F is another vector bundle on X, then

$$s_i(E)\cap(s_j(F)\cap(-)) \;=\; s_j(F)\cap(s_i(E)\cap(-)).$$

d) Vanishing and normalization: We have

$$s_i(E)\cap(-) \;=\; \begin{cases} 0 & \textit{for } i\notin\{0,1,\dots,\dim X\},\\ id & \textit{for } i=0.\end{cases}$$

Proof. Let $\mathscr{L}_E=\mathscr{O}_{\mathbb{P}(E)}(1)$. For *a), b)* we note that $\mathscr{O}_{\mathbb{P}(f^*(E)}(1)\simeq g^*\mathscr{L}_E$ where g is defined by the Cartesian square

$$\begin{tikzcd}
\mathbb{P}(f^*E) \arrow[r, "g"] \arrow[d, "q"'] & \mathbb{P}(E) \arrow[d, "p"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

So if f is proper, then for $\alpha\in A_*(X)$ and $r=\mathrm{rk}(E)$ we obtain

$$\begin{aligned}
f_*\left(s_i(f^*E)\cap\alpha\right) &= f_*q_*\big(c_1(\mathscr{O}_{\mathbb{P}(f^*(E)}(1))^{r+i}\cap q^*\alpha\big) && \text{by definition}\\
&= p_*g_*\big(c_1(g^*(\mathscr{L}_E))^{r+i}\cap q^*\alpha\big) && \text{since } f_*q_*=p_*g_*\\
&= p_*\big(c_1(\mathscr{L}_E)^{r+i}\cap g_*q^*\alpha\big) && \text{proj. formula for } c_1\cap(-)\\
&= p_*\big(c_1(\mathscr{L}_E)^{r+i}\cap p^*f_*\alpha\big) && \text{by base change}\\
&= s_i(E)\cap f_*\alpha && \text{by definition}
\end{aligned}$$

This proves *b)*, and a similar computation shows *a)*. For *c)* we reset notations and consider the Cartesian square

$$\begin{tikzcd}
\mathbb{P}(E)\times_X\mathbb{P}(F) \arrow[r, "p'"] \arrow[d, "q'"'] & \mathbb{P}(F) \arrow[d, "q"] \\
\mathbb{P}(E) \arrow[r, "p"] & X
\end{tikzcd}$$

Let $s=\mathrm{rk}(F)$ and $\mathscr{L}_F=\mathscr{O}_{\mathbb{P}(F)}(1)$. Then for $\alpha\in A_*(X)$ we obtain

$$\begin{aligned}
s_i(E)\cap(s_j(F)\cap\alpha) &= p_*\left[c_1(\mathscr{L}_E)^{r+i}\cap p^*q_*\big(c_1(\mathscr{L}_F)^{s+j}\cap q^*\alpha\big)\right]\\
&\overset{(1)}{=} p_*\left[c_1(\mathscr{L}_E)^{r+i}\cap q'_*p'^*\big(c_1(\mathscr{L}_F)^{s+j}\cap q^*\alpha\big)\right]\\
&\overset{(2)}{=} p_*q'_*\left[q'^*c_1(\mathscr{L}_E)^{r+i}\cap p'^*\big(c_1(\mathscr{L}_F)^{s+j}\cap q^*\alpha\big)\right]\\
&\overset{(3)}{=} p_*q'_*\left[c_1(q'^*\mathscr{L}_E)^{r+i}\cap\big(c_1(p'^*\mathscr{L}_F)^{s+j}\cap p'^*q^*\alpha\big)\right]\\
&\overset{(4)}{=} q_*p'_*\left[c_1(p'^*\mathscr{L}_F)^{s+j}\cap\big(c_1(q'^*\mathscr{L}_E)^{r+i}\cap q'^*p^*\alpha\big)\right]
\end{aligned}$$

where we have used

(1) $p^*q_* = q'_*p'^*$ by the base change lemma 6.7,

(2) the projection formula for $q'\colon \mathbb{P}(E)\times_X \mathbb{P}(F) \to \mathbb{P}(E)$,

(3) the compatibility of flat pullback with c_1 and with the cap product,

(4) commutativity of the intersection with Cartier divisors and functoriality.

The claim now follows by reversing the role of E and F in the computation. For *d)* trivially $s_i(E)\cap(-) = 0$ for all $i > \dim X$, indeed

$$\mathrm{Hom}(A_*(X), A_{*-i}(X)) \;=\; 0 \quad \text{for} \quad i \;>\; \dim X$$

since $A_d(X) = 0$ for all $d \notin \{0,1,\dots,\dim X\}$. To show $s_i(E)\cap(-) = 0$ for $i < 0$, it will be enough to check that

$$s_i(E)\cap[Z] \;=\; 0 \quad \text{for any subvariety} \quad Z \subset X \quad \text{and all} \quad i \;<\; 0.$$

Applying the already proven projection formula to the inclusion of the subvariety, we may assume that $X = Z$. In this case, the claimed vanishing follows from the fact that $A_{\dim X - i}(X) = 0$ for all $i < 0$. To prove the remaining claim for $i = 0$, we can as before assume that $X = Z$ is irreducible. Then $A_{\dim X}(X) = \mathbb{Z}\cdot[X]$ and hence we know that

$$s_0(E)\cap[X] \;=\; m\cdot[X] \quad \text{for some} \quad m \;\in\; \mathbb{Z}.$$

To check that $m = 1$, we may replace X by an open subset and therefore assume that E is a trivial vector bundle. In this case the claim is clear from the description in example 4.4. □

We have seen in theorem 4.2 that for affine bundles the flat pullback from the base to the total space is surjective. Using Segre classes we can now easily show that for projective bundles it is injective. We will soon compute the Chow groups of all such bundles, but the following partial result is needed on the way:

Corollary 4.7. *Let E be a vector bundle of rank $r+1$ on X. Then the flat pullback gives a split monomorphism*

$$p^*\colon \quad A_d(X) \;\hookrightarrow\; A_{d+r}(\mathbb{P}(E)) \quad \textit{for each} \quad d \;\in\; \mathbb{Z}.$$

Proof. By part *d)* of the above proposition, the map $\gamma \mapsto p_*(c_1(\mathscr{O}_{\mathbb{P}(E)}(1))^r \cap \gamma)$ provides a left inverse of $p^*\colon A_d(X) \to A_{d+r}(\mathbb{P}(E))$. □

5 Chern classes of vector bundles

In the previous section we have defined Segre classes of a vector bundle E on X as endomorphisms

$$s_i(E) \;:=\; s_i(E)\cap(-) \;\in\; \mathrm{End}(A_*(X))$$

of the group $A_*(X)$, and we have seen that these endomorphisms commute with each other. Thus they generate a commutative subring of $\mathrm{End}(A_*(X))$. It is convenient to put them together in a single generating series:

Definition 5.1. The *Segre polynomial* of a vector bundle E on a scheme X is the power series

$$s_t(E) \;:=\; 1+s_1(E)t+s_2(E)t^2+\cdots \;\in\; \mathrm{End}(A_*(X))[[t]].$$

This is in fact a polynomial of degree $\leq \dim X$ by the properties of Segre classes.

Example 5.2. Let L be a line bundle on X. Then we have

$$\begin{aligned} s_t(L) \;&=\; \sum_{i=0}^{\infty}(-1)^i\cdot c_1(L)^i\cdot t^i \qquad && \text{since } s_i(L)=(-1)^i c_1(L)^i \text{ by lemma 4.5,} \\ &=\; (1+c_1(L)\cdot t)^{-1} && \text{as a formal power series in } \mathrm{End}(A_*(X))[[t]]. \end{aligned}$$

This suggests that in general we should look at the inverse of the Segre polynomial:

Definition 5.3. As a power series over a ring with pairwise commuting coefficients and with constant term one, the Segre polynomial has a formal inverse. We can therefore define the *Chern classes* $c_i = c_i(E) \in \mathrm{End}(A_*(X))$ to be the coefficients in this formal inverse:

$$(1+s_1t+s_2t^2+\cdots)^{-1} \;=\; 1+c_1t+c_2t^2+\cdots \quad \text{in} \quad \mathrm{End}(A_*(X))[[t]].$$

Explicitly

$$\begin{aligned} c_1 \;&=\; -s_1, \\ c_2 \;&=\; s_1^2-s_2, \\ c_3 \;&=\; -s_1^3+2s_1s_2-s_3, \quad \text{etc.} \end{aligned}$$

In particular $c_i \in \mathrm{Hom}(A_*(X), A_{*-i}(X))$ and hence $c_i = 0$ for all $i > \dim X$. So the power series

$$c_t(E) \;:=\; 1+c_1t+c_2t^2+\cdots \;\in\; \mathrm{End}(A_*(X))[[t]]$$

is again a polynomial of degree $\leq \dim X$ which is called the *Chern polynomial* of E.

The variable t is introduced only for notational convenience: The ring $\mathrm{End}(A_*(X))$ is graded via

$$\mathrm{End}(A_*(X)) \;=\; \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(A_*(X), A_{*-i}(X)),$$

hence the same information as in the Segre and Chern polynomials is also captured by the *total Segre resp. Chern class*

$$\begin{aligned} s(E) \;&:=\; 1+s_1+s_2+\cdots \;\in\; \mathrm{End}(A_*(X)), \\ c(E) \;&:=\; 1+c_1+c_2+\cdots \;\in\; \mathrm{End}(A_*(X)). \end{aligned}$$

Proposition 5.4. *For any vector bundle E on X we have:*

a) Pullback: If $f\colon Y \to X$ is flat, then

$$c_i(f^*E) \cap f^*(-) \;=\; f^*(c_i(E) \cap (-)).$$

b) Projection formula: If $f\colon Y \to X$ is proper, then

$$f_*(c_i(f^*E) \cap (-)) \;=\; c_i(E) \cap f_*(-).$$

c) Commutativity: If F is another vector bundle on X, then

$$c_i(E) \cap (c_j(F) \cap (-)) \;=\; c_j(F) \cap (c_i(E) \cap (-)).$$

d) Vanishing and normalization: We have

$$c_i(E) \cap (-) \;=\; \begin{cases} 0 & \textit{for } i \notin \{0,1,\dots,\dim X\}, \\ id & \textit{for } i = 0. \end{cases}$$

Proof. Follows from the same statements for Segre classes, see proposition 5.4. □

In fact the vanishing statement in *d)* holds also for all $i > \operatorname{rk}(E)$. To show this and many other properties of Chern classes, we will use the following important tool that allows to reduce the proof of Chern class identities for arbitrary vector bundles to the case of line bundles:

Proposition 5.5 (Splitting principle). *Let $\mathbb{S}$ be a finite set of vector bundles on a scheme X. Then there exists a proper flat morphism $f\colon Y \to X$ such that*

a) the pullback $f^*\colon A_*(X) \to A_{*+d}(Y)$ is injective, where $d = \dim(Y/X)$, and

b) for every $E \in \mathbb{S}$, the pullback $f^*(E)$ is an iterated extension of line bundles, i.e. it has a finite filtration

$$f^*(E) \;=\; E_r \;\supset\; E_{r-1} \;\supset\; \cdots \;\supset\; E_0 \;=\; 0$$

by vector subbundles such that all the quotients $L_i = E_i/E_{i-1}$ are line bundles.

Proof. It is enough to prove the claim for a single vector bundle E, since we can then repeat the construction to cover any finite set $\mathbb{S}$ of vector bundles. The proof for a single vector bundle E goes by induction on the rank $r = \operatorname{rk}(E)$: The case $r = 1$ is trivial. For the induction step, consider the projective bundle $p\colon \mathbb{P}(E) \to X$ and the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(E)}(-1) \longrightarrow p^*(E) \longrightarrow Q \longrightarrow 0$$

where $Q := p^*(E)/\mathscr{O}_{\mathbb{P}(E)}(-1)$ is a vector bundle of rank $r-1$. By corollary 4.7 the map

$$p^*\colon \quad A_*(X) \longrightarrow A_{*+r-1}(\mathbb{P}(E))$$

is injective. Moreover, by induction on the rank there is a proper flat $g\colon Y \to \mathbb{P}(E)$ such that

a) $g^*\colon A_*(\mathbb{P}(E)) \to A_{*+d}(Y)$ for $d = \dim(Y/\mathbb{P}(E))$ is injective, and

b) $g^*(Q)$ is an iterated extension of line bundles.

The claim now follows by taking the composite morphism $f = p \circ g$. □

This essentially reduces us to vector bundles which are iterated extensions of line bundles. For such vector bundles the Chern classes are easy to compute:

Theorem 5.6. *Let E be a vector bundle with a filtration $E = E_r \supset \cdots \supset E_0 = 0$ by vector subbundles such that the quotients $L_i = E_i/E_{i-1}$ are line bundles. Then we have*

$$c_t(E) \;=\; \prod_{i=1}^{r} (1 + c_1(L_i)\cdot t).$$

Proof. We first claim that if there exists a nowhere vanishing section $s \in H^0(X,E)$, then $c_1(L_1)\cdots c_1(L_r) = 0$. Indeed, for any $s \in H^0(X,E)$ with zero locus $V(s) \subset X$ we claim that

$$c_1(L_1)\cap\cdots\cap c_1(L_r)\cap\alpha \in \operatorname{image}\big(A_{d-r}(V(s)) \to A_{d-r}(X)\big) \quad \text{for all} \quad \alpha \in A_d(X).$$

To see this, consider the section $\bar{s} \in H^0(X,L_r)$ which is the image of s under the map $E \twoheadrightarrow L_r$. Let $i\colon V(\bar{s}) \hookrightarrow X$ be the inclusion of the zero locus of this section, and put

$$A_{n-1}(V(\bar{s})) \;\ni\; D_r \;:=\; \begin{cases} [V(\bar{s})] & \text{if } V(\bar{s}) \subset X \text{ is a Cartier divisor,} \\ c_1(i^*L_r) & \text{otherwise.} \end{cases}$$

Then $c_1(L_r)\cap\alpha = i_*(D_r\cdot\alpha)$. By the projection formula it follows that

$$c_1(L_1)\cap\cdots\cap c_1(L_r)\cap\alpha \;=\; i_*\big(c_1(i^*L_1)\cap\cdots\cap c_1(i^*L_{r-1})\cap(D_r\cdot\alpha)\big)$$

By induction the term in parenthesis on the right hand side is represented by a cycle on $V(s)$, since i^*E_{r-1} has a section with zero locus $V(s)$. This proves our claim.

Now consider $p\colon \mathbb{P}(E) \to X$. The tautological subbundle $\mathscr{O}_{\mathbb{P}(E)}(-1) \subset p^*(E)$ gives rise to a trivial line subbundle $\mathscr{O}_{\mathbb{P}(E)} \subset p^*(E)\otimes\mathscr{O}_{\mathbb{P}(E)}(1)$. Hence we have a nowhere vanishing section

$$s \;\in\; H^0(\mathbb{P}(E), p^*(E)\otimes\mathscr{O}_{\mathbb{P}(E)}(1)).$$

Applying the claim from the previous step to the vector bundle $p^*(E)\otimes\mathscr{O}_{\mathbb{P}(E)}(1)$, we find that

$$\prod_{j=1}^{r} c_1(p^*L_j\otimes\mathscr{O}_{\mathbb{P}(E)}(1)) \;=\; 0.$$

Since $c_1(p^*L_j \otimes \mathscr{O}_{\mathbb{P}(E)}(1)) = c_1(p^*L_j) + \zeta$ for the class $\zeta = c_1(\mathscr{O}_{\mathbb{P}(E)}(1))$, we can rewrite the previous displayed equation by expanding the product on the left hand side as

$$\zeta^r + \varepsilon_1 \zeta^{r-1} + \cdots + \varepsilon_r \;=\; 0$$

where ε_j is the j-th elementary symmetric function of $c_1(p^*L_1), \dots, c_1(p^*L_r)$. It follows that

$$\zeta^{r+i-1} + \varepsilon_1 \zeta^{r+i-2} + \cdots + \varepsilon_r \zeta^{i-1} \;=\; 0,$$

so for all $\alpha \in A_*(X)$ we get

$$p_*(\zeta^{r-1+i} \cap p^*\alpha + \varepsilon_1 \zeta^{r-1+2} \cap p^*\alpha + \cdots + \varepsilon_r \zeta^{i-1} \cap p^*\alpha) \;=\; 0.$$

The projection formula and the definition of Segre classes then imply

$$(s_i(E) + e_1 s_{i-1}(E) + \cdots + e_r s_{i-r}(E)) \cap \alpha \;=\; 0$$

where e_j are the j-the elementary symmetric functions of $c_1(L_1), \dots, c_1(L_r)$. In other words

$$(1 + e_1 t + \cdots + e_r t^r) \cdot s_t(E) \;=\; 1,$$

hence we have $c_i(E) = e_i$ and the desired formula follows. □

We can now see easily that the Chern classes vanish in all degrees above the rank of the bundle, which is one reason why Chern classes are computationally often easier to handle than Segre classes, although the definition of the latter is more natural from a geometric viewpoint. We also record a few other functorial properties of Chern classes:

Corollary 5.7. *Let E be a vector bundle of rank r on X.*

a) We have $c_i(E) = 0$ for all $i > \mathrm{rk}(E)$.

Moreover, if E has a nowhere vanishing section, then $c_r(E) = 0$.

b) For an extension $0 \to E' \to E \to E'' \to 0$ of vector bundles, we have the Whitney formula

$$c_t(E) \;=\; c_t(E') \cdot c_t(E'') \quad \textit{and} \quad s_t(E) \;=\; s_t(E') \cdot s_t(E'').$$

In particular, these classes not depend on the extension $[E] \in \mathrm{Ext}^1(E'', E')$.

c) For the dual bundle we have $c_i(E^\vee) = (-1)^i c_i(E)$ and $s_i(E^\vee) = (-1)^i s_i(E)$.

d) We have $c_1(E) = c_1(\det(E))$ for the determinant line bundle $\det(E) = \mathrm{Alt}^r(E)$.

e) For the tensor product with a line bundle L we have

$$c_i(E \otimes L) \;=\; \sum_{j=0}^{i} \tbinom{r-j}{i-j} c_1(L)^{i-j} c_j(E).$$

Proof. By the splitting principle we may assume that $E = E_r \supset E_{r-1} \supset \cdots \supset E_0 = 0$ where each $L_i = E_i/E_{i-1}$ is a line bundle, and the previous theorem then says that we have

$$c_t(E) \;=\; \prod_{i=1}^{r}(1+\alpha_i \cdot t) \quad \text{for the classes} \quad \alpha_i \;:=\; c_1(L_i).$$

The first claim in *a)* is then obvious from theorem 5.6, and the second claim follows from its proof. For claim *b)* we may choose our filtration to be compatible with the given subbundle in the sense that $E' = E_s$ and $E'' = E/E_s$ for $s = \mathrm{rk}(E')$. In that case we have

$$c_t(E') \;=\; \prod_{i=1}^{s}(1+\alpha_i \cdot t) \quad \text{and} \quad c_t(E'') \;=\; \prod_{i=s+1}^{r}(1+\alpha_i \cdot t)$$

by another application of the theorem, hence we obtain $c_t(E) = c_t(E')c_t(E'')$ and then also $s_t(E) = s_t(E')s_t(E'')$ by taking the inverse power series. This in particular shows that the Chern and Segre classes of an extension of vector bundles do not depend on the extension class of the bundle, so in what follows we may assume that

$$E \;=\; L_1 \oplus \cdots \oplus L_r$$

splits as a direct sum of line bundles. Part *c)* then follows because $E^\vee = L_1^\vee \oplus \cdots \oplus L_r^\vee$ so that

$$c_t(E^\vee) \;=\; \prod_{i=1}^{r}(1+c_t(L_i^\vee)\cdot t) \;=\; \prod_{i=1}^{r}(1-c_t(L_i)\cdot t) \;=\; c_{-t}(E).$$

Part *d)* similarly follows from

$$\begin{aligned} c_1(E) &= c_1(L_1)+\cdots+c_1(L_r) && \text{the first elem. symm. pol. in } c_1(L_1),\ldots,c_1(L_r),\\ &= c_1(L_1\otimes\cdots\otimes L_r) && \text{since } c_1:\ \mathrm{Pic}(X)\to A_*(X) \text{ is a homomorphism,}\\ &= c_1(\det(E)) && \text{since } \det(E)=L_1\otimes\cdots\otimes L_r. \end{aligned}$$

For part *e)* put $\alpha_\nu = c_1(L_\nu)$ and $\beta = c_1(L)$, then $c_1(L_\nu \otimes L) = \alpha_\nu + \beta$ and one finds

$$\begin{aligned} c_t(E\otimes L) = \prod_{\nu=1}^{r}(1+c_1(L_\nu\otimes L)\cdot t) &= \prod_{\nu=1}^{r}(1+(\alpha_\nu+\beta)\cdot t)\\ &= \prod_{\nu=1}^{r}((1+\beta\cdot t)+\alpha_\nu\cdot t)\\ &= \sum_{j=0}^{r}(1+\beta\cdot t)^{r-j}\sum_{1\le\nu_1<\cdots<\nu_j\le r}\alpha_{\nu_1}\cdots\alpha_{\nu_j}\cdot t^j\\ &= \sum_{i=0}^{r}\sum_{j=0}^{i}\tbinom{r-j}{i-j}\cdot c_1(L)^{i-j}\cdot c_j(E)\cdot t^i \end{aligned}$$

where in the last step we have expanded the first factor, rewritten the second factor as the elementary symmetric polynomial $c_j(E) = \sum_{1\le\nu_1<\dots<\nu_j\le r} \alpha_{\nu_1}\cdots\alpha_{\nu_j}$ and put together the powers of t. ☐

Definition 5.8. For a vector bundle E on X and a flat proper morphism $f\colon Y\to X$ such that

- $f^*\colon A_*(X)\to A_{*+d}(Y)$ with $d=\dim(Y/X)$ is injective,
- $f^*(E)$ is a successive extension of line bundles $L_1,\dots,L_r$,

the classes $\alpha_i = c_1(L_i)$ in $A_*(Y)$ (or in $\mathrm{End}(A_*(Y))$) are called *Chern roots* of the vector bundle. By the previous theorem the Chern classes of E are mapped to the elementary symmetric polynomials in the Chern roots via $f^*\colon A_*(X)\hookrightarrow A_{*+d}(Y)$, and by abuse of notation we write

$$c_t(E) \;=\; \prod_{i=1}^{r}(1+\alpha_i\cdot t), \quad \text{i.e.} \quad c_i(E) \;=\; \sum_{1\le\nu_1<\dots<\nu_i\le r}\alpha_{\nu_1}\cdots\alpha_{\nu_i}.$$

Notice however:

a) The Chern roots α_ν are classes on Y and usually will not descend to X.

b) The proper flat morphism $f\colon Y\to X$ in the splitting principle is not unique.

c) Even when f is given, the individual Chern roots are not well-defined:

Example 5.9. The trivial vector bundle $E=\mathscr{O}_X^{\oplus 2}$ on $X=\mathbb{P}^1$ can also be written as the extension

$$0 \longrightarrow \mathscr{O}_X(-1) \xrightarrow{\binom{x}{y}} E \xrightarrow{(-y\;x)} \mathscr{O}_X(1) \longrightarrow 0.$$

The assumptions of the splitting principle hold for $f=id\colon Y=X\to X$, hence E could be said to have Chern roots

$$\alpha_1 \;=\; \alpha_2 \;=\; 0 \quad \text{or} \quad \alpha_1' \;=\; -\alpha_2' \;=\; c_1(\mathscr{O}_X(1)).$$

This is no contradiction, the elementary symmetric polynomials in the Chern roots are well-defined:

$$c_1(\mathscr{E}) \;=\; \alpha_1+\alpha_2 \;=\; \alpha_1'+\alpha_2' \;=\; 0 \quad \text{and} \quad c_2(\mathscr{E}) \;=\; \alpha_1\cdot\alpha_2 \;=\; \alpha_1'\cdot\alpha_2' \;=\; 0.$$

Example 5.10. Let E be a vector bundle of rank r with Chern roots $\alpha_1,\dots,\alpha_r$, then we have

$$c_t(\mathrm{Sym}^d(E)) \;=\; \prod_{1\le i_1\le\dots\le i_d\le r}(1+(\alpha_{i_1}+\dots+\alpha_{i_d})\cdot t),$$

$$c_t(\mathrm{Alt}^d(E)) \;=\; \prod_{1\le i_1<\dots<i_d\le r}(1+(\alpha_{i_1}+\dots+\alpha_{i_d})\cdot t).$$

The interpretation of Chern classes as the descent of the elementary symmetric polynomials in the Chern roots has a useful consequence:

Definition 5.11. Let R be a commutative ring and $\Lambda_r = R[x_1,\dots,x_r]^{\mathfrak{S}_r}$ the ring of symmetric polynomials in r variables. By the fundamental theorem of symmetric functions we know this ring is a polynomial ring $\Lambda_r = R[e_1,\dots,e_r]$ in the elementary symmetric functions

$$e_i \ := \sum_{1\le\nu_1<\cdots<\nu_i\le r} x_{\nu_1}\cdots x_{\nu_i} \ \in \ \Lambda_r$$

Hence for any $P\in\Lambda_r$ and any vector bundle E of rank r with Chern roots $\alpha_1,\dots,\alpha_r$ the element

$$P(\alpha_1,\dots,\alpha_r) \ \in \ \mathrm{End}(A_*(X))\otimes R$$

is well-defined as a polynomial in the Chern classes $e_i(\alpha_1,\dots,\alpha_r) = c_i(E)$.

Example 5.12. By the Whitney formula, the total Chern class of a vector bundle is multiplicative with respect to direct sums; its behaviour with respect to tensor products is somewhat complicated. Using the above definition, we can find instead an expression in Chern classes which is additive for direct sums and multiplicative for tensor products: For a vector bundle E on X of rank r with Chern roots $\alpha_1,\dots,\alpha_r$ we define its *Chern character* by

$$\mathrm{ch}(E) \ := \ \sum_{i=1}^{r}\exp(\alpha_i) \quad \text{where} \quad \exp(\alpha_i) \ := \ \sum_{n=0}^{\infty}\frac{1}{n!}\alpha_i^n \ \in \ \mathrm{End}(A_*(X))\otimes\mathbb{Q}.$$

Note that the exponential series on the right is in fact finite because the summands are zero for all $n>\dim(X)$. Writing $r=\mathrm{rk}(E)$ and $c_i=c_i(E)$, one computes the first terms

$$\mathrm{ch}(E) \ = \ r+c_1+\tfrac{1}{2}(c_1^2-2c_2)+\tfrac{1}{6}(c_1^3-3c_1c_2+c_3)+\cdots$$

It is clear from the definition in terms of Chern roots that the Chern character is additive in the sense that

$$\mathrm{ch}(E) \ = \ \mathrm{ch}(E')+\mathrm{ch}(E'') \quad \text{for extensions} \quad 0 \to E' \to E \to E'' \to 0$$

of vector bundles. It is also multiplicative in the sense $\mathrm{ch}(E'\otimes E'') = \mathrm{ch}(E')\cdot\mathrm{ch}(E'')$, so the Chern character defines a ring homomorphism

$$\mathrm{ch}\colon \quad K(X) \longrightarrow A_*(X)\otimes\mathbb{Q}.$$

Here $K(X)$ denotes the *Grothendieck ring* which is defined as follows:

- Its additive group is the free abelian group on isomorphism classes $[E]$ of vector bundles E on X modulo the relations

$$[E] \ = \ [E']+[E''] \quad \text{for any extension} \quad 0 \to E' \to E \to E'' \to 0.$$

- Its product is defined by the tensor product of vector bundles $[E]\cdot[F] := [E\otimes F]$.

This ring will later become important in the Grothendieck-Riemann-Roch theorem.

Remark 5.13. The above discussion of the splitting principle shows that the Chern classes can be characterized axiomatically as follows: There exists a unique way to assign to each vector bundle E on a scheme X a polynomial $c_t(E) \in \operatorname{End}(A_*(X))[t]$ such that the following properties hold.

a) Functoriality: For any morphism $f\colon Y \to X$ we have $f^*(c_t(E)) = c_t(f^*(E))$.

b) Whitney formula: For any extension $0 \to E' \to E \to E'' \to 0$ of vector bundles we have
$$c_t(E) \;=\; c_t(E') \cdot c_t(E'').$$

c) Normalization: For line bundles L we have $c_t(L) = 1 + c_1(L) \cdot t$.

6 Example: Chern classes of varieties

One of the most frequent applications of Chern classes in algebraic geometry is the case of tangent bundles:

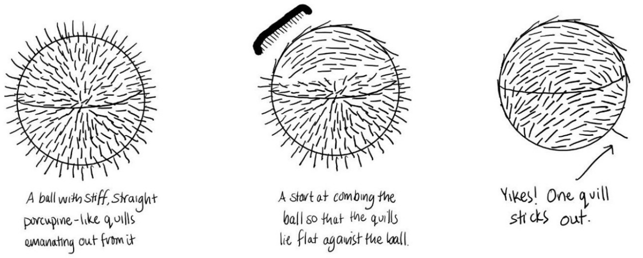
Definition 6.1. The *Chern classes* of a smooth variety X with tangent bundle $\mathscr{T}_X$ are defined by
$$c_i(X) \;:=\; c_i(\mathscr{T}_X) \;\in\; A_i(X).$$
We call $c(X) := 1 + c_1(X) + c_2(X) + \cdots \in A_*(X)$ the *total Chern class* of X.

Example 6.2. The tangent bundle of the projective space $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$ is given by the Euler sequence
$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \xrightarrow{\;i\;} \mathscr{O}_{\mathbb{P}^n}(1)^{n+1} \xrightarrow{\;p\;} \mathscr{T}_{\mathbb{P}^n} \longrightarrow 0$$
where
$$i(1) \;:=\; (x_0, \ldots, x_n) \quad \text{and} \quad p(f_0, \ldots, f_n) \;:=\; \sum_{i=0}^{n} f_i \cdot \tfrac{\partial}{\partial x_i}.$$
Applying the Whitney formula to the Euler sequence, we obtain the total Chern class
$$c(\mathbb{P}^n) \;=\; c(\mathscr{O}_{\mathbb{P}^n}(1))^{n+1} \;=\; (1+\zeta)^{n+1} \quad \text{for the class} \quad \zeta \;=\; c_1(\mathscr{O}_{\mathbb{P}^n}(1)).$$
This gives an algebraic version of the hairy ball theorem:

Corollary 6.3. *Every vector field $\xi \in H^0(\mathbb{P}^n, \mathscr{T}_{\mathbb{P}^n})$ has a zero.*

Proof. By the above we have $c_n(\mathscr{T}_{\mathbb{P}^n}) = (n+1)\zeta^n \neq 0$, so the result follows from the fact that a vector bundle with a nowhere vanishing global section has trivial top Chern class by corollary 5.7(a). □

The following cartoon by Susan D'Agostino illustrates the situation:



The Chern classes of smooth hypersurfaces, and more generally of smooth complete intersection in projective space, are easily deduced from this:

Lemma 6.4. *Consider a smooth complete intersection* $X = H_1 \cap \cdots \cap H_r \subset \mathbb{P}^n$ *of hypersurfaces* $H_i \subset \mathbb{P}^n$ *of degree* $d_i = \deg(H_i)$*. Then its total Chern class is given by the formula*

$$c(X) \;=\; \frac{(1+\zeta)^{n+1}}{\prod_{i=1}^{r}(1+d_i\zeta)} \quad \textit{for the class} \quad \zeta \;=\; c_1(\mathscr{O}_{\mathbb{P}^n}(1)|_X).$$

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_{\mathbb{P}^n}|_X \longrightarrow \mathscr{N}_{X/\mathbb{P}^n} \longrightarrow 0$$

where $\mathscr{N}_{X/\mathbb{P}^n} \;=\; (\mathscr{I}_X/\mathscr{I}_X^2)^\vee$ denotes the normal bundle. Since by assumption X is a complete intersection of hypersurfaces $H_i = V_+(f_i)$ with $f_i \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d_i))$, its ideal sheaf has the presentation

$$\bigoplus_{i,j=1}^{r} \mathscr{O}_{\mathbb{P}^n}(-d_i-d_j) \xrightarrow{\beta} \bigoplus_{i=1}^{r} \mathscr{O}_{\mathbb{P}^n}(-d_i) \xrightarrow{\alpha} \mathscr{I}_X \longrightarrow 0$$

where α, β are given on the standard basis vectors e_i and e_{ij} of the respective direct sums by

$$\alpha(e_i) \;=\; f_i \quad \text{and} \quad \beta(e_{ij}) \;=\; f_j e_i - f_i e_j.$$

Since $\beta|_X = 0$, it follows that $\mathscr{I}_X/\mathscr{I}_X^2 = (\mathscr{I}_X)|_X \simeq \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^n}(-d_i)|_X$. Hence the normal bundle is

$$\mathscr{N}_{X/\mathbb{P}^n} \;\simeq\; \bigoplus_{i=1}^{r} \mathscr{O}_{\mathbb{P}^n}(d_i)|_X$$

with total Chern class $c(\mathscr{N}_{X/\mathbb{P}^n}) = \prod_{i=1}^r (1+d_i\zeta)$. Applying the Whitney formula we get

$$c(X) \;=\; \frac{c(\mathbb{P}^n)}{c(\mathscr{N}_{X/\mathbb{P}^n})} \;=\; \frac{(1+\zeta)^{n+1}}{\prod_{i=1}^{r}(1+d_i\zeta)}$$

as claimed. □

For any smooth projective variety X we can attach a set of numerical invariants, the *Chern numbers*

$$\int_X P(c_1(X),\dots,c_n(X)) \;\in\; \mathbb{Z}$$

for $P \in \mathbb{Z}[x_1,\dots,x_n]$ weighted homogenous of degree n where $\deg(x_i) := i$. These Chern numbers and various inequalities between them play an important role in the classification of varieties. The most interesting one is the top Chern class:

Example 6.5. For a smooth projective curve C of genus g, the top Chern class has degree

$$\int_X c_1(C) \;=\; \deg(\mathscr{T}_C) \;=\; -\deg(\omega_C) \;=\; 2-2g.$$

Note that over the complex numbers the right hand side is equal to the topological Euler characteristic of the associated compact Riemann surface. This observation can be generalized to varieties of arbitrary dimension:

Theorem 6.6 (Gauss-Bonnet). *Let X be a smooth projective variety of dimension n over the complex numbers. Then we have*

$$\int_X c_n(X) \;=\; \sum_{i=0}^{2n} (-1)^i \dim_{\mathbb{Q}} H^i(X(\mathbb{C}),\mathbb{Q}).$$

We might see a proof of this later using the Hirzebruch-Riemann-Roch theorem and the Hodge decomposition. The right hand side in the above formula is usually denoted $\chi_{\mathrm{top}}(X(\mathbb{C}))$, the *topological Euler characteristic* of the associated complex manifold. For instance we obtain:

Example 6.7. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d over the complex numbers. By lemma 6.4 its top Chern class is the part of dimension $\dim(X) = n-1$ in the class

$$c(X) \;=\; \frac{(1+\zeta)^{n+1}}{(1+d_i\zeta)} \;=\; (1+\zeta)^{n+1}\cdot(1-d\zeta+d^2\zeta^2-\cdots) \;\in\; A_*(X).$$

Explicitly we have

$$c_{n-1}(X) \;=\; \sum_{i=0}^{n-1} (-1)^i \tbinom{n+1}{n-1-i} d^i \zeta^{n-1} \;\in\; A_{n-1}(X).$$

Since $\int_X \zeta^{n-1} = d$, it follows that the topological Euler characteristic of a smooth projective hypersurface is

$$\chi_{\mathrm{top}}(X(\mathbb{C})) \;=\; \sum_{i=0}^{n-1} (-1)^i \tbinom{n+1}{n-1-i} d^{i+1}.$$

As a sanity check, for smooth quadrics $X \subset \mathbb{P}^3$ we get $\chi_{\mathrm{top}}(X(\mathbb{C})) = 4$, which agrees with the value expected from the Künneth formula applied to $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

7 Chow groups of affine and projective bundles

Let E be a vector bundle of rank $r+1$ on a scheme X, and denote the projections from its total space and from the associated projective bundle by

$$q_E\colon \quad E \longrightarrow X \quad \text{and} \quad p_E\colon \quad \mathbb{P}(E) \longrightarrow X.$$

Some time ago we have seen that

- $q_E^*\colon A_d(X) \twoheadrightarrow A_{d+r+1}(E)$ is surjective,
- $p_E^*\colon A_d(X) \hookrightarrow A_{d+r}(\mathbb{P}(E))$ is injective.

Using Chern classes, we can now upgrade this to a complete description of the Chow groups of affine and projective bundles:

Theorem 7.1. *For any $d \in \mathbb{Z}$ we have:*

a) The morphism $q_E^*\colon A_d(X) \xrightarrow{\sim} A_{d+r+1}(E)$ is an isomorphism.

b) The morphism p_E^* induces an isomorphism

$$\begin{aligned} \theta_E\colon \quad \bigoplus_{\nu=0}^{r} A_{d-r+\nu}(X) \quad &\xrightarrow{\sim} \quad A_d(\mathbb{P}(E)), \\ (\alpha_{d-r},\dots,\alpha_d) \quad &\mapsto \quad \sum_{\nu=0}^{r} \zeta_E^\nu \cap p_E^*(\alpha_{d-r+\nu}) \quad \textit{for} \quad \zeta_E \;=\; c_1(\mathscr{O}_{\mathbb{P}(E)}(1)). \end{aligned}$$

Proof. Step 0. We first introduce some notation that will allow us to change the rank of the vector bundle: Suppose $E \simeq F \oplus \mathscr{O}_X$ for some vector bundle F on X. We then have a commutative diagram

$$\begin{tikzcd} \mathbb{P}(F) \arrow[r, hook, "i"] \arrow[d, "p_F"] & \mathbb{P}(E) \arrow[d, "p_E"] & F \arrow[l, hook', "j"'] \arrow[d, "q_F"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd}$$

where $\mathbb{P}(F) \subset \mathbb{P}(E)$ is viewed as the 'hyperplane at infinity' and $F = \mathbb{P}(E) \setminus \mathbb{P}(F)$ is identified with its complement. Passing to Chow groups we get the exact localization sequence

$$A_d(\mathbb{P}(F)) \xrightarrow{i_*} A_d(\mathbb{P}(E)) \xrightarrow{j^*} A_d(F) \longrightarrow 0.$$

Note that i_* cannot be compatible in the naive way with the pullbacks p_F^* and p_E^* since these two pullbacks involve different dimension shifts. However, we claim that it is compatible with the pullbacks in the sense that we have the following commutative diagram:

$$\begin{tikzcd}
A_{d-1}(\mathbb{P}(F)) \arrow[r, "i_*"] & A_{d-1}(\mathbb{P}(E)) & A_d(\mathbb{P}(E)) \arrow[l, "\zeta_E\cap(-)"'] \arrow[r, "j^*"] & A_d(F) \\
A_{d-r}(X) \arrow[u, "p_F^*"] \arrow[rr, equal] & & A_{d-r}(X) \arrow[u, "p_E^*"] \arrow[r, equal] & A_{d-r}(X) \arrow[u, "q_F^*"']
\end{tikzcd}$$

In other words, we claim that

$$\zeta_E \cap p_E^*(\alpha) \;=\; i_* p_F^*(\alpha) \quad \text{for all} \quad \alpha \,\in\, A_*(X).$$

Indeed, we may assume $\alpha = [Z]$ for a subvariety $Z \subset X$. The definition of the product with Cartier divisors implies

$$c_1(\mathscr{O}_{\mathbb{P}(E)}(1)) \cap [p_E^{-1}(Z)] \;=\; [p_F^{-1}(Z)]$$

since $\mathscr{O}_{\mathbb{P}(E)}(1)$ has a section with zero locus $\mathbb{P}(F) \subset \mathbb{P}(E)$, so the claim follows.

Step 1. We now show that θ_E is surjective. By Noetherian induction and the localization sequence, it will be enough to do this when the vector bundle is trivial and therefore $E \simeq F \oplus \mathscr{O}_X$ for some vector bundle F on X. Let $\beta \in A_d(\mathbb{P}(E))$. By the known result for affine bundles the pullback map q_F^* is surjective, hence we may write

$$j^*(\beta) \;=\; q_F^*(\alpha_{d-r}) \;=\; j^* p_E^*(\alpha_{d-r}) \quad \text{for some} \quad \alpha_{d-r} \,\in\, A_{d-r}(X).$$

Then $\beta - p_E^*(\alpha_{d-r}) \in \ker(j^*) = \operatorname{im}(i_*)$, so by induction on the rank r we may write

$$\beta - p_E^*(\alpha_{d-r}) \;=\; i_*\left(\sum_{\nu=1}^{r} \zeta_F^{\nu-1} \cap p_F^*(\alpha_{d-r-\nu})\right)$$

for certain $\alpha_{d-r+1}, \dots, \alpha_d \in A_*(X)$. Since $\zeta_F = i^*(\zeta_E)$, the projection formula and the claim from the previous step allow to rewrite this as

$$\beta - p_E^*(\alpha_{d-r}) \;=\; \sum_{\nu=1}^{r} \zeta_E^{\nu} \cap p_E^*(\alpha_{d-r-\nu})$$

which implies $\beta = \theta_E(\alpha_{d-r}, \dots, \alpha_d)$. Hence θ_E is surjective.

Step 2. Next we show that θ_E is injective: Let $a = (\alpha_{d-r}, \dots, \alpha_d) \in \ker(\theta_E)$ be an element in the kernel. If $a \neq 0$, we can take $e \in \{d-r, \dots, d\}$ to be the maximal index with $\alpha_e \neq 0$. Then

$$\begin{aligned}
0 \;&=\; p_{E*}\Big(\zeta_E^{d-e} \cap \theta(a)\Big) && \text{since } \theta_E(a) = 0 \\
&=\; \sum_{\nu\geq 0} p_{E*}\Big(\zeta_E^{d-e+\nu} \cap p_E^*(\alpha_{d-r+\nu})\Big) && \text{by definition of } \theta_E \\
&=\; \sum_{\nu\geq 0} s_{d-e+\nu-r}(E) \cap \alpha_{d-r+\nu} && \text{by definition of } s_i(E)\cap(-)
\end{aligned}$$

We claim that only a single summand survives: Indeed, put $i_\nu = d - e + \nu - r$, then the summands are

$$s_{i_\nu}(E) \cap \alpha_{i_\nu + e} \;=\; \begin{cases} 0 & \text{for } i_\nu > 0 \text{ since then } \alpha_{i_\nu + e} = 0, \\ 0 & \text{for } i_\nu < 0 \text{ since then } s_{i_\nu}(E) = 0, \\ \alpha_e & \text{for } i_\nu = 0 \text{ since we have } s_0(E) = id. \end{cases}$$

Hence the previous equality shows $\alpha_e = 0$, which contradicts our choice of e.

Step 3. Finally we show that q_E^* is injective: Let $G = E \oplus \mathscr{O}_X$, and consider the inclusion $j\colon E \hookrightarrow \mathbb{P}(G)$ as the complement of the hyperplane at infinity. Then for any class $\alpha \in A_d(X)$ with $q_E^*(\alpha) = 0$ we have

$$j^* p_G^*(\alpha) = 0.$$

Let $i\colon \mathbb{P}(E) \hookrightarrow \mathbb{P}(G)$. It follows that

$$p_G^*(\alpha) \;=\; i_*(\beta) \quad \text{for some} \quad \beta \;\in\; A_*(\mathbb{P}(E)).$$

The surjectivity of θ_E implies $\beta = \theta_E(\alpha_{d-r}, \dots, \alpha_d)$ and hence

$$p_E^*(\alpha) \;=\; i_*\left(\sum_{\nu=0}^{r} \zeta_E^\nu \cap p_E^*(\alpha_{d-r+\nu})\right) \;=\; \sum_{\nu=0}^{r} \zeta_G^{\nu+1} \cap p_G^*(\alpha_{d-r+\nu}),$$

where in the last equality we have again used the projection formula and the commutative diagram from step 0 (with (E,G) in place of (F,E)). But this means that

$$\theta_G(-\alpha, \alpha_{d-r}, \dots, \alpha_d) \;=\; 0,$$

hence $\alpha = 0$ by the injectivity of θ_G. □

The fact that the pullback homomorphism in the first part of the above theorem is an isomorphism has an important consequence:

Definition 7.2. Let $q\colon E \to X$ be a vector bundle of rank r on X, and let $s\colon X \to E$ be its zero section. Then we define the *Gysin homomorphism* to be the inverse of the pullback:

$$s^* \;:=\; (q^*)^{-1}\colon \quad A_d(E) \xrightarrow{\sim} A_{d-r}(X).$$

For $\alpha \in A_d(E)$ we call $s^*(\alpha) \in A_{d-r}(X)$ the *intersection of α with the zero section.*

Example 7.3. Let L be a line bundle on X. The zero section $s\colon X \hookrightarrow L$ embeds X as an effective Cartier divisor on the total space of the line bundle, and in these terms the homomorphism

$$s^*\colon \quad A_d(L) \longrightarrow A_{d-1}(X)$$

coincides with the Gysin morphism given by the intersection with a Cartier divisor as defined earlier: This follows from the fact that s^* is determined uniquely by the property that $s^*[q^{-1}(Z)] = [Z]$ for every subvariety $Z \subset X$.

Thus the intersection with the zero section on a vector bundle generalizes the intersection product with Cartier divisors that we introduced earlier. Note that for vector bundles of higher rank the zero section is not a divisor, so we have really moved a step forward; in fact this will be the basis for the general definition of the intersection product in the next chapter. The following gives an alternative formula for the Gysin homomorphism:

Proposition 7.4 (Gysin formula). *Let E be a vector bundle of rank r on X, and let $G = E \oplus 1$ be its direct sum with a trivial bundle of rank one. Let $p\colon \mathbb{P}(G) \to X$ be the projection, and let Q be the tautological quotient vector bundle in the exact sequence*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(G)}(-1) \longrightarrow p^*(G) \longrightarrow Q \longrightarrow 0.$$

Then $s^*: A_d(E) \to A_{d-r}(X)$ is given by

$$s^*(\beta) \;=\; p_*(c_r(Q) \cap \overline{\beta}) \quad \textit{for} \quad \beta \in A_*(E),$$

where $\overline{\beta} \in A_*(\mathbb{P}(G))$ is any class with $j^*(\overline{\beta}) = \beta$ for the inclusion $j\colon E \hookrightarrow \mathbb{P}(G)$.

Proof. Consider again the diagram

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[r, hook, "i"] \arrow[d, "p_E"] & \mathbb{P}(G) \arrow[d, "p_G"] & E \arrow[l, hook', "j"'] \arrow[d, "q_E"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

We want to show that

$$q_E^*\Big(p_{G*}(c_r(Q) \cap \overline{\beta})\Big) \;=\; j^*(\overline{\beta}) \quad \text{for all} \quad \overline{\beta} \in A_*(\mathbb{P}(G)).$$

From the diagram

$$\begin{tikzcd}
A_*(\mathbb{P}(E)) \arrow[r, "i_*"] & A_*(\mathbb{P}(G)) \arrow[r, "j^*"] & A_*(E) \arrow[r] & 0 \\
 & & A_{*-r}(X) \arrow[ul, "p_G^*"] \arrow[u, two heads, "q_E^*"'] &
\end{tikzcd}$$

where the top row is exact and q_E^* is surjective, we can write

$$\overline{\beta} \;=\; p_G^*(\gamma) + i_*(\delta) \quad \text{for some} \quad \gamma \in A_{*-r}(X),\ \delta \in A_*(\mathbb{P}(E)).$$

Since $j^*(\overline{\beta}) = q_E^*(\gamma)$, it will then be enough to show the following two statements:

a) $c_r(Q) \cap i_*(\delta) = 0$, and

b) $p_{G*}(c_r(Q) \cap p_G^*(\gamma)) = \gamma$.

The first statement follows from the projection formula and from the observation that $c_r(i^*(Q)) = 0$ since $i^*(Q)$ is a vector bundle of rank r which has a nowhere vanishing section. For the second statement notice that by the Whitney formula we have

$$c(Q) \;=\; \frac{c(p_G^*(G))}{c(\mathscr{O}_{\mathbb{P}(G)}(-1))} \;=\; c(p_G^*(G))\cdot\sum_{i=0}^{r}\zeta_G^i \quad \text{for} \quad \zeta_G \;=\; c_1(\mathscr{O}_{\mathbb{P}(G)}(1)).$$

For the top Chern class then

$$c_r(Q) \;=\; \sum_{i=0}^{r}\zeta_G^i c_{r-i}(p^P(G))$$

and it follows that

$$\begin{aligned} p_{G*}(c_r(Q)\cap p_G^*(\gamma)) \;&=\; \sum_{i=0}^{r} p_{G*}(\zeta_G^i\cap p_G^*(c_{r-i}(G)\cap\gamma)) \\ &=\; \sum_{i=0}^{r} s_{i-r}(G)\cap c_{r-i}(G)\cap\gamma \\ &=\; \gamma \end{aligned}$$

because $s_{i-r}(G)=0$ for $i<r$ and $s_0(G)=id$. □

Chapter III

The intersection product

1 Motivation: Why normal cones?

In the last chapter we have seen how on the total space of a vector bundle one can intersect arbitrary cycles with the zero section via the Gysin pullback. The main task of this chapter will be to generalize this to closed subschemes which locally look like the zero section in a vector bundle.

Definition 1.1. Let Y be a closed subscheme of a scheme X. The embedding $Y \hookrightarrow X$ is called a *regular embedding of codimension d* if the ideal sheaf $\mathscr{I}_Y \trianglelefteq \mathscr{O}_X$ is locally generated by a regular sequence of length d: We can cover X by open subsets $U \subset X$ such that

$$\mathscr{I}_Y(U) = (f_1, \dots, f_d) \quad \text{for certain} \quad f_1, \dots, f_d \in \mathscr{I}_Y(U)$$

with the property that each f_i maps to a nonzerodivisor in $\mathscr{O}_X(U)/(f_1, \dots, f_{i-1})$.

Remark 1.2. A regular embedding of codimension one is the same notion as the embedding of an effective Cartier divisor. For any regular embedding $Y \hookrightarrow X$ of codimension d, the properties of regular sequences in commutative algebra show that

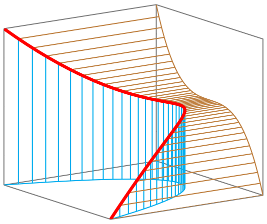
- the conormal sheaf $N^\vee_{Y/X} := \mathscr{I}_Y/\mathscr{I}_Y^2$ is locally free of rank d,
- the natural map $\mathrm{Sym}^n(\mathscr{I}_Y/\mathscr{I}_Y^2) \to \mathscr{I}_Y^n/\mathscr{I}_Y^{n+1}$ is an isomorphism for all n.

Regular embeddings can by definition be written locally as complete intersections of effective Cartier divisors, but usually not globally, even for smooth varieties:

Example 1.3. If X is a smooth variety, then for any smooth subvariety $Y \subset X$, its embedding is a regular embedding of codimension $d = \dim X - \dim Y$. But usually Y cannot be written globally as the scheme-theoretic intersection of d Cartier divisors on X: Take the embedding

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad [1:t] \mapsto [1:t:t^2:t^3]$$

whose image is the twisted cubic:



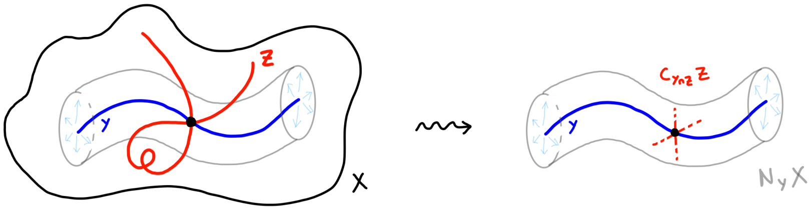
`https://enriqueacosta.github.io/blog/en/posts/tikz-twisted-cubic/`

This twisted cubic is the intersection of *three* quadrics

$$Y \;=\; V_+(xz-y^2)\cap V_+(yw-z^2)\cap V_+(xw-yz) \;\subset\; \mathbb{P}^3$$

but it cannot be the scheme-theoretic intersection of *two* surfaces, as one may see by looking at the space of homogenous polynomials of degree two in its ideal.

Thus we cannot directly invoke the intersection product with Cartier divisors to define an intersection product with all regularly embedded subschemes. We will overcome this problem via an algebraic analog of tubular neighborhoods: For any regular embedding $Y \hookrightarrow X$ we will construct a deformation[1] from that embedding to the one of the zero section in the normal bundle $N_Y X$. Any subvariety $Z \subset X$ will deform to the *normal cone* $C_{Y\cap Z}Z$, a certain subcone of the normal bundle $N_{Y/X}$:



Note that both Z and $Y\cap Z$ can be very singular and the normal cone $C_{Y\cap Z}Z$ will usually not be a vector bundle; this explains why a careful study of cones and their Segre classes is crucial for the development of intersection theory. Once we have done this, we will use the Gysin pullback for the zero section $i\colon Y \hookrightarrow N_Y X$ to get a well-defined class

$$Y\cdot Z \;:=\; i^*[C_{Y\cap Z}Z] \;\in\; A_*(Y\cap Z).$$

---

[1] Usually $N_{Y/X}$ does *not* embed into X (unlike tubular neighborhoods in differential geometry)!

If X is smooth, we can then easily define the intersection product of *arbitrary*, not necessarily regularly embedded subvarieties $Y, Z \subset X$ by considering the Cartesian diagram

$$\begin{tikzcd}
Y \cap Z \arrow[r, hook] \arrow[d, hook] & Y \times Z \arrow[d, hook] \\
\Delta \arrow[r, hook] & X \times X
\end{tikzcd}$$

where $\Delta \hookrightarrow X \times X$ is the embedding of the diagonal. For smooth X this is a regular embedding and we will get a well-defined intersection

$$Y \cdot Z \;:=\; \Delta \cdot (Y \times Z) \;\in\; A_*(Y \cap Z).$$

Carrying out this programme will occupy us for the rest of this chapter.

2 Cones and their Segre classes

The following definition generalizes our description of the total space of a vector bundle on a scheme:

Definition 2.1. Let X be a scheme. A *cone* over it is a scheme over X that can be written in the form

$$C \;=\; \operatorname{Spec}_X(\mathscr{S}^\bullet) \;\to\; X,$$

where $\mathscr{S}^\bullet$ is a sheaf of graded $\mathscr{O}_X$-algebras. We also consider the *projective cone* defined by

$$\mathbb{P}(C) \;=\; \operatorname{Proj}_X(\mathscr{S}^\bullet) \;\to\; X.$$

Note the abuse of notation here: When talking about cones we will always assume that we are given the sheaf of graded $\mathscr{O}_X$-algebras $\mathscr{S}^\bullet$ and not just the abstract scheme morphism $C \to X$, since the latter does not determine the action of $\mathbb{G}_m$ on the fibers. That being said, any cone C embeds as an open dense subscheme in its *projective completion* which is defined as $\mathbb{P}(C \oplus 1) := \operatorname{Proj}_X(\mathscr{S}^\bullet[z])$.

Example 2.2. To any closed subscheme $Y = V(\mathscr{I}) \hookrightarrow X$ with ideal sheaf $\mathscr{I} \trianglelefteq \mathscr{O}_X$, we can attach various cones:

a) The *blowup* of Y in X is the projective cone

$$\operatorname{Bl}_Y(X) \;:=\; \operatorname{Proj}_X \bigoplus_{n \geq 0} \mathscr{I}^n \;\longrightarrow\; X.$$

b) The *normal cone* to Y in X is the cone

$$C_Y X := \operatorname{Spec}_X \bigoplus_{n \geq 0} \mathscr{I}^n / \mathscr{I}^{n+1} \;\longrightarrow\; Y.$$

The projectivization of the normal cone can be identified with the exceptional fiber of the blowup via the Cartesian diagram

$$\begin{tikzcd}
\mathbb{P}(C_Y X) \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Y(X) \arrow[d] \\
Y \arrow[r, hook] & X
\end{tikzcd}$$

whose top row is induced by the surjections $\mathscr{I}^n \twoheadrightarrow \mathscr{I}^n/\mathscr{I}^{n+1}$.

Remark 2.3. If X is equidimensional, then for any subscheme $Y \subset X$ the normal cone $C = C_Y X$ is also equidimensional with $\dim C = \dim X$. Indeed, this follows from the fact that

- we have an open embedding $C \subset \mathbb{P}(C \oplus 1)$,
- $\mathbb{P}(C \oplus 1)$ is the exceptional divisor on $B = \mathrm{Bl}_{Y \times 0}(X \times \mathbb{A}^1)$,
- B is birational to $X \times \mathbb{A}^1$ (since $Y \times 0$ is nowhere dense in $X \times \mathbb{A}^1$).

Remark 2.4. We also have surjections $\mathrm{Sym}^n(\mathscr{I}/\mathscr{I}^2) \twoheadrightarrow \mathscr{I}^n/\mathscr{I}^{n+1}$ and hence an embedding of cones

$$C_Y X \hookrightarrow N_{Y/X} := \mathrm{Spec}_X \bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathscr{I}/\mathscr{I}^2).$$

If $Y \hookrightarrow X$ is a regular embedding, then this is an isomorphism $C_Y X \xrightarrow{\sim} N_{Y/X}$ and these cones are a vector bundle, called the *normal bundle* to the subvariety. Apart from that case the normal cone is the more interesting notion:

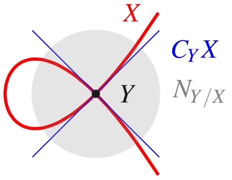
Example 2.5. Let $m, n > 0$, and let $X = V(f) \subset \mathbb{A}^n$ be a hypersurface cut out by a polynomial

$$f = \sum_{d \geq m} f_d \quad \text{with each } f_d \text{ homogenous of degree } d.$$

For $Y = \{0\} \hookrightarrow X$ one computes

$$C_Y X = \{0\} \times V(f_m) \subset N_{Y/X} = \{0\} \times T_0 X.$$

For $m \geq 2$ the scheme X has a singularity at the origin; then the first cone is the tangent cone to X at the singularity which keeps information about the leading term of the polynomial, while the second cone is just the tangent space $T_0 X = T_0 \mathbb{A}^n$ as shown in the following picture for a nodal cubic:



At this point we could directly start the construction of the intersection product via the deformation to the normal cone, but for later purposes it will be convenient to gather a few remarks about Segre classes here. The definition of Segre classes generalizes easily from vector bundles to arbitrary cones, with two caveats:

- The projection $p\colon \mathbb{P}(C) \to X$ is usually not flat, so we cannot define Segre classes as operations on Chow groups like for vector bundles. But we could still define them as classes by taking $p_*(c_1(\mathscr{O}_{\mathbb{P}(C)}(1)^i \cap [\mathbb{P}(C)]) \in A_*(X)$.
- While we only considered vector bundles of positive rank, we want to allow the zero cone $C = \operatorname{Spec}_X(\mathscr{O}_X)$ for instance in talking about the zero section in a vector bundle. But then $\mathbb{P}(C) = \varnothing$ is the empty scheme! One way to keep information about the zero section is to work on $\mathbb{P}(C \oplus 1)$ rather than on $\mathbb{P}(C)$.

Definition 2.6. The *total Segre class* of a cone $C = \operatorname{Spec}_X(\mathscr{S}^\bullet) \to X$ is defined as the class

$$s(C) \;:=\; \sum_{i \geq 0} p_*(c_1(\mathscr{O}_{\mathbb{P}(C\oplus 1)}(1))^i \cap [\mathbb{P}(C \oplus 1)]) \;\in\; A_*(X),$$

where $p\colon \mathbb{P}(C \oplus 1) \to X$ is the structure morphism. The component in dimension i of the total Segre class is called the *i-th Segre class* $s_i(C) \in A_i(X)$.

The total Segre class as defined above is compatible with the one we defined earlier for vector bundles, and it is compatible with the decomposition of cones into irreducible components:

Lemma 2.7. *Let $C = \operatorname{Spec}(\mathscr{S}^\bullet)$ be a cone on X.*

a) If $C = E$ is a vector bundle, then $s(C) = c(E)^{-1} \cap [X]$.

b) In general, if C has irreducible components C_i with multiplicities $m_i = \ell(\mathscr{O}_{C,C_i})$, then

$$s(C) \;=\; \sum_i m_i \cdot s(C_i).$$

Proof. If $C = E$ is a vector bundle, then we have $s(C) = c(E \oplus 1)^{-1} \cap [X]$ by the definitions. Here $c(E \oplus 1) = c(E)$ by the Whitney formula for Chern classes, so *a)* follows. For *b)* note that

$$[\mathbb{P}(C \oplus 1)] \;=\; \sum_i m_i \cdot [\mathbb{P}(C_i \oplus 1)]$$

since $C \subset \mathbb{P}(C \oplus 1)$ is an open dense subset. □

One consequence of adding the trivial bundle in the definition of Segre classes for cones is the following:

Exercise 2.8. The Segre classes as defined above are a *stable invariant* of the cone C in the sense that

$$s(C) \;=\; s(C \oplus 1) \;=\; s(C \oplus 1 \oplus 1) \;=\; \cdots$$

The most interesting cones for the purpose of intersection theory are normal cones to subschemes:

Definition 2.9. The *Segre class* of a subscheme $Y \subset X$ is $s(Y,X) = s(C_Y X) \in A_*(Y)$.

Example 2.10. If $Y \hookrightarrow X$ is a regular embedding, then the normal cone $C_{Y/X} = N_{Y/X}$ is a vector bundle and then lemma 2.7 a) gives

$$s(Y,X) \;=\; c(N_{Y/X})^{-1} \cap [Y].$$

For subschemes of equidimensional schemes the Segre classes are compatible with the decomposition into irreducible components:

Lemma 2.11. *Let X be an equidimensional scheme with irreducible components X_i and denote the multiplicities of the components by $m_i = \ell(\mathscr{O}_{X,X_i})$. Then for any subscheme $Y \subset X$ we have*

$$s(Y,X) \;=\; \sum_i m_i \cdot s(Y_i, X_i) \;\in\; A_*(Y) \quad \textit{where} \quad Y_i \;=\; Y \cap X_i.$$

Proof. We have $C_Y X \oplus 1 = C_{Y\times\{0\}}(X \times \mathbb{A}^1)$, so $E = \mathbb{P}(C_Y X \oplus 1)$ is the exceptional divisor of the blowup

$$B \;=\; \mathrm{Bl}_{Y\times\{0\}}(X \times \mathbb{A}^1) \;\longrightarrow\; X \times \mathbb{A}^1.$$

Likewise $E_i = \mathbb{P}(C_{Y_i} X_i \oplus 1)$ is the exceptional divisor of $B_i = \mathrm{Bl}_{Y_i\times\{0\}}(X_i \times \mathbb{A}^1)$. We also know

$$[B] \;=\; \sum_i m_i \cdot [B_i].$$

The equidimensionality of X implies that the blowup B is equidimensional. Since the restriction of Cartier divisors to equidimensional schemes is compatible with the decomposition in irreducible components in the sense of chapter II, remark 1.8, it follows that

$$[E] \;=\; \sum_i m_i \cdot [B_i \cap E] \;=\; \sum_i m_i \cdot [E_i]$$

and we are done by applying $p_*(\sum_j c_1(\mathscr{O}(1))^j \cap (-))$. □

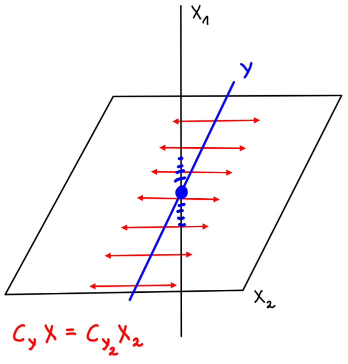
Remark 2.12. The equidimensionality is needed in the above: For instance, take the union $X = V(xz, yz) = X_1 \cup X_2 \subset \mathbb{A}^3$ of $X_1 = V(x,y)$ and $X_2 = V(z)$. Then the subscheme

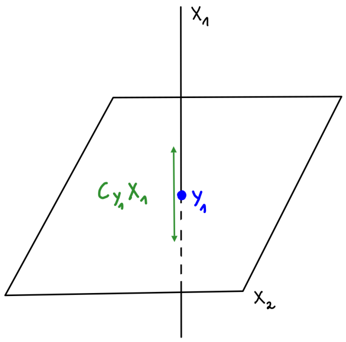
$$Y \;=\; V(f) \;\subset\; X \quad \text{cut out by} \quad f \;=\; (z-x)|_X \;\in\; \Gamma(X, \mathscr{O}_X)$$

is an effective Cartier divisor (since f is a nonzerodivisor). In particular $Y \subset X$ is regularly embedded of codimension one, so its normal cone is a vector bundle of rank one. It then follows easily that

$$C_Y X \;=\; N_{Y/X} \;=\; C_{Y_2} X_2 \quad \text{and} \quad s(Y,X) \;=\; s(Y_2, X_2) \;\neq\; s(Y_1, X_1) + s(Y_2, X_2).$$

The reason why $C_{Y_1}X_1$ does not enter $C_Y X$ is that the subscheme $Y \subset X$ has an embedded point at the origin, so that the subscheme is 'tangent' to the z-axis at the origin:





Note that this problem disappears if instead we took $X' = V(xz) = X_1' \cup X_2'$ to be the union of two planes: Then $Y' = V(f) \subset X'$ for f as above is still a Cartier divisor, but now it is a double line rather than a line with an embedded point. So $C_{Y'}X'$ is a vector bundle of rank one, but over a double line. For the underlying reduced closed subscheme $Y'_{\mathrm{red}} \subset Y'$ then

$$[C_{Y'}X'] \;=\; 2 \cdot [C_{Y'_{\mathrm{red}}}X] \quad \text{and} \quad s(Y',X') \;=\; 2s(Y'_{\mathrm{red}},X') \;=\; s(Y_1',X_1') + s(Y_2',X_2').$$

So equidimensionality avoids problems with embedded points on Cartier divisors.

Indeed, on equidimensional schemes Segre classes of closed subschemes have remarkable functorial properties — note that in the following statement the fiber product Y' may be very nasty and $\mathrm{codim}_{X'}(Y')$ may differ from $\mathrm{codim}_X(Y)$!

Proposition 2.13. *Let $f\colon X' \to X$ be a morphism of equidimensional schemes, and consider a Cartesian square*

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

where $Y \subset X$ is a closed subscheme with preimage $Y' = f^{-1}(Y)$, and $g = f|_{Y'}$.

a) If f is proper and every irreducible component of X' dominates X, then the Segre classes satisfy

$$g_*(s(Y',X')) \;=\; \deg(X'/X) \cdot s(Y,X).$$

b) If f is flat, then

$$g^*(s(Y,X)) \;=\; s(Y',X').$$

Proof. In *a)* in particular X is irreducible, and the degree in the statement is defined by

$$\deg(X'/X) \;=\; \sum_i m_i \cdot \deg(X'_i/X)$$

where X' has irreducible components X'_i with multiplicities m_i. Then by lemma 2.11 and by equidimensionality, we can argue for each of these irreducible components separately, hence we will assume for *a)* and *b)* that X' is irreducible. The universal property of blowups shows that there is a morphism F that restricts to a morphism G between the exceptional divisors as shown in the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[d, "g"'] & \mathbb{P}(C_{Y'}X' \oplus 1) \arrow[l, "p'"'] \arrow[r, hook] \arrow[d, "G"] & \mathrm{Bl}_{Y'\times\{0\}}(X' \times \mathbb{A}^1) \arrow[r] \arrow[d, "F"] & X' \times \mathbb{A}^1 \arrow[d, "f\times id"] \\
X & \mathbb{P}(C_Y X \oplus 1) \arrow[l, "p"'] \arrow[r, hook] & \mathrm{Bl}_{Y\times\{0\}}(X \times \mathbb{A}^1) \arrow[r] & X \times \mathbb{A}^1
\end{tikzcd}$$

By construction the morphism G is compatible with the tautological line bundles in the sense that

$$G^*(\mathscr{O}_{\mathbb{P}(C_Y X\oplus 1)}(1)) \;\simeq\; \mathscr{O}_{\mathbb{P}(C_{Y'}X'\oplus 1)}(1).$$

Let $d = \deg(X'/X)$. This is also the degree of the morphism F because it can be computed on an open dense subset where the blowup is an isomorphism. So by definition of the pushforward of cycles we have

$$F_*[\mathrm{Bl}_{Y'\times\{0\}}(X' \times \mathbb{A}^1)] \;=\; d \cdot [\mathrm{Bl}_{Y\times\{0\}}(X \times \mathbb{A}^1)].$$

Hence the projection formula for the intersection with Cartier divisors, applied to the exceptional divisors, implies

$$G_*[\mathrm{Proj}(C_{Y'}X' \oplus 1)] \;=\; d \cdot [\mathrm{Proj}(C_Y X \oplus 1)].$$

Now *a)* follows from

$$\begin{aligned}
g_*(s(Y',X')) \;&=\; g_* q'_* \left(\sum_i c_1(G^*(\mathscr{O}(1))^i \cap \left[\mathbb{P}(C_{Y'}X' \oplus 1)\right] \right) \\
&=\; q_* G_* \left(\sum_i c_1(G^*(\mathscr{O}(1))^i \cap \left[\mathbb{P}(C_{Y'}X' \oplus 1)\right] \right) \\
&=\; q_* \left(\sum_i c_1((\mathscr{O}(1))^i \cap d \cdot [\mathbb{P}(C_Y X \oplus 1)] \right) \\
&=\; d \cdot s(Y,X),
\end{aligned}$$

where the third step uses the projection formula for $c_1(\mathscr{O}(1)) \cap (-)$.

For *b)* notice that

$$\begin{aligned} g^*(s(Y,X)) &= g^*q_*\left(\sum_i c_1((\mathscr{O}(1))^i \cap [\mathbb{P}(C_YX \oplus 1)]\right) \\ &= q'_*G^*\left(\sum_i c_1((\mathscr{O}(1))^i \cap [\mathbb{P}(C_YX \oplus 1)]\right) \\ &= q'_*\left(\sum_i c_1((G^*\mathscr{O}(1))^i \cap G^*\,[\mathbb{P}(C_YX \oplus 1)]\right) \\ &= s(Y,X), \end{aligned}$$

where in the second step we have used base change. □

Corollary 2.14. *Assume that f (and hence g) is proper. If the base change $Y' \hookrightarrow X'$ is a regular embedding, then*

$$g_*(c(N_{Y'/X'})^{-1} \cap [Y']) \;=\; \deg(X'/X) \cdot s(Y,X).$$

Proof. Immediate from part *a)* of the proposition. □

This in particular means that we can compute Segre classes of arbitrarily nasty subschemes $Y \subset X$ by blowing up:

Corollary 2.15. *Let X be an equidimensional scheme, and $Y \subset X$ a subscheme of positive codimension. Let $p\colon X' = \mathrm{Bl}_Y(X) \to X$ be the blowup and $E = p^{-1}(Y) \subset X'$ the exceptional divisor. Then*

$$s(Y,X) \;=\; \sum_{i \geq 1} (-1)^{i-1} p_*(E^i),$$

where $E^i = E \cap \cdots \cap E$ is taken in the sense of intersections with Cartier divisors.

Proof. The exceptional divisor of a blowup is a Cartier divisor and hence regularly embedded. Now apply corollary 2.14 with $\deg(X'/X) = 1$ and $Y' = E$, using that here $N_{Y'/X'} = \mathscr{O}_{X'}(E)_{|E}$ by the adjunction formula for Cartier divisors. □

3 Two applications of Segre classes

Before we continue the development of intersection theory, let us briefly illustrate the use of Segre classes with two simple geometric applications. The first is the notion of the multiplicity of a scheme along a subvariety:

Definition 3.1. The *multiplicity* of a scheme X along a closed subvariety $Y \subset X$ is the coefficient $e = e_Y X \in \mathbb{N}$ with which the fundamental class $[Y]$ enters the top Segre class

$$s_d(Y,X) \;=\; e \cdot [Y] \;\in\; A_d(Y) \;=\; \mathbb{Z} \cdot [Y] \quad \text{for} \quad d \;=\; \dim Y.$$

Example 3.2. Let X be an equidimensional scheme, and let $Y \subset X$ be a subvariety of codimension $n > 0$. The multiplicity $e = e_Y X$ can be read off from $p\colon \mathrm{Bl}_Y(X) \to X$ via corollary 2.15:

$$e \cdot [Y] \;=\; (-1)^{n-1} p_*(E^n) \quad \text{for the exceptional divisor} \quad E \;\subset\; \mathrm{Bl}_Y(X).$$

We can also work directly from the definition of Segre classes: Let $C = C_Y X$ be the normal cone, then

$$\begin{aligned} e \cdot [Y] \;&=\; p_*(c_1(\mathscr{O}(1))^n \cap [\mathbb{P}(C \oplus 1)]) && \text{for the projection } p\colon \mathbb{P}(C \oplus 1) \to Y \\ &=\; q_*(c_1(\mathscr{O}(1))^{n-1} \cap [\mathbb{P}(C)]) && \text{for the projection } q\colon \mathbb{P}(C) \to Y \end{aligned}$$

where the second step uses our assumption $n > 1$. If $Y = \{P\}$ is a single closed point, then $C = C_P X$ is the tangent cone to X at that point, and the last displayed equation shows that

$$e_P X \;=\; \int_{\mathbb{P}(C)} c(\mathscr{O}(1))^{n-1} \cap [\mathbb{P}(C)] \;=\; \deg[\mathbb{P}(C)]$$

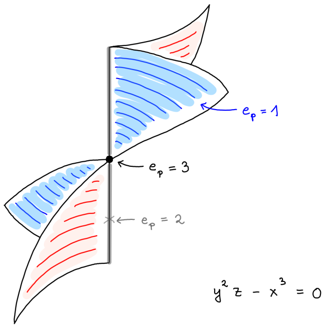
is the degree of the projective tangent cone to X at the point $P \in X$. For instance, for a hypersurface

$$X \;=\; V(f) \;\subset\; \mathbb{A}^n \quad \text{with} \quad f \;=\; f_0 + f_1 + f_2 + \cdots \;\in\; k[x_1, \dots, x_n]$$

where f_d is homogenous of degree d, the multiplicity at $P = (0, \dots, 0)$ is the degree of the initial term:

$$e_P \;=\; \min\{d \mid f_d \neq 0\}$$

Indeed, the initial term cuts out the tangent cone $C = C_P X$ as in example 2.5.



The second application of Segre classes that we want to discuss is a formula for the degree of rational maps. Given a line bundle $\mathscr{L} \in \operatorname{Pic}(X)$ on a variety X and a subspace $W \subset H^0(X, \mathscr{L})$ of sections, we have a rational map

$$\varphi_W : \quad X \dashrightarrow \mathbb{P}(W^\vee)$$

which is defined away from the base locus $B = \bigcap_{s \in W} V(s) \subset X$. We are interested in the following type of questions:

- If φ_W is dominant, what is its generic degree?
- If φ_W is birational onto a subvariety $Z \subset \mathbb{P}(W^\vee)$, what is the degree of Z?

More generally, let $Z \subset \mathbb{P}(W^\vee)$ be the closure of the image of the rational map φ_W and consider the generic degree $\deg(X/Z) = [k(X) : k(Z)]$ of this rational map over its image, then both questions above ask for the number $\deg(X/Z) \cdot \deg(Z)$. To compute this number, we first observe that the rational map φ_W is resolved via the blowup

$$p: \quad X' = \operatorname{Bl}_B(X) \longrightarrow X.$$

Indeed, let $\mathscr{I}_B \trianglelefteq \mathscr{O}_X$ be the ideal sheaf of $B \subset X$. The surjection $W \otimes \mathscr{O}_X \twoheadrightarrow \mathscr{I} \otimes \mathscr{L}$ induces a surjection $\operatorname{Sym}^\bullet(W \otimes \mathscr{L}^\vee) \twoheadrightarrow \bigoplus_{m \geq 0} \mathscr{I}^m$ and hence we obtain a closed immersion

$$i: \quad X' = \operatorname{Proj}_X \bigoplus_{m \geq 0} \mathscr{I}^m \hookrightarrow X \times \mathbb{P}(W^\vee) = \operatorname{Proj}_X \operatorname{Sym}^\bullet(W \otimes \mathscr{L}^\vee).$$

So we get a commutative diagram

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & X' \arrow[d, "p"'] \arrow[dr, "\exists ! f"] & \\
B \arrow[r, hook] & X \arrow[r, dashed, "\varphi_W"'] & \mathbb{P}(W^\vee)
\end{tikzcd}$$

where $f = \mathrm{pr}_2 \circ i$ and where $E = p^{-1}(B) \subset X'$ denotes the exceptional divisor. The number we want to compute can then be written as the degree

$$\deg f_*[X'] = \deg(X'/f(X')) \cdot \int_{\mathbb{P}(W)} c_1(\mathscr{O}(1))^n \cap [f(X')] \quad \text{where} \quad n = \dim X.$$

This degree can be computed in terms of Segre classes as follows:

Proposition 3.3. *With notation as above we have*

$$\deg f_*[X'] = \int_X c_1(\mathscr{L})^n - \int_B c_1(\mathscr{L})^n \cap s(B, X).$$

Proof. By construction $f^*(\mathscr{O}(1)) = p^*(\mathscr{L})(-E)$, hence

$$\begin{aligned}
\deg f_*[X'] &= \int_{X'} c_1(f^*(\mathscr{O}(1)))^n \\
&= \int_{X'} (c_1(p^*(\mathscr{L})) - c_1(\mathscr{O}_{X'}(E)))^n \\
&= \sum_{i=0}^{n} (-1)^i \tbinom{n}{i} \int_X c_1(\mathscr{L})^{n-i} p_*(c_1(\mathscr{O}_{X'}(E)^i \cap [X']) \\
&= \int_X c_1(\mathscr{L})^n \;-\; \int_X \sum_{i=1}^{n} \tbinom{n}{i} c_1(\mathscr{L})^{n-i} \cap (-1)^{i-1} p_*(E^i) \\
&= \int_X c_1(\mathscr{L})^n \;-\; \int_X \sum_{i=0}^{n} \tbinom{n}{i} c_1(\mathscr{L})^{n-i} \cap \sum_{j\geq 1} (-1)^{j-1} p_*(E^j) \\
&= \int_X c_1(\mathscr{L})^n \;-\; \int_B (1 + c_1(\mathscr{L}))^n \cap s(B,X),
\end{aligned}$$

where in the last step we used corollary 2.15. □

Example 3.4. Let $X = \mathbb{P}^2$ and $\mathscr{L} = \mathscr{O}(2)$.

a) For $W = H^0(X, \mathscr{L})$ the base locus is empty. The morphism $f = \varphi_W : \mathbb{P}^2 \to \mathbb{P}^5$ is the Veronese embedding, and the above proposition shows that the degree of its image is

$$\deg(f(\mathbb{P}^2)) \;=\; \int_X (c_1(\mathscr{L}))^2 \;=\; \int_X (c_1(\mathscr{O}(2)))^2 \;=\; 4 \int_X (c_1(\mathscr{O}(1))^2 \;=\; 4.$$

b) For $W = \langle x^2, xy, xz, y^2, yz \rangle \subset H^0(X, \mathscr{L})$ the base locus is the point $[0:0:1]$. We get a birational map

$$\varphi_W : \mathbb{P}^2 \dashrightarrow S \;\subset\; \mathbb{P}^4$$

to a projective surface S. As the Segre class of a point has degree $\int_X s(B,W) = 1$, we see that the surface has degree $\deg(S) = 4 - 1 = 3$.

c) For $W = \langle x^2, y^2, z^2 \rangle$ the base locus is again empty. However, $f\colon X \to \mathbb{P}^2$ is now a dominant morphism, so our formula computes its generic degree. As expected we get

$$\deg(f) \;=\; \int_X (c_1(\mathscr{L}))^2 \;=\; 4.$$

d) For $W = \langle x^2, xy, y^2 \rangle$ the base locus $B \subset W$ is a fat point. The rational map φ_W has positive generic fiber dimension, hence

$$0 \;=\; \deg f_*[X'] \;=\; 4 - \int_X s(B,X).$$

This gives a way to compute the multiplicity $e_B X = \int_X s(B,X) = 4$.

4 Deformation to the normal cone

In this section we want to construct for any subscheme $Y \subset X$ a natural specialization homomorphism

$$\sigma\colon \quad A_*(X) \longrightarrow A_*(C_Y X),$$

which will be an algebraic replacement for the restriction to tubular neighborhoods in analysis. The key ingredient is the following construction:

Proposition 4.1 (Deformation to the normal cone). *There is a scheme $M^\circ = M_Y^\circ X$ and a commutative diagram*

$$\begin{tikzcd} Y \times \mathbb{P}^1 \arrow[r, hook, "i"] \arrow[dr, "\mathrm{pr}_2"'] & M^\circ \arrow[d, "\rho"] \\ & \mathbb{P}^1 \end{tikzcd}$$

where i is a closed immersion and ρ is a flat morphism with the following properties:

a) over $U = \mathbb{P}^1 \setminus \{0\} \subset \mathbb{P}^1$ we have

$$\begin{aligned} \rho^{-1}(U) &= X \times U \\ i_{|Y\times U} &= \iota_Y \times id_U \quad \textit{for the inclusion} \quad \iota_Y : Y \hookrightarrow X. \end{aligned}$$

b) over $0 \in \mathbb{P}^1$ we have

$$\begin{aligned} \rho^{-1}(0) &= C_Y X \\ i_{|Y\times\{0\}} : Y &= Y \times \{0\} \hookrightarrow C_Y X \quad \textit{is the zero section.} \end{aligned}$$

Proof. The blowup $M = \mathrm{Bl}_{Y\times 0}(X \times \mathbb{P}^1)$ has the exceptional divisor $\mathbb{P}(C \oplus 1)$ for the cone $C = C_Y X$. From the closed embeddings $Y = Y \times 0 \hookrightarrow Y \times \mathbb{P}^1 \hookrightarrow X \times \mathbb{P}^1$ we get a closed embedding

$$Y \times \mathbb{P}^1 \;=\; \mathrm{Bl}_{Y\times 0}(Y \times \mathbb{P}^1) \;\hookrightarrow\; M \;=\; \mathrm{Bl}_{Y\times 0}(X \times \mathbb{P}^1).$$

Likewise, from the closed embeddings $Y = Y \times 0 \hookrightarrow X = X \times 0 \hookrightarrow X \times \mathbb{P}^1$ we get a closed embedding

$$\tilde{X} \;:=\; \mathrm{Bl}_Y(X) \;\hookrightarrow\; M \;=\; \mathrm{Bl}_{Y\times 0}(X \times \mathbb{P}^1)$$

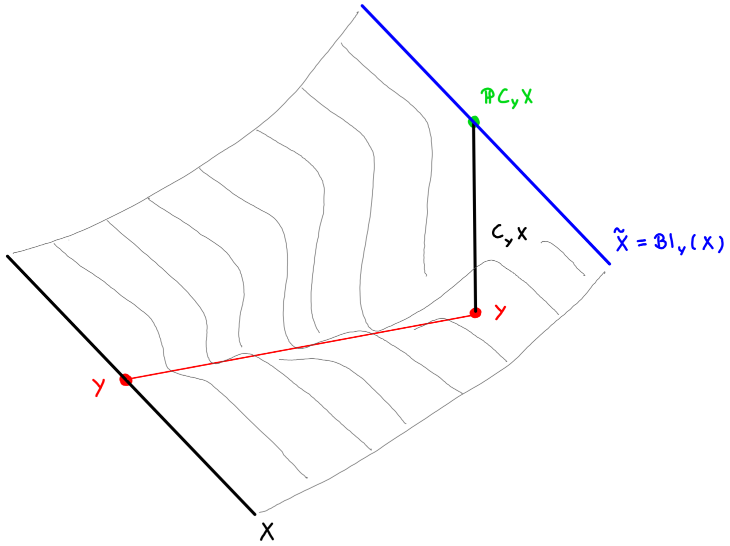
whose image is contained in the fiber $\pi^{-1}(0)$ of the projection $\pi\colon M \to \mathbb{P}^1$. Note that this projection is flat, because the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ is flat. Hence also its restriction

$$\rho\colon \quad M^\circ \;:=\; M \setminus \tilde{X} \;\longrightarrow\; \mathbb{P}^1$$

is flat, and claim *a)* is clear from the construction.

For the proof of *b)* it will be enough to show the following two statements which are illustrated in the picture below:

- The special fiber of the morphism $\pi\colon M \to \mathbb{P}^1$ is given as a Cartier divisor on M by the sum
$$\pi^{-1}(0) \;=\; E + \tilde{X}$$
of $\tilde{X}$ and the exceptional divisor $E = \mathbb{P}(C \oplus 1)$ of the blowup $M = \mathrm{Bl}_{Y\times 0}(X \times \mathbb{P}^1)$.
- The intersection of these two divisors is $E \cap \tilde{X} = \mathbb{P}(C)$, which can be regarded alternatively as the hyperplane at infinity in the projective completion $\mathbb{P}(C \oplus 1)$ or as the exceptional divisor of the blowup $\tilde{X} = \mathrm{Bl}_Y(X) \to X$.



Since all relevant embeddings are globally defined, both properties can be verified locally. Hence in what follows we we may assume that $X = \mathrm{Spec}(A)$ and $Y = V(I)$ for an ideal $J \trianglelefteq A$. It will be enough to work over $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} = \mathrm{Spec}\, k[t]$. Over this neighborhood of the origin we have

$$M_{|\mathbb{A}^1} \;=\; \mathrm{Proj}\,(\oplus_{n\geq 0}\,(J,t)^n) \quad \text{for the ideal} \quad (J,t) \;\trianglelefteq\; A[t].$$

The fiber of this scheme over the origin is given by

$$\pi^{-1}(0) \;=\; \mathrm{Proj}\,(\oplus_{n\geq 0}\,(J,t)^n /\; t \cdot \oplus_{n\geq 0}\,(J,t)^n)\,.$$

One easily verifies that this is the union of the closed subschemes

$$\tilde{X} \;=\; \mathrm{Proj}\,(\oplus_{n\geq 0} J^n) \quad \text{and} \quad E \;=\; \mathrm{Proj}\left(\oplus_{n\geq 0}\,(J,t)^n /\; \oplus_{n\geq 0}\;(J,t)^{n+1}\right)$$

and their scheme-theoretic intersection is $\tilde{X} \cap E = \mathrm{Proj}(\oplus_{n\geq 0} J^n / J^{n+1}) = C_Y X$. □

In what follows we fix a subscheme $Y \subset X$. For any other subscheme $Z \subset X$, the inclusions $Y \cap Z \subset Y \subset X$ induce a closed immersion $C_{Y\cap Z} Z \hookrightarrow C_Y X$. Moreover, if Z is equidimensional of dimension d, then by remark 2.3 the same also holds for the normal cone $C_{Y\cap Z}Z$, so we can define a specialization homomorphism on the level of cycles by

$$\sigma\colon \quad Z_d(X) \longrightarrow Z_d(C_Y X), \quad [Z] \mapsto [C_{Y\cap Z} Z].$$

The deformation to the normal cone implies that this homomorphism descends to rational equivalence classes:

Corollary 4.2 (Specialization to the normal cone). *The above homomorphism* σ *descends to a homomorphism*

$$\sigma\colon \quad A_d(X) \longrightarrow A_d(C_Y X), \quad [Z] \mapsto [C_{Y\cap Z} Z].$$

Proof. Let $M^\circ = M_Y^\circ X$ be the deformation to the normal cone $C = C_Y X$. Then the inclusions

$$i\colon C \hookrightarrow M^\circ \quad \text{and} \quad j\colon X \times \mathbb{A}^1 \hookrightarrow M^\circ \quad \text{for} \quad \mathbb{A}^1 = \mathbb{P}^1 \setminus \{0\}$$

fit in the following diagram whose row is the exact localization sequence:

$$\begin{tikzcd}
A_{d+1}(C) \arrow[r, "i_*"] \arrow[dr, "i^*i_*"'] & A_{d+1}(M^\circ) \arrow[r, "j^*"] \arrow[d, "i^*"] & A_{d+1}(X \times \mathbb{A}^1) \arrow[r] & 0 \\
& A_d(C) & A_d(X) \arrow[l, dashed, "\exists!\sigma"] \arrow[u, "\mathrm{pr}^*"'] &
\end{tikzcd}$$

Here pr^* is an isomorphism since it is the flat pullback for the structure morphism of a vector bundle. The morphism i^* is the Gysin map for the inclusion of the effective Cartier divisor $C \in \mathrm{Div}(M^\circ)$; since the latter has trivial normal bundle in M°, its first Chern class vanishes, hence we have $i^* i_* = 0$. It follows that there exists a uniqe homomorphism $\sigma\colon A_d(X) \to A_d(C)$ making the diagram commute. It only remains to check that this homomorphism is given on fundamental classes of subvarieties by the assignment $[Z] \mapsto [C_{Y\cap Z} Z]$. Indeed, we have

- $\mathrm{pr}^*[Z] = [Z \times \mathbb{A}^1] = j^*[M_{Y\cap Z}^\circ Z]$ for the subscheme $M_{Y\cap Z}^\circ Z \subset M_Y^\circ X$,
- $i^*[M_{Y\cap Z}^\circ Z] = [C_{Y\cap Z} Z]$ by construction of the deformation $M_{Y\cap Z}^\circ Z$.

Hence the claim follows. ☐

5 The intersection product

Note that our construction of the specialization to the normal cone works for any closed immersion. However, in the case of a regular embedding we can do more:

Definition 5.1. Let $i\colon Y \hookrightarrow X$ be a regular embedding of codimension d. We then define the *Gysin homomorphism* to be the composite $i^* = s^* \circ \sigma$ in the following diagram:

$$\begin{tikzcd}
A_*(X) \arrow[rr, "i^*"] \arrow[dr, "\sigma"'] & & A_{*-d}(Y) \\
& A_*(N_{Y/X}) \arrow[ur, "s^*"'] &
\end{tikzcd}$$

Here σ is the specialization to the normal cone $C_Y X = N_{Y/X}$, which in this case is a vector bundle, and s^* is the intersection with the zero section $s\colon Y \to N_{Y/X}$.

Remark 5.2. If $i\colon Y \hookrightarrow X$ is the embedding of the zero section in a vector bundle, then the above Gysin map coincides with the Gysin map that we defined earlier as the intersection with the zero section: Indeed, in this case we have a natural identification $X = N_{Y/X}$ via which $\sigma\colon A_*(X) \to A_*(N_{Y/X})$ becomes the identity.

We could now define the intersection product of Y with a subvariety $Z \subset X$ as the class $i^*[Z] \in A_{\dim Z - d}(Y)$, but we can do better: We want the intersection product to be a class in the Chow group of the scheme-theoretic intersection $Y \cap Z$ rather than just on Y or on X! In fact we can do this much more generally:

Definition 5.3. Let $i\colon Y \hookrightarrow X$ be a regular embedding of codimension d. Let Z be any equidimensional scheme endowed with a morphism $f\colon Z \to X$, and consider the Cartesian diagram

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "g"'] & Z \arrow[d, "f"] \\
Y \arrow[r, hook, "i"] & X
\end{tikzcd}$$

where $W = Y \times_X Z$ denotes the fiber product. Note that in general $W \hookrightarrow Z$ need not be a regular embedding of codimension d. But $N := g^*(N_{Y/X})$ is still a vector bundle of rank d, and we have a closed immersion of cones

$$C_W Z \hookrightarrow N \quad \text{induced by} \quad \bigoplus_n f^*(\mathscr{I}_Y^n/\mathscr{I}_Y^{n+1}) \twoheadrightarrow \bigoplus_n \mathscr{I}_W^n/\mathscr{I}_W^{n+1}$$

where $\mathscr{I}_Y \trianglelefteq \mathscr{O}_X$ resp. $\mathscr{I}_W \trianglelefteq \mathscr{O}_Z$ are the ideal sheaves of $Y \hookrightarrow X$ resp. $W \hookrightarrow Z$. Since Z is assumed equidimensional, we know from remark 2.3 that $C_W Z$ is equidimensional and $\dim C_W Z = \dim Z$. We can therefore define the *intersection product* of Y with Z by

$$Y \cdot Z := s^*[C_W Z] \in A_{\dim Z - d}(W) \quad \text{for the zero section} \quad s\colon W \hookrightarrow g^*(N_{Y/X}).$$

Remark 5.4. We will see soon that this generalizes our earlier intersection product with Cartier divisors. Moreover, it is linear in the second variable: If the scheme Z has irreducible components Z_i with multiplicities $m_i = \ell(\mathscr{O}_{Z,Z_i})$, then lemma 2.11 implies

$$Y \cdot Z = \sum_i Y \cdot Z_i \in A_{\dim W - d}(W).$$

In nice cases the intersection product can be computed as the fundamental class of the fiber product: In the above setup, let us say that Y and Z *intersect properly* if the scheme $W = Y \times_X Z$ is equidimensional of the expected dimension in the sense that

$$\operatorname{codim}_Z W' \;=\; \operatorname{codim}_X Y \quad \text{for every irreducible component} \quad W' \,\subset\, W.$$

If this is the case and if moreover Z is smooth or more generally Cohen-Macaulay, then $W \hookrightarrow X$ is a regular embedding of codimension d whenever $Y \hookrightarrow X$ is. In such cases we can compute the intersection product naively:

Lemma 5.5. *Let $Y \hookrightarrow X$ be a regular embedding of codimension d, and let $Z \to X$ be a morphism from an equidimensional scheme. If $W = Y \times_X Z \hookrightarrow Z$ is again a regular embedding of the same codimension d, then*

$$Y \cdot Z \;=\; [W].$$

Proof. Here $C_W Z = g^*(N_{Y/X}) = N$, hence $Y \cdot Z = s^*[C_W Z] = s^*[N] = [W]$. □

Note that the above includes cases of non-transverse intersections, as long as the intersection is regularly embedded of the expected codimension:

Example 5.6. Let $C, D \subset \mathbb{P}^2$ be two reduced curves with no common irreducible component over an algebraically closed field. Then $W = C \cap D \subset D$ is an effective Cartier divisor, i.e. a regularly embedded subscheme of codimension one. So the lemma gives

$$C \cdot D \;=\; [C \cap D] \;= \sum_{p \in W(k)} i_p(C,D) \cdot [p] \;\in\; A_0(W),$$

where $i_p(C,D) \in \mathbb{N}$ denotes the intersection multiplicity defined in the introduction.

Returning to the general case, we can always compute the intersection product in terms of Chern and Segre classes as follows, where for a class $\alpha \in A_*(W)$ we denote by $\{\alpha\}_i \in A_i(W)$ its component of dimension i.

Proposition 5.7 (Basic intersection formula). *Suppose as above that we are given a Cartesian diagram*

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "g"'] & Z \arrow[d, "f"] \\
Y \arrow[r, hook, "i"] & X
\end{tikzcd}$$

with i a regular embedding of codim d and Z equidimensional. Put $N = g^*(N_{Y/X})$ and consider the projection $p\colon \mathbb{P} = \mathbb{P}(N \oplus 1) \to W$. Let Q be the universal quotient bundle in the exact sequence

$$0 \;\to\; \mathscr{O}_{\mathbb{P}}(-1) \;\to\; p^*(N \oplus 1) \;\to\; Q \;\to\; 0.$$

Then we have

$$\begin{aligned} Y\cdot Z &= p_*(c_d(Q)\cap[\mathbb{P}(C_WV\oplus 1)]) \\ &= \{c(N)\cap s(W,Z)\}_{\dim Z-d} \end{aligned}$$

Proof. The first equality is the Gysin formula from chapter II, proposition 7.4, which expresses the intersection with the zero section as the pushforward of the cap product of $c_d(Q)$ with the extension $\overline{\beta}=[\mathbb{P}(C_WZ\oplus 1)]$ of $\beta=[C_WZ]$.

For the second equality note that $c(p^*N)=c(Q)\cdot c(\mathscr{O}(-1))$ by the Whitney formula, which implies that

$$c(Q) \;=\; c(p^*N)\cdot\sum_{i\geq 0}\zeta^i \quad \text{for the class} \quad \zeta \;=\; c_1(\mathscr{O}(1)).$$

Hence we get

$$\begin{aligned} p_*(c_d(Q)\cap[\mathbb{P}(C_WZ\oplus 1)]) &= \left\{p_*\left(c(p^*N)\cap\sum_{i\geq 0}\zeta^i\cap[\mathbb{P}(C_WZ\oplus 1)]\right)\right\}_{\dim Z-d} \\ &= \left\{c(N)\cap p_*\left(\sum_{i\geq 0}\zeta^i\cap[\mathbb{P}(C_WZ\oplus 1)]\right)\right\}_{\dim Z-d} \\ &= \{c(N)\cap s(C_WZ)\}_{\dim Z-d} \end{aligned}$$

and the claim follows. □

Note that in the proposition we allow $\mathrm{codim}_Z W < \mathrm{codim}_X Y$. The following result shows how to deal with such excess intersections as long as they are still regularly embedded:

Corollary 5.8 (Excess intersection formula). *If in the above situation $W\subset Z$ is also a regular embedding, say of codimension e with normal bundle N', then we have*

$$Y\cdot Z \;=\; c_{d-e}(N/N')\cap[W] \;\in\; A_n(W), \quad n \;=\; \dim W-(d-e) \;=\; \dim Z-d.$$

Proof. If $W\hookrightarrow Z$ is a regular embedding, the subcone $N'=C_WZ\subset N=g^*(N_{Y/X})$ is a vector subbundle. By the Whitney formula then

$$c(N)\cap s(W,Z) \;=\; c(N)\cap c(N')^{-1}\cap[W] \;=\; c(N/N')\cap[W]$$

and hence the claim follows from the second formula in proposition 5.7. □

For $e=d$ we recover the formula for proper intersections in lemma 5.5, with the convention that the zero vector bundle has total Chern class 1 as imposed by the Whitney formula. The other extreme is the case $e=0$ which happens for instance for self-intersections:

Example 5.9. Let Y be an equidimensional scheme, and let $Y \hookrightarrow X$ be a regular embedding of codimension d with normal bundle $N = N_{Y/X}$. Then we obtain the self-intersection formula

$$Y \cdot Y \;=\; c_d(N) \cap [Y] \;\in\; A_n(Y), \quad \text{where} \quad n \;=\; \dim Y - d.$$

If Y is a smooth variety, then the diagonal $Y = \Delta \subset X = Y \times Y$ is a regular embedding of codimension $d = \dim Y$ and we obtain that the top Chern class is the degree of the self-intersection of the diagonal:

$$\int_Y c_d(Y) \;=\; \deg(\Delta \cdot \Delta).$$

Over the complex numbers this gives another way of computing $\chi_{\mathrm{top}}(Y(\mathbb{C}))$.

Example 5.10. If $Y \subset X$ is an effective Cartier divisor, then for any subvariety $Z \subset X$ there are two cases:

- If $Z \not\subset Y$, then $Y \cap Z$ is an effective Cartier divisor on Z, hence regularly embedded of the expected codimension. Then lemma 5.5 gives $Y \cdot Z = [Y \cap Z]$.
- If $Z \subset Y$, then the embedding $Y \cap Z \hookrightarrow Z$ is the identity, a regular embedding of codimension $e = 0$. Then the excess intersection formula with $N = \mathscr{O}_X(Y)_{|Z}$ shows that the class of the intersection is $Y \cdot Z = c_1(\mathscr{O}(Y)_{|Z})$.

In both cases the above definition of the intersection product $Y \cdot Z \in A_*(Y \cap Z)$ agrees with our earlier notion of the intersection product with Cartier divisors.

6 Refined Gysin maps and compatibilities

We now want to establish some functorial properties of the intersection product. It will be convenient to do so in a more general setting of Gysin maps. Recall that for a regular embedding $i\colon Y \hookrightarrow X$ of codimension d we have only defined the Gysin map as a homomorphism

$$i^*\colon \quad A_*(X) \longrightarrow A_{*-d}(Y).$$

For the intersection product we wanted more:

- We defined $Y \cdot Z$ for *any morphism* $Z \to X$ from an equidimensional scheme Z.
- We defined $Y \cdot Z$ as a class *on the fiber product* $W = Y \times_X Z$. Even if $Z \subset X$ is a closed subscheme, this gives us more refined information than the previous Gysin pullback in the sense that we get a class in $A_{*-d}(Y \cap Z)$, not just in $A_{*-d}(Y)$.

If we similarly try to put as much information as possible in the Gysin map, we arrive at the following refined notion:

Definition 6.1. Let $i\colon Y \hookrightarrow X$ be a regular embedding of codimension d. For any morphism $f\colon Z \to X$, consider the Cartesian diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "g"'] & Z \arrow[d, "f"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

where $W = Y \times_X Z$ is the fiber product. We define the *refined Gysin map* as the group homomorphism

$$i^!\colon \quad A_*(Z) \longrightarrow A_{*-d}(W), \quad [V] \mapsto X \cdot V$$

where $X \cdot V$ denotes the intersection product defined in the previous section. Note that this only depends on the rational equivalence class $[V] \in A_*(Z)$: Specialization to the normal cone gives a commutative diagram

$$\begin{tikzcd}
Z_*(Z) \arrow[r, "i^!"] \arrow[d, "\sigma"'] & A_{*-d}(W) \\
Z_*(C_W Z) \arrow[r] & Z_*(N) \arrow[u, "s^*"']
\end{tikzcd}$$

for the vector bundle $N = g^*(N_{Y/X})$ and its zero section $s\colon W \to N$, and σ descends by corollary 4.2 to the Chow group of cycles modulo rational equivalence.

The refined Gysin pullback exists not only for regular embeddings, but for *any* embedding $W \hookrightarrow Z$ which arises from a regular embedding via base change as above. In the special case of the identity morphism $f = id\colon Z = X \to X$ the refined Gysin morphism coincides with the naive Gysin morphism from above, i.e. $i^* = i^!$ in this case. However, in more general cases

$$i^!\colon \quad A_*(Z) \longrightarrow A_{*-d}(W)$$

is *not* determined by the embedding $W \hookrightarrow Z$ but usually depends on the regular embedding $i\colon Y \hookrightarrow X$, in fact already the dimension shift does so since in general the two embeddings can have different codimension. The next theorem gathers some basic properties of Gysin maps:

Theorem 6.2. *Consider a diagram of two Cartesian squares*

$$\begin{tikzcd}
W' \arrow[r] \arrow[d, "g"'] & Z' \arrow[d, "f"] \\
W \arrow[r, "i_Z"] \arrow[d, "h"] & Z \arrow[d] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

where i is a regular embedding of codimension d. Let $\alpha \in A_n(Z), \alpha' \in A_n(Z')$.

a) Compatibility: If i_Z is also a regular embedding of the same codimension d, then we have

$$i^! \alpha' \;=\; i_Z^! \alpha' \;\in\; A_{n-d}(W').$$

b) Pushforward: If f is proper, then

$$i^! f_* \alpha' \;=\; g_* i^! \alpha' \;\in\; A_{n-d}(W).$$

c) Pullback: If f is flat of relative dimension r, then

$$i^! f^* \alpha \;=\; g^* i^! \alpha \;\in\; A_{n+r-d}(W').$$

d) Excess intersection: If i_Z is a regular embedding of any codimension e, then the normal bundle $N' = N_{W/Z}$ embeds in the vector bundle $N = h^* N_{Y/X}$ and we have the formula

$$i^! \alpha' \;=\; c_{d-e}(g^*(N/N')) \cap i_Z^! \alpha' \;\in\; A_{n-d-e}(W').$$

Proof. For *a)* note that for the vanishing ideals $\mathscr{I}_Y \trianglelefteq \mathscr{O}_X$ and $\mathscr{I}_W \trianglelefteq \mathscr{O}_Z$ we have an epimorphism $h^*(\mathscr{I}_Y/\mathscr{I}_Y^2) \twoheadrightarrow \mathscr{I}_W/\mathscr{I}_W^2$ of vector bundles. By assumption these two vector bundles have the same rank, hence the epimorphism is an isomorphism. So we have $N_{W/Z} = h^* N_{Y/X}$ and the claim follows.

For *b)* consider the vector bundle $N = h^*(N_{Y/X})$. We may assume that $\alpha' = [V']$ for a subvariety $V' \subset Z'$. From the basic intersection formula in proposition 5.7 we get

$$i^![V'] \;=\; \big\{ c(g^*N) \cap s(W' \cap V', V') \big\}_{n-d}.$$

The pushforward under g then gives

$$\begin{aligned} g_* i^! [V'] \;&=\; g_* \big\{ c(g^*N) \cap s(W' \cap V', V') \big\}_{n-d} \\ &=\; \big\{ c(N) \cap g_*(s(W' \cap V', V')) \big\}_{n-d} && \text{by the projection formula} \\ &=\; \deg(V'/V) \cdot \{ c(N) \cap s(W \cap V, V) \}_{n-d} && \text{by prop. 2.13 for } V = f(V') \\ &=\; \deg(V'/V) \cdot i^! [V] && \text{by definition of } i^! \\ &=\; i^! g_* [V']. && \text{by definition of } g_* \end{aligned}$$

The proof of *c)* follows by the same type of computations.

For part *d)* we consider the embedding $g^*N' \hookrightarrow g^*N$ and the induced commutative diagram

$$\begin{tikzcd} \mathbb{P}' := \mathbb{P}(g^*N' \oplus 1) \arrow[r, hook] \arrow[dr, "p"'] & \mathbb{P} := \mathbb{P}(g^*N \oplus 1) \arrow[d] \\ & W' \end{tikzcd}$$

where p denote the projection. Then the tautological sections of the pullback vector bundles fit in a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{\mathbb{P}'} \arrow[r] \arrow[d, equal] & p^*(g^*N' \oplus 1) \arrow[r] \arrow[d, hook] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{P}'} \arrow[r] & p^*(g^*N \oplus 1) \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

where Q' and Q denote the quotient bundles. It follows that we have a short exact sequence

$$0 \longrightarrow Q' \longrightarrow Q \longrightarrow p^*(g^*(N/N')) \longrightarrow 0$$

and hence

$$c_d(Q) \;=\; c_{d-e}(p^*g^*(N/N')) \cap c_e(Q')$$

by the Whitney formula. Then for a class $\alpha = [V']$ and the cone $C = C_{W' \cap V'}V'$ we get

$$\begin{aligned}
i^![V'] &= Y \cdot V' && \text{by definition of } i^! \\
&= p_*(c_d(Q) \cap [\mathbb{P}(C \oplus 1)] && \text{by the basic formula prop. 5.7} \\
&= p_*(c_{d-e}(p^*g^*(N/N')) \cap c_e(Q') \cap [\mathbb{P}(C \oplus 1)]) && \text{by inserting } c_d(Q) \\
&= c_{d-e}(g^*(N/N') \cap p_*(c_e(Q') \cap [\mathbb{P}(C \oplus 1)]) && \text{by the projection formula} \\
&= c_{d-e}(g^*(N/N')) \cap i_Z^![V'] && \text{by the basic formula for } i_Z^!
\end{aligned}$$

and the claim *d)* follows. □

Corollary 6.3 (Compatibility with Chern classes). *Assume as above that we have a Cartesian square*

$$\begin{tikzcd}
W \arrow[r, hook, "i_Z"] \arrow[d] & Z \arrow[d] \\
Y \arrow[r, hook, "i"] & X
\end{tikzcd}$$

where i is a regular embedding of codimension d. Let E be a vector bundle on Z, then for all $\alpha \in A_n(Z)$, $m \in \mathbb{N}$ we have

$$i^!(c_m(E) \cap \alpha) \;=\; c_m(E_{|W}) \cap i^! \alpha \;\in\; A_{n-d-m}(W).$$

Proof. We proceed in three steps:

1) It is enough to find a proper morphism $\pi\colon \tilde{Z} \to Z$ and $\tilde{\alpha} \in A_n(\tilde{Z})$ with $\pi_*(\tilde{\alpha}) = \alpha$ such that

$$i^!(c_m(\tilde{E}) \cap \tilde{\alpha}) \;=\; c_m(\tilde{E}_{|\tilde{W}}) \cap i^! \tilde{\alpha}$$

for $\tilde{E} = \pi^*(E)$ and $\tilde{W} = W \times_Z \tilde{Z}$. Indeed, this follows from the compatibility of the refined Gysin maps with pushforward in theorem 6.2 and from the projection formula for the cap product with Chern classes.

2) The claim holds if $E = L$ is a line bundle and $m = 1$. Indeed, it is enough to verify this when $\alpha = [V]$ for some subvariety $V \subset Z$. By the previous step we may replace Z by the blowup $\mathrm{Bl}_{V \cap W} V$. After this replacement Z will be a variety and one of the following two cases occurs: Either $W = Z$, in which case the claim follows easily from the excess intersection formula in theorem 6.2. Or $W \subset Z$ is a Cartier divisor. In the latter case put $N = N_{W/Z} \supset N' = h^* N_{Y/X}$ as before, then we have

$$\begin{aligned} i^!(c_1(L) \cap \alpha) &= c_{d-1}(N/N') \cap i_Z^!(c_1(L) \cap \alpha) && \text{by excess intersection formula} \\ &= c_{d-1}(N/N') \cap c_1(L_{|W}) \cap i_Z^! \alpha) && \text{by Gysin for Cartier divisor} \\ &= c_1(L_{|W}) \cap c_{d-1}(N/N') \cap i_Z^! \alpha && \text{by commutativity of } \cap \\ &= c_1(L_{|W}) \cap i^! \alpha && \text{by excess intersection formula} \end{aligned}$$

Note that in the second step we have used the compatibility of Chern classes with the Gysin pullback to Cartier divisors, which we already know from the previous chapter. This is why step 2) only works for the case of divisors.

3) The claim holds for arbitrary E and m. To see this, put $F = E_{|W}$. Since the total Chern class of a vector bundle is the inverse of the total Segre class, it will be enough to show that $i^!(s_j(E) \cap \alpha) = s_j(F) \cap i^! \alpha$ for all $j \in \mathbb{N}_0$. For this we look at the diagram

$$\begin{tikzcd} \mathbb{P}(F) \arrow[r, hook] \arrow[d, "q"'] & \mathbb{P}(E) \arrow[d, "p"] \\ W \arrow[r, hook, "i_Z"] & Z \end{tikzcd}$$

Then one computes

$$\begin{aligned} i^!(s_j(E) \cap \alpha) &= i^! p_*((c_1(\mathscr{O}_{\mathbb{P}E}(1))^{e+j} \cap p^* \alpha) && \text{by definition} \\ &= q_* i^! \left(c_1(\mathscr{O}_{\mathbb{P}E}(1))^{e+j} \cap p^* \alpha\right) && \text{by theorem 6.2b)} \\ &= q_* \left(c_1(\mathscr{O}_{\mathbb{P}F}(1))^{e+j} \cap i^! p^* \alpha\right) && \text{by step 2} \\ &= q_* \left(c_1(\mathscr{O}_{\mathbb{P}F}(1))^{e+j} \cap q^* i^! \alpha\right) && \text{by theorem 6.2c)} \\ &= q_* \left(c_1(\mathscr{O}_{\mathbb{P}F}(1))^{e+j}\right) \cap i^! \alpha && \text{by the proj. formula} \\ &= s_j(F) \cap i^! \alpha && \text{by definition} \end{aligned}$$

and the claim follows. □

The intersection product is also commutative. Since we are writing everything in terms of *refined* Gysin maps, the statement requires three Cartesian squares:

Theorem 6.4 (Commutativity). *Consider a diagram of Cartesian squares*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & Y_2 \arrow[d, "i_2"] \\ W_1 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X_2 \\ Y_1 \arrow[r, "i_1"] & X_1 & \end{tikzcd}$$

where i_ν are regular embeddings of codimension d_ν for $\nu = 1,2$. Then for $\alpha \in A_n(Z)$ we have

$$i_1^! i_2^! \alpha \;=\; i_2^! i_1^! \alpha \;\in\; A_{n-d_1-d_2}(W).$$

Proof. The proof goes by reduction to the case of Cartier divisors in a way similar to the previous proof. For details we refer to Fulton, th. 6.4. □

Theorem 6.5 (Functoriality). *Consider a diagram of Cartesian squares*

$$\begin{tikzcd} W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & Z \arrow[d] \\ Y_1 \arrow[r, "i_1"] & Y_2 \arrow[r, "i_2"] & X \end{tikzcd}$$

where i_ν are regular embeddings of codimension d_ν for $\nu = 1,2$. Then for $\alpha \in A_n(Z)$ we have

$$(i_2 \circ i_1)^! \alpha \;=\; i_1^! i_2^! \alpha \;\in\; A_{n-d_1-d_2}(W_1).$$

Proof. We omit the proof for the sake of time; it can be found in Fulton, th. 6.5. □

7 The Chow ring of a smooth variety

From now on let us assume that X is a smooth variety. Then the diagonal

$$\delta\colon \quad X \hookrightarrow X \times X, \quad p \mapsto (p,p)$$

is a regular embedding of codimension $n = \dim(X)$: Indeed, its composite with the projection $\mathrm{pr}\colon X \times X \to X$ is the identity, and since the identity is a smooth morphism, it follows that the conormal sequence for the diagonal embedding is a split exact sequence $0 \to N^\vee_{X/X\times X} \to \delta^* \Omega^1_{X\times X/X} \to \Omega^1_{X/k} \to 0$. The smoothness of X implies that the two rightmost terms are locally free of rank $2n$ resp. n, hence the conormal sheaf $N^\vee_{X/X\times X} = \mathscr{I}/\mathscr{I}^2$ is locally free of rank n.

Definition 7.1. The *intersection product* on the Chow groups of a smooth variety X is defined as the composite of the Künneth map

$$\times: \quad A_*(X)\otimes A_*(X) \longrightarrow A_*(X\times X), \quad [Y]\otimes[Z] \mapsto [Y\times Z]$$

with the Gysin pullback δ^* under the regular embedding δ as shown in the following diagram:

$$\begin{tikzcd}
A_*(X)\otimes A_*(X) \arrow[rr, "\cdot"] \arrow[dr, "\times"'] & & A_*(X) \\
& A_*(X\times X) \arrow[ur, "\delta^*"'] &
\end{tikzcd}$$

Note that this uses only the ordinary Gysin pullback, not the refined one from the previous section. If we want to intersect 'physical' cycles $\alpha\in Z_{n-d}(X)$, $\beta\in Z_{n-e}(X)$ with support $Y=|\alpha|$ and $Z=|\beta|$, we can define the intersection product in the refined sense by

$$\alpha\cdot\beta \;:=\; \delta^!(\alpha\times\beta) \;\in\; A_{n-d-e}(Y\cap Z).$$

Note that the dimension of this cycle is by our construction of the Gysin pullback equal to

$$\dim(Y\times Z)-\operatorname{codim}_{X\times X}(X) \;=\; (n-d)+(n-e)-n \;=\; n-d-e.$$

In particular, if we view the intersection in the refined sense, then to any irreducible component W of $Y\cap Z$ of the expected dimension $\dim W=n-d-e$ we may attach a unique multiplicity

$$i_W(\alpha,\beta) \;\in\; \mathbb{Z}$$

as follows: Let $W'\subset Y\cap Z$ be the union of all other irreducible components, then since W is an irreducible component of dimension $\dim W=n-d-e$, we have a natural isomorphism

$$A_{n-d-e}(Y\cap Z) \;\simeq\; A_{n-d-e}(W')\oplus\mathbb{Z}\cdot[W],$$

and we define $i_W(\alpha,\beta)\in\mathbb{Z}$ to be the coefficient of $[W]$ in $\alpha\cdot\beta$. If $\alpha=[Y]$, $\beta=[Z]$ are fundamental classes of subvarieties, we also denote this intersection multiplicity by $i_W(Y,Z)$. In nice cases these multiplicities can be computed naively:

Remark 7.2. The compatibility of the refined Gysin map with flat pullback shows that intersection multiplicities can be computed locally: In the above situation we have

$$i_W(Y,Z) \;=\; i_{W\cap U}(Y\cap U, Z\cap U) \quad \text{for any open} \quad U\subset X \quad \text{with} \quad W\cap U\neq\varnothing.$$

Hence if $W\subset Z$ is cut out on some open subset by a regular sequence of length d, for instance if the local ring $\mathscr{O}_{Z,W}$ is regular or more generally Cohen-Macaulay, then by lemma 5.5 the intersection multiplicity is the length $\ell(\mathscr{O}_{Y\cap Z,W})$.

We have defined the intersection product via the pullback under the diagonal morphism. This construction has an important generalization that allows to define pullbacks under arbitrary morphisms:

Definition 7.3. Let $f\colon Y \to X$ be a morphism from an arbitrary scheme Y to a smooth scheme X. The graph morphism $\gamma_f\colon Y \to X \times Y, p \mapsto (f(p), p)$ is a regular embedding of codimension $n = \dim(X)$ by the same argument as for the diagonal, and we define the *cap product*

$$A_d(X) \otimes A_e(Y) \longrightarrow A_{d+e-n}(Y), \quad x \otimes y \mapsto f^*(x) \cap y := \gamma_f^*(x \times y).$$

Again there is also a refined version of this cap product using $\gamma_f^!$ instead.

We have already seen above that in dealing with intersection products it is more convenient to index cycles by their *codimension* rather than their dimension, hence in what follows we write

$$A^*(X) := \bigoplus_{d=0}^{n} A^d(X) \quad \text{with} \quad A^d(X) := A_{n-d}(X)$$

for $n = \dim X$. The intersection product preserves the grading by codimension, i.e. it can be written as

$$A^d(X) \otimes A^e(X) \longrightarrow A^{d+e}(X), \quad \alpha \otimes \beta \mapsto \alpha \cdot \beta,$$

Thus $A^*(X)$ becomes a commutative graded ring called the *Chow ring* of X:

Theorem 7.4. *Let X be a smooth variety.*

a) The group $A^*(X)$ is a commutative graded ring with respect to the intersection product; the unit of this ring is the fundamental class $1 = [X] \in A^0(X)$.

b) For any scheme Y with a morphism $f\colon Y \to X$ we have:

- *The group $A_*(Y)$ is a module under the ring $A^*(X)$ via*

$$A^m(X) \otimes A_e(Y) \longrightarrow A_{e-m}(Y), \quad x \otimes y \mapsto f^*(x) \cap y.$$

- *If Y is also smooth, then for all $x \in A^*(X)$ and all $y_1, y_2 \in A^*(Y)$ we have the formula*

$$(f^*(x) \cap y_1) \cdot y_2 = f^*(x) \cap (y_1 \cdot y_2),$$

  and we get a ring homomorphism

$$f^*\colon \quad A^*(X) \longrightarrow A^*(Y), \quad x \mapsto f^*(x) \cap [Y].$$

- *If $f\colon Y \to X$ is a proper morphism between smooth varieties, then we have the projection formula*

$$f_*(f^*(x) \cap y) = x \cdot f_*(y).$$

Proof. We know from theorem 6.4 that the intersection product is commutative, and it is clear from the definitions that intersection with the fundamental class $[X]$ acts as the identity on $A^*(X)$. The associativity of the intersection product is best checked together with the associativity of the scalar multiplication for the module structure in *b)*, so let $f\colon Y \to X$ be any morphism to a smooth variety X. From the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "\gamma_f"] \arrow[d, "\gamma_f"'] & X \times Y \arrow[d, "id\times\gamma_f"] \\
X \times Y \arrow[r, "\delta\times id"] & X \times X \times Y
\end{tikzcd}$$

so by functoriality of the Gysin pullback we compute for $x_1, x_2 \in A^*(X), y \in A_*(Y)$:

$$\begin{aligned}
f^*(x_1) \cap (f^*(x_2) \cap y) &= \gamma_f^!(x_1 \times \gamma_f^!(x_2 \times y)) \\
&= \gamma_f^!(id \times \gamma_f)^!(x_1 \times x_2 \times y) \\
&= \gamma_f^!(\delta \times id)^!(x_1 \times x_2 \times y) \\
&= \gamma_f^!(\delta^!(x_1 \times x_2) \times y) \\
&= f^*(x_1 \cdot x_2) \cap y
\end{aligned}$$

Taking $Y = X$ and $f = id$ gives the associativity of the intersection product on $A^*(X)$, which finishes the proof of *a)*. Taking Y and f arbitrary, we get the associativity of the scalar multiplication in the first item of *b)*, which shows that $A_*(Y)$ is a module over $A^*(X)$. If Y is also smooth, then we also know that the horizontal arrows in the commutative square

$$\begin{tikzcd}
Y \arrow[r, "\delta"] \arrow[d, "\gamma_f"'] & Y \times Y \arrow[d, "\gamma_f\times id"] \\
X \times Y \arrow[r, "id\times\delta"] & X \times Y \times Y
\end{tikzcd}$$

are regular embeddings, hence

$$\begin{aligned}
(f^*(x) \cap y_1) \cdot y_2 \;=\; \delta^!(\gamma_f \times id)^!(x \times y_1 \times y_2) &= \gamma_f^!(id \times \delta)^!(x \times y_1 \times y_2) \\
&= f^*(x) \cap (y_1 \cdot y_2).
\end{aligned}$$

In particular $f^*(x) \cap (-) \in \operatorname{End}(A^*(Y))$ is determined by $f^*(x) \cap [Y] \in A^*(Y)$ in this case, since the last formula shows $f^*(x) \cdot y = (f^*(x) \cap [Y]) \cdot y$. Altogether we then obtain

$$\begin{aligned}
(f^*(x_1) \cap [Y]) \cdot (f^*(x_2) \cap [Y]) &= f^*(x_1) \cap ([Y] \cdot (f^*(x_2) \cap [Y]) \\
&= f^*(x_1) \cap (f^*(x_2) \cap [Y]) \\
&= f^*(x_1 \cdot x_2) \cap [Y],
\end{aligned}$$

so $f^*\colon A^*(X) \to A^*(Y)$ is a ring homomorphism. For the projection formula take the Cartesian square

$$\begin{tikzcd} Y \arrow[r, "\gamma_f"] \arrow[d, "f"'] & X \times Y \arrow[d, "id\times\gamma_f"] \\ X \arrow[r, "\delta"] & X \times X \end{tikzcd}$$

If f is proper, then the compatibility of Gysin pullbacks with proper pushforward in theorem 6.2 shows that

$$f_*(f^*(x)\cap y) \;=\; f_*\gamma_f^!(x\times y) \;=\; \delta^!(id\times f)_*(x\times y) \;=\; \delta^!(x\times f_*(y)) \;=\; x\cdot f_*(y)$$

for all $x\in A^*(X), y\in A^*(Y)$ as claimed. □

As we observed in the proof, for smooth varieties Y the operation $f^*(x)\cap(-)$ is determined uniquely by the class $f^*(x)\cap[Y]$; in this case we also abuse notation and write

$$f^*(x) \;:=\; f^*(x)\cap[Y] \;\in\; A^*(Y).$$

A similar result holds for the Chern class operations:

Lemma 7.5. *Let $f\colon Y\to X$ be a morphism of smooth varieties, and E a vector bundle on X. Then*

$$(c(E)\cap x)\cdot y \;=\; c(E)\cap(f^*(x)\cdot y) \quad \textit{for all} \quad x\in A^*(X),\, y\in A^*(Y).$$

Proof. Let $p\colon X\times Y\to Y$ be the projection. Then the definitions easily imply the formula

$$(c(E)\cap x)\times y \;=\; c(p^*(E))\cap(x\times y) \;\in\; A^*(X\times Y).$$

From this we compute

$$\begin{aligned}(c(E)\cap x)\cdot y \;&=\; \delta^!((c(E)\cap x)\times y)\\ &=\; \delta^!(c(p^*(E))\cap(x\times y))\\ &=\; c(\delta^*(p^*(E)))\cap\delta^!(x\times y)\\ &=\; c(E)\cap(x\cdot y),\end{aligned}$$

where the third step uses the compatibility of Chern classes with Gysin pullack. □

Applying the above to the identity $f=id\colon Y=X\to X$ and the fundamental class $x=[X]$, we see that on any smooth varieties the Chern class operations are determined by their values on the fundamental class. Again by abuse of notation we simply write

$$c(E) \;:=\; c(E)\cap[X] \;\in\; A^*(X)$$

in this case. One can summarize the above by saying that on smooth varieties, all notions of intersection product and pullback that we defined are compatible with each other, and all intersection operations that we defined are determined by their values on the fundamental class. Now finally all the foundations are settled, and in the next chapters we will consider some applications.

Before doing so, let us look at a few simple examples. The following computation gives another characterization of Chern classes:

Lemma 7.6 (Projective bundles). *Let E be a vector bundle of rank $r+1$ on a smooth variety. Then the Chow ring of the associated projective bundle is given as an $A^*(X)$-algebra by*

$$A^*(\mathbb{P}(E)) \;\simeq\; A^*(X)[t]/(f(t))$$

where $f(t) = t^{r+1} + c_1 t^r + \cdots + c_{r+1}$ for the Chern classes $c_i = c_i(E)$.

Proof. Let $\zeta = c_1(\mathscr{O}_{\mathbb{P}(E)}(1))$. The ring homomorphism $f^*\colon A^*(X) \to A^*(\mathbb{P}(E))$ gives rise to a homomorphism of $A^*(X)$-algebras

$$\varphi\colon \quad A^*(X)[t] \twoheadrightarrow A^*(\mathbb{P}(E)), \quad t \mapsto \zeta.$$

In the discussion of Chow groups of projective bundles in the previous chapter we have seen that as an additive group

$$A^*(\mathbb{P}(E)) \;\simeq\; \bigoplus_{i=0}^{r} A^*(X)\cdot\zeta^i$$

On the other hand, for the pullback of the vector bundle under $p\colon \mathbb{P}(E) \to X$ we have the tautological sequence $0 \to \mathscr{O}_{\mathbb{P}(E)}(-1) \to p^*(E) \to Q \to 0$, hence by the Whitney formula

$$c(Q) \;=\; c(p^*(E))/(1-\zeta) \;=\; c(p^*(E))\cdot(1+\zeta+\zeta^2+\cdots).$$

Since Q has rank r, it follows that

$$0 \;=\; c_{r+1}(Q) \;=\; \zeta^{r+1} + c_1(p^*(E))\zeta^r + \cdots + c_{r+1}(E).$$

Hence $f(t) = t^{r+1} + c_1 t^r + \cdots + c_{r+1} \in \ker(\varphi)$. Given the above decomposition of the total Chow group as a direct sum, there can be no further relations, hence the algebra homomorphism φ induces an isomorphism $A^*(\mathbb{P}(E)) \simeq A^*(X)[t]/f(t)$. □

Example 7.7 (Hirzebruch surfaces). For $n \in \mathbb{Z}$, let $f\colon \Sigma_n \to \mathbb{P}^1$ denote the fibered surface obtained by the following glueing with respect to the standard open affine cover $\mathbb{P}^1 = U_0 \cup U_\infty$:

$$\begin{array}{ccccccc} f^{-1}(U_0) := U_0\times\mathbb{P}^1 & \supset & \mathbb{G}_m\times\mathbb{P}^1 & \xrightarrow{\sim} & \mathbb{G}_m\times\mathbb{P}^1 & \subset & U_\infty\times\mathbb{P}^1 =: f^{-1}(U_\infty) \\ & & (t,[x:y]) & \mapsto & (t^{-1},[x:t^n y]) & & \end{array}$$

Such surfaces are known as *Hirzebruch surfaces*. They can be written as projective bundles

$$\Sigma_n \;\simeq\; \mathbb{P}(E_n) \;\longrightarrow\; \mathbb{P}^1 \quad \text{for the vector bundle} \quad E_n \;:=\; \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n).$$

By the Whitney formula $c_2(E_n) = 0$ and $c_1(E_n) = n \cdot \eta$ for the class $\eta = c_1(\mathscr{O}_{\mathbb{P}^1}(1))$, so from the previous example we get an explicit presentation of the Chow ring as the quotient

$$A^*(\Sigma_n) \;\simeq\; A^*(X)[t]/(t^2 + n\eta t) \;\simeq\; \mathbb{Z}[s,t]/(s^2, t^2 + nst)$$

where the second isomorphism comes from $A^*(\mathbb{P}^1) \simeq \mathbb{Z}[s]/s^2$. Note that the additive groups

$$\begin{aligned} A^0(\Sigma_n) \;&\simeq\; \mathbb{Z} \\ A^1(\Sigma_n) \;&\simeq\; \mathbb{Z} \oplus \mathbb{Z} \\ A^2(\Sigma_n) \;&\simeq\; \mathbb{Z} \end{aligned}$$

do not depend on n, but the intersection pairing on $A^1(\Sigma_n) = \mathbb{Z}\eta \oplus \mathbb{Z}\zeta$ does:

$\cdot$	η	ζ
η	0	1
ζ	1	$-n$

Proposition 7.8 (Chow ring of a blowup). *Let $Y \hookrightarrow X$ be a regular embedding of codimension d between smooth varieties. Consider the blowup $f \colon \tilde{X} = \mathrm{Bl}_Y(X) \to X$ with exceptional divisor $E = f^{-1}(Y)$ as shown below:*

$$\begin{tikzcd} E \arrow[r, hook, "i_E"] \arrow[d, "f_E"'] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

Then $\tilde{X}$ is again a smooth variety. Moreover, every element of the Chow ring $A^*(\tilde{X})$ has the form

$$f^*(x) + i_{E*}(e) \quad \textit{for some} \quad x \,\in\, A^*(X) \quad \textit{and} \quad e \,\in\, A^*(E),$$

and the intersection product is given as follows for $x, x' \in A^*(X)$, $e, e' \in A^*(E)$:

$$\begin{aligned} f^*(x) \cdot f^*(x') \;&=\; f^*(x \cdot x'), \\ f^*(x) \cdot i_{E*}(e) \;&=\; i_{E*}(e \cdot p^* i^* x), \\ i_{E*}(e) \cdot i_{E*}(e') \;&=\; -i_*(e \cdot e' \cdot \zeta), \end{aligned}$$

where we have put $\zeta = c_1(\mathscr{O}_{\mathbb{P}(N_{Y/X})}(1))$ using the identification $E = \mathbb{P}(N_{Y/X})$.

Proof. To see that every element of $A^*(\tilde{X})$ has the given form, we consider the localization sequence

$$\begin{tikzcd}
A_*(E) \arrow[r, "i_{E*}"] & A_*(\tilde{X}) \arrow[r, "j_E^*"] & A_*(\tilde{X} \setminus E) \arrow[r] \arrow[d, equal] & 0 \\
& A_*(X) \arrow[u, "f^*"] \arrow[r, "j^*"] & A_*(X \setminus Y) &
\end{tikzcd}$$

The diagram shows that for any $\tilde{x} \in A^*(\tilde{X})$ we have $\tilde{x} - f^* f_*(\tilde{x}) \in \ker(j_E^*) = \operatorname{im}(i_{E*})$, hence $\tilde{x} = f^* f_*(\tilde{x}) + i_{E*}(e)$ for some $e \in A^*(E)$ as claimed.

The first formula $f^*(x) \cdot f^*(x') = f^*(x \cdot x')$ is clear since $f^*\colon A^*(X) \to A^*(\tilde{X})$ is a ring homomorphism. For the second formula one computes

$$f^*(x) \cdot i_{E*}(e) \;=\; i_{E*}(e \cdot i_E^* f^*(x)) \;=\; i_{E*}(e \cdot p^* i^*(x))$$

by the projection formula and functoriality of pullback. For the last formula note that $c_1(N_{E/\tilde{X}}) = -\zeta$ and hence $i_E^* i_{E*}(e) = -e \cdot \zeta$, which gives

$$i_{E*}(e) \cdot i_{E*}(e') \;=\; i_{E*}(i_E^* i_{E*}(e) \cdot e') \;=\; -i_{E*}(e \cdot e' \cdot \zeta)$$

again by the projection formula. □

One can also describe all relations between $f^*(A^*(X))$ and $i_{E*}(A^*(E))$ in the above presentation of the Chow ring of the blowup, they all arise from the Chow ring $A^*(Y)$; see Fulton, sect. 6.7. In favorable cases it is easier to check for relations by hand, using the intersection pairing:

Example 7.9. Let S be a smooth surface and $Y = \{p_1, \dots, p_r\} \subset S$ a finite set of closed points. Then for the blowup $f\colon \tilde{S} = \mathrm{Bl}_Y(S) \to S$ and the divisors $E_i = f^{-1}(p_i)$ we get

$$A^1(\tilde{S}) \;=\; A^1(S) \oplus \bigoplus_{i=1}^{r} \mathbb{Z}[E_i]$$

where the directness of the sum on the right hand side follows from the explicit form of the intersection pairing:

- $[E_i] \cdot [E_j] = 0$ for $i \neq j$,
- $[E_i] \cdot [E_i] = -[q_i]$ for any point $q_i \in E_i$,
- $f^*[D] \cdot [E_i] = 0$ for all $D \in A^1(S)$,
- $f^*[D] \cdot f^*[D'] = f^*[D \cdot D']$ for all $D, D' \in A^1(S)$.

Chapter IV
The Grothendieck-Riemann-Roch theorem

1 Motivation: Why the Todd class?

The Riemann-Roch theorem says that for any line bundle $\mathscr{L} \in \operatorname{Pic}(C)$ on a smooth projective curve C of genus g, we have

$$\dim H^0(C, \mathscr{L}) - \dim H^1(C, \mathscr{L}) \;=\; \deg(\mathscr{L}) + 1 - g.$$

Note that the right hand side can be written in terms of Chern classes. It easily generalizes to vector bundles of higher rank:

Lemma 1.1. *For any vector bundle $\mathscr{E}$ on a smooth projective curve* C *we have the formula*

$$\dim H^0(C, \mathscr{E}) - \dim H^1(C, \mathscr{E}) \;=\; \int_C \left(c_1(\mathscr{E}) + \operatorname{rk}(\mathscr{E}) \cdot \frac{c_1(\mathscr{T}_C)}{2} \right).$$

Proof. Both sides are additive for extensions of vector bundles. Since for $n \gg 0$ the vector bundle $\mathscr{E}(n)$ has a global section, the vector bundle $\mathscr{E}$ has a line subbundle of the form $\mathscr{O}_C(-n)$. Let $\mathscr{L} \subset \mathscr{E}$ be the saturation of this subbundle, i.e. the preimage of the maximal torsion subsheaf of the quotient $\mathscr{E}/\mathscr{O}_C(-n)$. Then $\mathscr{F} = \mathscr{E}/\mathscr{L}$ is a torsion-free sheaf on a smooth curve, hence locally free. So we have a short exact sequence of vector bundles $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{F} \to 0$ and the claim follows by induction on the rank. □

More generally, we will see that for any smooth projective variety X one can express the Euler characteristic

$$\chi(X, \mathscr{E}) \;=\; \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathscr{E})$$

of a locally free sheaf $\mathscr{E}$ by a universal formula in terms of the Chern classes of $\mathscr{E}$ and of the tangent bundle to the variety. Let us see how to guess the shape of such a formula. First, we know that the Euler characteristic of coherent sheaves has the following properties:

a) Additivity: For any $\mathscr{E},\mathscr{F}\in\mathrm{Coh}(X)$ we have $\chi(X,\mathscr{E}\oplus\mathscr{F})=\chi(X,\mathscr{E})+\chi(X,\mathscr{F})$.

b) Multiplicativity: For any $\mathscr{E}\in\mathrm{Coh}(X),\mathscr{F}\in\mathrm{Coh}(Y)$ we have

$$\chi(X\times Y,\mathscr{E}\boxtimes\mathscr{F})=\chi(X,\mathscr{E})\cdot\chi(Y,\mathscr{F})\quad\text{for}\quad \mathscr{E}\boxtimes\mathscr{F}:=\mathrm{pr}_1^*(\mathscr{E})\otimes\mathrm{pr}_2^*(\mathscr{F}).$$

We have already seen one expression in Chern classes of vector bundles $\mathscr{E}$ that has these two properties: The *Chern character*, which is given in terms of the Chern roots $\alpha_1,\dots,\alpha_r$ by

$$\mathrm{ch}(\mathscr{E}) \;=\; \sum_{i=1}^{r}\exp(\alpha_i) \;\in\; A^*(X).$$

However, this is not quite what we want. Looking at the case of curves, we make the ansatz

$$\chi(X,\mathscr{E}) \;\overset{?}{=}\; \int_X \mathrm{ch}(\mathscr{E})\cdot\mathrm{td}(\mathscr{T}_X)$$

where $\mathrm{td}(\mathscr{T}_X)$ should be some universal power series in the Chern classes of the tangent bundle. With this ansatz the additivity property follows directly from the additivity of the Chern character. For the multiplicativity, notice that the tangent bundle of a product is given by a direct sum

$$\mathscr{T}_{X\times Y} \;=\; \mathrm{pr}_1^*(\mathscr{T}_X)\oplus\mathrm{pr}_1^*(\mathscr{T}_Y),$$

not a tensor product. Hence if we assume that the Todd class is defined as a power series on the Chern classes of arbitrary vector bundles, the desired multiplicativity would hold if

$$\mathrm{td}(\mathscr{E}\oplus\mathscr{F}) \;\overset{?}{=}\; \mathrm{td}(\mathscr{E})\cdot\mathrm{td}(\mathscr{F}).$$

Finally, we need some normalization: The trivial bundle has no higher Chern classes, so in this case the Todd class should be a scalar. Looking at the Riemann-Roch theorem in the case of elliptic curves where the tangent bundle is trivial, we want to take $\mathrm{td}(\mathscr{O}_X)=1$. If we moreover want that the Riemann-Roch formula holds for the tautological sheaf on projective space, we have no more choice:

Proposition 1.2. *There exists a unique power series $F\in\mathbb{Q}[[x_1,x_2,\dots]]$ in infinitely many variables such that for all vector bundles $\mathscr{E}$ on smooth projective varieties X the classes $\mathrm{td}(\mathscr{E})=F(c_1(\mathscr{E}),c_2(\mathscr{E}),\dots)\in A^*(X)\otimes\mathbb{Q}$ satisfy*

a) $\mathrm{td}(\mathscr{E}\oplus\mathscr{F})=\mathrm{td}(\mathscr{E})\cdot\mathrm{td}(\mathscr{F})$,

b) $\mathrm{td}(\mathscr{O}_X)=1$, and

c) $\chi(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n})=\int_{\mathbb{P}^n}\mathrm{ch}(\mathscr{O}_{\mathbb{P}^n})\cdot\mathrm{td}(\mathscr{T}_{\mathbb{P}^n})$ for all $n\in\mathbb{N}$.

Explicitly we have

$$\mathrm{td}(\mathscr{E}) \;=\; \prod_{i=1}^{r}\frac{\alpha_i}{1-\exp(-\alpha_i)}\quad \textit{for the Chern roots } \alpha_1,\dots,\alpha_r \textit{ of } \mathscr{E}.$$

Proof. If there exists a power series F with the above properties, then by the splitting principle and by *a)* we have

$$\operatorname{td}(\mathscr{E}) \;=\; \prod_{i=1}^{r} f(\alpha_i) \quad \text{for the power series} \quad f(t) \;=\; F(t,0,0,\dots) \;\in\; \mathbb{Q}[[t]].$$

Hence for the uniqueness we only need to determine the power series $f(t)$. From the decomposition $\mathscr{T}_{\mathbb{P}^n} \oplus \mathscr{O}_{\mathbb{P}^n} \simeq (\mathscr{O}_{\mathbb{P}^n}(1))^{\oplus n+1}$ together with the properties *a)*, *b)* we get that

$$\operatorname{td}(\mathscr{T}_{\mathbb{P}^n}) \;=\; \operatorname{td}(\mathscr{O}_{\mathbb{P}^n}(1))^{n+1} \;=\; (f(\zeta))^{n+1} \quad \text{for the class} \quad \zeta \;=\; c_1(\mathscr{O}_{\mathbb{P}^n}(1)).$$

Now by *c)* for $a = 0$, with $\operatorname{ch}(\mathscr{O}_{\mathbb{P}^n}) = 1$, we have

$$1 \;=\; \chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}) \;=\; \int_{\mathbb{P}^n} \operatorname{ch}(\mathscr{O}_{\mathbb{P}^n}) \cdot \operatorname{td}(\mathscr{T}_{\mathbb{P}^n}) \;=\; \int_X (f(\zeta))^{n+1}.$$

Thus we see that the power series $f(t) \in \mathbb{Q}[[t]]$ must satisfy the condition that the coefficient of t^n in $(f(t))^{n+1}$ is equal to one for all natural numbers $n \in \mathbb{N}$. We write this as

$$\{f(t)^{n+1}\}_n \;=\; 1 \quad \text{for all} \quad n.$$

Now there is a convenient way to recover a power series $f(t) \in \mathbb{Q}[[t]]$ with $f(0) \neq 0$ from the coefficients of t^{n-1} in $f(t)^n$ which is known as Lagrange inversion. It goes as follows:

- Consider the power series $g(t) = t/f(t) \in \mathbb{Q}[[t]]$.
- Since $g'(0) \neq 0$, there exists a power series $h(s) \in \mathbb{Q}[[s]]$ with $t = h(g(t))$.
- Write this inverse as a power series $h(s) = a_1 s + a_2 s^2 + \cdots$, then for every n the desired coefficient is the residue

$$\begin{aligned} \{(f(t))^n\}_{n-1} \;&=\; \operatorname{Res} \frac{(f(t))^n}{t^n} dt \\ &=\; \operatorname{Res} \frac{1}{(g(t))^n} dt \\ &=\; \operatorname{Res} \frac{h'(s)}{s^n} ds \\ &=\; n \cdot a_n \end{aligned}$$

where in the third equality we have used the relation $dt = h'(s)ds$ of differentials coming from the substitution $t = h(s)$. Starting from the coefficients $\{(f(t))^n\}_{n-1}$ we find by the above the coefficients a_n, hence the power series $h(s)$ and therefore the formal inverse $g(t)$ with respect to the composition of power series. From this we recover the desired power series $f(t) = t/g(t)$.

In our case our assumption on the coefficients says $1 = \{(f(t))^n\}_{n-1} = n \cdot a_n$, so we get $a_n = 1/n$ for all n. Then

$$h(s) \;=\; \sum_{n\geq 1} \frac{1}{n} s^n \;=\; -\log(1-s),$$

is the logarithm series, whose formal inverse is the series $g(t) = 1 - \exp(-t)$. Thus we find that

$$f(t) \;=\; \frac{t}{1-\exp(-t)},$$

which proves the uniqueness and the desired formula. Conversely, one easily verifies that the class $\mathrm{td}(\mathscr{E})$ defined by this formula has all the mentioned properties. □

Let us write out the first terms of the Todd class more explicitly. The power series in the proposition is given by

$$f(t) \;=\; \frac{t}{1-\exp(-t)} \;=\; 1 + \tfrac{1}{2}t + \sum_{i\geq 1} (-1)^{i-1} \tfrac{B_{2i}}{(2i)!} \cdot t^{2i} \;=\; 1 + \tfrac{1}{2}t + \tfrac{1}{12}t^2 - \tfrac{1}{720}t^4 + \cdots$$

where $B_{2i} \in \mathbb{Q}$ are the Bernoulli numbers. For $\mathscr{E}$ with Chern roots $\alpha_1, \dots, \alpha_r$ one then computes

$$\mathrm{td}(\mathscr{E}) \;=\; \prod_{i=1}^{r} f(\alpha_i) \;=\; 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \cdots$$

where the Chern classes $c_i = c_i(\mathscr{E})$ are the elementary symmetric polynomials in the Chern roots. We can also express the Todd class in terms of Chern classes of wedge powers as follows:

Lemma 1.3. *For any vector bundle $\mathscr{E}$ of rank r we have*

$$\sum_{i=0}^{r} (-1)^i \,\mathrm{ch}(\wedge^i \mathscr{E}^\vee) \;=\; c_r(\mathscr{E}) \cdot \mathrm{td}(\mathscr{E})^{-1}.$$

Proof. Let $\alpha_1, \dots, \alpha_r$ be the Chern roots of $\mathscr{E}$. Then $c_r(\mathscr{E}) = \alpha_1 \cdots \alpha_r$ while the Chern roots of the exterior powers $\wedge^i \mathscr{E}^\vee$ are the sums of the form $-\alpha_{\nu_1} - \cdots - \alpha_{\nu_i}$ with $1 \leq \nu_1 < \cdots < \nu_i \leq r$. Now use

$$\begin{aligned} \sum_{i=0}^{r} \sum_{\nu \in I_i} (-1)^i e^{-\alpha_{\nu_1} - \cdots - \alpha_{\nu_i}} \;&=\; \prod_{i=1}^{r} (1 - e^{-\alpha_i}) \\ &=\; \alpha_1 \cdots \alpha_r \cdot \prod_{i=1}^{r} \frac{1 - e^{-\alpha_i}}{\alpha_i} \end{aligned}$$

where I_i denotes the set of tuples $\nu = (\nu_1, \dots, \nu_i)$ with $1 \leq \nu_1 < \cdots < \nu_i \leq n$. □

For the Hirzebruch-Riemann-Roch theorem we also need the Chern character. In terms of the Chern roots it is given by

$$\mathrm{ch}(\mathscr{E}) \;=\; \sum_{i=1}^{r} \exp(\alpha_i) \;=\; \sum_{n\geq 0} \frac{p_n}{n!} \quad \text{for the power sums} \quad p_n \;=\; \alpha_1^n + \cdots + \alpha_r^n.$$

In the theory of symmetric functions one learns that the power sum functions can be expressed as a certain determinant in the elementary symmetric polynomials. In concrete terms

$$p_n \;=\; \det \begin{pmatrix} c_1 & 1 & 0 & \cdots & \\ 2c_2 & c_1 & 1 & 0 & \cdots \\ 3c_3 & c_2 & c_1 & 1 & \cdots \\ \vdots & \vdots & & \ddots & \ddots \\ nc_n & c_{n-1} & \cdots & & c_1 \end{pmatrix}$$

which gives

$$\mathrm{ch}(\mathscr{E}) \;=\; r + c_1 + \frac{c_1^2 - 2c_2}{2} + \frac{c_1^3 - 3c_1c_2 + 3c_3}{6} + \cdots$$

We will soon prove that for all vector bundles on smooth projective varieties the Hirzebruch-Riemann-Roch theorem $\chi(X,\mathscr{E}) = \int_X \mathrm{ch}(\mathscr{E})\mathrm{td}(X)$ holds. Let us see how this generalizes what we already know:

Example 1.4. For any vector bundle $\mathscr{E}$ of rank r on a smooth projective variety X we have

$$\begin{aligned} &\mathrm{ch}(\mathscr{E}) \cdot \mathrm{td}(\mathscr{T}_X) \;= \\ &= \left(r + c_1(\mathscr{E}) + \frac{c_1^2(\mathscr{E}) - 2c_2(\mathscr{E})}{2} + \cdots\right)\left(1 + \frac{c_1(X)}{2} + \frac{c_1^2(X) + c_2(X)}{12} + \cdots\right)\end{aligned}$$

Let us take a look at low dimensions:

a) For $\dim X = 1$ the HRR theorem predicts

$$\chi(X,\mathscr{E}) \;=\; \int_X \left(c_1(\mathscr{E}) + r \cdot \frac{c_1(X)}{2}\right)$$

which is precisely the classical Riemann-Roch formula.

b) For $\dim X = 2$ the HRR theorem predicts

$$\chi(X,\mathscr{E}) \;=\; \int_X \left(\frac{c_1^2(\mathscr{E}) - 2c_2(\mathscr{E})}{2} + \frac{c_1(\mathscr{E})c_1(X)}{2} + r \cdot \frac{c_1^2(X) + c_2(X)}{12}\right)$$

For $\mathscr{E} = \mathscr{O}_X$ this gives Noether's formula

$$\chi(X,\mathscr{O}_X) \;=\; \frac{c_1^2(X) + c_2(X)}{12}$$

from the theory of algebraic surfaces. More generally, if $\mathscr{E} = \mathscr{O}_X(D)$ is a line bundle given by a Cartier divisor $D \in \operatorname{Div}(X)$ and if we denote by $K_X = c_1(\omega_X)$ the first Chern class of the canonical line bundle on the smooth surface X, then we get

$$\chi(X, \mathscr{O}_X(D)) \;=\; \frac{D \cdot (D-K)}{2} + \chi(X, \mathscr{O}_X),$$

the Riemann-Roch formula for line bundles on smooth projective surfaces.

2 Some remarks about Grothendieck groups

We have seen some time ago that the Chern character of vector bundles is additive in short exact sequences, and the same holds for the Euler characteristic of coherent sheaves. In general, the study of such additive functionals on an abelian category leads to the following definition:

Definition 2.1. By the *Grothendieck group* $K(\mathscr{A})$ of an abelian category $\mathscr{A}$ we mean the quotient of the free abelian group generated by isomorphism classes $[E]$ of objects $E \in \mathscr{A}$ modulo the relations

$$[E] \;=\; [E'] + [E''] \quad \text{for each exact sequence} \quad 0 \to E' \to E \to E'' \to 0 \quad \text{in} \quad \mathscr{A}.$$

For a scheme X, let $\mathrm{VB}(X)$ resp. $\mathrm{Coh}(X)$ be the abelian category of vector bundles resp. coherent sheaves on the scheme. We denote their Grothendieck groups by

$$K^\circ(X) \;:=\; K(\mathrm{VB}(X)) \quad \text{and} \quad K_\circ(X) \;:=\; K(\mathrm{Coh}(X)).$$

They behave in a way similar to cohomology resp. Borel-Moore homology:

Remark 2.2. For any scheme X we have:

a) $K^\circ(X)$ is a ring with respect to the product

$$K^\circ(X) \times K^\circ(X) \;\longrightarrow\; K^\circ(X), \quad [E]\cdot[F] \;:=\; [E \otimes F].$$

Note that this is well-defined because on vector bundles the tensor product is exact in both variables. Moreover, for any morphism $f\colon Y \to X$ we get a ring homomorphism

$$f^*\colon K^\circ(X) \;\longrightarrow\; f^*(Y), \quad [E] \;\mapsto\; [f^*(E)],$$

since the pullback of coherent sheaves is an exact functor on vector bundles.

b) $K_\circ(X)$ is a module under the ring $K^\circ(X)$, with the scalar multiplication defined by

$$K^\circ(X) \times K_\circ(X) \;\longrightarrow\; K_\circ(X), \quad [E] \times [F] \;:=\; [E \otimes \mathscr{F}],$$

since the tensor product of vector bundles with coherent sheaves is exact in both variables. Moreover, for any proper morphism $f\colon Y \to X$ we have a group homomorphism

$$f_*\colon \quad K_\circ(Y) \longrightarrow K_\circ(X), \quad [E] \mapsto \sum_{i \geq 0} (-1)^i [R^i f_*(E)].$$

and the projection formula holds:

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta) \quad \text{for all} \quad \alpha \in K^\circ(X),\, \beta \in K_\circ(Y).$$

At this point one may wonder whether we really need to care about the distinction between $K^\circ(X)$ and $K_\circ(X)$. The forgetful functor $\mathrm{VB}(X) \to \mathrm{Coh}(X)$ is exact and hence induces a homomorphism

$$K^\circ(X) \longrightarrow K_\circ(X)$$

called the *duality map* by analogy with Poincaré duality. If X is singular, then this duality map is not an isomorphism:

Remark 2.3. A local ring A with maximal ideal $\mathfrak{m}$ is regular iff $A/\mathfrak{m}_A$ admits a finite resolution by free A-modules. Hence if X is a variety with a singular point $p \in X$, then the structure sheaf of the singular point is a coherent sheaf

$$\mathscr{E} = \mathscr{O}_X/\mathfrak{m}_{X,p} \in \mathrm{Coh}(\mathscr{O}_X)$$

which does not admit a finite resolution by locally free $\mathscr{O}_X$-modules. From this one can deduce that $[\mathscr{E}] \in K_\circ(X)$ is not in the image of the duality map $K^\circ(X) \to K_\circ(X)$, i.e. for singular varieties the duality map cannot be surjective. Examples show that in general it is also not injective. For smooth varieties the situation is better:

Lemma 2.4. *For any smooth quasiprojective variety X the map $K^\circ(X) \to K_\circ(X)$ is an isomorphism.*

Proof. The quasiprojectivity ensures that every coherent sheaf has a resolution by locally free sheaves, and the smoothness then implies that it even has a finite such resolution. Now one checks that the map sending the class of a coherent sheaf to the alternating sum of the terms in any finite locally free resolution is well-defined as a map $K_\circ(X) \to K^\circ(X)$ and gives an inverse to the duality map. □

For the rest of this chapter, we will only deal with smooth varieties X and denote their Grothendieck ring simply by $K(X) := K^\circ(X) \simeq K_\circ(X)$.

Example 2.5. The Grothendieck ring $K(\mathbb{P}^n)$ is generated by the classes $[\mathscr{O}_{\mathbb{P}^n}(i)]$ with $i \in \mathbb{Z}$. Indeed, it follows from the graded version of the Hilbert syzygy theorem that every coherent sheaf on projective space has a finite resolution by direct sums of line bundles $\mathscr{O}_{\mathbb{P}^n}(i)$ for various $i \in \mathbb{Z}$.

3 The Grothendieck-Riemann-Roch theorem

Let X be a smooth projective variety. Since the Chern character of vector bundles is additive for short exact sequences and multiplicative for tensor products, it induces a well-defined ring homomorphism

$$\mathrm{ch}\colon \quad K(X) \longrightarrow A^*(X)_{\mathbb{Q}}, \quad [E] \mapsto \mathrm{ch}(E)$$

from the Grothendieck ring to the Chow ring $A^*(X)_{\mathbb{Q}} = A^*(X) \otimes \mathbb{Q}$ with rational coefficients. We have already seen that in order to compute the Euler characteristic of coherent sheaves, we should multiply the Chern character by the Todd class of the variety. So consider the additive group homomorphisms

$$\tau_X\colon \quad K(X) \longrightarrow A^*(X)_{\mathbb{Q}}, \quad \tau(E) := \mathrm{ch}(E) \cdot \mathrm{td}(X)$$

and

$$\chi\colon \quad K(X) \longrightarrow \mathbb{Z}, \quad [E] \mapsto \chi(X,E) = \sum_{i \geq 0} (-1)^i \dim H^i(X,E).$$

We want to show that the diagram

$$\begin{tikzcd} K(X) \arrow[r, "\chi"] \arrow[d, "\tau_X"'] & \mathbb{Z} \arrow[d, hook] \\ A^*(X)_{\mathbb{Q}} \arrow[r, "\int_X"] & \mathbb{Q} \end{tikzcd}$$

commutes. For projective space this is a simple computation:

Proposition 3.1. *We have*

$$\chi(E) = \int_{\mathbb{P}^n} \mathrm{ch}(E) \cdot \mathrm{td}(\mathbb{P}^n) \quad \textit{for all} \quad E \in K(\mathbb{P}^n).$$

Proof. Since $K(\mathbb{P}^n)$ is generated by classes of line bundles $\mathscr{O}_{\mathbb{P}^n}(m)$ with $m \in \mathbb{Z}$, it is enough to check the formula when E is such a line bundle. In this case the Chern character is

$$\mathrm{ch}(\mathscr{O}_{\mathbb{P}^n}(m)) = \exp(m\zeta)$$

where $\zeta = c_1(\mathscr{O}_{\mathbb{P}^n}(1))$, and the Todd class is

$$\begin{aligned} \mathrm{td}(\mathbb{P}^n) &= \mathrm{td}\left((\mathscr{O}_{\mathbb{P}^n}(1))^{n+1}\right) \\ &= \mathrm{td}\left(\mathscr{O}_{\mathbb{P}^n}(1)\right)^{n+1} \\ &= \left(\frac{\zeta}{1-\exp(-\zeta)}\right)^{n+1} \end{aligned}$$

where the first equality uses the Euler sequence. Now one computes

$$\begin{aligned}
\int_{\mathbb{P}^n} \mathrm{ch}(\mathscr{O}(m)) \cdot \mathrm{td}(\mathbb{P}^n) &= \left\{ e^{m\zeta} \cdot \frac{\zeta^{n+1}}{(1-e^{-\zeta}))^{n+1}} \right\}_{\zeta^n} \\
&= \left\{ e^{mx} \cdot \frac{x^{n+1}}{(1-e^{-x})^{n+1}} \right\}_{x^n} \\
&= \mathrm{Res}_{x=0} \frac{e^{mx}}{(1-e^{-x})^{n+1}} dx \\
&= \mathrm{Res}_{y=0} \frac{(1-y)^{-(m+1)}}{y^{n+1}} dy \\
&= (-1)^n \cdot \tbinom{-(m+1)}{n} \\
&= \tbinom{m+n}{n} \\
&= \chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m)).
\end{aligned}$$

where in the fourth step we used the substitution $y = 1 - e^{-x}$ so that $dy = e^x dx$. $\square$

The case of arbitrary smooth projective varieties X can be reduced to the above case as follows. Let $i \colon X \hookrightarrow \mathbb{P}^n$ be an embedding in projective space, and consider the diagram

$$\begin{tikzcd}
K(X) \arrow[r, "i_*"] \arrow[d, "\tau_X"'] & K(\mathbb{P}^n) \arrow[r, "\chi"] \arrow[d, "\tau_{\mathbb{P}^n}"'] & \mathbb{Z} \arrow[d, hook] \\
A^*(X)_{\mathbb{Q}} \arrow[r, "i_*"] & A^*(\mathbb{P}^n)_{\mathbb{Q}} \arrow[r, "\int_{\mathbb{P}^n}"] & \mathbb{Q}
\end{tikzcd}$$

The commutativity of the right hand square is what we checked above. Hence the Hirzebruch-Riemann-Roch theorem will follow if we can show that the left hand square also commutes. This is a special case of the following much more general result, which can be seen as a relative version of the Hirzebruch-Riemann-Roch theorem:

Theorem 3.2 (Grothendieck-Riemann-Roch). *Let $f \colon X \to Y$ be any projective morphism between smooth quasiprojective varieties. Then the following diagram commutes:*

$$\begin{tikzcd}
K(X) \arrow[r, "f_*"] \arrow[d, "\tau_X"'] & K(Y) \arrow[d, "\tau_Y"] \\
A^*(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A^*(Y)_{\mathbb{Q}}
\end{tikzcd}$$

Proof. Step 1. We first discuss the special case where $f \colon X \hookrightarrow Y$ is the embedding of the zero section in the projective completion of a vector bundle. More precisely, suppowe that there is a vector bundle N of rank d on X such that $Y = \mathbb{P}(N \oplus 1)$

and we have the following commutative diagram where the middle row is the zero section $X \hookrightarrow N$ followed by the projective completion $N \hookrightarrow \mathbb{P}(N \oplus 1)$:

$$\begin{tikzcd}
 & & Y \arrow[d, equal] \\
X \arrow[r, hook] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "id"'] & N \arrow[r, hook] & \mathbb{P}(N \oplus 1) \arrow[d, "p"] \\
 & & X
\end{tikzcd}$$

We first express $\mathrm{ch}(f_*E)$ in terms of tautological bundles: Let $S \subset p^*(N \oplus 1)$ denote the tautological line subbundle and $Q = p^*(N \oplus 1)/S$ the tautological quotient of rank d. Note that this quotient bundle comes with a natural section σ defined by the following diagram:

$$\begin{tikzcd}
\mathscr{O}_Y \arrow[rr, hook, "\sigma"] \arrow[d, equal] & & Q \\
p^*(0 \oplus 1) \arrow[rr, hook] & & p^*(N \oplus 1) \arrow[u, two heads]
\end{tikzcd}$$

By construction the zero locus of this section is $V(\sigma) = X \hookrightarrow Y$. Since the vanishing locus of any global setion of a vector bundle represents the top Chern class of the vector bundle, we get

$$c_d(Q) \;=\; [X] \;\in\; A^*(Y).$$

Now consider the Koszul resolution

$$0 \;\to\; \wedge^d Q^\vee \;\to\; \cdots \;\to\; \wedge^2 Q^\vee \;\to\; Q^\vee \;\to\; \mathscr{O}_Y \;\to\; f_*\mathscr{O}_X \;\to\; 0$$

for the structure sheaf on the closed subscheme $f\colon X \hookrightarrow Y$. The arrows in this Koszul resolution are induced by $s^\vee\colon Q^\vee \to \mathscr{O}_Y$, and a local computation shows that this Koszul resolution is an exact sequence. The multiplicativity of the Chern character then gives

$$\begin{aligned}
\mathrm{ch}(f_*\mathscr{O}_X) \;&=\; \sum_{i=0}^{d} (-1)^i \,\mathrm{ch}(\wedge^i Q^\vee) && \text{by the Koszul resolution} \\
&=\; c_d(Q) \cdot \mathrm{td}(Q)^{-1} && \text{by the Todd class formula in lemma 1.3} \\
&=\; [X] \cap \mathrm{td}(Q)^{-1} && \text{since } c_d(Q) = [X] \text{ by the above.}
\end{aligned}$$

With this formula for the Chern character $\mathrm{ch}(f_*\mathscr{O}_X)$, we can now easily compute $\mathrm{ch}(f_*E)$ for any vector bundle E on X as follows: We know from the projection formula that

$$f_*E \;\simeq\; f_*(f^*p^*E) \;\simeq\; f_*(\mathscr{O}_X) \otimes E$$

because $p \circ f = id$. Hence

$$\begin{aligned}
\mathrm{ch}(f_*(E)) &= \mathrm{ch}(f_*(\mathscr{O}_X))\cdot\mathrm{ch}(p^*(E)) && \text{by multiplicativity of } \mathrm{ch}(-)\\
&= [X]\cap(\mathrm{td}(Q)^{-1}\cdot\mathrm{ch}(p^*E)) && \text{by the above formula for } \mathrm{ch}(f_*\mathscr{O}_X)\\
&= f_*f^*(\mathrm{td}(Q)^{-1}\cdot\mathrm{ch}(p^*E)) && \text{by the projection formula}\\
&= f_*(\mathrm{td}(f^*Q)^{-1}\cdot\mathrm{ch}(f^*p^*E)) && \text{by naturality of td and ch}\\
&= f_*(\mathrm{td}(N)^{-1}\cdot\mathrm{ch}(E)) && \text{since } f^*Q\simeq N \text{ and } p\circ f=id\\
&= f_*(f^*(\mathrm{td}(Y))^{-1}\cdot\mathrm{td}(X)\cdot\mathrm{ch}(E)) && \text{since } N\simeq f^*(T_Y)/T_X\\
&= \mathrm{td}(Y)^{-1}\cdot f_*(\mathrm{td}(X)\cdot\mathrm{ch}(E)) && \text{by the projection formula}
\end{aligned}$$

This proves the Grothendieck-Riemann-Roch theorem for the embedding of the zero section in the projective completion of a vector bundle.

Step 2. Next we prove the theorem in the case when $f\colon X\hookrightarrow Y$ is an arbitrary closed immersion (by the discussion preceding the theorem, this will in particular imply the Hirzebruch-Riemann-Roch formula $\chi(X,E)=\int_X \mathrm{ch}(E)\cdot\mathrm{td}(X)$). The idea is to reduce the statement to the previous case, using the deformation to the normal cone. Let $N=N_{X/Y}$ be the normal bundle. Like in the construction of the deformation to the normal cone, consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g_\infty"] \arrow[d, hook, "i_\infty"'] & \mathbb{P}(N\oplus 1)\cup\tilde{Y} \arrow[r, equal] & M_\infty \arrow[r] \arrow[d, hook, "I_\infty"] & \{\infty\} \arrow[d, hook] \\
X\times\mathbb{P}^1 \arrow[r, "g"] \arrow[d, bend right, "p"'] & \mathrm{Bl}_{X\times\{\infty\}}(Y\times\mathbb{P}^1) \arrow[r, equal] & M \arrow[r] & \mathbb{P}^1 \\
X \arrow[u, hook, "i_0"'] \arrow[r, hook, "g_0=f"] & Y \arrow[r, equal] & M_0 \arrow[r] \arrow[u, hook, "I_0"'] & \{0\} \arrow[u, hook]
\end{tikzcd}$$

Given a vector bundle E on X, consider its pullback $\tilde{E}=p^*(E)$ on $X\times\mathbb{P}^1$ and pick a finite locally free resolution of the coherent sheaf obtained by pushforward of $\tilde{E}$ to M, i.e. an exact sequence

$$0\to G_n\to G_{n-1}\to\cdots\to G_0\to g_*(\tilde{E})\to 0$$

where all G_i are locally free sheaves on M. Since $X\times\mathbb{P}^1$ and M are flat over $\mathbb{P}^1$, the above exact sequence remains exact shen restricted to the fibers M_0 and M_∞. Restricting to M_0 we get a resolution

$$0\to G_n|_{M_0}\to G_{n-1}|_{M_0}\to\cdots\to G_0|_{M_0}\to f_*(E)\to 0.$$

With the shorthand notation $\mathrm{ch}(H_*)=\sum_{i\in\mathbb{Z}}(-1)^i\mathrm{ch}(H_i)$ for finite complexes H_* of locally free sheaves, we then compute

$$\begin{aligned}
I_{0*}(\mathrm{ch}(f_*E)) &= I_{0*}(\mathrm{ch}(G_*|_{M_0})) && \text{by the previous resolution} \\
&= \mathrm{ch}(G_*)\cdot[M_0] && \text{by the projection formula} \\
&= \mathrm{ch}(G_*)\cdot[M_\infty] && \text{since } [M_\infty]=[M_0] \text{ in } A^*(M) \\
&= \mathrm{ch}(G_*)\cdot\big([\mathbb{P}(N\oplus 1)]+[\tilde{Y}]\big) && \text{since } M_\infty=\mathbb{P}(N\oplus 1)\cup\tilde{Y} \\
&= a_*(\mathrm{ch}(G_*|_{\mathbb{P}(N\oplus 1)}))+b_*(\mathrm{ch}(G_*|_{\tilde{Y}})) && \text{for } a=I_\infty|_{\mathbb{P}(N\oplus 1)},\ b=I_\infty|_{\tilde{Y}}.
\end{aligned}$$

By our earlier flatness observation we know that $G_*|_{M_\infty}$ is a resolution of $\tilde{E}|_{M_\infty}$, hence for the two terms on the right hand side of the previous chain of equalities we obtain:

- $G_*|_{\mathbb{P}(N\oplus 1)}$ is a resolution of $g_{\infty *}(E)$, hence $\mathrm{ch}(G_*|_{\mathbb{P}(N\oplus 1)})=\mathrm{ch}(g_{\infty *}(E))$,
- $G_*|_{\tilde{Y}}$ is a resolution of $g_{\infty *}(E)|_{\tilde{Y}}=0$, hence $\mathrm{ch}(G_*|_{\tilde{Y}})=0$.

It follows that

$$\begin{aligned}
I_{0*}(\mathrm{ch}(f_*E)) &= a_*(\mathrm{ch}(g_{\infty *}(E)) \\
&= a_*g_{\infty *}(\mathrm{td}(N)^{-1}\mathrm{ch}(E))
\end{aligned}$$

where the last equality applies step 1 for the embedding $g_\infty\colon X\hookrightarrow\mathbb{P}(N\oplus 1)$, noting that $\mathrm{td}(N)=\mathrm{td}(\mathbb{P}(N\oplus 1))|_X\cdot\mathrm{td}(X)^{-1}$. Taking the pushforward of the last displayed identity under

$$q\colon\quad M \;=\; \mathrm{Bl}_{X\times\{\infty\}}(Y\times\mathbb{P}^1) \;\to\; Y\times\mathbb{P}^1 \;\to\; Y$$

with $q\circ I_0=id_Y$ and $q\circ a\circ g_\infty=g_0=f$, we get $\mathrm{ch}(f_*E)=f_*(\mathrm{td}(N)^{-1}\cdot\mathrm{ch}(E))$ and the claim follows via the projection formula as in step 1.

Step 3. To prove the Grothendieck-Riemann-Roch theorem for $f\colon X\to Y$ an arbitrary projective morphism of smooth quasi-projective varieties, we write the morphism as the composite

$$X \;\hookrightarrow\; Y\times\mathbb{P}^n \;\longrightarrow\; Y$$

of a closed immersion in a relative projective space and the projection. The case of a closed immersion has been treated in the previous step, so it only remains to deal with the case when $f\colon X=Y\times\mathbb{P}^n\to Y$ is a projection. In this case we have a commutative diagram

$$\begin{tikzcd}
K(Y)\otimes K(\mathbb{P}^n) \arrow[r, two heads, "\times"] \arrow[d, "\tau_Y\otimes\tau_{\mathbb{P}^n}"'] & K(Y\times\mathbb{P}^n) \arrow[d, "\tau_{Y\times\mathbb{P}^n}"] \\
A^*(Y)_{\mathbb{Q}}\otimes A^*(\mathbb{P}^n)_{\mathbb{Q}} \arrow[r, two heads, "\times"] & A^*(Y\times\mathbb{P}^n)_{\mathbb{Q}}
\end{tikzcd}$$

where the surjectivity of the Künneth map on Grothendieck rings in the top row can be shown in a similar way to the one on Chow rings, using that one of the factors is a projective space. This reduces us to the case where $Y = \operatorname{Spec}(k)$ and $X = \mathbb{P}^n$ which has been treated in proposition 3.1. □

Corollary 3.3 (Hirzebruch-Riemann-Roch). *Let X be a smooth projective variety, then*

$$\chi(X,E) \;=\; \int_X \operatorname{ch}(E)\cdot \operatorname{td}(X) \quad \textit{for all} \quad E \;\in\; K(X).$$

Proof. Apply Grothendieck-Riemann-Roch to $f\colon X \to Y = \operatorname{Spec}(k)$. Note that for this we only need the first two steps of the above proof. □

4 Example: How to compute Hodge numbers

Recall that the cohomology of any smooth complex projective variety X admits the Hodge decomposition

$$H^i(X(\mathbb{C}),\mathbb{C}) \;\simeq\; \bigoplus_{p+q=i} H^q(X(\mathbb{C}),\Omega_X^p).$$

Assuming this, the Hirzebruch-Riemann-Roch formula gives a very simple proof of the Gauss-Bonnet formula:

Theorem 4.1 (Gauss-Bonnet). *Let X be smooth projective of dimension d over $\mathbb{C}$, then*

$$\chi_{\mathrm{top}}(X(\mathbb{C})) \;=\; \int_X c_d(X).$$

Proof. We have

$$\begin{aligned}
\chi_{\mathrm{top}}(X) \;&=\; \sum_{i\geq 0} (-1)^i \dim H^i(X(\mathbb{C}),\mathbb{C}) && \text{by definition of } \chi_{\mathrm{top}}(X)\\
&=\; \sum_{p,q\geq 0} (-1)^{p+q} \dim H^q(X,\Omega_X^p) && \text{by the Hodge decomposition}\\
&=\; \sum_{p\geq 0} (-1)^p \chi(X,\Omega_X^p) && \text{by definition of } \chi(X,-)\\
&=\; \int_X \sum_{p\geq 0} (-1)^p \operatorname{ch}(\Omega_X^p)\cdot \operatorname{td}(X) && \text{by Hirzebruch-Riemann-Roch}\\
&=\; \int_X c_d(T_X)\cdot \operatorname{td}(X)^{-1}\cdot \operatorname{td}(X) && \text{by lemma 1.3 for } \mathscr{E} = T_X
\end{aligned}$$

and hence the claim follows. □

To study the individual Euler characteristics $\chi(X, \Omega_X^p)$ it is convenient to consider the polynomial

$$\chi_y(X) \;:=\; \sum_{p\geq 0} \chi(X, \Omega_X^p)\cdot y^p \;\in\; \mathbb{Z}[y],$$

sometimes called the *Hirzebruch genus*. This polynomial unifies various invariants related to the Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$:

- For $y = 0$ we get $\chi_0(X) = \chi(X, \mathscr{O}_X)$.
- For $y = -1$ we get $\chi_{-1}(X) = \sum_{p,q}(-1)^{p+q}h^{p,q}(X)$. The Hodge decomposition shows that this coincides with the topological Euler characteristic.
- For $y = +1$ we get $\chi_{+1}(X) = \sum_{p,q}(-1)^q h^{p,q}(X)$. If $d = \dim X$ is even, then this number is by the Hodge index theorem equal to the signature of the intersection form on the middle cohomology $H^d(X(\mathbb{C}), \mathbb{R})$.

As an example, let us now compute the Hirzebruch genus for smooth projective hypersurfaces. The result does not depend on the specific choice of the hypersurface but only on its degree and dimension. The formula is best stated as a generating series over all dimensions:

Theorem 4.2. *Fix $a \in \mathbb{N}$, and for $n \in \mathbb{N}$ denote by $Z_n \subset \mathbb{P}^n$ any smooth degree a hypersurface. Then we have*

$$\sum_{n\geq 0} \chi_y(Z_n)\cdot z^n \;=\; \frac{1}{(1-z)(1+yz)} \cdot \frac{(1+yz)^a - (1-z)^a}{(1+yz)^a + y(1-z)^a}.$$

Proof. Let $\zeta = c_1(\mathscr{O}_{\mathbb{P}^n}(1))|_{Z_n}$. From the sequence $0 \to T_Z \to T_{\mathbb{P}^n}|_{Z_n} \to \mathscr{O}_{\mathbb{P}^n}(d)|_{Z_n} \to 0$ we get

$$\begin{aligned}
\operatorname{td}(Z_n) \;&=\; \operatorname{td}(\mathbb{P}^n)|_{Z_n}\cdot \operatorname{td}(\mathscr{O}_{\mathbb{P}^n}(d)|_{Z_n})^{-1} \\
&=\; \operatorname{td}(\mathscr{O}_{\mathbb{P}^n}(1)|_{Z_n})^{n+1}\cdot \operatorname{td}(\mathscr{O}_{\mathbb{P}^n}(d)|_{Z_n})^{-1} \\
&=\; \left(\frac{\zeta}{1-e^{-\zeta}}\right)^{n+1}\cdot\left(\frac{a\zeta}{1-e^{-a\zeta}}\right)^{-1}
\end{aligned}$$

For the Chern character we need some more notation: For any vector bundle E on X consider the polynomial

$$\operatorname{ch}_y(E) \;:=\; \sum_{p\geq 0} \operatorname{ch}(\wedge^p E)\cdot y^p \;\in\; A^*(X)[y].$$

One easily sees from the Whitney formula that these polynomials are multiplicative in the sense that

$$\operatorname{ch}_y(E) \;=\; \operatorname{ch}_y(E')\cdot \operatorname{ch}_y(E'')$$

for any short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$. This being said, we can compute the polynomial $\mathrm{ch}_y(\Omega^1_{Z_n})$ by looking at the two short exact sequences

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-a)|_{Z_n} \to \Omega^1_{\mathbb{P}^n}|_{Z_n} \to \Omega^1_{Z_n} \to 0 \quad \text{and} \quad 0 \to \Omega^1_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n} \to 0$$

to get:

$$\begin{aligned} \mathrm{ch}_y(\Omega^1_{Z_n}) &= \mathrm{ch}_y(\Omega^1_{\mathbb{P}^n})|_{Z_n} \cdot \mathrm{ch}_y(\mathscr{O}_{\mathbb{P}^n}(-a)|_{Z_n})^{-1} \\ &= \frac{\mathrm{ch}_y(\mathscr{O}_{\mathbb{P}^n}(-1)^{n+1})}{\mathrm{ch}_y(\mathscr{O}_{\mathbb{P}^n})}\Big|_{Z_n} \cdot \mathrm{ch}_y(\mathscr{O}_{\mathbb{P}^n}(-a)|_{Z_n})^{-1} \\ &= \frac{(1+ye^{-\zeta})^{n+1}}{1+y} \cdot (1+ye^{-a\zeta})^{-1} \end{aligned}$$

By Hirzebruch-Riemann-Roch then

$$\begin{aligned} \chi_y(Z_n) &= \sum_{p\geq 0} \chi(Z_n, \Omega^p_{Z_n}) \cdot y^p \\ &= \int_{Z_n} \mathrm{ch}_y(Z_n) \cdot \mathrm{td}(Z_n) \\ &= \int_{Z_n} \left(\frac{(1+ye^{-\zeta})^{n+1}}{1+y} \cdot \frac{1}{1+ye^{-a\zeta}} \cdot \left(\frac{\zeta}{1-e^{-\zeta}} \right)^{n+1} \cdot \frac{1-e^{-a\zeta}}{a\zeta} \right) \\ &= \left[\frac{(1+ye^{-w})^{n+1}}{1+y} \cdot \frac{1}{1+ye^{-aw}} \cdot \left(\frac{w}{1-e^{-w}} \right)^{n+1} \cdot \frac{1-e^{-aw}}{aw} \right]_{w^n} \\ &= \mathrm{Res}_{w=0} \left(\frac{(1+ye^{-w})^{n+1}}{1+y} \cdot \frac{1}{1+ye^{-aw}} \cdot \left(\frac{1}{1-e^{-w}} \right)^{n+1} \cdot \frac{1-e^{-aw}}{aw} \cdot dw \right) \end{aligned}$$

To simplify the residue on the right hand side, we introduce a new variable z by substituting $e^{-w} = (1-z)/(1+yz)$, which after a short computation leads to the identities

$$\frac{dw}{1+y} = \frac{dz}{(1-z)(1+yz)} \quad \text{and} \quad \frac{1+ye^{-w}}{1-e^{-w}} = \cdots = \frac{1}{z}.$$

Substituting these expressions, we can rewrite the above formula in a simpler way as

$$\chi_y(Z_n) = \mathrm{Res}_{z=0} \ \frac{1}{z^{n+1}} \cdot \left(\frac{1}{(1-z)(1+yz)} \cdot \frac{(1+yz)^a - (1-z)^a}{(1+yz)^a + y(1-z)^a} \right)$$

which is the coefficient of z^n of the term in brackets. Hence the claim follows. □

As the above illustrates, the Euler characteristics of sheaves of differential forms are very easily computed using the Hirzebruch-Riemann-Roch theorem. In nice cases we can then even compute the Hodge numbers:

Remark 4.3. Let Y be a smooth complex projective variety and $X \subset Y$ an effective ample divisor. Then the Lefschetz hyperplane theorem says that on cohomology the restriction homomorphism

$$H^i(Y(\mathbb{C}),\mathbb{Z}) \longrightarrow H^i(X(\mathbb{C}),\mathbb{Z}) \quad \text{is} \quad \begin{cases} \text{bijective for } i < \dim Y, \\ \text{injective for } i = \dim Y. \end{cases}$$

If Y and X are both smooth, then the restriction map is compatible with the Hodge decompositions and hence

$$h^{p,q}(X) \;=\; h^{p,q}(Y) \quad \text{for} \quad p+q \;<\; \dim Y.$$

By Serre duality the Hodge numbers of X are then determined by those of Y except in the middle degree, and the Hodge numbers in the middle degree can be computed from the other Hodge numbers and from the Euler characteristics $\chi(X,\Omega_X^p)$ via the formula

$$h^{p,q}(X) \;=\; (-1)^q \cdot \left(\chi(X,\Omega_X^p) \;-\; 2\sum_{i=0}^{n-p-1} (-1)^i h^{p,i}(X) \right).$$

Example 4.4. If $Y = \mathbb{P}^n$ is a projective space, we know that for $0 \le p,q \le n$ the Hodge numbers are

$$h^{p,q}(\mathbb{P}^n) \;=\; \delta_{pq} \;=\; \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

If $X \subset \mathbb{P}^n$ is a smooth hypersurface, it follows that for $0 \le p,q \le n-1$ the Hodge numbers are

$$h^{p,q}(X) \;=\; \begin{cases} \delta_{pq} & \text{if } p+q \neq n-1, \\ (-1)^q \chi(X,\Omega_X^p) \;+\; (-1)^n \;+\; \delta_{pq} & \text{if } p+q = n-1. \end{cases}$$

Here $\chi(X,\Omega_X^p)$ can be computed via the previous theorem. For instance, to compute the Hodge numbers of cubic hypersurfaces $Z_n \subset \mathbb{P}^n$ we take $a = 3$ in the previous theorem, which gives the power series

$$\begin{aligned} \sum_{n\ge 0} \chi_y(Z_n)\cdot z^n \;&=\; \frac{1}{(1-z)(1+yz)} \cdot \frac{(1+yz)^3-(1-z)^3}{(1+yz)^3+y(1-z)^3} \\ &=\; 3z + (y^2-7y+1)z^3 + (-y^3-4y^2+4y+1)z^4 + \cdots \end{aligned}$$

Let us look at a few examples:

- $\chi_y(Z_1) = 3$: A smooth cubic $Z_1 \subset \mathbb{P}^1$ consists of three reduced points.
- $\chi_y(Z_2) = 0$: A smooth cubic $Z_2 \subset \mathbb{P}^2$ is an elliptic curve, with Hodge diamond

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

- $\chi_y(Z_3) = y^2 - 7y + 1$: A smooth cubic surface $Z_3 \subset \mathbb{P}^3$ has the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

- $\chi_y(Z_4) = -y^3 - 4y^2 + 4y + 1$: A smooth cubic $Z_4 \subset \mathbb{P}^4$ has the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Chapter V
Grassmann varieties and Schubert calculus

1 Plücker coordinates on Grassmann varieties

A important class of varieties that are ubiquitous in enumerative geometry and in many other applications are Grassmann varieties parametrizing subspaces of a given dimension in a given vector space. In this final chapter we want to discuss their intersection theory in detail, but let us first riefly recall some basic definitions.

Let V be a vector space of dimension $\dim_k(V) = n$. For $d \in \{1, \dots, n-1\}$, we want to endow the set

$$\mathrm{Gr}(d,V) \;=\; \{\text{subspaces } W \subset V \text{ of dimension } \dim_k(W) = d\}$$

with a natural structure of an algebraic variety. For this we embed it in a projective space via the following map which is called the *Plücker embedding*:

Lemma 1.1. *We have an injective map*

$$\imath\colon \quad G(d,V) \hookrightarrow \mathbb{P}(\wedge^d V), \quad [W] \mapsto [\wedge^d W]$$

Proof. For any subspace $W \subset V$ of dimension d, the top wedge power $\wedge^d W \subset \wedge^d V$ is a line, i.e. a point

$$[\wedge^d W] \;\in\; \mathbb{P}(\wedge^d V).$$

In order to verify that the Plücker map is injective, pick any basis $v_1, \dots, v_d$ of W and note that

$$\imath(W) \;=\; [w] \;\in\; \mathbb{P}(\wedge^d V) \quad \text{for the vector} \quad w \;=\; v_1 \wedge \cdots \wedge v_d \;\in\; \wedge^d V.$$

Extending the chosen basis to a basis of V and working in coordinates with respect to this basis, one easily sees that

$$W \;=\; \{\, v \in V \mid v \wedge w \;=\; 0 \text{ in } \wedge^{d+1} V \,\}.$$

Hence the subspace $W \subset V$ is determined uniquely by the point $\imath(W) = [w]$. $\square$

We next claim that the image of the Plücker embedding is a Zariski closed subset, which will endow it with a natural structure of an algebraic variety. For this it will be convenient to introduce coordinates: Fix a basis $e_1, \dots, e_n$ of V. Then $\wedge^d V$ has a basis consisting of the vectors

$$e_I \;:=\; e_{i_1} \wedge \cdots \wedge e_{i_d} \quad \text{with} \quad I \;=\; (1 \leq i_1 < \cdots < i_d \leq n).$$

Thus any vector in $\wedge^d V$ has a unique expansion as $\sum_I p_I \cdot e_I$ where $p_I = p_{i_1,\dots,i_d} \in k$ are called the *Plücker coordinates* of the vector. By varying the vector we can regard the Plücker coordinates as homogenous coordinates on the projective space $\mathbb{P}(\wedge^d V)$ and write

$$\mathbb{P}(\wedge^d V) \;=\; \operatorname{Proj} R \quad \text{for the graded ring} \quad R \;=\; k[p_I : I = (1 \leq i_1 < \cdots i_d \leq n)],$$

where the generators $p_I \in R$ are put in degree one.

Example 1.2. For $V = k^4$ and $d = 2$, we have seen earlier that

$$\operatorname{Gr}(2,4) = V_+(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subset \mathbb{P}^5 = \operatorname{Proj} k[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}].$$

Explicitly, given a point

$$a \;=\; [a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34}] \in \mathbb{P}^5(k)$$

with coordinates $a_{ij} = p_{ij}(a)$, say $a_{12} = 1$, we can write it as $a = \iota(W)$ for the subspace

$$W \;\subset\; V \quad \text{spanned by the rows of the matrix} \quad A \;=\; \begin{pmatrix} 0 & 1 & a_{13} & a_{14} \\ -1 & 0 & a_{23} & a_{24} \end{pmatrix}.$$

This example generalizes as follows:

Proposition 1.3. *The image $\iota(\operatorname{Gr}(d,V)) \subset \mathbb{P}(\wedge^d V)$ is a Zariski closed subset. It is cut out by the following set of homogenous quadratic equations which are called the Plücker relations:*

$$\sum_{\nu=1}^{d+1} (-1)^\nu \, p_{i_1,\dots,i_{d-1},j_\nu} \cdot p_{j_1,\dots,\widehat{j_\nu},\dots,j_{d+1}} \;=\; 0 \quad \text{for all} \quad \begin{cases} 1 \leq i_1 < \cdots < i_{d-1} \leq n, \\ 1 \leq j_1 < \cdots < j_{d+1} \leq n. \end{cases}$$

Proof. First we comment on the notation in the Plücker relations: The hat $\widehat{\cdot}$ over an index means as usual that this index is omitted. For the insertion of indices, note that depending on ν the tuple $(i_1, \dots, i_{d-1}, j_\nu)$ might not be increasing and might even contain an index twice. To take care of this, we extend the notation as follows: For arbitrary $(k_1, \dots, k_d)$ we put

$$p_{k_1,\dots,k_d} \;:=\; \begin{cases} \operatorname{sgn}(\sigma) \cdot p_{k_{\sigma(1)},\dots,k_{\sigma(d)}} & \text{if } k_{\sigma(1)} < \cdots < k_{\sigma(d)} \text{ for some } \sigma \in \mathfrak{S}_d, \\ 0 & \text{otherwise.} \end{cases}$$

Step 1. Let us now verify the inclusion $\imath(\mathrm{Gr}(d,V)) \subset V_+(\mathrm{PR})$ for the set PR of Plücker relations. Let $W \subset V$ be any subspace of dimension d, corresponding to a point $[W] \in \mathrm{Gr}(d,V)$. With respect to the chosen basis $e_1,\ldots,e_n$ of V from above, we may represent the subspace as the span of the columns of some matrix

$$A \;=\; (a_{ij}) \;\in\; \mathrm{Mat}(d \times n) \quad \text{of maximal rank} \quad \mathrm{rk}(A) \;=\; d,$$

as in the above example. The matrix A is of course not unique, it is determined only up to left multiplication by a matrix in $\mathrm{GL}(d)$ (corresponding to a change of basis in W). In these terms the Plücker coordinates of the point $\imath(W) \in \mathbb{P}(\wedge^d V)$ are given by the $d \times d$ minors

$$p_{k_1,\ldots,k_d}(W) \;=\; \det \begin{pmatrix} a_{1k_1} & \cdots & a_{1k_d} \\ \vdots & \ddots & \vdots \\ a_{dk_1} & \cdots & a_{dk_d} \end{pmatrix}$$

i.e. the determinants of the matrix given by a subset of columns of A. One then computes

$$\begin{aligned} &\sum_{\nu=1}^{d+1} (-1)^{\nu} \cdot p_{i_1,\ldots,i_{d-1},j_\nu}(W) \cdot p_{j_1,\ldots,\widehat{j_\nu},\ldots,j_{d+1}}(W) \\ &= \sum_{\nu=1}^{d+1} (-1)^{\nu} \cdot \det \begin{pmatrix} a_{1i_1} & \cdots & a_{1i_{d-1}} & a_{1j_\nu} \\ \vdots & & \vdots & \vdots \\ a_{di_1} & \cdots & a_{di_{d-1}} & a_{dj_\nu} \end{pmatrix} \cdot \det \begin{pmatrix} a_{1j_1} & \cdots & \widehat{a}_{1j_\nu} & \cdots & a_{1j_d} \\ \vdots & & \vdots & & \vdots \\ a_{dj_1} & \cdots & \widehat{a}_{dj_\nu} & \cdots & a_{dj_d} \end{pmatrix} \\ &= \sum_{\nu=1}^{d+1} \sum_{\mu=1}^{d} (-1)^{\nu+d+\mu} \cdot a_{\mu j_\nu} \cdot \det \begin{pmatrix} \vdots & & \vdots \\ \widehat{a}_{\mu i_1} & \cdots & \widehat{a}_{\mu i_{d-1}} \\ \vdots & & \vdots \end{pmatrix} \cdot \det \begin{pmatrix} \cdots & \widehat{a}_{1j_\nu} & \cdots \\ & \vdots & \\ \cdots & \widehat{a}_{dj_\nu} & \cdots \end{pmatrix} \\ &= \sum_{\nu=1}^{d+1} \sum_{\mu=1}^{d} (-1)^{d+\mu} \cdot \det \begin{pmatrix} \vdots & & \vdots \\ \widehat{a}_{\mu i_1} & \cdots & \widehat{a}_{\mu i_{d-1}} \\ \vdots & & \vdots \end{pmatrix} \cdot \det \begin{pmatrix} \cdots & \widehat{a}_{\mu j_\nu} & \cdots \\ \cdots & \widehat{a}_{1j_\nu} & \cdots \\ & \vdots & \\ \cdots & \widehat{a}_{dj_\nu} & \cdots \end{pmatrix} \;=\; 0 \end{aligned}$$

where in the second equality we have developed the first of the two determinants with respect to its last column, in the third equality we have developed the second of the determinants, and the vanishing in the last equality comes from the fact that in the last occuring determinant the first and the $(\mu+1)$-st rows coincide. Thus the image $\imath(W)$ satisfies the Plücker relations as desired.

Step 2. As a preparation for proving the opposite inclusion $V(\mathrm{PR}) \subset \iota(\mathrm{Gr}(d,V))$, we need an auxiliary statement: Let $a \in V(\mathrm{PR})$ be a point in the vanishing locus of the Plücker relations. Let $p_I(a)$ be the Plücker coordinates of the given points; they are only well-defined up to multiplication with a common nonzero scalar, but we do not care since the Plücker relations are homogenous. Fix $K = (1 \leq k_1 < \cdots < k_d \leq n)$ with $p_K(a) \neq 0$. Then we claim that *all* Plücker coordinates of the given point are determined already if we know the Plücker coordinates $p_I(a)$ only for the tuples I which differ from K in at most one entry:

We show this by induction on the number of entries in which a given tuple differs from K. Let $I = (1 \leq i_1 < \cdots < i_d \leq n)$ be any tuple, and suppose it differs from K in precisely $m > 1$ entries. Pick any μ such that the index $j = i_\mu$ does not appear in K. Then by the Plücker relations we have

$$p_{i_1,\dots,\widehat{i_\mu},\dots,i_d,j}(a) \cdot p_{k_1,\dots,k_d}(a) \;=\; \sum_{\nu=1}^{d-1} (-1)^\nu \cdot p_{i_1,\dots,\widehat{i_\nu},\dots,i_d,k_\nu}(a) \cdot p_{k_1,\dots,\widehat{k_\nu},\dots,k_d,j}(a).$$

After dividing by $p_{k_1,\dots,k_d}(a) \neq 0$, this expresses $p_{i_1,\dots,\widehat{i_\mu},\dots,i_d,j}(a) = \pm p_I(a)$ in terms of Plücker coordinates for index sets which differ from the reference index K in fewer than m terms. Hence the claim of step 2 follows.

Step 3. Now we can prove the inclusion $V(\mathrm{PR}) \subset \iota(\mathrm{Gr}(d,V))$. Let $a \in V(\mathrm{PR})$ be any point in the vanishing locus of the Plücker relations, and as in the previous step pick a tuple $K = (1 \leq k_1 < \cdots < k_d \leq n)$ with the property that $p_K(a) \neq 0$. We want to show

$$a \;=\; \iota(W) \quad \text{for some} \quad [W] \;\in\; \mathrm{Gr}(d,V).$$

Consider for $i = 1,\dots,d$ the vectors

$$w_i \;:=\; \sum_{\nu=1}^{n} a_{ij} e_j \;\in\; V \quad \text{with} \quad a_{ij} \;:=\; p_{k_1,\dots,k_{i-1},j,k_{i+1},\dots,k_d}(a)/p_K(a).$$

The vectors $w_1,\dots,w_d \in V$ are linearly independent, because for all indices i and ν we have

$$(\text{coefficient of } e_{k_\nu} \text{ in } w_i) \;=\; a_{ik_\nu} \;=\; \begin{cases} 0 & \text{for } i \neq \nu, \\ 1 & \text{for } i = \nu. \end{cases}$$

Therefore $W = \langle w_1,\dots,w_d \rangle$ is a subspace of dimension $\dim_k W = d$, giving a point

$$[W] \;\in\; \mathrm{Gr}(d,V).$$

We claim that this point has the given Plücker coordinates $p_I(a)$. Indeed, by the previous step it suffices to verify this for tuples I which differ from K by at most one entry, i.e. tuples of the form

$$I \;=\; (k_1,\dots,k_{i-1},j,k_{i+1},\dots,k_d).$$

The corresponding Plücker coordinate of the point $[W] \in \mathrm{Gr}(d,V)$ is then given by the determinant

$$p_I(W) \;=\; \det \begin{pmatrix} a_{1k_1} & \cdots & a_{1k_{i-1}} & a_{1j} & a_{1k_{i+1}} & \cdots a_{1k_d} \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{dk_1} & \cdots & a_{dk_{i-1}} & a_{dj} & a_{dk_{i+1}} & \cdots a_{dk_d} \end{pmatrix} \;=\; a_{ij} \;=\; p_I(a)/p_K(a)$$

because the displayed matrix coincides with the identity matrix outside of its i-th column. As Plücker coordinates only matter up to multiplication by an overall scalar, we can clear the denominator $p_K(a)$. Then $p_I(W) = p_I(a)$ for all I as desired. □

2 Schubert varieties

We have seen earlier that the Grassmann variety $\mathrm{Gr}(2,4)$ is cellular, i.e. it admits a decomposition as a disjoint union of locally closed subvarieties isomorphic to affine spaces. We will now show that such a cell decomposition exists for every Grassmann variety $\mathrm{Gr}(d,V)$. As before we fix a basis $e_1,\dots,e_n$ of V and write

$$\mathrm{Gr}(d,V) \;=\; V(\mathrm{PR}) \;\subset\; \mathbb{P}(\wedge^d V) \;=\; \mathrm{Proj}\, k[p_I : I = (1 \le i_1 < \cdots < i_d \le n)].$$

We first show that $\mathrm{Gr}(d,V)$ is covered by open charts which are affine spaces:

Lemma 2.1. *For any index I, let $U_I = \mathrm{Gr}(d,V) \setminus V_+(p_I) \subset \mathrm{Gr}(d,V)$ be the basic open subset of the Grassmannian where the corresponding Plücker coordinate p_I does not vanish. Then we have*

$$U_I \;\simeq\; \mathbb{A}^{d(n-d)}$$

In particular the Grassmannian $\mathrm{Gr}(d,V)$ is a smooth variety of dimension $d(n-d)$.

Proof. Let $[W] \in \mathrm{Gr}(d,V)$. As above we write $W \subset V$ as the span of the columns of a matrix

$$A \;\in\; \mathrm{Mat}(d \times n) \quad \text{of full rank} \quad \mathrm{rk}(A) \;=\; d.$$

In these terms we have $[W] \in U_I$ iff $\det(A_I) \neq 0$ for the matrix $A_I \in \mathrm{Mat}(d \times d)$ which is obtained from the given matrix by taking only the columns indexed by the entries of I. The row span $W \subset V$ of a matrix A does not change if we multiply the matrix from the left by an invertible matrix. If $[W] \in U_I$, then the matrix A_I is itself invertible, so after left multiplication by its inverse we may assume that $A_I = \mathbf{1}$. This normalization leaves no freedom for further left multiplication: Every $[W] \in U_I$ is the row span of a unique $A \in \mathrm{Mat}(d \times n)$ with $A_I = \mathbf{1}$. Since for the entries of the remaining $n-d$ columns there are no restrictions, we see that

$$U_I \;\simeq\; \{A \in \mathrm{Mat}(d \times n) \mid A_I = \mathbf{1}\} \;\simeq\; \mathbb{A}^{d(n-d)}$$

and the claim follows. □

From the viewpoint of normal forms of matrices we have done a bad job in fixing the chart in advance: Most points of $\mathrm{Gr}(d,V)$ lie in more than one charts of our open cover, so there is in general no unique choice for the index set I. Instead of taking an open cover, we now want to obtain a cell decomposition of the Grassmannian into pairwise disjoint locally closed subsets. The building blocks arise directly from the Gauss algorithm in linear algebra:

Definition 2.2. For any matrix $A \in \mathrm{Mat}(d \times n)$ of full rank $\mathrm{rk}(A) = d$, Gaussian elimination by elementary row operations shows that there exists a unique invertible matrix $S \in \mathrm{GL}(d)$ such that left multiplication by S brings the given matrix in the following *reduced echelon form*[1]

$$S\cdot A \;=\; \left(\begin{array}{l} * \cdots * \; 1 \\ * \cdots * \; 0 \; * \cdots * \; 1 \\ * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \; 1 \\ \;\;\vdots \quad\; \vdots \quad\; \vdots \quad\; \vdots \quad\; \vdots \quad\; \vdots \\ * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \;\; \cdots \;\; * \cdots * \; 1 \end{array}\qquad\qquad\right)$$

Here the blank entries in the upper triangular part stand for zeroes, whereas the stars stand for arbitrary elements. Note that on the right end of the reduced echelon form there may be some zero columns, depending on the matrix A. For generic A the square matrix formed by the last d columns of A will be invertible, in which case the reduced echelon form is a block matrix

$$\left(\begin{array}{ccc|ccc} * & \cdots & * & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ * & \cdots & * & 0 & & 1 \end{array}\right)$$

formed by an arbitrary matrix of size $d \times (n-d)$ juxtaposed with the identity matrix of size $d \times d$. In this case the number of trailing zeroes at the end of row i is $d-i$. To measure how far the reduced echelon form of a given matrix A is from this generic case, we define $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$ by the formula

$$\begin{aligned} \lambda_i \;&=\; (\text{number of trailing zeroes at the end of row } i \text{ in } S\cdot A) - (d-i) \\ &=\; \max\{\ell \mid S\cdot A = (b_{ij}) \text{ has entries } b_{ij} = 0 \text{ for all } j > n-d+i-\ell\}, \end{aligned}$$

where $S \in \mathrm{GL}(d)$ denotes the unique matrix such that $S\cdot A$ is in reduced echelon form. We also say that the reduced echelon form of A has *type* $\lambda = (\lambda_1, \dots, \lambda_d)$. As usual we will drop zeroes at the end of partitions when there is no risk of confusion, writing $\lambda = (3,2) = (3,2,0) = (3,2,0,0) = \cdots$ etc.

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[1] In linear algebra one usually considers echelon forms which look like the above rotated by 180 degrees; for us the above shape is more convenient since it will allow us to label flags increasingly, though later we will in fact need both types of echelon forms.

Example 2.3. For a generic matrix $A \in \mathrm{Mat}(d \times n)$ the reduced echelon form has type $\lambda = (0, \dots, 0)$; matrices whose reduced echelon form is 'more special' will correspond to partitions with larger entries. The matrix

$$A = \begin{pmatrix} 4\ 1\ 0\ 0\ 0\ 0 \\ 5\ 0\ 7\ 1\ 0\ 0 \\ 6\ 0\ 3\ 0\ 2\ 1 \end{pmatrix}$$

is in reduced echelon form and its type is the partition $\lambda = (2,1,0) = (2,1)$.

Definition 2.4. We say that a subspace $W \subset V$ has type $\lambda = (\lambda_1, \dots, \lambda_d)$ if it can be written as the span of the rows of a matrix $A \in \mathrm{Mat}(d \times n)$ in reduced echelon form of type λ. To include limiting cases, we endow the set of partitions with the partial order defined by

$$\mu \ \geq \ \lambda \quad \Longleftrightarrow \quad \forall i: \ \mu_i \ \geq \ \lambda_i.$$

and we say that a d-dimensional subspace $W \subset V$ is of type $\geq \lambda$ if it is of type μ for some partition $\mu \geq \lambda$. This being said, we define the *Schubert variety* of type λ to be the subset

$$\Sigma_\lambda \ := \ \{[W] \in \mathrm{Gr}(d,V) \mid W \text{ is of type} \geq \lambda\} \ \subset \ \mathrm{Gr}(d,V).$$

By the *Schubert cell* of type λ we mean the subset

$$\Sigma_\lambda^\circ \ := \ \{[W] \in \mathrm{Gr}(d,V) \mid W \text{ is of type } \lambda\} \ = \ \Sigma_\lambda \setminus \bigcup_{\mu > \lambda} \Sigma_\mu \ \subset \ \mathrm{Gr}(d,V),$$

where we write $\mu > \lambda$ if $\mu \geq \lambda$ but $\mu \neq \lambda$.

Remark 2.5. The above conditions on subspaces can be formulated geometrically as follows: Consider the flag

$$V_*^\lambda: \quad V_1^\lambda \ \subset \ V_2^\lambda \ \subset \ \cdots \ \subset \ V_d^\lambda \ \subset \ V \quad \text{with} \quad V_i^\lambda \ = \ \langle e_1, e_2, \dots, e_{n-d+i-\lambda_i} \rangle,$$

then W is of type $\geq \lambda$ iff $\dim W \cap V_i^\lambda \geq i$ for all i. More generally, for any flag of subspaces

$$V_*: \quad V_1 \ \subset \ V_2 \ \subset \ \cdots \ \subset \ V_d \ \subset \ V$$

we can consider the associated *Schubert variety*

$$\Sigma_{V_*} \ := \ \{[W] \in \mathrm{Gr}(d,V) \mid \dim W \cap V_i \geq i \text{ for all } i\,\} \ \subset \ \mathrm{Gr}(d,V).$$

This is just the Schubert variety in the old sense but with $e_1, \dots, e_n$ replaced by a different basis adapted to the given flag: The group $\mathrm{GL}(V)$ acts transitively on the Grassmannian $\mathrm{Gr}(d,V)$ as well as on the set of flags and Schubert varieties of a given type λ. We will later compute intersection products between cycles by moving them into general position via this transitive group action.

Example 2.6. Let $d = 2$ and $V = k^4$ with the standard basis $e_1, \dots, e_5$. In $\mathrm{Gr}(2,4)$ we have

$$\begin{aligned}\Sigma_\lambda &= \big\{ [W] \in \mathrm{Gr}(2,4) \mid \dim W \cap \langle e_1, \dots, e_{3-\lambda_1} \rangle \geq 1,\, \dim W \cap \langle e_1, \dots, e_{4-\lambda_2} \rangle \geq 2 \big\} \\ &= \big\{ [W] \in \mathrm{Gr}(2,4) \mid W \cap \langle e_1, \dots, e_{3-\lambda_1} \rangle \neq 0,\, W \subset \langle e_1, \dots, e_{4-\lambda_2} \rangle \big\},\end{aligned}$$

so explicitly we find the following Schubert varieties:

$$\begin{aligned}\Sigma_{0,0} &= \mathrm{Gr}(2,4), \\ \Sigma_{1,0} &= \{W \mid W \cap \langle e_1, e_2 \rangle \neq 0\} = V_+(p_{34}), \\ \Sigma_{1,1} &= \{W \mid W \subset \langle e_1, e_2, e_3 \rangle\} = V_+(p_{14}, p_{24}, p_{34}), \\ \Sigma_2 &= \{W \mid e_1 \in W\} = V(p_{23}, p_{24}, p_{34}), \\ \Sigma_{2,1} &= \{W \mid e_1 \in W \subset \langle e_1, e_2, e_3 \rangle\} = V_+(p_{14}, p_{23}, p_{24}, p_{34}), \\ \Sigma_{2,2} &= \{\langle e_1, e_2 \rangle\} = V_+(p_{13}, p_{14}, p_{23}, p_{24}, p_{34}).\end{aligned}$$

At this point some general observations are in order:

- The inclusion relations are given by $\Sigma_\lambda \supset \Sigma_\mu$ iff $\mu \geq \lambda$.
- The codimension of Σ_λ in the Grassmannian is the degree $|\lambda| = \sum_i \lambda_i$.
- Each Schubert variety is cut out by certain Plücker coordinates, hence it is an intersection of $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ with a certain collection of hyperplanes.

All three facts remain true in general. Let us start with the last one, which endows the Schubert variety with the structure of an algebraic variety:

Proposition 2.7. *Schubert varieties are Zariski closed: Put $d_i = n - d + i - \lambda_i$, then we have*

$$\Sigma_\lambda = V_+\left(p_{j_1, \dots, j_d} \mid 1 \leq j_1 < \dots < j_d \leq n \text{ and } \exists i : j_i > d_i \right).$$

Proof. Before starting the proof, you may want to compare with the above example to get a feeling for the indices. Back to the general case, let S be the vanishing locus of Plücker coordinates on the right hand side of the statement. We first show the inclusion $\Sigma_\lambda \subset S$: Let $[W] \in \Sigma_\lambda$. By definition

$$\dim(W \cap V_i) \geq i \quad \text{for the subspaces} \quad V_i := \langle e_1, \dots, e_{d_i} \rangle,$$

so inductively we can find a basis $w_1, \dots, w_d$ of W with $w_i \in W \cap V_i$ for all i. We then have

$$w_i = \sum_{j=1}^n a_{ij} \cdot e_j \quad \text{with} \quad a_{ij} = 0 \quad \text{for all} \quad j > d_i.$$

and the corresponding Plücker coordinates are given for $I = (1 \leq j_1 < \cdots < j_d \leq n)$ by the minors

$$p_I(W) \;=\; \det \begin{pmatrix} a_{1j_1} & \cdots & a_{1j_d} \\ \vdots & \ddots & \vdots \\ a_{dj_1} & \cdots & a_{dj_d} \end{pmatrix}.$$

If the index set $I = (j_1, \dots, j_d)$ satisfies $j_\nu > d_\nu$ for some index ν, then this minor has the block form

$$p_I(W) \;=\; \det \left(\begin{array}{cccc|ccc} a_{1j_1} & \cdots & a_{1j_{\nu-1}} & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\nu j_1} & \cdots & a_{\nu j_{\nu-1}} & 0 & 0 & \cdots & 0 \\ \hline & * & & & & * & \\ & * & & & & * & \end{array}\right)$$

where the two diagonal blocks are square matrices. Now this determinant vanishes due to the zeroes in the rightmost column of the first diagonal block.

It remains to show the reverse inclusion $S \subset \Sigma_\lambda$. Let $a \in S$. Then by definition a is a point of the Grassmannian whose Plücker coordinates satisfy

$$p_I(a) \;=\; 0 \quad \text{for all index sets} \quad I \;=\; (j_1, \dots, j_d) \quad \text{such that} \quad \exists\, \nu\colon \;\; j_\nu \;>\; d_\nu.$$

Now among all index sets $K = (1 \leq k_1 < \cdots < k_d \leq n)$ with $p_K(a) \neq 0$, fix one for which $k_1 + \cdots k_d$ is maximal. Up to multiplication with an overall nonzero scalar we may assume that $p_K(a) = 1$. Then $a \in \mathrm{Gr}(d,V)$ is the point corresponding to the subspace

$$W \;=\; \langle w_1, \dots, w_d \rangle, \quad \text{where} \quad w_i \;=\; \sum_j a_{ij} \cdot e_j \quad \text{for} \quad a_{ij} \;=\; p_{k_1,\dots,k_{i-1},j,k_{i+1},\dots,k_n}$$

as we have seen at the end of the proof of proposition 1.3. We claim that $w_i \in V_i$ for all i, or equivalently

$$a_{ij} \;=\; 0 \quad \text{for all} \quad j \;>\; d_i.$$

Indeed, since by assumption $a \in S \setminus V(p_K)$, the definition of S implies $k_i \leq d_i$. If $j > d_i$, then it follows that $k_i < j$. But then

$$\sum_\nu k_\nu \;<\; j + \sum_{\nu \neq j} k_\nu,$$

hence by our choice of K we have $p_{k_1,\dots,k_{\nu-1},j,k_{\nu+1},\cdots,k_d}(a) = 0$. This proves $a_{ij} = 0$ for all $j > d_i$, hence $w_i \in V_i$ for all i. It follows that

$$w_j \;\in\; V_j \;\subset\; V_i \quad \text{for} \quad j \;=\; 1, 2, \dots, i.$$

Since $w_1, \dots, w_d$ are linearly independent, we get $\dim(W \cap V_i) \geq \dim\langle w_1, \dots, w_i \rangle = i$ for all i, which means that $[W] \in \Sigma_\lambda$ as desired. □

In particular, the Schubert cells give a partition of the Grassmann variety into locally closed subsets:

Corollary 2.8. *The Schubert cells* $\Sigma_\lambda^\circ \subset \mathrm{Gr}(d,V)$ *are locally closed subvarieties and we have*

$$\mathrm{Gr}(d,V) \;=\; \bigsqcup_{\lambda} \Sigma_\lambda^\circ.$$

Proof. By the above $\Sigma_\lambda^\circ = \Sigma_\lambda \setminus \bigcup_{\mu>\lambda} \Sigma_\mu$ is the complement of a closed subset in another closed subset, so it is locally closed. To see that these locally closed subsets are pairwise disjoint, note that Σ_λ° is the set of spaces spanned by matrices with reduced echelon of type λ. The uniqueness of the reduced echelon form of matrices with a given row span therefore implies $\Sigma_\lambda^\circ \cap \Sigma_\mu^\circ = \varnothing$ for $\mu \neq \lambda$. □

In fact the same argument shows that the Grassmann variety is a cellular variety, because the Schubert cells are affine spaces:

Proposition 2.9. *The Schubert cells* $\Sigma_\lambda^\circ \subset \mathrm{Gr}(d,V)$ *are smooth of codimension* $|\lambda|$, *more precisely*

$$\Sigma_\lambda^\circ \;\simeq\; \mathbb{A}^{n(d-n)-|\lambda|}.$$

The tangent spaces to the Grassmannian and to the Schubert cell at a point $[W] \in \Sigma_\lambda^\circ$ *are given by*

$$\begin{aligned} T_{[W]}\mathrm{Gr}(d,V) &= \mathrm{Hom}\,(W,\, V/W) \\ &\cup \\ T_{[W]}\Sigma_\lambda^\circ &= \{f \in \mathrm{Hom}(W,V/W) \colon\, f(W_i^\lambda) \subset (V/W)_i^\lambda \textit{ for all } i\} \end{aligned}$$

where we put

$$W_i^\lambda \;:=\; W \cap V_i^\lambda \quad \textit{and} \quad (V/W)_i^\lambda \;:=\; V_i^\lambda / W_i^\lambda \quad \textit{with} \quad V_i^\lambda \;:=\; V_{n-d+i-\lambda_i}.$$

Proof. Put $K = (k_1,\dots,k_d)$ with $ki = n-d+i-\lambda_i$. On $U_K = \mathrm{Gr}(d,V) \setminus V_+(p_K)$ we have the isomorphism

$$U_K \;\simeq\; \mathbb{A}^{d(n-d)} \;=\; \{A \in \mathrm{Mat}(d \times n) : A_K \;=\; \mathbf{1}\}$$

sending a subspace $W \subset V$ to the unique matrix A whose rows span the subspace and whose columns labelled by the indices in K are the standard basis vectors. Via this isomorphism the Schubert cell $\Sigma_\lambda^\circ \subset U_K$ corresponds to those matrices A which are moreover in reduced row echelon form of type λ. This condition imposes the vanishing of $|\lambda|$ further matrix entries. Looking at the position of those entries one obtains the given description of tangent spaces. □

Remark 2.10. The closure Σ_λ of the cells Σ_λ° is usually singular, but one can show that it is always Cohen-Macaulay.

3 The Chow ring of Grassmann varieties

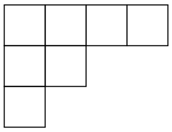
We have seen that the Grassmann variety is cellular, with a disjoint decomposition into Schubert cells. In particular, its Chow ring is generated as an additive group by the fundamental classes

$$\sigma_\lambda \;=\; [\Sigma_\lambda] \;\in\; A^{|\lambda|}(\mathrm{Gr}(d,V)),$$

called *Schubert classes*. Note that up to now we have not put any restrictions on the shape of the partition $\lambda = (\lambda_1,\dots,\lambda_d)$, but for λ of the 'wrong shape' the Schubert variety Σ_λ will be empty; more precisely, one easily sees from the definitions that

$$\begin{aligned} \Sigma_\lambda \;\neq\; \varnothing \quad &\iff \quad \lambda \;=\; (\lambda_1,\dots,\lambda_d) \quad \text{satisfies} \quad 0 \le \lambda_i \le n-d \quad \text{for all} \quad i \\ &\iff \quad \text{the Young diagram of } \lambda \text{ is contained in a } d \times (n-d) \text{ rectangle,} \end{aligned}$$

where for the last characterization we represent partitions by their *Young diagram* consisting of a left-aligned table of d rows with λ_i boxes in row i. For instance, the partition $\lambda = (4,2,1)$ has the Young diagram



so it defines a nonempty Schubert variety in $\mathrm{Gr}(3,7)$ but not in $\mathrm{Gr}(3,6)$.

We want to see that as an additive group $A^*(\mathrm{Gr}(d,V))$ is freely generated by the cycles σ_λ for the partitions λ whose Young diagram fits inside a $d \times (n-d)$ rectangle. We will proceed like for projective space: To show that there exist no additive relations between Schubert cycles, we look at suitable intersection products with complementary cycles. In general Schubert varieties Σ_λ and Σ_μ do not meet transversely, so to find the intersection product $\sigma_\lambda \cdot \sigma_\mu$ we first need to move them in general position. We will do this using the transitive action of $\mathrm{GL}(V)$.

Lemma 3.1. *Let X be a variety endowed with an action of $G = \mathrm{GL}(V)$. Then for any subvariety $Z \subset X$ and any $g \in G(k)$ we have*

$$[Z) \;=\; [gZ] \;\in\; A_*(X).$$

Proof. View G as an open subset of the affine space $\mathrm{Mat}(n \times n)$. Let $L \subset \mathrm{Mat}(n \times n)$ be the line connecting the two points $id, g \in G(k)$. We pick an isomorphism $L \simeq \mathbb{A}^1$ such that $id \mapsto 0$ and $g \mapsto 1$. Let

$$W \;=\; \{(x,g) \in X \times (G \cap L) | g^{-1}x \in Z\} \;\subset\; X \times L \;=\; X \times \mathbb{A}^1.$$

Then $W \to \mathbb{A}^1$ has the fibers $W_0 = Z$ and $W_1 = gZ$, so the claim follows. $\square$

When applying the above to the Grassmann variety $X = \mathrm{Gr}(d,V)$, we are led to considering Schubert varieties with respect to different flags. We have so far used the notation

$$\Sigma_\lambda \;=\; \{[W] \in \mathrm{Gr}(d,V) | \forall i\colon\; \dim W \cap V_i^\lambda \;\geq\; i\}$$

for the flag

$$V_*^\lambda\colon \quad V_1^\lambda \subset \cdots \subset V_d^\lambda \subset V \quad \text{with} \quad V_i^\lambda \;:=\; \langle e_1, \dots, e_{n-d+i-\lambda_i} \rangle,$$

which is convenient when dealing only with one partition at a time. In studying intersection products between two or more Schubert cycles, we need to deal with several partitions at the same time, so we cannot stick to flags of length d with dimension jumps adapted to a fixed partition. Instead, we consider *complete* flags, i.e. flags of the form

$$F_*\colon \quad 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V \quad \text{with} \quad \dim F_i/F_{i-1} = 1 \quad \text{for all } i$$

and use the notation

$$\Sigma_\lambda(F_*) \;=\; \{[W] \in \mathrm{Gr}(d,V) | \forall i\colon\; \dim W \cap F_{n-d+i-\lambda_i} \geq i\}$$

for Schubert varieties. For any complete flag F_* and any partition $\lambda = (\lambda_1, \dots, \lambda_d)$ we have

$$\sigma_\lambda \;=\; [\Sigma_\lambda(F_*)] \;\in\; \mathrm{Gr}(d,V)$$

by the above lemma, since $\mathrm{GL}(V)$ acts transitively on the set of complete flags. The freedom to use arbitrary complete flags easily allows to compute intersection numbers between Schubert cycles of complementary degree. To state the result we need one more definition:

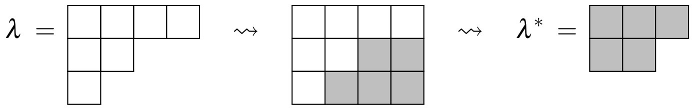
Definition 3.2. For fixed n and d, the *dual* of a partition $\lambda = (\lambda_1, \dots, \lambda_d)$ is defined to be the partition

$$\lambda^* \;=\; (n-d-\lambda_d, n-d-\lambda_{d-1}, \dots, n-d-\lambda_1),$$

i.e. the partition whose Young diagram is the complement of the one of λ inside a box of size $d \times (n-d)$, rotated by 180 degrees so that the sizes of the rows again form a weakly decreasing sequence from top to bottom. Note that the degrees of λ and λ^* satisfy

$$|\lambda| + |\lambda^*| \;=\; d(n-d).$$

Example 3.3. Let $n = 7$ and $d = 3$. Then for $\lambda = (4,2,1)$ we have $\lambda^* = (2,1,0)$ as shown in the following picture:



Proposition 3.4. *Let* $\lambda = (\lambda_1, \dots, \lambda_d)$ *and* $\mu = (\mu_1, \dots, \mu_d)$ *be two partitions such that the associated Schubert cycles have complementary codimension in* $\mathrm{Gr}(d,V)$, *i.e.* $|\lambda| + |\mu| = d(n-d)$. *Then their intersection number is*

$$(\sigma_\lambda, \sigma_\mu) \; = \; \begin{cases} 1 & \textit{if } \mu = \lambda^* \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Fix a basis $e_1, \dots, e_n$ of V and let F_* and G_* be the two complete flags defined by

$$F_i \; = \; \langle e_1, \dots, e_i \rangle \quad \text{and} \quad G_i \; = \; \langle e_n, \dots, e_{n-i+1} \rangle.$$

Since the rational equivalence classes of Schubert cycles can be represented using any flag, we have

$$\sigma_\lambda \; = \; [\Sigma_\lambda(F_*)], \;\; \sigma_\mu \; = \; [\Sigma_\mu(G_*)] \; \in \; A^*(\mathrm{Gr}(d,V)).$$

If $\Sigma_\lambda(F_*) \cap \Sigma_\mu(G_*) \neq \varnothing$, pick any $[W] \in \Sigma_\lambda(F_*) \cap \Sigma_\mu(G_*)$; then for all $i \in \{1, \dots, d\}$ we have

$$\dim W \cap F_{n-d+i-\lambda_i} \; \geq \; i \quad \text{and} \quad \dim W \cap G_{n-d+(d+1-i)-\mu_{d+1-i}} \; \geq \; d+1-i,$$

which for dimension reasons forces $W \cap F_{n-d+i-\lambda_i} \cap G_{n+1-i-\mu_{d+1-i}} \; \neq \; 0$. By our choice of F_* and G_* this implies

$$\langle e_1, \dots, e_{n-d+i-\lambda_i} \rangle \cap \langle e_n, \dots, e_{n+1-(n+1-i-\mu_{d+1-i})} \rangle \; \neq \; 0$$

which forces $n - d + i - \lambda_i \geq i + \mu_{d+1-i}$, i.e. $\lambda_i + \mu_{d+1-i} \geq n - d$ for all i. Since by assumption the degrees of the two partitions satisfy $|\lambda| + |\mu| = d(n-d)$, it follows that equality holds for all i. Thus

$$\Sigma_\lambda(F_*) \cap \Sigma_\mu(G_*) \; \neq \; \varnothing \quad \text{only if} \quad \mu \; = \; \lambda^*$$

Moreover, for $\mu = \lambda^*$ the previous estimates give

$$\dim W \cap \langle e_1, \dots, e_{n-d+i-\lambda_i} \rangle \; \geq \; i$$

$$\dim W \cap \langle e_{n-d+i-\lambda_i}, \cdots, e_n \rangle \; \geq \; d+1-i$$

for all i. This forces $\dim W \cap \langle e_{n-d+i-\lambda_i} \rangle = 1$ for all i, and since $\dim W = d$ we get

$$W \; = \; \langle e_{n-d+1-\lambda_1}, e_{n-d+2-\lambda_2}, \cdots, e_{n-\lambda_d} \rangle.$$

So as a set the intersection $\Sigma_\lambda(F_*) \cap \Sigma_\mu(G_*)$ consists of a single point. Moreover, the intersection is transverse and hence is a reduced point as a scheme, because the tangent spaces to the Schubert varieties satisfy $T_W(\Sigma_\lambda(F_*)) \cap T_W(\Sigma_\mu(G_*)) = 0$ inside $T_W(\mathrm{Gr}(d,V)) = \mathrm{Hom}(W, V/W)$ by proposition 2.9. Hence the intersection number is one, and the claim follows. □

Remark 3.5. The above arguments may become more suggestive if the points of the Grassmann variety are represented as the row spans of matrices $A \in \mathrm{Mat}(d \times n)$ whose column indices refer to the chosen basis vectors $e_1, \dots, e_n$. In these terms, the points of $\Sigma_\lambda^\circ(F_*) = \Sigma_\lambda(F_*) \setminus \bigcup_{\nu > \lambda} \Sigma_\nu(F_*)$ are spans of matrices in 'lower' echelon form

$$S \cdot A \;=\; \left(\begin{array}{l} * \cdots * \; 1 \\ * \cdots * \; 0 \; * \cdots * \; 1 \\ * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \; 1 \\ \;\;\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \;\; \cdots \;\; * \cdots * \; 1 \end{array}\qquad\qquad\qquad\right)$$

while points of $\Sigma_\mu^\circ(G_*)$ are spans of matrices in 'upper' echelon form

$$T \cdot A \;=\; \left(\qquad\qquad\qquad \begin{array}{r} 1 \; * \cdots * \; 0 \; * \cdots * \; 0 \; * \cdots * \;\; \cdots \;\; * \cdots * \;\; \cdots \;\; * \cdots * \\ 1 \; * \cdots * \; 0 \; * \cdots * \;\; \cdots \;\; * \cdots * \;\; \cdots \;\; * \cdots * \\ 1 \; * \cdots * \;\; \cdots \;\; * \cdots * \;\; \cdots \;\; * \cdots * \\ \vdots \qquad \vdots \quad \\ 1 \;\; * \cdots * \end{array}\right)$$

The condition $\mu = \lambda^*$ precisely says that the two echelon forms have the pivot columns at the same positions, say $1 \leq i_1 < \cdots < i_d \leq n$. We can then compute the intersection between the two Schubert varieties in the affine chart $p_{i_1,\dots,i_d} = 1$ of the Grassmann variety by equating the above two matrices: The intersection is the locus where all entries $*$ vanish; this shows that the intersection consists of a single point and that the two Schubert varieties meet transversely at that point.

While the above still only deals with Schubert cycles of complementary degree, it is already enough to determine the additive structure of the Chow ring:

Corollary 3.6. *The Chow groups of* $\mathrm{Gr}(d,V)$ *are free abelian groups generated by the Schubert cycles,*

$$A^r(\mathrm{Gr}(d,V)) \;\simeq\; \bigoplus_{|\lambda|=r} \mathbb{Z} \cdot \sigma_\lambda \quad \textit{for all} \quad r \;\geq\; 0.$$

Proof. We already know from the cell decomposition that the Chow groups are generated by the Schubert cycles. Moreover, for any collection of integers $n_\lambda \in \mathbb{Z}$ with

$$\gamma \;:=\; \sum_\lambda n_\lambda \cdot \sigma_\lambda \;=\; 0$$

we get $n_\lambda = (\sigma_\mu, \gamma) = 0$ for $\mu = \lambda^*$ because of proposition 3.4, hence there are no nontrivial additive relations between Schubert cycles. □

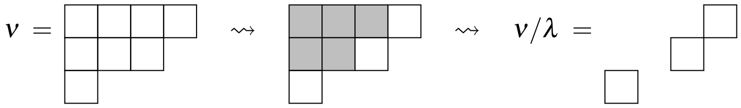
Of course this is not enough for intersection theory: In order to understand the ring structure on $A^*(\mathrm{Gr}(d,V))$ we want to determine the coefficients $c_{\lambda,\mu}^\nu \in \mathbb{N}_0$ in the expansion

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \cdot \sigma_\nu$$

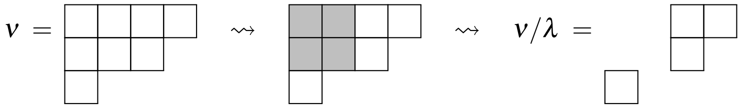
for partitions λ, μ of arbitrary degree, i.e. also when $|\lambda| + |\mu| < d(n-d)$. The $c^\nu_{\lambda\mu}$ play an important role not only in enumerative geometry but also in representation theory and combinatorics; they are called *Littlewood-Richardson coefficients*. To understand them we need some more notation:

Definition 3.7. Let λ, ν be two partitions with $\nu \geq \lambda$. The Young diagram of λ is a subset of the Young diagram of ν when both are written in left-aligned form, and we call the complement of the former in the latter a *skew diagram of shape* ν/λ. A skew diagram is a *horizontal strip* if no column of it contains more than one box.

Example 3.8. Let $\nu = (4,3,1)$. For $\lambda = (3,2)$ the skew shape ν/λ is a horizontal strip:



On the other hand, for $\lambda = (2,2)$ the skew shape ν/λ is *not* a horizontal strip:



Proposition 3.9 (Pieri's formula). *For any partition $\lambda = (\lambda_1, \dots, \lambda_d)$ and any $r \in \mathbb{N}$ we have*

$$\sigma_\lambda \cdot \sigma_{(r)} = \sum_\nu \sigma_\nu \quad \textit{in} \quad \mathrm{Gr}(d,V),$$

where the sum runs over all partitions $\nu \geq \lambda$ with ν/λ a horizontal strip of size r.

Proof. Taking intersection products with Schubert cycles of complementary degree, we see by proposition 3.4 that the claim is equivalent to the statement that for any partition ν and the dual partition $\nu^* = (n-d-\nu_d, \dots, n-d-\nu_1)$ we have

$$\sigma_\lambda \cdot \sigma_{(r)} \cdot \sigma_{\nu^*} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is a horizontal strip of size } r, \\ 0 & \text{otherwise.} \end{cases}$$

To prove the above formula, we pick a basis $e_1, \dots, e_n$ of V and let F_* and G_* be the complete flags defined as above by

$$F_i = \langle e_1, \dots, e_i \rangle \quad \text{and} \quad G_i = \langle e_n, \dots, e_{n-i+1} \rangle.$$

Again $\sigma_\lambda = [\Sigma_\lambda(F_*)]$ and $\sigma_{\nu^*} = [\Sigma_{\nu^*}(G_*)]$, but since the dimensions of these two Schubert cycles are not complementary, the corresponding reduced lower resp. upper echelon matrices will no longer have their pivot elements in the same columns, so we need a bit more care. By definition we have

$$[W] \in \Sigma_\lambda(F_*) \iff \forall i\colon \ \dim W \cap F_{n-d+i-\lambda_i} \geq i$$

$$[W] \in \Sigma_{\nu^*}(G_*) \iff \forall i\colon \ \dim W \cap G_{d+1-i-\nu_i} \geq d+1-i$$

where for the second inequality we have inserted the definition $\nu^*_{d+1-i} = n-d-\nu_i$ of the dual partition. In particular, any $[W] \in \Sigma_\lambda(F_*) \cap \Sigma_{\nu^*}(G_*)$ must have a non-zero intersection with

$$A_i = F_{n-d+i-\lambda_i} \cap G_{d+1-i-\nu_i} = \langle e_{n-d+i-\nu_i}, e_{n-d+i-\nu_i+1}, \dots, e_{n-d+i-\lambda_i} \rangle.$$

for each $i \in \{1, \dots, d\}$. Since

$$\dim A_i = \begin{cases} \nu_i - \lambda_i + 1 & \text{if } \nu_i \geq \lambda_i, \\ 0 & \text{otherwise,} \end{cases}$$

we already see that $\Sigma_\lambda(F_*) \cap \Sigma_{\nu^*}(G_*) = \varnothing$ unless $\nu \geq \lambda$, which is in line with the prediction of Pieri's formula. So in what follows we assume $\nu \geq \lambda$. From our choice of F_* and G_* as 'opposite' flags and from the above inequalities defining the Schubert varieties, one sees by linear algebra that any $[W] \in \Sigma_\lambda(F_*) \cap \Sigma_{\nu^*}(G_*)$ satisfies

$$W \subset A := A_1 + \cdots + A_d \subset V$$

A priori the sum on the right hand side need not be direct, and we only have the estimate

$$\dim A \leq \sum_{i=1}^r \dim A_i = \sum_{i=1}^r (\nu_i - \lambda_i + 1) = |\nu| - |\lambda| + d = d + r$$

The directness of the sum can be characterized in several ways:

$$\begin{aligned} \nu/\lambda \text{ is a horizontal strip} \quad &\iff \quad \forall i\colon \nu_i \leq \lambda_{i-1} \\ &\iff \quad A = A_1 \oplus \cdots \oplus A_d \\ &\iff \quad \dim A = d + r \end{aligned}$$

Now let H_* be another complete flag in V that has been chosen generically. By definition

$$[W] \in \Sigma_{(r)}(H_*) \quad \iff \quad W \cap U \neq 0$$

where $U = H_{n-d+1-r} \subset V$ is a generic subspace of dimension $n-d+1-r$. Hence any element

$$[W] \in \Sigma_\lambda(F_*) \cap \Sigma_{\nu^*}(G_*) \cap \Sigma_{(r)}(H_*)$$

in the intersection of all three Schubert varieties satisfies $W \subset A$ and $W \cap U \neq 0$, which can happen only if $A \cap U \neq 0$. Since $U \subset V$ has been chosen to be a *generic* subspace of dimension

$$\dim U \;=\; n+1-(d+r) \;\leq\; n+1-\dim A$$

the three Schubert varieties can intersect only if $\dim A \geq d-r$. By the previous discussion we then have

- $\dim A = d-r$,
- $\dim U \cap A = 1$,
- $A = A_1 \oplus \cdots \oplus A_d$.

Let $v \in U \cap A \setminus \{0\}$. By the last item we can decompose this vector uniquely as a sum

$$v \;=\; v_1 + \cdots v_r \quad \text{with} \quad v_i \;\in\; A_i.$$

For $[W] \in \Sigma_\lambda(F_*) \cap \Sigma_{\nu^*}(G_*) \cap \Sigma_{(r)}(H_*)$ we have $0 \;\neq\; A \cap W \;\subset\; A \cap U \;=\; \langle v \rangle$, hence $v \in W$. On the other hand, the dimension inequalities in the definition of Schubert varieties also imply that

$$W \;=\; W_1 \oplus \cdots \oplus W_r \quad \text{for the intersections} \quad W_i \;=\; W \cap A_i,$$

so it follows that $v_i \in W$ for all i. But then for dimension reasons we have

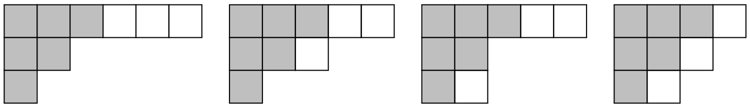
$$W \;=\; \langle v_1, \ldots, v_d \rangle,$$

hence there exists a *unique* point $[W] \in \Sigma_\lambda(F_*) \cap \Sigma_{\nu^*}(G_*) \cap \Sigma_{(r)}(H_*)$ in the intersection of the three Schubert varieties. To conclude the proof, one easily verifies in affine charts that the three Schubert varieties intersect transversely at this point if the flag H_* is chosen generically, so the intersection number is one. □

Example 3.10. Let $n = 9$ and $d = 3$. For $\lambda = (3,2,1)$ and $r = 3$ we get from Pieri's formula that

$$\sigma_{(3,2,1)} \cdot \sigma_{(3)} \;=\; \sigma_{(6,2,1)} + \sigma_{(5,3,1)} + \sigma_{(5,2,2)} + \sigma_{(4,3,2)}$$

as illustrated in the following picture where the white boxes depict the strip ν/λ for the four partitions ν that appear in the sum:



In shat follows we will often omit the brackets around partitions indexing Schubert cycles, writing $\sigma_{3,2,1}$ instead of $\sigma_{(3,2,1)}$ to simplify notations.

Already the case $\mu = (1)$ in Pieri's formula is interesting. Using it repeatedly we can answer questions in enumerative geometry such as the following:

Example 3.11. How many lines in the projective space $\mathbb{P}^3$ intersect four given lines in general position? To answer this question we have to compute the intersection number $\sigma_1^4 \in \mathbb{Z}$ for the Schubert cycle $\sigma_1 \in A^1(\mathrm{Gr}(2,4))$. Applying Pieri's formula three times we get

$$\begin{aligned}
\sigma_1^2 &= \sigma_2 + \sigma_{1,1} \\
\sigma_1^3 &= (\sigma_{3,0} + \sigma_{2,1}) + \sigma_{2,1} \;=\; 2\sigma_{2,1} \\
\sigma_1^4 &= 2(\sigma_{3,1} + \sigma_{2,2}) \;=\; 2\sigma_{2,2},
\end{aligned}$$

where $\sigma_{2,2}$ is the class of a point. Thus in $\mathbb{P}^3$ there are precisely two lines that intersect four given lines in general position. Try to check this by hand!

More generally, how many subspaces of dimension $d-1$ are there in $\mathbb{P}^n$ that intersect $d(n-d)$ subspaces of dimension $n-d-1$? To answer this we need to find the intersection number

$$N_{n,d} \;=\; \sigma_1^{d(n-d)} \;\in\; \mathbb{Z} \quad \text{for the class} \quad \sigma_1 \;\in\; A^1(\mathrm{Gr}(d,n)).$$

Note that the class of a point is $\sigma_{n-d,\cdots n-d}$, corresponding to the Young diagram which is a box of size $d \times (n-d)$. Proceeding via Pieri's formula as above, we see that $N_{n,d}$ is the number of ways to successively build this Young diagram by appending boxes at the end of existing rows or columns, starting from the top left corner. Putting a label i in the i-th appended box, we get a *standard Young tableau* of shape λ, i.e. a filling of the Young diagram of shape λ with the labels $1,2,\dots,|\lambda|$ such that all rows and columns are increasing. For $n=4$ and $d=2$ there are precisely two standard tableaux of shape $(2,2)$:

1	2
3	4

1	3
2	4

In general, the above shows that the intersection number $N_{n,d}$ is the number of standard tableaux of shape $d \times (n-d)$. In combinatorics one shows that the number $N(\lambda)$ of standard tableaux of any shape λ is given by the *hook length formula*

$$N(\lambda) \;=\; \frac{|\lambda|!}{\prod_{s\in\lambda} \mathrm{hook}(s)}$$

where the hook length $\mathrm{hook}(s)$ is defined to be the number of boxes in the Young diagram that lie to the right or below the box s, including s. The following picture shows the hook lengths for $\lambda = (3,3)$:

4	3	2
3	2	1

Hence

$$N_{5,2} \;=\; \frac{6!}{4\cdot 3\cdot 2\cdot 3\cdot 2\cdot 1} \;=\; 5.$$

Note that Pieri's formula for the intersection with the Schubert cycles $\sigma_{(r)}$ for singleton partitions determines the intersection product on the Chow ring of the Grassmann variety completely:

Lemma 3.12. *The ring $A^*(\mathrm{Gr}(d,V))$ is generated by the classes $\sigma_{(r)}$ with $r \in \mathbb{N}_0$.*

Proof. It suffices to show that for each partition $\lambda = (\lambda_1, \dots, \lambda_d)$ the class σ_λ can be written as a polynomial in classes $\sigma_{(r)}$ for various $r \in \mathbb{N}$. We do this by lexicographic induction on the pair

$$N(\lambda) \;:=\; (\ell, \lambda_\ell) \quad \text{where} \quad \ell \;=\; \max\{\, i \in \mathbb{N}_0 \,|\, \lambda_i > 0\,\}.$$

For $\ell = 1$ the partition $\lambda = (r)$ is already a singleton partition and there is nothing to show. The next case are partitions of length $\ell = 2$, say $\lambda = (r,s)$, where Pieri's formula says

$$\sigma_\lambda \;=\; \sigma_{(r,s)} \;=\; \sigma_{(r)} * \sigma_{(s)} - \sum_{i=0}^{s-1} \sigma_{(r-i,i)}$$

and we are done by induction because the right hand side only involves terms σ_μ with $N(\mu) < N(\lambda)$. The general case works similarly: For any λ of length ℓ Pieri's formula gives

$$\sigma_\lambda \;\equiv\; \sigma_{(\lambda_1,\dots,\lambda_{\ell-1})} \cdot \sigma_{(\lambda_\ell)} \quad \text{modulo} \quad \bigoplus_{N(\mu)<N(\lambda)} \mathbb{Z}\cdot\sigma_\mu,$$

so the claim follows by induction. □

In principle this allows us to compute arbitrary products of Schubert cycles by writing the cycles as polynomials in classes $\sigma_{(r)}$ and using Pieri's formula, but doing this by hand can be quite tedious; convenient methods come from the theory of symmetric functions which we briefly review now:

Definition 3.13. A *semistandard tableau* of shape $\lambda = (\lambda_1, \dots, \lambda_d)$ is a filling of the Young diagram of shape λ with natural numbers such that in each row the entries weakly increase from left to right and in each column the entries strictly increase from top to bottom. We denote by $\mathrm{SST}(\lambda)_{\le d}$ the set of semistandard tableaux of shape λ with entries in $\{1, \dots, d\}$, and to any such tableau $T \in \mathrm{SST}(\lambda)_{\le d}$ we attach the monomial

$$x^T \;=\; x_1^{n_1} \cdots x_d^{n_d} \;\in\; \mathbb{Q}[x_1, \dots, x_d]$$

where n_i denotes the number of occurances of the entry i in the tableau. We define the *Schur polynomial* of shape $\lambda = (\lambda_1, \dots, \lambda_d)$ in d variables as the sum of all such monomials:

$$s_\lambda(x) \;=\; \sum_{T \in \mathrm{SST}(\lambda)_{\le d}} x^T \;\in\; \mathbb{Q}[x_1, \dots, x_d].$$

Example 3.14. The semistandard tableaux of shape $\lambda = (r)$ with entries $\leq d$ are precisely the tableaux

$$\begin{array}{|c|c|c|c|c|}\hline i_1 & i_2 & \cdots & \cdots & i_r \\ \hline\end{array} \quad \text{with} \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_r.$$

They give the complete symmetric polynomial $s_{(r)}(x) = h_r(x) \in \mathbb{Q}[x_1, \ldots, x_d]$.

Example 3.15. For $\lambda = (1^r) = (1, \ldots, 1)$ the tableaux in $\mathrm{SST}(\lambda)_{\leq d}$ are precisely those of the form

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_r \\ \hline\end{array} \quad \text{with} \quad 1 \leq i_1 < \cdots < i_r \leq d.$$

They give the elementary symmetric polynomial $s_{(1^r)}(x) = e_r(x) \in \mathbb{Q}[x_1, \ldots, x_d]$.

Thus Schur polynomials can be seen as symmetric polynomials that 'interpolate' between complete and elementary symmetric polynomials. The following theorem recalls some of their basic properties:

Theorem 3.16. *The Schur polynomials $s_\lambda(x)$ with $\lambda = (\lambda_1, \ldots, \lambda_d)$ form a $\mathbb{Z}$-basis for the ring*

$$\Lambda_d \;=\; \mathbb{Z}[x_1, \ldots, x_d]^{\mathfrak{S}_d}$$

of symmetric polynomials in d variables. They can be expressed as follows.

a) Weyl character formula: For any $e = (e_1, \ldots, e_d)$ put $D_\alpha(x) = \det(x_i^{e_j})_{1 \leq i,j \leq d}$, then

$$s_\lambda(x) \;=\; \frac{D_{\lambda+\delta}(x)}{D_\delta(x)} \quad \textit{for the vector} \quad \delta \;=\; (d-1, d-2, \ldots, d_1, 0).$$

b) Jacobi-Trudi formula: Let $\mu = \lambda^t$ be the transpose of the partition λ, obtained by interchanging rows with columns in the Young diagram. Then in terms of the complete symmetric polynomials and the elementary symmetric polynomials we have

$$s_\lambda(x) \;=\; \det(h_{\lambda_i+j-i})_{1 \leq i,j \leq d} \;=\; \det(e_{\mu_i+j-i})_{1 \leq i,j \leq d}.$$

Proof. Omitted, see any book on symmetric functions. □

We are now prepared to describe the ring structure on $A^*(\mathrm{Gr}(d,V))$ completely using Schur polynomials. Put $n = \dim(V)$ as above.

Theorem 3.17. *Inside the ring of symmetric polynomials in d variables, consider the ideal*

$$I_{d,n} \;=\; (\sigma_\lambda(x) \colon \lambda_1 > n-d) \;\trianglelefteq\; \Lambda_d \;=\; \mathbb{Z}[x_1, \ldots, x_d]^{\mathfrak{S}_d}.$$

Then we have a ring isomorphism

$$\Lambda_d / I_{n,d} \xrightarrow{\sim} A^*(\mathrm{Gr}(d,V)), \quad s_\lambda(x) \;\mapsto\; \sigma_\lambda.$$

Proof. Since the Schur polynomials for a $\mathbb{Z}$-basis for the free abelian group Λ_d, there is a unique group homomorphism

$$\varphi\colon \quad \Lambda_r \longrightarrow A^*(\mathrm{Gr}(d,V)) \quad \text{with} \quad s_\lambda(x) \mapsto \sigma_\lambda.$$

This is an isomorphism of groups since the Chow groups are free abelian on the set of Schubert cycles σ_λ with $\lambda_1 \leq n-d$ whereas $\sigma_\lambda = 0$ for $\lambda_1 > n-d$. So the only thing that remains to be verified is that the above group homomorphism φ sends the product of symmetric functions to the intersection product on the Chow ring. To see this we recall from the theory of symmetric functions that the product of Schur functions also satisfies Pieri's formula

$$s_\lambda(x)\cdot s_{(r)}(x) \;=\; \sum_\nu s_\nu(x)$$

where the sum is over all partitions $\nu \geq \lambda$ with ν/λ a horizontal strip of size r. It follows that

$$\varphi(s_\lambda(x)\cdot s_{(r)}(x)) \;=\; \sum_\nu \varphi(s_\nu(x)) \;=\; \sum_\nu \sigma_\nu \;=\; \sigma_\lambda\cdot\sigma_{(r)} \;=\; \varphi(s_\lambda(x))\cdot\varphi(s_{(r)}(x))$$

for any λ. It follows as desired that

$$\varphi(s_\lambda(x)\cdot s_\mu(x)) \;=\; \varphi(s_\lambda(x))\cdot\varphi(s_\mu(x))$$

for all λ, μ because Λ_d is generated as a ring by the polynomials $s_{(r)}(x) = h_r(x)$. □

The upshot is that the intersection theory on Grassmann varieties is completely determined by combinatorics. In particular, in the intersection product

$$\sigma_\lambda\cdot\sigma_\mu \;=\; \sum_\nu c^\nu_{\lambda\mu}\cdot\sigma_\nu$$

the $c^\nu_{\lambda\mu} \in \mathbb{Z}$ coincide with the Littlewood-Richardson coefficients defined by the expansion

$$s_\lambda(x)\cdot s_\mu(x) \;=\; \sum_\nu c^\nu_{\lambda\mu}\cdot s_\nu(x)$$

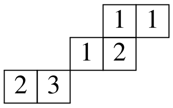
of the product of Schur polynomials. There is a well-known way to compute them combinatorially. To describe it we need some more notation:

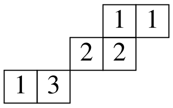
Definition 3.18. A *semistandard skew tableau* of shape ν/λ is a filling of the skew diagram of shape ν/λ such that in each row the entries increase weakly from left to right and in each column they increase strictly from top to bottom. The tableau has *content* μ if it contains each entry i precisely μ_i times. The associated *word* is the sequence obtained by reading the entries of each row from right to left, starting with the top row and continuing until the bottom row. We say that the skew tableau is a *Littlewood-Richardson tableau* if the associated word has the property that each of its initial segments contain each entry i at least as often as the entry $i+1$.

Theorem 3.19 (Littlewood-Richardson rule). *The coefficient* $c^{\nu}_{\lambda\mu}$ *is equal to the number of Littlewood-Richardson tableaux of shape* ν/λ *with content* μ.

Proof. See any textbook on combinatorics and symmetric functions. □

Example 3.20. Let $\lambda = (3,2)$, $\mu = (3,2,1)$ and $\nu = (5,4,2)$. One easily checks that there are precisely two Littlewood-Richardson tableaux of shape ν/λ and content μ:





Hence by the Littlewood-Richardson rule it follows that $c^{(5,4,2)}_{(3,2),(3,2,1)} = 2$.

4 Degeneracy loci

In this section we will apply Schubert calculus to find formulas for the fundamental class of degeneraci loci

$$D_r(\varphi) \;=\; \{x \in X \mid \mathrm{rk}(\varphi(x)\colon F(x) \to G(x)) \leq r\} \;\subset\; X$$

for morphisms $\varphi\colon F \to G$ between vector bundles, and more generally degeneracy loci where the restriction of the morphism to a given flag of subbundles of F has prescribed ranks. For this we need to work with Grassmannians in a relative setting, replacing vector spaces by vector bundles:

Let V be a vector bundle of rank n on a smooth quasiprojective variety X. Then there exists a variety $\mathrm{Gr}(d,V)$ with a smooth projective morphism

$$p\colon \quad \mathrm{Gr}(d,V) \;\longrightarrow\; X$$

parametrizing subbundles of V of rank d in the sense that for any scheme $f\colon T \to X$ over X we have a natural bijection

$$\mathrm{Hom}_X(T,\mathrm{Gr}(d,V)) \;\simeq\; \{\,\text{subbundles } W \subset f^*(V) \text{ of rank } d\,\}.$$

Note that by this universal property the variety $\mathrm{Gr}(d,E)$ and the morphism p are determined uniquely up to isomorphism. To show the existence, one can start with the case where $V = X \times \mathbb{A}^n$ is a trivial bundle and take $\mathrm{Gr}(d,V) = X \times \mathrm{Gr}(d,n)$; the general case is obtained by gluing such models on charts via local trivializations of the vector bundle. Generalizing the previous section, consider a non-complete flag of subbundles

$$V_*\colon \quad V_1 \;\subset\; \cdots \;\subset\; V_d \;\subset\; V \quad \text{with} \quad \dim V_i \;=\; n-d+i-\lambda_i,$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$ is a partition of length at most d. The relative analog of Schubert varieties are the loci $\Sigma_{V_*} \subset \mathrm{Gr}(d,V)$ whose points with values in $f\colon T \to X$ are given by

$$\begin{aligned}\Sigma_{V_*}(T) \;&=\; \{\text{subbundles } W \subset f^*(V) \text{ of rank } d \mid \forall i\colon\; \mathrm{rk}(W \cap V_i) \;\geq\; i\} \\ &=\; \{\text{subbundles } W \subset f^*(V) \text{ of rank } d \mid \forall i\colon\; \mathrm{rk}(W \to V/V_i) \;\leq\; d-i\}.\end{aligned}$$

Our main goal in this section is to find a formula for the fundamental class

$$[\Sigma_{V_*}] \;\in\; A^*(\mathrm{Gr}(d,V))$$

in terms of Chern classes of the bundles $p^*(V_i)$ and the tautological quotient bundle on the Grassmann variety; this will then easily imply formulas for the degeneracy loci between arbitrary morphisms of vector bundles by using the universal property of the Grassmann variety. The first step is to pass from the Schubert variety to a suitable flag variety:

Lemma 4.1. *There exists a variety* $\mathrm{Fl}(V_*)$ *with a morphism* $\pi\colon \mathrm{Fl}(V_*) \to X$ *which parametrizes subflags of the given flag where the pieces have minimal rank, in the sense that*

$$\mathrm{Hom}_X(T, \mathrm{Fl}(V_*)) \;=\; \{\, W_1 \subset \cdots \subset W_d \;\mid\; W_i \subset f^*(V_i) \textit{ subbundle of rank } \mathrm{rk}(W_i) = i \,\}$$

for any $f\colon T \to S$. *The variety* $\mathrm{Fl}(V_*)$ *is irreducible and the morphism* $\mathrm{Fl}(V_*) \to X$ *is smooth of relative dimension*

$$\dim(\mathrm{Fl}(V_*)) - \dim(X) \;=\; d(n-d) - |\lambda|.$$

We have a forgetful morphism

$$p\colon \quad \mathrm{Fl}(V_*) \;\longrightarrow\; \Sigma_{V_*} \;\subset\; \mathrm{Gr}(d,V), \quad [W_1 \subset \cdots \subset W_d] \;\mapsto\; [W_d]$$

which restricts to an isomorphism over the open dense subset

$$U \;=\; \{\, [W] \in \mathrm{Gr}(d,V) \mid \forall i\colon\; \mathrm{rk}(W \cap V_i) \;=\; i\} \;\subset\; \Sigma_{V_*}.$$

Proof. Starting from $\pi_0 = id\colon G_0 = X \to X$ with the zero vector bundle $D_0' = 0$, we construct for $i = 1, \dots, d$ inductively a sequence of

- varieties G_i,
- morphisms $\pi_i\colon G_i \to X$,
- subbundles $D_i' \subset \pi_i^*(V_i)$ of rank $\mathrm{rk}(D_i') = i$,

as follows: On G_{i-1} we have the vector bundle $E_{i-1} = \pi_{i-1}^*(V_i)/D_{i-1}'$ and take G_i to be the projective bundle

$$\varphi_i\colon \quad G_i \;:=\; \mathbb{P}(E_{i-1}) \;\longrightarrow\; G_{i-1}.$$

Let $\pi_i = \pi_{i-1} \circ \varphi_i \colon G_i \to X$ denote the projection. On $G_i = \mathbb{P}(E_{i-1})$ we have the tautological subbundle

$$S_i \;\subset\; \varphi_i^*(E_{i-1}) \;=\; \pi_i^*(V_i)/\varphi_i^*(D'_{i-1})$$

and define

$$D'_i \;:=\; (\text{preimage of } S_i) \;\subset\; \pi_i^*(V_i).$$

Let $\mathrm{Fl}(V_*) = G_d$ with its natural morphism $\pi \colon \mathrm{Fl}(V_*) \to X$ and $\psi_i = \varphi_{i+1} \circ \cdots \circ \varphi_d$ as shown in the diagram

$$\begin{tikzcd}
\mathrm{Fl}(V_*) \arrow[rrrrrr, "\pi"] \arrow[d, equal] \arrow[rrr, "\psi_i"] & & & & & & X \arrow[d, equal] \\
G_d \arrow[r] & \cdots \arrow[r] & G_i \arrow[r, "\varphi_i"] & G_{i-1} \arrow[r] \arrow[urrr, "\pi_{i-1}"] & \cdots \arrow[r] & & G_0
\end{tikzcd}$$

where each φ_i is a projective bundle of relative dimension

$$\begin{aligned}
\dim(G_i) - \dim(G_{i-1}) \;&=\; \mathrm{rk}(E_{i-1}) - 1 \\
&=\; \mathrm{rk}(V_i) - \mathrm{rk}(D'_{i-1}) - 1 \\
&=\; (n-d+i-\lambda_i) - (i-1) - 1 \\
&=\; n-d-\lambda_i.
\end{aligned}$$

In particular $\pi \colon \mathrm{Fl}(V_*) \to X$ is smooth of relative dimension $d(n-d) - |\lambda|$ and $\mathrm{Fl}(V_*)$ is irreducible. Moreover, on the flag variety $\mathrm{Fl}(V_*)$ the pullback of the given vector bundle comes with a universal flag

$$D_* \colon \quad D_1 \subset \cdots \subset D_d \subset \pi^*(V) \quad \text{defined by} \quad D_i \;:=\; \psi_i^*(D'_i) \;\subset\; \pi^*(V_i),$$

and one easily verifies that this implies the desired universal property. Moreover it is clear from the definitions that the forgetful morphism $p \colon \mathrm{Fl}(V_*) \to \mathrm{Gr}(d,V)$ has its image contained in the Schubert variety Σ_{V_*} and restricts to an isomorphism over the open subset where the lower bounds on the rank are attained. □

The idea is now to write the birational morphism $p \colon \mathrm{Fl}(V_*) \to \Sigma_{V_*} \subset \mathrm{Gr}(d,V)$ as the graph embedding into $Y = \mathrm{Gr}(d,V) \times_X \mathrm{Fl}(V_*)$ followed by the projection onto the first factor:

$$\begin{tikzcd}
F(V_*) \arrow[r, hook] \arrow[dr, "p"'] & Y = \mathrm{Gr}(d,V) \times_X \mathrm{Fl}(V_*) \arrow[d, "p_1"] \\
& \mathrm{Gr}(d,V)
\end{tikzcd}$$

We can then compute the fundamental class $[\Sigma_{V_*}]$ in two steps:

- Compute the fundamental class $[\mathrm{Fl}(V_*)] \in A^*(Y)$.
- Compute the pushforward $p_{1*}[\mathrm{Fl}(V_*)] \in A^*(\mathrm{Gr}(d,V))$.

We begin with the first task. To state the result we need some notation for certain determinants that will appear repeatedly:

Definition 4.2. Suppose we are given for $i = 1, \dots, d$ a power series $P_i = \sum_j a_{ij} t^j$ with coefficients in some fixed commutative ring. Then for any sequence of natural numbers $\mu = (\mu_1, \dots, \mu_d)$ we put

$$\Delta_\mu(P_1, \dots, P_d) \ := \ \det(a_{i,\mu_i+j-i})_{1 \le i,j \le d}.$$

When all the power series are equal, we simply write $\Delta_\mu(P) = \Delta_\mu(P, \cdots, P)$.

Example 4.3. The expansion of Schur polynomials for a partition $\lambda = (\lambda_1, \dots, \lambda_d)$ in terms of elementary symmetric polynomials via the Jacobi-Trudi identity can be written as

$$s_\lambda(x) \ = \ \Delta_\mu(P) \quad \text{where} \quad \mu \ = \ \lambda^t \quad \text{and} \quad P \ = \ \sum_j e_j(x) t^j \ \in \ \Lambda_d[t].$$

Example 4.4. For any polynomials $a(t) = \prod_{i=1}^d (1 + \alpha_i t)$ and $b(t) = \prod_{j=1}^e (1 + \beta_j t)$ one defines their *resultant* by

$$\operatorname{Res}(a,b) \ = \ \prod_{i=1}^{d} \prod_{j=1}^{e} (\beta_j - \alpha_i).$$

This resultant vanishes iff the polynomials $a(t)$ and $b(t)$ have a common zero. What makes it interesting is that it can be computed directly from the coefficients of $a(t)$ and $b(t)$ without knowing the zeroes α_i and β_j. This can be seen for instance from the first identity in the following result:

Lemma 4.5. *Let $a(t), b(t)$ be as above, and take $\mu = (e, \dots e)$ with e repeated d times. Then we have*

$$\operatorname{Res}(a,b) \ = \ \Delta_\mu(P) \ = \ \Delta_\mu(P_1, \dots, P_d)$$

for the power series $P(t) = b(t)/a(t)$ and $P_k(t) = \prod_{j=1}^e (1 + \beta_j t) / \prod_{i=1}^k (1 + \alpha_i t)$.

Proof. These are equalities between universal polynomials in the elements α_i, β_j with integer coefficients, so it suffices to prove them over $\mathbb{Z}[\alpha_1, \dots, \alpha_d, \beta_1, \dots \beta_e]$ where the α_i, β_j are free variables: If we can show the claimed equalities over this universal ring, they will also hold over any other commutative ring by specializing the variables. So from now on we will assume to work over an integral domain.

For the first equality $\operatorname{Res}(a,b) = \Delta_\mu(P)$, one then easily checks that both sides are homogenous polynomials of total degree de in the variables α_i and β_j, and they agree if all β_j vanish. Hence it will be enough to show that the determinant $\Delta_\mu(P)$ vanishes if the polynomials $a(t)$ and $b(t)$ have a common zero. To see this, note that if $a(t)$ and $b(t)$ have a common zero, then $P(t) = b(t)/a(t)$ is a rational function with denominator of degree $< d$, which means that the coefficients in the power series development

$$P(t) \;=\; \sum_{i\geq 0} c_i t^i$$

satisfy a linear recurrence relation of length $< d$. This recurrence gives an element in the kernel of the matrix $(c_{\mu_i+j-i})_{1\leq i,j\leq d}$ and consequently $\Delta_\mu(P) = 0$. The second equality $\Delta_\mu(P) = \Delta_\mu(P_1,\dots,P_d)$ follows by applying elementary row operations to the matrices, using the relations between rows given by $(1+\alpha_k)P_k = P_{k-1}$. □

The following simple example explains why determinants of the above shape are relevant for the study of degeneracy loci:

Example 4.6. Let $\varphi\colon F \to G$ be a morphism between vector bundles on X. Then the locus

$$D_0(\varphi) \;=\; \{x \in X \mid \varphi(x) = 0\} \;\subset\; X$$

is the zero locus of a section of the vector bundle

$$V \;=\; \mathscr{H}om(F,G) \;=\; F^\vee \otimes G \quad \text{of rank} \quad ef \quad \text{where} \quad d \;=\; \mathrm{rk}(F), e = \mathrm{rk}(G).$$

If it has the expected codimension, then the fundamental class of this zero locus is given by the top Chern class of the vector bundle. Let α_i and β_j be the Chern roots of F and G, respectively, then the Chern roots of V are the differences $\beta_j - \alpha_i$ and the top Chern class is

$$c_{de}(V) \;=\; \prod_{i,j}(\beta_j - \alpha_i) \;=\; \mathrm{Res}(c_t(F), c_t(G))$$

Let $\mu = (e,\dots,e)$ with e repeated d times, then lemma 4.5 gives

$$c_{de}(V) \;=\; \Delta_\mu(P) \quad \text{where} \quad P \;=\; c_t(G)/c_t(F).$$

If F has a filtration by subbundles $F_1 \subset \cdots \subset F_d = F$ with $\mathrm{rk}(F_i/F_{i-1}) = 1$ for all i, then the same lemma also gives

$$c_{de}(V) \;=\; \Delta_\mu(P_1,\dots,P_d) \quad \text{where} \quad P_i \;:=\; c_t(G)/c_t(F_i).$$

We can now compute the fundamental class of the Flag variety as follows:

Proposition 4.7. *The image of the graph* $\mathrm{Fl}(V_*) \hookrightarrow Y = \mathrm{Gr}(d,V) \times_X \mathrm{Fl}(V_*)$ *has the fundamental class*

$$[\mathrm{Fl}(V_*)] \;=\; \Delta_{n-d,\dots,n-d}(P_1,\dots,P_d) \;\in\; A^*(Y) \quad \textit{for} \quad P_i \;=\; c_t(Q)/c_t(D_i),$$

where

- $R = p_1^*(Q)$ *is the pullback of the universal quotient bundle* Q *on* $\mathrm{Gr}(d,V)$*,*
- $C_i = p_2^*(D_i)$ *are the pullbacks of the universal bundles* D_i *on* $\mathrm{Fl}(V_*)$*.*

Proof. By definition the image of the graph embedding is

$$\begin{aligned}\mathrm{Fl}(V_*) \;&=\; \{(W, W_1 \subset \cdots \subset W_d) \in Y = \mathrm{Gr}(d,V) \times \mathrm{Fl}(V_*) \,|\, W = W_d\} \\ &=\; \text{zero set of the composite morphism } p_2^*(D_d) \to p_2^* \pi^* V = p_1^* \pi^* V \to p_1^* Q \\ &=\; \text{zero set of a section of the vector bundle } p_2^*(D_d^\vee) \otimes p_1^*(Q) \;=\; C_d^\vee \otimes R.\end{aligned}$$

Now the rank of this vector bundle is $\mathrm{rk}(C_d^\vee \otimes R) = d(n-d) = \mathrm{codim}_Y(\mathrm{Fl}(V_*))$, hence the fundamental class of the vanishing locus of its section is given by the top Chern class $[\mathrm{Fl}(V_*)] \;=\; c_{d(n-d)}(C_d^\vee \otimes R) \;=\; \Delta_{n-d,\dots,n-d}(P_1, \dots, P_d)$ where the last step uses example 4.6. ☐

It remains to compute the pushforward of the above class under the projection to the Grassmann variety, which leads to the main result of this section:

Theorem 4.8 (Kempf-Laksov formula for the universal Schubert cycles). *The subvariety $\Sigma_{V_*} \subset \mathrm{Gr}(d,V)$ is irreducible of codimension $|\lambda|$ with fundamental class*

$$[\Sigma_{V_*}] \;=\; \Delta_\lambda(c(Q - V_1), \dots, c(Q - V_d)) \quad \textit{where} \quad c(Q - V_i) \;:=\; c_t(Q)/c_t(V_i).$$

Proof. Consider again the graph embedding $\mathrm{Fl}(V_*) \subset Y = \mathrm{Gr}(d,V) \times \mathrm{Fl}(V_*)$. We know that the morphism

$$p_1 : \quad \mathrm{Fl}(A) \longrightarrow \Sigma_{V_*} \;\subset\; \mathrm{Gr}(d,V)$$

is an isomorphism over an open dense subset the Schubert variety Σ_{V_*}, hence the statement about the irreducibility and the codimension follows from the analogous statement about the flag variety. Moreover, we compute

$$\begin{aligned}[\Sigma_{V_*}] \;&=\; p_{1*}[\mathrm{Fl}(V_*)] && \text{by birationality of } p_1 \\ &=\; p_{1*}\Delta_{n-d,\dots,n-d}(P_1, \dots, P_d) && \text{by proposition 4.7,}\end{aligned}$$

where

$$P_i \;=\; c_t(R)/c_t(C_i) \;\in\; A^*(\mathrm{Gr}(d,V) \times_X \mathrm{Fl}(V_*))[t].$$

We will now compute the pushforward step by step using the description of the flag variety in the proof of lemma 4.1: Putting $H_i = \mathrm{Gr}(d,V) \times_X G_i$ we factor the projection p_1 as

$$\begin{tikzcd}
\mathrm{Gr}(d,V) \times_X \mathrm{Fl}(V_*) \arrow[rrrrr, "p_1"] \arrow[d, equal] \arrow[drr, "\psi_i"] & & & & & \mathrm{Gr}(d,V) \arrow[d, equal] \\
H_d \arrow[r] & \cdots \arrow[r] & H_i \arrow[r, "\varphi_i"] & H_{i-1} \arrow[r] \arrow[urr, "\pi_{i-1}"'] & \cdots \arrow[r] & H_0
\end{tikzcd}$$

The idea is to work downwards by descending induction on $i = d, d-1, \dots, 1$ to replace $\mu = (n-d, \dots, n-d)$ by the partitions

$$\mu^{(i)} \;=\; (n-d, \cdots, n-d, \lambda_{i+1}, \dots, \lambda_d)$$

and the P_j by the polynomials defined by

$$P_j^{(i)} \;=\; \begin{cases} c_t(Q)/c_t(V_j) & \text{for } j > i, \\ c_t(Q)/c_t(D'_j) & \text{for } j \leq i, \end{cases}$$

where abusively we again denote by Q, V_j, D'_j the pullback of the respective bundles to G_i. We claim that

$$\varphi_{i,*}\Delta_{\mu^{(i)}}(P_1^{(i)}, \dots, P_d^{(i)}) \;=\; \Delta_{\mu^{(i-1)}}(P_1^{(i-1)}, \dots, P_d^{(i-1)}) \;\in\; A^*(H_i)$$

for all i. To see this one can proceed as follows:

- We have $c_t(Q)/c_t(D'_i) = c_t(Q_i) \cdot c_t(Q)/c_t(V_i)$ for $Q_i = V_i/D'_i$.
- We have $\varphi_{i*}(c_t(Q_i)) = t^{r_i}$ with $r_i = \mathrm{rk}(Q_i) = n - d - \lambda_i$ because

$$\varphi_{i*}(c_j(Q_i)) \;=\; \begin{cases} 0 & \text{for } j > r_i \text{ because then } c_j(Q_i) = 0 \text{ as } r_i = \mathrm{rk}(Q_i), \\ 0 & \text{for } j < r_i \text{ for dimension reasons since } r_i = \dim(H_i/H_{i-1}), \\ 1 & \text{for } j = r_i \text{ since } H_i \to H_{i-1} \text{ is a } \mathbb{P}^{r_i}\text{-bundle.} \end{cases}$$

- The claim then follows from the projection formula since putting together the above we have

$$\varphi_{i,*}(P_i^{(i)}) \;=\; \varphi_{i,*}(c_t(Q_i) \cdot c_t(Q)/c_t(V_i)) \;=\; t^{r_i} \cdot c_t(Q)/c_t(V_i) \;=\; P_i^{(i-1)}.$$

Since $p_1 = \varphi_1 \circ \cdots \circ \varphi_d$, it then follows by applying the above formula inductively that

$$p_{1,*}\Delta_{n-d,\dots,n-d}(P_1, \dots, P_d) \;=\; \Delta_\lambda(c(Q - V_1), \dots, c(Q - V_d))$$

as required, because $\mu^{(1)} = \lambda$ and $P_j^{(1)} = c(Q - V_j)$. □

Theorem 4.9 (Kempf-Laksov formula for degeneracy loci). *Let* $\varphi\colon F \to G$ *be a morphism between vector bundles on a smooth quasiprojective variety, and denote their ranks by* $\mathrm{rk}(F) = d$ *and* $\mathrm{rk}(G) = e$. *Suppose that we are given a flag of vector subbundles*

$$V_1 \subset \cdots \subset V_c \subset F \quad \text{for some} \quad c \;\geq\; d - e,$$

and put $a_i = \mathrm{rk}(F_i)$. *Let* $\lambda = (\lambda_1, \dots, \lambda_c)$ *where* $\lambda_i = e + i - a_i$. *Then the degeneracy locus*

$$D(\varphi, V_*) \;:=\; \{x \in X \mid \forall i\colon \mathrm{rk}(\varphi(x)\colon F_i(x) \to G_i(x)) \leq a_i - i\} \;\subset\; X$$

is either empty or of codimension $\leq |\lambda|$. *If it is empty or of codimension* $= |\lambda|$, *then its fundamental class is*

$$[D(\varphi, V_*)] \;=\; \Delta_\lambda(c(G - F_1), \dots, c(G - F_c)).$$

Proof. Consider the vector bundle $V = F \oplus G$. We regard F as a subbundle of V via the graph of φ, with quotient G. By the universal property of the Grassmann variety there exists a unique morphism $\delta\colon X \to \mathrm{Gr}(d,V)$ such that $G \simeq \delta^*(Q)$ where Q denotes the universal quotient bundle on the Grassmann variety. Now by the splitting principle we may assume that the given flag extends to a flag

$$V_*: \quad V_1 \subset \cdots V_c \subset V_{c+1} \subset \cdots V_d = V \quad \text{with} \quad \mathrm{rk}(V_j) = e + j \quad \text{for} \quad j = c+1, \dots, d.$$

Then the associated Schubert variety $\Sigma_{V_*} \subset \mathrm{Gr}(d,V)$ satisfies

$$D(\varphi, V_*) \; = \; \delta^{-1}(\Sigma_{V_*}).$$

One can show that Σ_{V_*} is Cohen-Macaulay of the right codimension, and that this implies that the fundamental class of the scheme-theoretic inverse image coincides with the pullback between Chow rings in the sense that $[D(\varphi, V_*)] = \delta^*[\Sigma_{V_*}]$. Hence the result follows from the previous theorem. □

Corollary 4.10 (Porteous' formula). *Let* $\varphi\colon F \to G$ *be a morphism between vector bundles on a smooth quasiprojective variety, and denote their ranks by* $\mathrm{rk}(F) = d$ *and* $\mathrm{rk}(G) = e$. *For* $0 \le r \le e$ *the degeneracy locus*

$$D_r(\varphi) \; = \; \{x \in X \mid \mathrm{rk}(\varphi(x)) \le r\} \; \subset \; X$$

is either empty or of codimension $\le (d-r)(e-r)$. *If it is empty or of codimension equal to* $(d-r)(e-r)$, *then its fundamental class is*

$$[D_r(\varphi)] \; = \; \Delta_{e-r,\dots,e_r}(c(G-F))$$

where the index $(e-r,\dots,e-r)$ *consists of precisely* $d-r$ *parts.*

Proof. By the splitting principle we may assume that there exists a flag

$$F_1 \; \subset \; \cdots \; \subset \; F_{d-r} \; = \; F \quad \text{with} \quad a_i \; = \; \mathrm{rk}(F_i) \; = \; r+i.$$

Then $\mathrm{rk}(\varphi(x)) \le r$ iff $\mathrm{rk}(\varphi(x)|_{F_i(x)}) \le a_i - i = r$ for all i, which means that

$$D_r(\varphi) \; = \; D(\varphi, F_*)$$

and we can apply the previous theorem with $\lambda_i = e - a_i + i = e - r$ for all i. In the case where the degeneracy locus is empty or of the expected dimension, we get

$$[D_r(\varphi)] \; = \; \Delta_{e-r,\dots,e-r}(c(G-F_1),\dots,c(G-F_1) \; = \; \Delta_{e-r,\dots,e-r}(c(G-F))$$

where the last equality uses the identity between the two determinantal formulas for the resultant in lemma 4.5. □