1. The finiteness theorem for $h^1(X, \mathscr{E})$

The main point in the proof of the Riemann-Roch theorem for holomorphic vector bundles E on a compact Riemann surface X is that the sheaf $\mathscr{E} = \mathscr{O}_X(E)$ satisfies

$$h^i(X, \mathscr{E}) \; := \; \dim_{\mathbb{C}} H^i(X, \mathscr{E}) \; < \; \infty \quad \text{for} \quad i \; = \; 0, 1.$$

For the proof of this finiteness result we follow the very clear and concise exposition by Narasimhan in [2, sect. 7]. In class we have already treated the case $i = 0$ by a simple argument using the maximum principle. For $i = 1$ we rely on the following consequence of the open mapping theorem in functional analysis:

Proposition 1.1. *If $f : X \twoheadrightarrow Y$ is a surjective compact operator between Banach spaces, then*

$$\dim Y \; < \; \infty.$$

Proof. Consider the unit ball $B = \{x \in X \mid \|x\| < 1\}$. The image $f(B) \subset Y$ is relatively compact since f is a compact operator. On the other hand, the open mapping theorem in functional analysis says that any surjective continuous linear operator between Banach spaces is an open mapping, hence the image $f(B) \subset Y$ is also open and therefore a relatively compact neighborhood of the origin. But it is a general fact (exercise) that $\dim Y < \infty$ iff the origin $0 \in Y$ admits a relatively compact open neighborhood. $\square$

We now return to complex analysis. For an open $U \subset \mathbb{C}$ let $\mathscr{O}_b(U) \subset \mathscr{O}(U)$ be the vector subspace of all *bounded* holomorphic functions. This is a Banach space with respect to the norm $\|f\| := \sup_{x \in U} |f(x)|$. A sequence $f_1, f_2, \dots \in \mathscr{O}_b(U)$ is called

- *uniformly bounded* if there is a constant $C > 0$ with $\|f_i\| < C$ for all i.
- *uniformly convergent* to a function $f \in \mathscr{O}_b(U)$ if $\lim_{n \to \infty} \|f_n - f\| = 0$.
- *locally uniformly convergent* if for every relatively compact open $U_0 \Subset U$ the restrictions $f_n|_{U_0} \in \mathscr{O}_b(U_0)$ converge uniformly on U_0.

Locally uniform convergence implies pointwise convergence to a holomorphic limit but this convergence need not be uniform; for example, consider $f_n(z) = z^n$ on the open disk $U = \{z \in \mathbb{C} \mid |z| < 1\}$. We need the following powerful result from complex analysis:

Theorem 1.2 (Montel)**.** *Any uniformly bounded sequence $f_1, f_2, \dots \in \mathscr{O}_b(U)$ has a locally uniformly convergent subsequence. Thus for every relatively compact open subset $U_0 \Subset U$ the map $\mathscr{O}_b(U) \to \mathscr{O}_b(U_0), f \mapsto f|_{U_0}$ is a compact operator.*

The proof is elementary and can be found for instance in [1, th. IV.4.9]. We then obtain

Theorem 1.3. $h^1(X, \mathscr{E}) < \infty$.

Proof. The proof essentially combines proposition 1.1 and theorem 1.2, but the Cech description for cohomology makes it a bit technical and so we divide it into several steps:

Step 1. Fixing a nice cover. Put $D(r) = \{z \in \mathbb{C} \mid |z| < r\}$ for $r > 0$. We can find an open cover

$$X \; = \; \bigcup_{i=1}^{N} U_i \quad \text{with charts} \quad z_i : \quad U_i \xrightarrow{\sim} D(2)$$

on which the given vector bundle becomes trivial, and on each chart we choose a trivialization

$$h_i: \quad E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^n$$

Putting $U_i(r) = z_i^{-1}(D(r))$, we can furthermore assume that shrinking our cover we still have

$$X \;=\; \bigcup_{i=1}^{N} U_i(1/2),$$

which will leave enough space for arguments requiring relative compactness.

Step 2. The bounded Cech complex. We now introduce a variant of the Cech complex using only bounded holomorphic sections. For $U \Subset V \subset X$ open with a trivialization $h = (h_1, \dots, h_n) : E|_V \simeq V \times \mathbb{C}^n$ we define the space of bounded sections

$$\mathscr{E}_b(U) \;=\; \Big\{ s \in \mathscr{E}(U) \mid h_i(s) \in \mathscr{O}_b(U) \text{ for } i = 1, \dots, n \Big\}.$$

Note that this definition does not depend on the chosen trivialization h or V as long as $U \Subset V$ is relatively compact in the latter. We apply this to the intersections of open subsets in the cover

$$\mathfrak{U}(r) \;=\; \big(U_i(r)\big)_{i=1,\dots,N} \quad \text{for} \quad r \;\in\; (1/2, 2).$$

Consider the bounded Cech complex

$$\begin{aligned}
\mathscr{C}^0(r) \;&=\; \{\xi = (\xi_i) \in \mathscr{C}^0(\mathfrak{U}(r), \mathscr{E}) \mid \xi_i \in \mathscr{E}_b(U_i(r)) \text{ for all } i\} \\
{\scriptstyle \delta^0} \big\downarrow \; & \\
\mathscr{C}^1(r) \;&=\; \{\eta = (\eta_{ij}) \in \mathscr{C}^1(\mathfrak{U}(r), \mathscr{E}) \mid \eta_{ij} \in \mathscr{E}_b(U_{ij}(r)) \text{ for all } i, j\} \\
{\scriptstyle \delta^1} \big\downarrow \; & \\
\vdots \;\; &
\end{aligned}$$

and put

$$\mathscr{Z}^1(r) \;=\; \ker(\delta^1) \;\subseteq\; \mathscr{C}^1(r).$$

Then

- $\mathscr{C}^0(r)$ is a Banach space with $\|\xi\|_r = \max_i \sup_{x \in U_i(r)} |(h_i \xi_i)(x)|$,
- $\mathscr{Z}^1(r)$ is a Banach space with $\|\eta\|_r = \max_{i,j} \sup_{x \in U_{ij}(r)} |(h_i \eta_{ij})(x)|$.

The comparison of these norms for various radii r will be the main next point.

Step 3. Bounded versus usual Cech cohomology. Comparing the above bounded with the usual Cech complex, we claim that for any $r \in (1/2, 1)$ the following properties hold:

(a) We have natural isomorphisms

$$H^1(r) := \mathscr{Z}^1(r)/\delta^0(\mathscr{C}^0(r)) \xrightarrow[\varphi]{\sim} H^1(\mathfrak{U}(r), \mathscr{E}) \xrightarrow{\sim} H^1(X, \mathscr{E}).$$

(b) The composition $\psi : \mathscr{Z}^1(1) \longrightarrow \mathscr{Z}^1(r) \twoheadrightarrow H^1(r)$ is surjective.

Indeed, the second isomorphism in part (a) comes from the Leray theorem as $\mathfrak{U}(r)$ is an acyclic cover. Furthermore, shrinking the radius from 2 to r induces a natural map $H^1(\mathfrak{U}(2), \mathscr{E}) \to H^1(\mathfrak{U}(r), \mathscr{E})$ on Cech cohomology, and since the $U_i(r) \Subset U_i(2)$

are relatively compact, this map factors over $H^1(r)$. So we obtain the following diagram:

$$\begin{tikzcd}
H^1(\mathfrak{U}(2),\mathscr{E}) \arrow[r, dotted, "\exists"] \arrow[d, "\simeq"'] & H^1(r) \arrow[r, "\varphi"] & H^1(\mathfrak{U}(r),\mathscr{E}) \arrow[d, "\simeq"] \\
H^1(X,\mathscr{E}) \arrow[rr, equal] & & H^1(X,\mathscr{E})
\end{tikzcd}$$

Since the lower row is the identity, it follows that the morphism φ from part (a) is surjective, and similarly ψ from (b) is surjective.

It remains to check that φ is injective. Thus we want to show: If $\xi \in \mathscr{C}(\mathfrak{U}(r),\mathscr{E})$, then

$$\delta^0(\xi) \in \mathscr{Z}^1(r) \quad \Longrightarrow \quad \xi \in \mathscr{C}^0(r),$$

i.e. boundedness of the differential of a cochain implies that the cochain itself was bounded. This latter statement follows from the more precise estimate that for $1/2 < \rho < r < 1$ there exists a constant $C = C(r,\rho) > 0$, independent of ξ, such that

$$\|\xi\|_r \;\leq\; \|\delta^0(\xi)\|_r \;+\; C \cdot \|\xi\|_\rho.$$

To verify this last inequality, consider any point $x \in U_i(r)$, pick j with $U_j(\rho) \ni x$ and write

$$\xi_i(x) \;=\; (\xi_i - \xi_j)(x) + \xi_j(x) \;=\; \delta^0(\xi)_{ij}(x) + \xi_j(x).$$

Then

$$|(h_i\xi_i)(x)| \;\leq\; |(h_i\delta^0(\xi)_{ij})(x)| + |(h_i\xi_j)(x)|$$

and so the desired inequality follows with

$$C \;=\; \sup_{x\in U_{ij}(r)} \|(h_i h_j^{-1})(x)\|,$$

the supremum over the pointwise operator norms of the transition matrices.

Step 4. Cech cohomology as a Banach space. We next claim $\delta^0(\mathscr{C}^0(r)) \subseteq \mathscr{Z}^1(r)$ is a closed subspace, hence

$$H^1(r) \;:=\; \mathscr{Z}^1(r)/\delta^0(\mathscr{C}^0(r))$$

inherits from $\mathscr{Z}^1(r)$ the structure of a Banach space. Indeed, let $1/2 < \rho < r < 1$ and for $N \in \mathbb{N}$ put

$$\mathscr{C}^0(r,N) \;:=\; \left\{\xi \in \mathscr{C}^0(r) \mid \mathrm{ord}_{a_i}(\xi_i) \geq N\right\}$$

where $a_i \in U_i(r)$ denotes the point corresponding to the origin, i.e. $z_i(a_i) = 0$. As in the proof of the finiteness theorem for $h^0(X,\mathscr{E})$, it follows from the maximum principle that

$$\|\xi\|_\rho \;\leq\; \left(\tfrac{\rho}{r}\right)^N \cdot \|\xi\|_r \quad \text{for} \quad \xi \;\in\; \mathscr{C}^0(r,N).$$

So the inequality in step 3 gives

$$\|\xi\|_r \;\leq\; \|\delta^0\xi\|_r + C \cdot \left(\tfrac{\rho}{r}\right)^N \cdot \|\xi\|_r$$

and therefore

$$\|\xi\|_r \;\leq\; 2 \cdot \|\delta^0\xi\|_r \quad \text{for} \quad N \gg 0.$$

It follows that

$$\delta^0(\mathscr{C}^0(r,N)) \;\hookrightarrow\; Z^1(r) \quad \text{is closed,}$$

hence

$$H^1(r,H) \;:=\; Z^1(r)/\delta^0(\mathscr{C}^0(r,N)) \quad \text{is a Banach space.}$$

But

$$\dim \mathscr{C}^0(r)/\mathscr{C}^0(r,N) < \infty \quad \Longrightarrow \quad \delta^0(\mathscr{C}^0(r)) \;\subseteq\; H^1(r,N) \text{ is closed,}$$

since any finite-dimensional subspace of a normed vector space is closed. Therefore the claim of step 4 follows.

Step 5. An application of Montel's theorem. We now put everything together; since $U_i(r) \cap U_j(r) \Subset U_i(1) \cap U_j(1)$ is relatively compact, Montel's theorem 1.2 shows that

$$\mathscr{Z}^1(1) \;\longrightarrow\; \mathscr{Z}^1(r)$$

is a compact operator for $r < 1$. The composite

$$\mathscr{Z}^1(1) \;\longrightarrow\; \mathscr{Z}^1(r) \;\twoheadrightarrow\; H^1(r) \;\simeq\; H^1(X,\mathscr{E})$$

is then a compact and surjective operator, hence it follows that $\dim H^1(X,\mathscr{E}) < \infty$ by proposition 1.1. □

References

[1] E. Freitag and R. Busam, *Complex Analysis*, Springer (2005).
[2] R. Narasimhan, *Compact Riemann Surfaces*, Lectures in Math. ETH Zürich, Springer (1992).