Let k be an algebraically closed field.

Problem 1.1. Let $C \subset \mathbb{P}^2_k$ be an integral curve of degree d. Show that C has at most $r = \binom{d-1}{2}$ singular points:

(a) First reduce to the case $d \geq 3$.

(b) Then argue by contradiction: Pick $r+1$ singular points and $d-3$ other distinct points on the curve. Show by a dimension count that there exists a curve of degree $d-2$ passing through all the chosen points, and deduce a contradiction to Bézout's theorem.

Problem 1.2. Let $i_p(f,g) := \dim_k \mathscr{O}_{\mathbb{A}^2,p}/(f,g)$ be the intersection multiplicity of $f,g \in k[x,y]$ at $p \in \mathbb{A}^2(k)$ as in the lecture.

(a) Show that $i_p(f, g+fh) = i_p(f,g)$ and $i_p(f,gh) = i_p(f,g) + i_p(f,h)$ for all f,g,h.

(b) Find an algorithm that computes in finitely many steps the intersection multiplicity of arbitrary polynomials $f,g \in k[x,y] \setminus \{0\}$ at the origin $p = (0,0)$, assuming that the two curves $V(f), V(g)$ have no common components passing through the origin.

Problem 1.3. Let X be a variety with normalization $\pi\colon \tilde{X} \to X$, and $Z \subset X$ a codimension one subvariety.

(a) Show that

$$\operatorname{ord}_Z(f) \;=\; \sum_{\tilde{Z} \subset \tilde{X}} [k(\tilde{Z}) : k(Z)] \cdot \operatorname{ord}_{\tilde{Z}}(f \circ \pi) \quad \text{for all} \quad f \;\in\; k(X)^\times$$

where the sum runs over all subvarieties $\tilde{Z} \subset \tilde{X}$ that dominate the subvariety $Z \subset X$.

(b) Illustrate the above formula by looking at a cuspidal and a nodal cubic in the plane.

Problem 1.4. Recall that the *Borel-Moore homology* $H_*^{\mathrm{BM}}(X)$ of a topological space X is by definition the homology of the complex consisting of all locally finite formal $\mathbb{Z}$-linear combinations of singular simplices in X. Here a linear combination is called *locally finite* if for any compact $K \subset X$ it involves only finitely many simplices whose image intersects K.

(a) Compute $H_*^{\mathrm{BM}}(X)$ for the real line $X = \mathbb{R}$ (with the usual topology).

(b) Show that Borel-Moore homology forms a covariant functor for *proper* maps, but that there is no way to make it a covariant functor for *all* continuous maps.

Intersection Theory — Problem Set 02

HU Berlin, Prof. Dr. T. Krämer — Summer Term 2023

Problem 2.1. Let $X \subset \mathbb{P}^2$ be an irreducible nodal cubic.

(a) Show that $\mathscr{O}_X(p) \not\simeq \mathscr{O}_X(q)$ for any two smooth points $p \neq q$ on C.

(b) Deduce that $c_1 \colon \operatorname{Pic}(X) \longrightarrow A_1(X)$ is not injective. Can you describe its kernel?

Problem 2.2. Let $X \subset \mathbb{P}^3$ be a quadric cone with vertex p.

(a) Let $C \subset X$ be a curve of odd degree as a curve in $\mathbb{P}^3$. Show that if C meets a general line of the ruling of X at δ points away from p and has multiplicity m at p, then $\deg(C) = 2\delta + m$. Deduce that the cycle $[C]$ cannot be a Cartier divisor.

(b) Show that $A_1(X) \simeq \mathbb{Z}$, generated by the class of a line, and determine the image of the homomorphism

$$c_1 \colon \quad \operatorname{Pic}(X) \longrightarrow A_1(X).$$

Problem 2.3. Consider a Cartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\;i\;} & X \end{array}$$

where i is a closed immersion, p is proper and $p \colon X' \setminus Y' \longrightarrow X \setminus Y$ is an isomorphism. Show that for each $d \geq 0$ we have an exact sequence

$$A_d(Y') \longrightarrow A_d(Y) \oplus A_d(X') \longrightarrow A_d(X) \longrightarrow 0.$$

Problem 2.4. Let $f \colon X' \to X$ be a finite birational morphism of n-dimensional varieties. For each codimension one subvariety $V \subset X$, let d_V be the greatest common divisor of the degrees $\deg(W/V)$, where $W \subset X'$ runs through all subvarieties with $f(W) = V$. Show that we have an exact sequence

$$A_{n-1}(X') \longrightarrow A_{n-1}(X) \longrightarrow \bigoplus_V \mathbb{Z}/d_V\mathbb{Z} \longrightarrow 0.$$

Use this to determine the Chow groups of

(a) the Whitney umbrella $X = V(x^2 - yz^2) \subset \mathbb{A}^3$ using a suitable $f \colon \mathbb{A}^2 \to X$.

(b) the scheme $X = \operatorname{Spec} k[s^n, st, s^2t, \ldots, s^{n-1}t, t]$ seen as a quotient of $\mathbb{A}^2 = \operatorname{Spec} k[s,t]$.

Problem 3.1. Let Y be a quasi-projective variety with an action of a finite group G.

(a) Show that every $p \in Y$ has an affine open neighborhood $U = \mathrm{Spec}(A)$ which is stable under the group action. Use this fact to show that there is a quotient variety $X = Y/G$ with a morphism $\pi\colon Y \to X$ such that

$$\mathrm{Hom}(X, W) \xrightarrow{\sim} \mathrm{Hom}(Y, W)^G \quad \text{for any scheme } W$$

where we put $\mathrm{Hom}(Y, W)^G := \{f \in \mathrm{Hom}(Y, W) \mid f \circ g = f \text{ for all } g \in G\}$.

(b) Show that for the Chow groups with rational coefficients, the morphism π induces an isomorphism

$$\pi^*\colon \quad A_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} (A_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q})^G$$

where the right hand side denotes the invariants with respect to the action of G.

Problem 3.2. A cycle on a scheme X is called *effective* if every subvariety $Z \subset X$ enters in it with a nonnegative coefficient. Show that for any $\alpha, \beta \in Z_d(X)$ the following are equivalent:

(a) The cycles α and β are rationally equivalent.

(b) There are *effective* cycles $\gamma \in Z_d(X)$, $Z \in Z_{d+1}(X \times \mathbb{P}^1)$ such that all components of V dominate $\mathbb{P}^1$ via the projection on the second factor and the fibers over $0, \infty \in \mathbb{P}^1$ are given by

$$Z_0 = \alpha + \gamma \quad \text{and} \quad Z_\infty = \beta + \gamma.$$

Problem 3.3. Let X be a projective scheme over an algebraically closed field.

(a) Show that two zero-cycles $\alpha, \beta \in Z_0(X)$ are rationally equivalent iff for some $n \in \mathbb{N}$ there is a morphism

$$f\colon \quad \mathbb{P}^1 \longrightarrow \mathrm{Sym}^n(X) = X^n/\mathfrak{S}_n$$

such that $f(0) = \alpha + \gamma$ and $f(\infty) = \beta + \gamma$ for some effective zero-cycle $\gamma \in Z_0(X)$.

(b) Show that even if α, β are effective, the cycle γ cannot be omitted in the above: Let C be a smooth non-rational curve, fix a closed point $p \in C$, and choose α, β to be suitable points on the blowup

$$X = \mathrm{Bl}_{(p,0)}(C \times \mathbb{P}^1).$$

Problem 4.1. Let X, Y be schemes.

(a) Show that we have a well-defined homomorphism

$$\times\colon \quad A_d(X) \otimes A_e(Y) \longrightarrow A_{d+e}(X \times Y), \quad [Z] \times [W] := [Z \times W].$$

(b) Show that if X is a cellular variety, then the homomorphisms

$$\times\colon \quad \bigoplus_{d+e=m} A_d(X) \otimes A_e(Y) \longrightarrow A_m(X \times Y)$$

are surjective for all $m \in \mathbb{N}$. Is this still true without the cellular assumption?

Problem 4.2. Let $r, s \in \mathbb{N}$, and let $X \subset \mathbb{P}^r \times \mathbb{P}^s$ be a subvariety of codimension e.

(a) Let $\alpha, \beta \in A_{r+s-1}(\mathbb{P}^r \times \mathbb{P}^s)$ be the pullback of the hyperplane classes on $\mathbb{P}^r$, $\mathbb{P}^s$ under the two projections. Show that

$$[X] = c_e\alpha^e + c_{e-1}\alpha^{e-1}\beta + \cdots + c_0\beta^e \in A_{r+s-e}(\mathbb{P}^r \times \mathbb{P}^s) \quad \text{for certain} \quad c_i \in \mathbb{N}_0.$$

(b) Show that if X is the graph of a morphism $f\colon \mathbb{P}^r \to \mathbb{P}^s$ given by $s+1$ homogenous polynomials of degree d in $r+1$ variables without common zeroes, then we have $s \geq r$ and

$$c_i = d^i \quad \text{for} \quad i = 0, 1, \ldots, s.$$

(c) Now take $r = s$. How many fixed points does a general $f\colon \mathbb{P}^r \to \mathbb{P}^r$ as above have?

Problem 4.3. Let $\sigma\colon \mathbb{P}^r \times \mathbb{P}^s \hookrightarrow \mathbb{P}^n$ be the Segre embedding, with $n = (r+1)(s+1) - 1$.

(a) What is the degree of the image $\sigma(X) \subset \mathbb{P}^n$ of a subvariety $X \subset \mathbb{P}^r \times \mathbb{P}^s$ as in 4.2(a)?

(b) Deduce that any linear subspace of the image $\Sigma = \sigma(\mathbb{P}^r \times \mathbb{P}^s) \subset \mathbb{P}^n$ is contained in a fiber of one of the projections

$$\mathrm{pr}_1\colon \quad \Sigma \simeq \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^r \quad \text{or} \quad \mathrm{pr}_2\colon \quad \Sigma \simeq \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^s$$

Problem 4.4. Let X be a scheme with a Cartier divisor $D \in \mathrm{Div}(X)$. If $\mathscr{O}_X(D)$ restricts to the trivial line bundle on $|D|$, we can define its intersection with *any* subvariety $Z \subset X$ of dimension d as a *cycle* by

$$Z_{d-1}(|D| \cap Z) \ni D \cdot [Z] := \begin{cases} [i^*(D)] & \text{if } Z \not\subset |D|, \\ 0 & \text{otherwise,} \end{cases}$$

where $i\colon Z \hookrightarrow X$ is the inclusion. Compute $D \cdot [D']$ and $D' \cdot [D]$ on the blowup $X = \mathrm{Bl}_0(\mathbb{A}^2)$ for the effective Cartier divisors $D, D' \subset X$ that are the preimages of the coordinate axes.

Problem 5.1. Let E be a vector bundle of rank 3.

(a) Express the Chern classes of $\mathrm{Alt}^2 E$ via those of E using the splitting principle.

(b) Check your answer by noting that we have a natural isomorphism $\mathrm{Alt}^2 E \simeq E^\vee \otimes \det(E)$.

Problem 5.2. Consider a smooth complete intersection

$$S \;=\; H_1 \cap H_2 \;\subset\; \mathbb{P}^4$$

of hypersurfaces $H_i \subset \mathbb{P}^4$ of degrees $d_i = \deg(H_i)$. Show by looking at the exact sequence of normal bundles

$$0 \to N_{S/H} \to N_{S/\mathbb{P}^4} \to N_{H/\mathbb{P}^4} \to 0$$

that every smooth hypersurface $H \subset \mathbb{P}^4$ containing S must have degree $\deg(S) \in \{d_1, d_2\}$.

Problem 5.3. Let $X \subset \mathbb{P}^m$ be a smooth subvariety. Let $\check{\mathbb{P}}^m$ be the dual projective space, and put

$$\Lambda_X \;=\; \{(x, H) \in X \times \check{\mathbb{P}}^m \mid T_x X \subset H\} \;\subset\; X \times \check{\mathbb{P}}^m$$

Let $p \colon \Lambda_X \to X$ and $f \colon \Lambda_X \to \check{\mathbb{P}}^m$ be the projections. We call $f(\Lambda_X)$ the *dual variety* of X.

(a) Show that for the normal bundle $N = N_{X/\mathbb{P}^m}$ we have an isomorphism $\Lambda_X \simeq \mathbb{P}(N^\vee)$ and

$$f^*(\mathscr{O}_{\mathbb{P}^m}(1)) \;\simeq\; \mathscr{O}_{\mathbb{P}(N^\vee)}(1) \otimes p^*(L) \quad \text{for the line bundle} \quad L \;:=\; \mathscr{O}_{\mathbb{P}^m}(-1)|_X.$$

(b) Deduce that

$$\deg f_*[\Lambda_X] \;=\; (-1)^n \cdot \int_X \frac{c(\mathscr{T}_X)}{(1+h)^2} \quad \text{for the class} \quad h \;=\; c_1(\mathscr{O}_X(1)).$$

(c) Let $X \subset \mathbb{P}^m$ be a smooth hypersurface of degree d, and assume that $f \colon \Lambda_X \to f(\Lambda_X)$ is birational (one can show that this is always true in characteristic zero). Compute the degree of the dual variety

$$f(\Lambda_X) \;\subset\; \check{\mathbb{P}}^m$$

via the above formula. Can you find a more direct geometric argument in this case?

Problem 6.1. Let $m, n \in \mathbb{N}$.

(a) Compute the Chern classes $c_i(\mathbb{P}^m \times \mathbb{P}^n)$ for all $i \in \mathbb{N}$.

(b) Give a formula for the topological Euler characteristic $\chi_{\mathrm{top}}(X(\mathbb{C}))$ when $X \subset \mathbb{P}^m \times \mathbb{P}^n$ is a smooth hypersurface of bidegree (d, e) over the complex numbers.

Problem 6.2. Let E and F be two vector bundles of rank r and s on a scheme X.

(a) Show that

$$
\begin{aligned}
c_1(E \otimes F) &= rc_1(F) + sc_1(E), \\
c_2(E \otimes F) &= rc_2(F) + sc_2(E) + \tbinom{r}{2}c_1(F)^2 + \tbinom{s}{2}c_1(E)^2 + (rs-1)c_1(E)c_1(F).
\end{aligned}
$$

(b) Optional: Try to get formulas in higher degree using a computer algebra system.

Problem 6.3. Let $X = \mathrm{Gr}(2, V)$ be the Grassmann variety parametrizing planes in $V = \mathbb{A}^4$.

(a) Let $S \subset V \otimes \mathscr{O}_X$ be the universal subbundle of rank two, and let Q be the quotient in the exact sequence

$$0 \longrightarrow S \longrightarrow V \otimes \mathscr{O}_X \longrightarrow Q \longrightarrow 0.$$

Express the Chern classes of Q and of the dual $S^\vee$ in terms of fundamental classes of the loci

$$
\begin{aligned}
X_3 &:= \{W \in X \mid \langle e_3, e_4 \rangle \cap W \neq 0\}, \\
X_2' &:= \{W \in X \mid e_4 \in W\}, \\
X_2'' &:= \{W \in X \mid W \subset \langle e_2, e_3, e_4 \rangle\} \quad \text{for the standard basis vectors } e_i \in V.
\end{aligned}
$$

You may use without proof the interpretation of Chern classes of globally generated vector bundles as linear dependence loci of generic sections (see the first lecture).

(b) Show that $\mathscr{T}_X \simeq S^\vee \otimes Q$ and use this to compute the Chern classes $c_1(X)$ and $c_2(X)$.

(c) Compare your result with the Chern classes of a smooth quadric hypersurface in $\mathbb{P}^5$.

Problem 7.1. Let E be a vector bundle and C a cone, both over a scheme X.

(a) Observe that the fiber product $C \times_X E \to X$ is again a cone.

(b) Let $p\colon \mathbb{P} = \mathbb{P}(C \times_X E) \to X$ be the associated projective cone. Show that there is a section

$$\sigma \in H^0(\mathbb{P}, p^*(E) \otimes \mathscr{O}_{\mathbb{P}}(1)) \quad \text{with zero locus} \quad V(\sigma) = \mathbb{P}(C),$$

and deduce that for any C we have

$$s(C \times_X E) = c(E)^{-1} \cap s(C).$$

Problem 7.2. Let E be a vector bundle and $C \subset E$ an equidimensional closed subcone.

(a) Let $r = \mathrm{rk}(E)$, $e = \dim(C)$, and let $\sigma\colon X \hookrightarrow E$ be the zero section. Deduce from the Gysin formula that

$$\sigma^*[C] = \{c(E) \cap s(C)\}_{e-r}\,.$$

(b) Now let X be smooth, and let $E = T_X^\vee$ be its cotangent bundle. Let $Y \subset X$ be a smooth subvariety. Show that

$$\sigma^*[N_{Y/X}^\vee] = (-1)^d \cdot c_d(Y) \cap [Y] \quad \text{where} \quad d = \dim(Y).$$

Over the complex numbers this gives another view on the Euler characteristic $\chi_{\mathrm{top}}(Y)$.

Problem 7.3. Determine the following normal cones:

(a) The cones $C_Y X$ and $C_Y \mathbb{A}^2$ where $Y = V(x^2, xy) \subset X = V(xy) \subset \mathbb{A}^2 = \operatorname{Spec} k[x, y]$.

Can you explain your result geometrically?

(b) The cone $C_Z \mathbb{A}^3$ for $Z = V(xz, yz) \subset \mathbb{A}^3 = \operatorname{Spec} k[x, y, z]$.

Here $Z = L \cup H$ for a line L and a plane H, but $C_L \mathbb{A}^3 \to C_Z \mathbb{A}^3$ is not an embedding.

Problem 8.1. Let $Y = V(w,x), Z = V(y,z) \subset X = \operatorname{Spec} k[w,x,y,z]/(wz - xy)$.

(a) Compute the codimensions of Y, Z and $Y \cap Z$ in the scheme X.

(b) Deduce that the diagonal $\Delta \hookrightarrow X \times X$ is not a regular embedding.

Problem 8.2. Let $Y = V_+(x) \subset Z = V_+(xy) \subset X = \mathbb{P}^2$.

(a) Compute the normal cones $C_{Z \cap Y} Y$ and $C_{Y \cap Z} Z$.

(b) Determine the cycles $Z \cdot Y$ and $Y \cdot Z$ directly from the definitions.

Problem 8.3. Let C be a smooth projective curve of genus g. Let $C_d = (C \times \cdots \times C)/\mathfrak{S}_d$ be its d-th symmetric power, parametrizing degree d effective divisors on the curve. Fix $p_0 \in C$ and consider the map

$$u_d\colon \quad C_d \longrightarrow A = \operatorname{Pic}^0(C), \quad D \mapsto \mathscr{O}_C(D - dp_0).$$

You may assume without proof that

- the fibers of u_d are linear series $|D| = \{\text{effective divisors linearly equivalent to } D\} \simeq \mathbb{P}^r$,
- for $d > 2g - 2$ the morphism $C_d \to A$ is a projective bundle over A,
- for $1 \leq d \leq g$ the morphism $C_d \to A$ is birational onto its image $W_d = u_d(C_d) \subset A$.

Starting from the case $d \gg 0$, show that for $D \in C_d$ the Segre class of $|D| \subset C_d$ is given by the formula

$$s(|D|, C_d) = (1+h)^{g-d+r} \cap [|D|]$$

where h denotes the first Chern class of the canonical line bundle on $|D|$ and $r = \dim|D|$.

Deduce by functoriality of Segre classes that the multiplicity of the subscheme $W_d \subset A$ at a point $u_d(D)$ is given by

$$e = \binom{g-d+r}{r} \quad \text{for} \quad r = \dim|D|.$$

Problem 9.1. Let $p\colon Y = \mathbb{P}(E) \to X$ be a projective bundle.

(a) Show that the decomposition $A^*(Y) \simeq \bigoplus_{i=0}^r A^*(X) \cdot \zeta^i$ depends on E.

(b) However, show that for the group homomorphisms

$$\psi_i\colon \quad A^*(Y) \longrightarrow A^*(X)^{\oplus(i+1)} \quad \text{with} \quad \psi_i(y) := (\pi_*(y), \pi_*(\zeta y), \dots, \pi_*(\zeta^i y)),$$

the filtration $A^*(Y) \supset \ker(\psi_0) \supset \cdots \supset \ker(\psi_r) = 0$ is independent of the chosen E.

Problem 9.2. Let $X = \mathrm{Gr}(2, V)$ be the Grassmannian of planes in $V = \mathbb{A}^4$.

(a) Determine the intersection product on the Chow ring $A^*(X)$.

(b) Consider the flag variety $F = \{(p, H) \in \mathbb{P}(V) \times \mathrm{Gr}(2, V) \mid p \in \mathbb{P}(H)\}$. Show that

- $\mathrm{pr}_1\colon F \to \mathbb{P}(V)$ is a $\mathbb{P}^2$-bundle,
- $\mathrm{pr}_2\colon F \to \mathrm{Gr}(2, V)$ is a $\mathbb{P}^1$-bundle,

and use this to compute the Chow ring $A^*(F)$ in two ways. Compare your two results.

Problem 9.3. Let $S_1, S_2 \subset \mathbb{P}^4$ be two smooth surfaces of degree d_1, d_2 whose intersection is a disjoint union

$$S_1 \cap S_2 \;=\; C \sqcup \Gamma$$

where $C \subset \mathbb{P}^4$ is a smooth curve of degree d and genus g, and Γ is a finite reduced subscheme.

(a) Deduce from the excess intersection formula that the number of points in Γ is

$$\deg(\Gamma) \;=\; d_1 d_2 - 2g + 2 - 5d + \sum_{i=1}^{2} \deg\, [C]^2_{S_i}$$

where $[C]^2_{S_i}$ denotes the self-intersection of the curve C inside the Chow ring $A^*(S_i)$.

(b) What does this formula say when C is a line? Check it explicitly in the following two cases:

(1) S_1 is a smooth surface in a hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing a line ℓ, and S_2 is a general plane containing that line.

(2) S_1 and S_2 are smooth quadrics.