Tropical methods in diophantine geometry

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June 20, 2008

Main references

- [Gu1] W. Gubler: Tropical varieties for non-archimedean analytic spaces. Invent. Math. 169, 321–376 (2007)
- [Gu2] W. Gubler: The Bogomolov conjecture for totally degenerate abelian varieties. Invent. Math. 169, 377–400 (2007)
- [Gu3] W. Gubler: Non-archimedean canonical measures on abelian varieties. ArXiv(2008)
- [Gu4] W. Gubler: Equidistribution over function fields. ArXiv (2008)

- A complete list of references is located at the end of the talk.
- In the frametitle, there is usually a reference where one finds additional material.

Notation

- Let K be a field with a non-trivial non-archimedean absolute value $|\ |$.
- $v(\ ) := -\log|\ |$ is the associated valuation.
- The valuation ring $K^\circ := \{\alpha \in K \mid v(\alpha) \geq 0\}$ has the unique maximal ideal $K^{\circ\circ} := \{\alpha \in K \mid v(\alpha) > 0\}$ and residue field $\widetilde{K} := K^\circ / K^{\circ\circ}$.
- We have completion K_v and algebraic closure $\overline{K}$ of K.
- $\mathbb{C}_K := (\overline{K_v})_v$ is the smallest algebraically closed field extension of K which is complete with respect to an extension of $|\ |$ to a complete absolute value.
- By abuse of notation, we use also v and $|\ |$ on $\mathbb{C}_K$.
- Let κ be the residue field of $\mathbb{C}_K$. One can easily show that κ is algebraically closed.

Tate-algebra [BGR,Ch.5]

- All analytic considerations will be done over $\mathbb{C}_K$.
- Idea: Proceed as in the theory of affine varieties or complex spaces.
- For $f = \sum a_m x^m \in \mathbb{C}_K[x_1, \ldots, x_n]$, we have the Gauss-norm

$$|f| := \sup |a_m|.$$

By the Gauss-Lemma, this is a multiplicative norm.

Definition

The completion $\mathbb{C}_K\langle x_1, \ldots, x_n\rangle$ of $\mathbb{C}_K[x_1, \ldots, x_n]$ with respect to the Gauss-norm is called the *Tate-algebra*.

The elements of $\mathbb{C}_K\langle x_1, \ldots, x_n\rangle$ are the power series $f = \sum a_m x^m$ characterized by $\lim_{|m|\to\infty} |a_m| = 0$. They are called strictly convergent on the closed cube $\mathbb{B}^n := \{\alpha \in \mathbb{C}_K^n \mid |\alpha| \le 1\}$. Here, $|m|$ and $|\alpha|$ are the max-norms.

Affinoid algebras [BGR,Ch.6]

Definition

A $\mathbb{C}_K$-algebra $\mathscr{A}$ is called an *affinoid algebra* if there is an ideal I in $\mathbb{C}_K\langle x_1, \ldots, x_n\rangle$ with $\mathscr{A} \cong \mathbb{C}_K\langle x_1, \ldots, x_n\rangle / I$.

$a \in \mathscr{A}$ is an analytic function on $Z(I) := \{\alpha \in \mathbb{B}^n \mid f(\alpha) = 0 \ \ \forall f \in I\}$.

Definition

- The *supremum-seminorm* for $f \in \mathscr{A}$ is $|f|_{\sup} := \sup_{x \in Z(I)} |f(x)|$.
- We get the $(\mathbb{C}_K)^\circ$-algebra $\mathscr{A}^\circ := \{f \in \mathscr{A} \mid |f|_{\sup} \leq 1\}$ with ideal $\mathscr{A}^{\circ\circ} := \{f \in \mathscr{A} \mid |f|_{\sup} < 1\}$ and *residue algebra* $\widetilde{\mathscr{A}} := \mathscr{A}^\circ / \mathscr{A}^{\circ\circ}$.

Example

If $\mathscr{A} = \mathbb{C}_K\langle x_1, \ldots, x_n\rangle$, then $|\ |_{\sup}$ is the Gauss-norm and hence

$$\widetilde{\mathscr{A}} = \kappa[x_1, \ldots, x_n].$$

Properties of affinoid algebras [BGR,Ch.6-7]

Proposition

Similarly to the coordinate ring of an affine variety, the affinoid algebra $\mathscr{A} \cong \mathbb{C}_K\langle x_1, \ldots, x_n\rangle / I$ satisfies the following properties:

- $\mathscr{A}$ is noetherian.
- Hilbert's Nullstellensatz holds.
- $\widetilde{\mathscr{A}}$ is a finitely generated reduced algebra over the residue field κ.
- $\dim(\mathscr{A}) = \dim(\widetilde{\mathscr{A}})$
- The reduction map

$$\pi : Z(I) \to \mathrm{Max}(\widetilde{\mathscr{A}}), x \mapsto \{f \in \mathscr{A}^\circ \mid |f(x)| < 1\}/\mathscr{A}^{\circ\circ}$$

is surjective.

More notation

For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, we use the notation

$$x \cdot y := x_1 y_1 + \cdots + x_n y_n$$

and

$$x^y := x_1^{y_1} \cdots x_n^{y_n}.$$

In the following, $\Gamma := v(\mathbb{C}_K^\times)$ denotes the value group.

Definition

A *polyhedron* Δ in $\mathbb{R}^n$ is a finite intersection of half spaces of the form $\{u \in \mathbb{R}^n \mid u \cdot m \geq c\}$. We call Δ Γ-*rational* if we may choose all $m \in \mathbb{Z}^n$ and all $c \in \Gamma$. A *polytope* is a bounded polyhedron.

The valuation extends to the multiplicative torus by

$$\mathrm{val} : (\mathbb{C}_K^\times)^n \to \mathbb{R}^n, x \mapsto (v(x_1), \ldots, v(x_n)).$$

Polytopal domains [Gu1,§4]

Let Δ be a Γ-rational polytope. We set $U_\Delta := \mathrm{val}^{-1}(\Delta)$ and

$$\mathbb{C}_K\langle U_\Delta\rangle := \left\{ f := \sum_{m\in\mathbb{Z}^n} a_m x^m \,\middle|\, \lim_{|m|\to\infty} |a_m| e^{-u\cdot m} = 0 \;\; \forall u \in \Delta \right\}.$$

By construction, the elements of $\mathbb{C}_K\langle U_\Delta\rangle$ are convergent Laurent series on the *polytopal domain* U_Δ in $(\mathbb{C}_K^\times)^n$. More precisely, we have:

Proposition

$\mathbb{C}_K\langle U_\Delta\rangle$ is an affinoid algebra with supremum norm

$$|f|_{\sup} := \sup_{m\in\mathbb{Z}^n, u\in\Delta} |a_m| e^{-u\cdot m}$$

and maximal spectrum U_Δ.

Berkovich spectra [Ber1,Ch.1]

If X is an affine variety, then $\mathrm{Spec}(K[X])$ is a "compactification" of X. Berkovich has given a similar construction for an affinoid $\mathbb{C}_K$-algebra $\mathscr{A}$:

Definition

The *Berkovich spectrum* $\mathscr{M}(\mathscr{A})$ is the set of multiplicative bounded seminorms p on $\mathscr{A}$, i.e.

- $p : \mathscr{A} \to \mathbb{R}_+$
- $p(a+b) \leq p(a) + p(b)$ for $a, b \in \mathscr{A}$
- $p(\lambda a) = |\lambda| p(a)$ for $\lambda \in \mathbb{C}_K$ and $a \in \mathscr{A}$
- $p(1) = 1$ and $p(ab) = p(a)p(b)$ for $a, b \in \mathscr{A}$
- $p(a) \leq |a|_{\sup}$ for $a \in \mathscr{A}$

We endow $\mathscr{M}(\mathscr{A})$ with the coarsest topology such that the maps $p \mapsto p(a)$ are continuous for all $a \in \mathscr{A}$.

Properties [Ber1,Ch.1-2]

- Multiplicative bounded seminorms p satisfy the ultrametric triangle inequality.
- p induces a non-archimedean absolute value $|\ |$ on the completion $\mathscr{H}(p)$ of the quotient field of $\mathscr{A}/\{a \in \mathscr{A} \mid p(a) = 0\}$ and a bounded character $\chi : \mathscr{A} \to \mathscr{H}(p)$.
- Conversely, every bounded character on $\mathscr{A}$ to a complete extension of $\mathbb{C}_K$ induces a bounded multiplicative seminorm.
  $\Rightarrow$ Analogy to the Gelfand spectrum of a C^*-algebra.
- We have a canonical embedding $Z(I) = \mathrm{Max}(\mathscr{A}) \to \mathscr{M}(\mathscr{A})$, mapping $x \in Z(I)$ to the seminorm $p_x(f) := |f(x)|$.

Theorem

$\mathscr{M}(\mathscr{A})$ is a compactification of $\mathrm{Max}(\mathscr{A})$.

Reduction [Ber1,Ch.2]

Definition

The *reduction* of the Berkovich spectrum $X := \mathscr{M}(\mathscr{A})$ is $\widetilde{X} := \operatorname{Spec}(\widetilde{\mathscr{A}})$.

The reduction map $\pi : Z(I) \to \operatorname{Max}(\widetilde{\mathscr{A}})$ extends to a map $\pi : X \to \widetilde{X}, p \mapsto \{f \in \mathscr{A}^\circ \mid p(f) < 1\}/\mathscr{A}^{\circ\circ}$.

Proposition

- $\pi : X \to \widetilde{X}$ is surjective.
- For every irreducible component Y of $\widetilde{X}$, there is a unique $\xi_Y \in X$ with $\pi(\xi_Y)$ dense in Y.

In fact, $\{\xi_Y \mid Y \text{ irred. comp. of } \widetilde{X}\}$ is the *Shilov boundary* of X, i.e. the minimal subset S of X such that $|f|_{\sup} = \sup_{p \in S} p(f)$ for all $f \in \mathscr{A}$.

Examples

Example

- We redefine the closed unit ball by $\mathbb{B}^n := \mathscr{M}(\mathbb{C}_K\langle x_1, \ldots, x_n\rangle)$. Then $\widetilde{\mathbb{B}^n} = \mathrm{Spec}(\kappa[x_1, \ldots, x_n])$ is the affine n-space over the residue field κ and hence it is irreducible.
- The generic point of the reduction corresponds to $\{0\}$. If $p \in \mathbb{B}^n$ satisfies $\pi(p) = \{0\}$, then $\{f \in \mathscr{A}^\circ \mid p(f) < 1\} = \mathscr{A}^{\circ\circ}$ and hence $p = |\ |_{\sup}$. Obviously, the Gauss-norm is the Shilov-boundary of $\mathbb{B}^n$.

Example

Let $U_\Delta := \mathscr{M}(\mathbb{C}_K\langle U_\Delta\rangle)$ and $u \in \Delta$. We get a multiplicative norm

$$|f|_u := \sup_{m\in\mathbb{Z}^n} |a_m| e^{-u\cdot m}, \quad f = \sum_{m\in\mathbb{Z}^n} a_m x^m \in \mathbb{C}_K\langle U_\Delta\rangle.$$

Obviously, $\{|\ |_u \mid u \text{ vertex of } \Delta\}$ is the Shilov boundary of U_Δ.

Analytic spaces [Ber1], [Ber2]

- The category of Berkovich spectra is antiequivalent to the category of affinoid spaces.
- An *analytic space* X is given by an atlas of Berkovich spectra (see [Ber2], §1, for the precise definition). Technical difficulties arise as the charts are not open in X but compact. We look only at the relevant examples:

Example

The analytic space $(\mathbb{A}^n)^{\mathrm{an}}$ associated to the affine space $\mathbb{A}^n$ is

$$\{p : \mathbb{C}_K[x_1, \ldots, x_n] \to \mathbb{R}^n \mid p \text{ multiplicative seminorm}\}$$

endowed with the coarsest topology such that $p \mapsto p(f)$ is continuous for all $f \in K[x_1, \ldots, x_n]$. The cuboids $\mathbb{B}^n_r := \{p \in (\mathbb{A}^n)^{\mathrm{an}} \mid p(x_i) \leq r_i \ \ \forall i\}$ with $r \in \Gamma^n$ form an atlas.

GAGA principle [Ber1,Ch.3]

Let $X = \mathrm{Spec}(A)$ be a scheme of finite type over K, i.e. $A = K[x_1, \dots, x_n]/I$ for an ideal I.

Definition

The *analytic space* X^{an} associated to X is

$$\{p : A \otimes K^\circ \mathbb{C}_K^\circ \to \mathbb{R}^n \mid p \text{ multiplicative seminorm}\}$$

endowed with the coarsest topology such that $p \mapsto p(f)$ is continuous for all $f \in A$. The charts are given by $\mathbb{B}_r^n \cap X^{\mathrm{an}}$, $r \in \Gamma^n$.

- By a glueing process, we get an analytic space X^{an} associated to every scheme X of finite type over K.
- The complex GAGA theorems hold here as well (e.g. X is separated/proper over $K \iff X^{\mathrm{an}}$ is hausdorff/compact).

Tropicalization [Gu1,§5]

Of major importance for the course is the following:

Example

Let $\mathbb{G}_m^n$ be the multiplicative torus $\mathrm{Spec}(K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. In simpler terms, it is $(K^\times)^n$. Note that

$$\mathrm{val} : (\mathbb{C}_K^\times)^n \to \mathbb{R}^n, x \mapsto (v(x_1), \ldots, v(x_n))$$

extends to a continuous map

$$\mathrm{val} : (\mathbb{G}_m^n)^{\mathrm{an}} \to \mathbb{R}^n, p \mapsto (-\log p(x_1), \ldots, -\log p(x_n)).$$

If X is a closed subscheme of $\mathbb{G}_m^n$, then $\mathrm{val}(X^{\mathrm{an}})$ is called the *tropical variety associated to* X. If $X_{\overline{K}}$ is connected, then X^{an} is connected by GAGA and hence $\mathrm{val}(X^{\mathrm{an}})$ is connected.

Admissible formal schemes [BL2]

Definition

An *admissible formal scheme* $\mathscr{X}$ over $\mathbb{C}_K^\circ$ is a locally finite union of admissible formal affine schemes over $\mathbb{C}_K^\circ$ of the form $\mathrm{Spf}(A)$ for $A \cong \mathbb{C}_K^\circ\langle x_1, \ldots, x_n\rangle/I$ without $\mathbb{C}_K^\circ$-torsion (i.e. A is flat over $\mathbb{C}_K^\circ$).

- $\mathscr{X}$ has a *generic fibre* $\mathscr{X}^{\mathrm{an}}$ which is an analytic space over $\mathbb{C}_K$ locally given by the Berkovich spectrum of $A \otimes_{\mathbb{C}_K^\circ} \mathbb{C}_K$.
- $\mathscr{X}$ has a *special fibre* $\widetilde{\mathscr{X}}$ which is a scheme over κ locally given by $\mathrm{Spec}(A \otimes_{\mathbb{C}_K^\circ} \kappa)$.

Example

The formal completion of the affine space $\mathbb{A}^n$ over $\mathbb{C}_K^\circ$ along the special fibre is $\mathscr{X} := \mathrm{Spf}(\mathbb{C}_K^\circ\langle x_1, \ldots, x_n\rangle)$. Then $\mathscr{X}^{\mathrm{an}} = \mathbb{B}^n$ and $\mathscr{X}^{\mathrm{an}}(\mathbb{C}_K) = \mathbb{A}^n(\mathbb{C}_K^\circ)$.

$\mathbb{C}_K^\circ$-models

- This generalizes to any flat scheme $\mathfrak{X}$ of finite type over $\mathbb{C}_K^\circ$. Then the formal completion $\mathscr{X}$ of $\mathfrak{X}$ along the special fibre is an admissible formal scheme and $\mathscr{X}^{\mathrm{an}}(\mathbb{C}_K)$ is the set of $\mathbb{C}_K^\circ$-integral points of $\mathfrak{X}_K$.
- If $\mathfrak{X}$ is proper over $\mathbb{C}_K^\circ$ (e.g. projective), then $\mathscr{X}^{\mathrm{an}} = (\mathfrak{X}_K)^{\mathrm{an}}$.

Definition

A $\mathbb{C}_K^\circ$*-model* $\mathscr{X}$ for the scheme X of finite type over K is an admissible formal scheme over $\mathbb{C}_K^\circ$ with $X^{\mathrm{an}} = \mathscr{X}^{\mathrm{an}}$ and similarly for line bundles.

As a working definition, you may think about algebraic models which is okay as we deal with projective varieties (formal GAGA-principle).

Strictly semistable models [dJ]

Definition

A $\mathbb{C}_K^\circ$-*model* $\mathscr{X}$ is called *strictly semistable* if $\mathscr{X}$ is covered by formal open subsets $\mathscr{U}$ with an étale morphism

$$\psi : \mathscr{U} \longrightarrow \mathrm{Spf}\,(\mathbb{C}_K^\circ\langle x_0, \dots, x_n\rangle / \langle x_0 \cdots x_r - \pi\rangle)$$

for some $r \leq n$ and $\pi \in \mathbb{C}_K^{\circ\circ}$.

i.e. the special fibre of $\mathscr{X}$ is a divisor with normal crossings in $\mathscr{X}$. The importance of strictly semistable models comes from the *semistable alteration theorem*:

Theorem (de Jong)

If K° is a complete discrete valuation ring, then every variety X over K has a generically finite covering by a variety X' with a strictly semistable $\mathbb{C}_K^\circ$-model.

Strictly semistable examples

Example

Let $\Delta := \{u \in \mathbb{R}^n_+ \mid u_1 + \cdots + u_n \leq v(\pi)\}$ (standard simplex). Then $\mathrm{Spf}((\mathbb{C}_K\langle\Delta\rangle)^\circ) \cong \mathrm{Spf}\,(\mathbb{C}_K^\circ\langle x_0, \ldots, x_n\rangle / \langle x_0 \cdots x_r - \pi\rangle)$ is a strictly semistable $\mathbb{C}_K^\circ$-model for U_Δ with special fibre $x_0 \cdots x_r = 0$ in $(\mathbb{G}_m^n)_\kappa$. Up to étale coverings, these are the building blocks in the definition.

Example

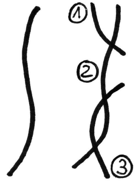

Figure: $\mathscr{X}^{\mathrm{an}}$ and $\widetilde{\mathscr{X}}$ for a strictly semistable model $\mathscr{X}$

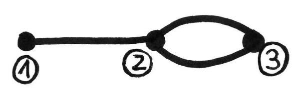

Figure: corresponding dual graph $S(\mathscr{X})$

Skeleton

- Let $(Y_i)_{i \in I}$ be the irreducible components of the special fibre $\widetilde{\mathscr{X}}$ of a strictly semistable $\mathbb{C}_K^\circ$-model $\mathscr{X}$. By definition, they are smooth.
- For $p \geq 1$, let $Y^{(p)} := \bigcup_{J \subset I, |J| = p} \bigcap_{j \in J} Y_j$.
- Then $Y^{(p)} \setminus Y^{(p+1)}$ is smooth and the irreducible components are called *strata* of $\widetilde{\mathscr{X}}$.
- The strata form a partition of $\widetilde{\mathscr{X}}$.

Definition

The *skeleton* $S(\mathscr{X})$ is an abstract simplicial set given as the union of canonical simplices Δ_S which are in bijective correspondence to the strata S of $\mathscr{X}$ subject to the following rules:

- $\overline{S} \subset \overline{T}$ if and only if Δ_T is a closed face of Δ_S. Moreover, every closed face of Δ_S is of this form.
- $\Delta_R \cap \Delta_S$ is the union of all Δ_T with $\overline{R} \cup \overline{S} \subset \overline{T}$.

Realization of the skeleton [Ber3], [Ber4]

We may realize the skeleton as an abstact metrized simplicial set:

- There is a formal affine open covering $\mathscr{U}$ of $\mathscr{X}$ such that we have an étale map ψ from $\mathscr{U}$ to $\mathrm{Spf}\,(\mathbb{C}_K^\circ\langle x_0,\ldots,x_n\rangle/\langle x_0\ldots x_r-\pi\rangle)$.
- By passing to a subcovering, we may assume that $\bigcap_{Y_i\cap\widetilde{\mathscr{U}}\neq\emptyset} Y_i\cap\widetilde{\mathscr{U}}$ is a stratum S. Then $\Delta_S := \{u\in\mathbb{R}_+^{r+1}\mid u_0+\cdots+u_r=v(\pi)\}$.
- The coordinates u_j correspond to Y_i with $Y_i\cap\widetilde{\mathscr{U}}\neq\emptyset$ and hence $S(\mathscr{X})$ may be glued according to the rules.
- There is a canonical $\mathrm{Val}:\mathscr{X}^{\mathrm{an}}\to S(\mathscr{X})$, given on $\mathscr{U}^{\mathrm{an}}$ by $\mathrm{Val}(p) := (-\log p(\psi^* x_0),\ldots,-\log p(\psi^* x_r))\in\Delta_S$.
- Berkovich has shown that the skeleton $S(\mathscr{X})$ may be identified with a subset of $\mathscr{X}^{\mathrm{an}}$ given by certain maximal points.

Theorem (Berkovich)

There is a continuous deformation retraction $d:\mathscr{X}^{\mathrm{an}}\times[0,1]\to\mathscr{X}^{\mathrm{an}}$ with $d(x,0)=x$, $d(x,1)=Val(x)$ and $d(u,t)=u$ for all $u\in S(\mathscr{X}), t\in[0,1]$.

Abelian varieties

- *Abelian varieties* are projective group varieties.
- An abelian variety of dimension 1 is called an *elliptic curve*.

Definition

An abelian variety A over K is called of *potentially good reduction* with respect to v if A^{an} is the generic fibre of an admissible formal group scheme $\mathscr{B}$ over $\mathbb{C}_K^{\mathrm{an}}$ such that $\widetilde{\mathscr{B}}$ is an abelian variety over κ.

- Algebraically, this is equivalent to the existence of an abelian scheme over $\mathbb{C}_K^\circ$ with generic fibre $A_{\mathbb{C}_K}$.
- For an elliptic curve E, this is equivalent to $|j(E)| \leq 1$.

Definition

An abelian variety A has *totally degenerate reduction* with respect to v if A^{an} is isomorphic as an analytic group to $(\mathbb{G}_m^n)^{\mathrm{an}}/M$ for a discrete subgroup M of $\mathbb{C}_K^\times$ such that $\mathrm{val}(M)$ is a lattice in $\mathbb{R}^n$.

Tate elliptic curve [BG,§9.5]

Theorem (Tate)

For an elliptic curve E the following properties are equivalent:

(i) $|j(E)| > 1$.

(ii) $E^{\mathrm{an}} \cong \mathbb{C}_K^\times / q^{\mathbb{Z}}$ *for some* $q \in K^{\circ\circ}$.

(iii) E *is totally degenerate with respect to* v.

Remark

For q in (ii), the elliptic curve E can be defined by the Weierstrass equation $y^2 + xy = x^3 + a_4 x + a_6$ where a_4 and a_6 are convergent power series given by

$$a_4(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad a_6(q) = -\frac{1}{12} \sum_{n=1}^{\infty} \frac{(7n^5 + 5n^3) q^n}{1 - q^n}.$$

Properties

Remark

- The isomorphism $\mathbb{C}_K^\times/q^\mathbb{Z} \to E$ in (ii) is given by the convergent power series

$$x(\zeta, q) = \sum_{n=-\infty}^{\infty} \frac{q^n\zeta}{(1-q^n\zeta)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$$

$$y(\zeta, q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}\zeta^2}{(1-q^n\zeta)^3} + \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

- Furthermore $j(E) = \frac{1}{q} + 744 + 196884q + \dots$
- The reduction of E is given by $y^2 + xy = x^3$.

Raynaud extension [BL1]

For higher dimensional abelian varieties A, a mixture of good reduction and total degeneration is possible. It is given by the *Raynaud extension*

$$1 \to (\mathbb{G}_m^n)^{\mathrm{an}} \to E \to B^{\mathrm{an}} \to 0.$$

- This is a short exact sequence of analytic groups with B an abelian variety of good reduction. We omit the construction which is canonical. E is locally trivial over B^{an} such that the $|x_j|$ are well-defined on E for the coordinates x_j on $(\mathbb{G}_m^n)^{\mathrm{an}}$.
- This leads to a continuous map $\mathrm{val} : E \to \mathbb{R}^n, p \mapsto (-\log p(x_j))$.
- We have a *uniformization* of A, i.e. $A^{\mathrm{an}} \cong E/M$ for a discrete subgroup M of $E(\mathbb{C}_K)$ such that $\mathrm{val}(M)$ is a lattice in $\mathbb{R}^n$.
- If A has potentially good reduction, then $A = B$.
- If A has totally degenerate reduction, then $B = 0$.

Notation

- Let K be a field with a non-trivial non-archimedean absolute value $|\ |$.
- $v(\ ) := -\log|\ |$ is the associated valuation.
- The valuation ring $K^\circ := \{\alpha \in K \mid v(\alpha) \geq 0\}$ has the unique maximal ideal $K^{\circ\circ} := \{\alpha \in K \mid v(\alpha) > 0\}$ and residue field $\widetilde{K} := K^\circ / K^{\circ\circ}$.
- We have the completion K_v and algebraic closure $\overline{K}$ of K.
- $\mathbb{C}_K := (\overline{K_v})_v$ is the smallest algebraically closed field extension of K which is complete with respect to an extension of $|\ |$ to a complete absolute value.
- By abuse of notation, we use also v and $|\ |$ on $\mathbb{C}_K$.
- Let κ be the residue field of $\mathbb{C}_K$.
- Let $\Gamma := v(\mathbb{C}_K^\times)$ be the value group of $\mathbb{C}_K$.

Tropical algebraic geometry

We consider the multiplicative torus $\mathbb{G}_m^n$ with $\mathbb{G}_m^n(\mathbb{C}_K) = (\mathbb{C}_K^\times)^n$ and

$$\mathrm{val} : (\mathbb{C}_K^\times)^n \to \mathbb{R}^n, \quad \mathrm{val}(x_1, \ldots, x_n) = (v(x_1), \ldots, v(x_n)) \ .$$

Let X be a closed algebraic subvariety of $\mathbb{G}_m^n$ and $d := \dim(X)$.

Definition

The closure of $\mathrm{val}(X)$ in $\mathbb{R}^n$ is denoted by $\mathrm{trop}(X)$ and is called the *tropical variety* associated to X.

Theorem (Einsiedler, Kapranov, Lind)

$\mathrm{trop}(X)$ *is a finite connected union of d-dimensional Γ-rational polyhedrons.*

Indeed, we have seen that $\mathrm{trop}(X) = \mathrm{val}(X^{\mathrm{an}})$ which is the set of valuations on $K[X]$ extending v. It was shown by Bieri and Groves that this set has the required properties.

Examples [RST]

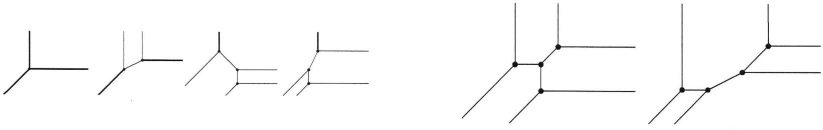

Figure: Plane conics

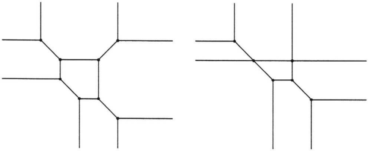

Figure: Plane biquadratic curves

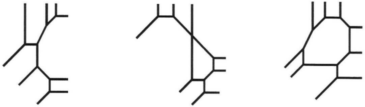

Figure: Plane cubics

Toric varieties [KKMS,Ch.I]

Definition

A $\mathbb{G}_m^n$-*toric variety* over the arbitrary field F is a normal variety Y with an algebraic $\mathbb{G}_m^n$-action containing a dense n-dimensional orbit.

Proposition

There are bijective correspondences between

(a) rational polyhedral cones σ in $\mathbb{R}^n$ which do not contain a linear subspace $\neq \{0\}$,

(b) finitely generated saturated semigroups S in $\mathbb{Z}^n$ which generate $\mathbb{Z}^n$ as a group,

(c) affine $\mathbb{G}_m^n$-toric varieties Y over F (up to equivariant isomorphisms).

They are given by $S = \check{\sigma} \cap \mathbb{Z}^n$ and $Y = \mathrm{Spec}(F[x^S])$, where $\check{\sigma}$ is the dual cone $\{u' \in \mathbb{R}^n \mid u \cdot u' \geq 0 \ \forall u \in \sigma\}$ and $x^S := \{x^m \mid m \in S\}$.

Fans and toric varieties, [KKMS,Ch.I]

Definition

A *rational polyhedral fan* $\mathscr{C}$ in $\mathbb{R}^n$ is a set of rational polyhedral cones such that

(a) $\sigma \in \mathscr{C} \Rightarrow$ all closed faces of σ are in $\mathscr{C}$;

(b) $\sigma, \rho \in \mathscr{C} \Rightarrow \sigma \cap \rho$ is either empty or a closed face of ρ and σ.

(c) No $\sigma \in \mathscr{C}$ contains a linear subspace $\neq \{0\}$.

Remark

- If ρ is a closed face of $\sigma \in \mathscr{C}$, then $\check{\sigma} \subset \check{\rho}$ induced a canonical *open* immersion $\mathrm{Spec}(F[x^{\check{\rho}\cap\mathbb{Z}^n}]) \to \mathrm{Spec}(F[x^{\check{\sigma}\cap\mathbb{Z}^n}])$.
- Hence one can glue the affine toric varieties corresponding to the elements of $\mathscr{C}$ and we get a $\mathbb{G}_m^n$-toric variety.
- Every $\mathbb{G}_m^n$-toric variety is of this form.
- The toric variety is proper over F if and only if $\bigcup_{\sigma\in\mathscr{C}} \sigma = \mathbb{R}^n$.

Reduction of a polytopal domain [Gu1,§4]

- Let Δ be a Γ-rational polytope in $\mathbb{R}^n$. Then we have seen the polytopal domain $U_\Delta := \mathrm{val}^{-1}(\Delta)$ in $(\mathbb{G}_m^n)^{\mathrm{an}}$.
- The affinoid torus $\mathbb{T}_1^{\mathrm{an}}$ acts on U_Δ. It is defined by $\mathbb{T}_1^{\mathrm{an}} := \{p \in (\mathbb{G}_m^n)_{\mathbb{K}}^{\mathrm{an}} \mid p(x_j) = 1 \textit{ for } j = 1, \ldots, n\} = val^{-1}(0)$.
- Passing to reductions, we get a torus action of $(\mathbb{G}_m^n)_\kappa$ on $\widetilde{U}_\Delta$.

Proposition

(a) There is a bijective order reversing correspondence between torus orbits Z of $\widetilde{U_\Delta}$ and open faces τ of Δ, given by $Z_\tau = \pi(\mathrm{val}^{-1}(\tau))$ and $\tau_Z = \mathrm{val}(\pi^{-1}(Z))$.

(b) $\dim(\tau) + \dim(Z_\tau) = n$.

(c) If Y_u is the irreducible component of $\widetilde{U_\Delta}$ corresponding to the vertex u of Δ by (a), then the natural $(\mathbb{G}_m^n)_\kappa$-action of $\widetilde{U_\Delta}$ makes Y_u into an affine toric variety with polyhedral cone generated by $\Delta - u$.

Polytopal decompositions [Gu1,§4]

Definition

A *polytopal decomposition* of $\mathbb{R}^n$ is a locally finite set $\mathscr{C}$ of polytopes with

(a) $\Delta \in \mathscr{C} \Rightarrow$ all closed faces of Δ are in $\mathscr{C}$;

(b) $\Delta, \sigma \in \mathscr{C} \Rightarrow \Delta \cap \sigma$ is either empty or a closed face of Δ and σ.

(c) $\bigcup_{\Delta \in \mathscr{C}} \Delta = \mathbb{R}^n$.

Remark

- If Δ' is a closed face of $\Delta \in \mathscr{C}$, then the canonical morphism $U_{\Delta'} \to U_\Delta$ induces an open immersion of the reductions.
- Hence one can glue the formal affine schemes $\mathrm{Spf}(\mathbb{C}_K\langle U_\Delta\rangle)$ to get a $\mathbb{C}_K^\circ$-model $\mathscr{X}$ of $\mathbb{G}_m^n$.
- Clearly, $(\mathbb{G}_m^n)_\kappa$ acts on the special fibre $\widetilde{\mathscr{X}}$.

Properties of these models $\mathscr{X}$ of $\mathbb{G}_m^n$ [Gu1,§4]

Proposition

(a) There is a bijective correspondence between torus orbits of $\widetilde{\mathscr{X}}$ and open faces of $\mathscr{C}$.

(b) The irreducible components of $\widetilde{\mathscr{X}}$ match with the vertices of $\mathscr{C}$.

(c) If Y_u is the irreducible component of $\widetilde{\mathscr{X}}$ corresponding to the vertex u, then Y_u is a toric variety with fan given by the cones σ in $\mathbb{R}^n$ which are generated by $\Delta - u$ for $\Delta \in \mathscr{C}$ with vertex u.

Example

We pave $\mathbb{R}^2$ by squares of length $v(\pi)$ for a fixed $\pi \in \mathbb{C}_K^{\circ\circ}$ and then we choose in every square a diagonal. This gives a simplex decomposition $\mathscr{C}$ of $\mathbb{R}^2$. The associated $\mathbb{C}_K^\circ$-model $\mathscr{X}$ of $\mathbb{G}_m^n$ is strictly semistable since the local pieces are $\mathrm{Spf}(\mathbb{C}_K\langle U_\Delta\rangle^\circ) \cong \mathrm{Spf}(\mathbb{C}_K^\circ\langle x_0, x_1, x_2\rangle / \langle x_0 x_1 x_2 - \pi\rangle)$. The torus orbits are equal to the strata and hence the skeleton is $\mathbb{R}^n$.

Analytic subvarieties of polytopal domains [Gu1,§5]

It will be important in the sequel to generalize the tropicalization to analytic subvarieties of $\mathbb{G}_m^n$. We start locally:

- Let U_Δ be a Γ-rational polytopal domain in $\mathbb{R}^n$.
- A closed analytic subvariety X of U_Δ is given by a unique ideal I in $\mathscr{A} := \mathbb{C}_K\langle U_\Delta\rangle$ such that $X = \mathscr{M}(\mathscr{A}/I)$.
- Note that I is not assumed to be reduced or prime.
- $\mathrm{trop}(X) := \mathrm{val}(X)$ is called the *tropical variety* associated to X.

Theorem (Gu1)

$\mathrm{trop}(X)$ *is a finite union of* Γ*-rational polytopes of dimension* $\leq \dim(X)$.

This can be deduced from de Jong's alteration theorem.

The dimension theorem [Gu1,§5]

Theorem (Gu1)

Suppose that Δ is n-dimensional and that X is an analytic subvariety of U_Δ of pure dimension d. If $\mathrm{val}(X)$ contains an interior point of Δ, then $\mathrm{trop}(X) \cap \mathrm{int}(\Delta)$ is of pure dimension d.

Remark

This proves also the dimension theorem for the tropical variety of a closed subscheme of $\mathbb{G}_m^n$. Indeed, we may use a polytopal decomposition of $\mathbb{R}^n$ to deduce it from the local dimension theorem above.

For the proof, we need the following result for affinoid algebras.

Proposition

Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a homomorphism of affinoid algebras. Then φ is finite (i.e. $\mathscr{B}$ is a finite $\mathscr{A}$-algebra) if and only if $\widetilde{\varphi} : \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ is finite.

Proof of the dimension theorem for $N = 0$

- It is enough to prove $d \leq N := \dim(\mathrm{trop}(X))$.
- We may assume X irreducible and therefore $\mathrm{val}(X)$ is connected.
- We handle $N = 0$ on this slide, hence $\mathrm{val}(X)$ is a point in Γ^n.
- Since the embedding $i : X \hookrightarrow U_\Delta$ of Berkovich spectra is finite, the reduction $\widetilde{i} : \widetilde{X} \to \widetilde{U_\Delta}$ is also finite.
- Since $\mathrm{val}(X) \subset \mathrm{int}(\Delta)$, we deduce that $\widetilde{X}$ is mapped to the closed orbit of $\widetilde{U_\Delta}$.
- The latter is a point and hence $\widetilde{X}$ is finite.
- We conclude $\dim(X) = \dim(\widetilde{X}) = 0$.

Proof of the dimension theorem for $N > 0$.

- By shrinking Δ, we may assume that $\mathrm{val}(X)$ is pure dimensional.
- There is $x \in X(\mathbb{C}_K)$ with $u := \mathrm{val}(x) \in \mathrm{int}(\Delta)$.
- Using x for a change of coordinates, we may assume $u = 0$.
- There is $m \in \mathbb{Z}^n$ such that the hyperplane $\{u \cdot m = 0\}$ intersects $\mathrm{val}(X)$ transversally.
- We apply the induction hypothesis to $X' := X \cap \{x^m = 1\}$.
- Hence we get $d - 1 = \dim(X') \leq \dim(\mathrm{val}(X'))$.
- Using $\mathrm{val}(X') \subset \mathrm{val}(X) \cap \{u \cdot m = 0\}$, we deduce $d - 1 \leq N - 1$. $\square$

Periodical tropical geometry I [Gu1,§6]

Let X be a d-dim. algebraic subvariety of a totally degenerate abelian variety A wrt. v, i.e. $A^{\mathrm{an}} = T/M$ and $\Lambda = \mathrm{val}(M)$ is a lattice in $\mathbb{R}^n$.

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{val}} & \mathbb{R}^n \\ \downarrow p & & \downarrow \\ A^{\mathrm{an}} & \xrightarrow{\overline{\mathrm{val}}} & \mathbb{R}^n/\Lambda \end{array}$$

Definition

$\overline{\mathrm{val}}(X^{\mathrm{an}})$ is called the *tropical variety* and is denoted by $\mathrm{Trop}(X)$.

Applying the dimension theorem to the analytic subvariety $p^{-1}(X)$, we get:

Theorem (Gu1)

$\mathrm{Trop}(X)$ *is a finite union of* d *dimensional* Γ*-rational polytopes in* $\mathbb{R}^n/\Lambda$.

Periodical tropical geometry II [Gu3,§3]

Let A be an abelian variety with uniformization E from the Raynaud extension $1 \to (\mathbb{G}_m^n)^{\mathrm{an}} \to E \to B^{\mathrm{an}} \to 0$ such that $A^{\mathrm{an}} = E/M$.

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{val}} & \mathbb{R}^n \\ \downarrow p & & \downarrow \\ A^{\mathrm{an}} & \xrightarrow{\overline{\mathrm{val}}} & \mathbb{R}^n/\Lambda \end{array}$$

Definition

$\overline{\mathrm{val}}(X^{\mathrm{an}})$ is called the *tropical variety* associated to the algebraic subvariety X of A and is denoted by $\mathrm{Trop}(X)$.

Theorem (Gu3)

There is $e \in \{0, 1, \ldots, \min\{\dim(X), \dim(B)\}\}$ *such that* $\mathrm{Trop}(X)$ *is a finite union of* Γ*-rational polytopes of dimension* $\dim(X) - e$ *in* $\mathbb{R}^n/\Lambda$.

Illustration of tropical excess

Example

- Assume $A = B_1 \times B_2$ with B_1 of potentially good reduction and B_2 totally degenerate.
- Then $(B_2)^{\mathrm{an}} = (\mathbb{G}_m^n)^{\mathrm{an}}/M$ and the Raynaud extension is given by $1 \to (\mathbb{G}_m^n)^{\mathrm{an}} \to B_1^{\mathrm{an}} \times (\mathbb{G}_m^n)^{\mathrm{an}} \to B_1^{\mathrm{an}} \to 0$.
- If X is a d-dimensional algebraic subvariety of A, then $\mathrm{Trop}(X) = \overline{\mathrm{val}}(p_2(X^{\mathrm{an}}))$ and hence $\dim(\mathrm{Trop}(X)) = \dim(p_2(X))$.
- This dimension is $d - e$, where any $e \in \{0, 1, \ldots, \min\{d, \dim(B)\}\}$ can be achieved.

By using the local triviality of the Raynaud extension, essentially the same argument proves the previous theorem in general.

Mumford's construction [Gu1,§6]

Let A be a totally degenerate abelian variety with respect to v, i.e. $A^{\mathrm{an}} = T/M$ and $\Lambda = \mathrm{val}(M)$ is a lattice in $\mathbb{R}^n$.

Definition

- A *polytope* $\overline{\Delta}$ in $\mathbb{R}^n/\Lambda$ is given by a polytope Δ in $\mathbb{R}^n$ such that Δ maps bijectively onto $\overline{\Delta}$.
- A *polytopal decomposition* of $\mathbb{R}^n/\Lambda$ is a finite family $\overline{\mathscr{C}}$ of polytopes in $\mathbb{R}^n/\Lambda$ induced by a Λ-periodic polytopal decomposition $\mathscr{C}$ of $\mathbb{R}^n$.

- Glueing the polytopal domains, we get a $\mathbb{C}_K^\circ$-model $\mathscr{U}$ of $\mathbb{G}_m^n$.
- By Λ-periodicity, $\mathscr{U}$ has a canonical action of M. We get a $\mathbb{C}_K^\circ$-model $\mathscr{A} := \mathscr{U}/M$ of A.
- $\mathscr{A}$ is obtained by glueing the formal affine $\mathscr{U}_\Delta := \mathrm{Spf}(\mathbb{C}_K\langle U_\Delta\rangle)$ along commen faces and by identifying $\mathscr{U}_\Delta$ and $\mathscr{U}_{\Delta+\lambda}$ for all $\lambda \in \Lambda$.

Mumford models [Gu1,§6]

Definition

We call $\mathscr{A}$ the *Mumford model* associated to A.

Proposition

- There is a bijective order reversing correspondence between torus orbits Z of $\widetilde{\mathscr{A}}$ and open faces $\overline{\tau}$ of $\overline{\mathscr{C}}$.
- The irreducible components Y of $\widetilde{\mathscr{A}}$ are toric varieties and correspond to the vertices of $\overline{\mathscr{C}}$.

- The Mumford model is strictly semistable if there is $\pi \in \mathbb{C}_K^{\circ\circ}$ such that every maximal $\Delta \in \mathscr{C}$ is $GL(n, \mathbb{Z})$-isomorphic to the standard simplex $\{u \in \mathbb{R}_+^n \mid u_1 + \ldots + u_n \leq v(\pi)\}$.
- Then the associated skeleton is the fundamental domain $\mathbb{R}^n/\Lambda$ and the canonicial simplices are the elements of $\overline{\mathscr{C}}$.

Notation

- Let K be a field with a non-trivial non-archimedean absolute value $| \ |$.
- $v(\ ) := -\log | \ |$ is the associated valuation.
- The valuation ring $K^\circ := \{\alpha \in K \mid v(\alpha) \geq 0\}$ has the unique maximal ideal $K^{\circ\circ} := \{\alpha \in K \mid v(\alpha) > 0\}$ and residue field $\widetilde{K} := K^\circ / K^{\circ\circ}$.
- We have the completion K_v and algebraic closure $\overline{K}$ of K.
- $\mathbb{C}_K := (\overline{K_v})_v$ is the smallest algebraically closed field extension of K which is complete with respect to an extension of $| \ |$ to a complete absolute value.
- By abuse of notation, we use also v and $| \ |$ on $\mathbb{C}_K$.
- Let κ be the residue field of $\mathbb{C}_K$.
- Let $\Gamma := v(\mathbb{C}_K^\times)$ be the value group of $\mathbb{C}_K$.

Metrics

- Let X be a projective variety over $\mathbb{C}_K$. By GAGA, X^{an} is compact.
- We consider a line bundle L on X, i.e. a family of 1-dimensional vector spaces $(L_x)_{x\in X}$ with a continuity condition.
- A *metric* $\| \ \|$ on L^{an} is a norm on each fibre $L_x^{\mathrm{an}} \cong \mathbb{C}_K$.
- A section of L on the open subset U of X is a family $s(x) \in L_x$, $x \in U$, which gives a morphism $s : U \to L$.
- We consider only continuous metrics, i.e. $x \mapsto \|s(x)\|$ is continuous with respect to the analytic topology for every local section.
- For continuous metrics $\| \ \|, \| \ \|'$ on L, we have the *distance of uniform convergence*

$$d(\| \ \|, \| \ \|') := \sup_{x\in X^{\mathrm{an}}} \left|\log\left(\|s(x)\|/\|s(x)\|'\right)\right|.$$

- Clearly, the definition is independent of the choice of $s(x) \in L_x \setminus \{0\}$.

Formal metrics [Gu4,§2]

- Let $\mathscr{L}$ be a formal $\mathbb{C}_K^\circ$-model of L, i.e. $\mathscr{L}$ is a line bundle on the $\mathbb{C}_K^\circ$-model $\mathscr{X}$ with $L = \mathscr{L}|_{X^{\mathrm{an}}}$.
- The associated metric $\| \ \|_{\mathscr{L}}$ on L is defined as follows: Every $x \in X^{\mathrm{an}}$ is contained in $\mathscr{U}^{\mathrm{an}}$ for a trivialization $\mathscr{U}$ of $\mathscr{L}$. The latter means that there is a section s of $\mathscr{L}$ without zeros.
- We set $\|s(x)\|_{\mathscr{L}} := 1$. This is well-defined as $s(x)$ is determined up to units in $\mathbb{C}_K^\circ$ and determines the metric completely.
- On $\mathscr{U}^{\mathrm{an}}$, every section t of L corresponds to an analytic function f with respect to the trivialization and $\|t(x)\| = |f(x)|$, therefore the metric is continuous.

Definition

- Metrics of the form $\| \ \|_{\mathscr{L}}$ are called *formal*.
- A *root of a formal metric* is a metric $\| \ \|$ on L such that $\| \ \|^{\otimes m}$ is a formal metric for some non-zero $m \in \mathbb{N}$.

Semipositive admissible metrics [Gu4,§2]

Theorem (Gu, 1998)

The roots of formal metrics are dense in the space of continuous metrics on L^{an}. *In particular, the set of roots of formal metrics on* $O_{X^{\mathrm{an}}}$ *is embedded onto a dense subset of* $C(X^{\mathrm{an}})$ *by the map* $\| \ \| \mapsto -\log \|1\|$.

- A line bundle F on a projective variety Y is called *nef* if $\deg_F(C) \geq 0$ for all closed curves C in Y.
- Then one can show that the degree of any closed subvariety with respect to F is non-negative.

Definition

- A metric $\| \ \|_{\mathscr{L}}$ induced by the line bundle $\mathscr{L}$ on the $\mathbb{C}_K^\circ$-model $\mathscr{X}$ is called *semipositive* if the reduction $\widetilde{\mathscr{L}}$ is a nef line bundle on the special fibre $\widetilde{\mathscr{X}}$.
- A *semipositive admissible metric* $\| \ \|$ on L is a uniform limit of roots of semipositive formal metrics $\| \ \|_n$ on L.

Canonical metrics [BG,§9.5]

- Now let (L, ρ) be a rigidified line bundle on the abelian variety A over K, i.e. $\rho \in L_0(K) \setminus \{0\}$.
- Then there is a *canonical metric* $\| \ \|_\rho$ for (L, ρ) which behaves well with respect to tensor product and homomorphic pull-back.
- We restrict to the case that L is ample and symmetric, then $\| \ \|_\rho$ is given by the following variant of Tate's limit argument.
- The rigidification and the theorem of the cube yield an identification $[m]^*L = L^{\otimes m^2}$ for $m \in \mathbb{Z}$.
- The canonical metric is characterized by $[m]^*\| \ \|_{\mathrm{can}} = \| \ \|_{\mathrm{can}}{}^{\otimes m^2}$ and it is given by

$$\| \ \|_{\mathrm{can}} = \lim_{m \to \infty} \left([m]^*\| \ \|\right)^{1/m^2},$$

  where $\| \ \|$ is any continuous metric on L^{an}.
- In particular, we may choose $\| \ \|$ as a root of a semipositive formal metric and hence $\| \ \|_\rho$ is a semipositive admissible metric.

Examples of canonical metrics [BG,§9.5]

Example

If A is an abelian variety with potentially good reduction, then $L^{\otimes 2}$ has an ample symmetric $\mathbb{C}_K^\circ$-model $\mathscr{L}$ and hence $\| \ \|_{\mathrm{can}} = \| \ \|_{\mathscr{L}}^{1/2}$ is a root of a semipositive formal metric.

If A has bad reduction, then $\| \ \|_{\mathrm{can}}$ is no longer a root of a formal metric.

Example

- Let E be a Tate elliptic curve, i.e. $E^{\mathrm{an}} = \mathbb{C}_K^\times / q^{\mathbb{Z}}$ and let $L = O([P])$ be the line bundle for the 2-torsion point P given by $\tilde{q} := q^{1/2}$.
- Then P is the divisor of the global section of L corresponding to the theta function $\theta(\zeta, q) := \sum_{n=-\infty}^{\infty} \tilde{q}^{n^2} \zeta^n$.
- The pull-back of the even ample line bundle L to $\mathbb{C}_K^\times$ is trivial and we can easily compute $-\log(\|1\|_{\mathrm{can},\zeta}) = \frac{v(\zeta)^2}{2v(q)}$.

Chambert-Loir's measures [Gu1,§3]

Theorem (Chambert-Loir)

For a d-dimensional projective variety X *and* $\overline{L} = (L, \| \ \|)$ *an ample line bundle endowed with a semipositive admissible metric, there is a unique positive regular Borel measure* $c_1(\overline{L})^{\wedge d}$ *on* X^{an} *with the properties:*

(a) $c_1(\overline{L}^{\otimes m})^{\wedge d} = m^d c_1(\overline{L})^d$ *and* $c_1(\overline{L})^{\wedge d}$ *is continuous in* $\| \ \|$.

(b) *If* $\varphi : Y \to X$ *is a morphism of d-dimensional projective varieties, then the projection formula* $\varphi_* \left(c_1(\varphi^* \overline{L})^{\wedge d} \right) = \deg(\varphi) c_1(\overline{L})^{\wedge d}$ *holds.*

(c) $c_1(\overline{L})^{\wedge d}$ *has total measure* $\deg_L(X)$.

(d) *If* $\mathscr{X}$ *is a formal* $\mathbb{C}_K^\circ$*-model of* X *with reduced special fibre and if the metric of* $\overline{L}$ *is induced by a formal* $\mathbb{C}_K^\circ$*-model* $\mathscr{L}$ *of* L *on* $\mathscr{X}$, *then* $c_1(\overline{L})^{\wedge d} = \sum_Y \deg_{\widetilde{\mathscr{L}}}(Y) \delta_{\xi_Y}$, *where* Y *ranges over the irreducible components of* $\widetilde{\mathscr{X}}$ *and* δ_{ξ_Y} *is the Dirac measure in the unique point* ξ_Y *of* X^{an} *which reduces to the generic point of* Y.

Canonical measures

Now we consider an ample symmetric line bundle L on an abelian variety A and a d-dimensional subvariety X of A.

Definition

We call $\mu := c_1(L|_X, \| \ \|_{\mathrm{can}})^{\wedge d}$ the *canonical measure* on X associated to L.

Example

If $X = A$ and if A has potentially good reduction, then (d) from the above theorem shows that $\mu = \deg_L(A)\delta_\xi$, where ξ is the unique point of A^{an} which reduces to the generic point of the Néron-model $\mathscr{A}$.

Setup

- We assume that v is a discrete valuation of K and hence $\Gamma = \mathbb{Q}$.
- Let X be a closed d-dimensional variety of the abelian variety A.
- The tropical excess e was defined by $\dim(\mathrm{Trop}(X)) = d - e$.
- We assume for simplicity that X has a strictly semistable $\mathbb{C}_K^\circ$-model $\mathscr{X}$, otherwise we have to use a strictly semistable alteration.
- Recall that the skeleton $S(\mathscr{X})$ of $\mathscr{X}$ is a subset of X^{an} given as the union of canonical simplices Δ_S corresponding to the strata S of $\widetilde{\mathscr{X}}$.
- Let $b := \dim(B)$ for the abelian variety B of good reduction in the Raynaud extension $1 \to (\mathbb{G}_m^n)^{\mathrm{an}} \to E \to B^{\mathrm{an}} \to 0$ of A.

Explicit description of canonical measures [Gu3]

Theorem (Gu3)

There is a list of canonical simplices $(\Delta_S)_{S\in I}$ *with the properties:*

- *The maximal simplices* $(\Delta_S)_{S\in J}$ *from this list are* $(d-e)$*-dimensional.*
- $\overline{\mathrm{val}}$ *is one-to-one on every* Δ_S, $S \in I$, *and* $\bigcup_{S\in J} \overline{\mathrm{val}}(\Delta_S) = \mathrm{Trop}(X)$.
- *For any ample line bundle* $\overline{L}$ *on* A*, the canonical measure* $\mu := c_1(L|_X, \|\ \|_{\mathrm{can}})^{\wedge d}$ *is supported in* $\bigcup_{S\in J} \Delta_S$.
- *The restriction of* μ *to the relative interior of* Δ_S *is a positive multiple of the relative Lebesgue measure which may be explicitly computed in terms of convex geometry.*

Remark

- If A is totally degenerate, then $\dim(\Delta_S) = d$ for all $S \in I$.
- In general, there are examples where simplices of all dimensions in $\{d-b, \dots, d-e\}$ may occur for a single canonical measure.

Sketch of proof I

- We sketch the proof in the special case $X = A$ totally degenerate.
- Hence $A^{\mathrm{an}} = (\mathbb{G}_m^n)^{\mathrm{an}}/M$ for a discrete subgroup M of $(\mathbb{C}_K^\times)^n$ such that $\Lambda := \mathrm{val}(M)$ is a lattice in $\mathbb{R}^n$.
- Since Λ is a subgroup of $\mathbb{Q}^n$ of rank n, there is a basis $b_1, \ldots, b_n$ of $\mathbb{Z}^n$, $k \in \mathbb{N}$ and $k_1 | k_2 | \cdots | k_n \in \mathbb{Z}$ such that $\frac{k_1}{k} b_1, \ldots, \frac{k_n}{k} b_n$ is a basis of Λ.
- The fundamental domain of Λ is a cuboid with respect to the basis $b_1, \ldots, b_n$ and hence we can easily pave $\mathbb{R}^n$ by translates of $\frac{1}{m} Q$, where Q is the unit cube and $m \in \mathbb{N}$ is fixed.
- We deduce that there is a rational Λ-periodic simplex decomposition $\mathscr{C}$ of $\mathbb{R}^n$ such that every n-dimensional $\Delta \in \mathscr{C}$ is $\mathrm{GL}(n, \mathbb{Z})$ isomorphic to a translate of $\frac{1}{m}\Delta_1$ for the standard simplex

$$\Delta_1 := \{u \in \mathbb{R}_+^n \mid u_1 + \cdots + u_n \leq 1\}.$$

Sketch of proof II

- We conclude that the Mumford model $\mathscr{A}$ of A associated to $\mathscr{C}$ is strictly semistable.
- Note that the skeleton $S(\mathscr{A})$ of $\mathscr{A}$ is $\mathbb{R}^n/\Lambda$ with canonical simplices given by $\overline{\mathscr{C}} := \mathscr{C}/\Lambda$. Moreover, $S(\mathscr{A})$ is a subset of A^{an}.
- By a result of Künnemann, we may assume that L has a $\mathbb{C}_K^\circ$-model $\mathscr{L}$ on $\mathscr{A}$ such that the formal affine open subsets $\mathscr{U}_\Delta := \mathrm{Spf}(\mathbb{C}_K\langle U_\Delta\rangle^\circ)$ form a trivialization of $\mathscr{L}$.
- We identify the pull-back p^*L to $(\mathbb{G}_m^n)^{\mathrm{an}}$ with $(\mathbb{G}_m^n)^{\mathrm{an}} \times \mathbb{C}_K$. Then the section 1 corresponds to a $\gamma \in K\langle U_\Delta\rangle^\times$ with respect to the trivialization $\mathscr{U}_\Delta$.
- It is easy to show that γ is equal to $a_\Delta x^{m_\Delta}$ up to smaller terms.
- We conclude that $f_{\mathscr{L}} := -\log p^*\|1\|_{\mathscr{L}}$ is a continuous function on $\mathbb{R}^n$ with $f_{\mathscr{L}}(u) = m_\Delta \cdot u + v(a_\Delta)$ on Δ.

Dual complex

- $f_{\mathscr{L}}$ induces a canonical *dual complex* $\mathscr{C}^{f_{\mathscr{L}}}$ on $\mathbb{R}^n$.
- The vertices of $\mathscr{C}^{f_{\mathscr{L}}}$ are given by m_Δ, $\Delta \in \mathscr{C}$.
- Every k-dimensional polytope σ of $\mathscr{C}$ induces an $(n-k)$-dimensional polytope $\sigma^{f_{\mathscr{L}}}$ given by the vertices m_Δ, $\Delta \supset \sigma$.
- By results of Mac Mullen, $\mathscr{C}^{f_{\mathscr{L}}}$ is a polytopal decomposition of $(\mathbb{R}^n)^* = \mathbb{R}^n$ for a suitable lattice Λ^L not depending on $\mathscr{L}$.

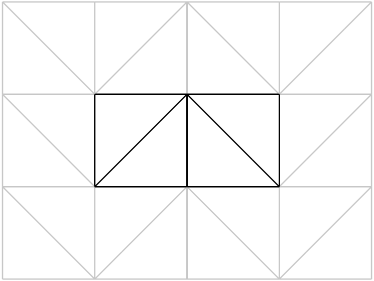

Figure: simplex decomposition $\mathscr{C}$

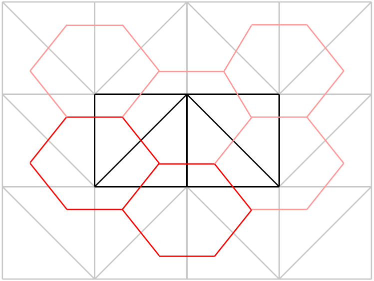

Figure: dual complex

Facts from the theory of toric varieties [KKMS,Ch.I]

- We know that every vertex $\overline{u}$ of $\overline{\mathscr{C}}$ corresponds to an irreducible component $Y_{\overline{u}}$ of $\widetilde{\mathscr{A}}$.
- We have seen that $Y_{\overline{u}}$ is a $(\mathbb{G}_m^n)_\kappa$-toric variety.
- $\mathscr{L}$ is ample on $\mathscr{A}$ if and only if $f_{\mathscr{L}}$ is strictly convex with respect to $\mathscr{C}$, i.e. a convex function such that the maximal domains of "linearity" are the n-dimensional polytopes in $\mathscr{C}$.
- $\deg_{\mathscr{L}}(Y_{\overline{u}}) = n!\ \mathrm{vol}(\{u\}^{f_{\mathscr{L}}})$

Sketch of proof III

- Chambert-Loir's measure with respect to the formal metric $\| \ \|_{\mathscr{L}}$ is given by
$$c_1(L, \| \ \|_{\mathscr{L}})^{\wedge n} = \sum_{\overline{u}} \deg_{\mathscr{L}}(Y_{\overline{u}}) \delta_{\xi_{\overline{u}}} ,$$
where $\overline{u}$ ranges over the vertices of $\overline{\mathscr{C}}$ in $\overline{\Omega}$ and $\xi_{\overline{u}}$ is the unique point of A^{an} with reduction equal to the generic point of the irreducible component $Y_{\overline{u}}$.
- For $\overline{\Omega}$ measurable in $\mathbb{R}^n/\Lambda$, we have
$$\mu_1(\overline{\Omega}) := \int_{\overline{\Omega}} c_1(L, \| \ \|_{\mathscr{L}})^{\wedge n} = \sum_{\overline{u} \in \overline{\Omega}} \deg_{\mathscr{L}}(Y_{\overline{u}}) = n! \sum_{\overline{u} \in \overline{\Omega}} \mathrm{vol}(\{\overline{u}\}^{f_{\mathscr{L}}})$$
where $\overline{u}$ is always supposed to be a vertex in $\overline{\mathscr{C}}$.

Sketch of proof IV

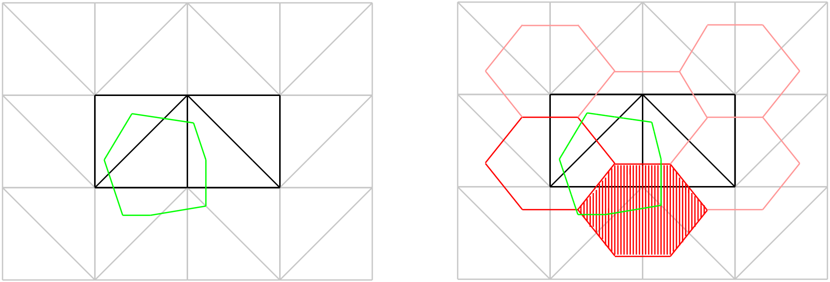

Figure: Ω Figure: $\mu_1(\overline{\Omega})$

- By Tate's limit argument, we have

$$\| \ \|_{\mathrm{can}} = \lim_{m\to\infty} \left([m]^* \| \ \|_{\mathscr{L}}\right)^{1/m^2}.$$

Sketch of proof V

- Let $\mathscr{A}_m$ be the Mumford model of A associated to $\mathscr{C}_m := \frac{1}{m}\mathscr{C}$. $[m]$ extends to a morphism $\mathscr{A}_m \to \mathscr{A}_1$ and hence $[m]^*\mathscr{L}$ is a $\mathbb{C}_K^\circ$-model of $[m]^*L = L^{\oplus m^2}$ an A.

$$\Rightarrow \|\ \|_{\mathrm{can}} = \lim_{m\to\infty} (\|\ \|_{[m]^*\mathscr{L}})^{1/m^2}.$$

- The canonical measure $\mu := c_1(L, \|\ \|_{\mathrm{can}})^{\wedge n}$ is given by

$$\mu = \lim_{m\to\infty} m^{-2n} c_1(L, \|\ \|_{[m]^*\mathscr{L}}) = \lim_{m\to\infty} m^{-2n}\mu_m$$

with

$$\mu_m(\overline{\Omega}) = n! \sum_{\overline{u_m}\in\overline{\Omega}} \mathrm{vol}(\{u_m\}^{mf_{\mathscr{L}}}) = n!\ m^n \sum_{\overline{u_m}\in\overline{\Omega}} \mathrm{vol}(\{u_m\}^{f_{\mathscr{L}}})$$

where $\overline{u_m}$ is supposed to be a vertex of $\overline{\mathscr{C}_m}$.

Sketch of proof VI

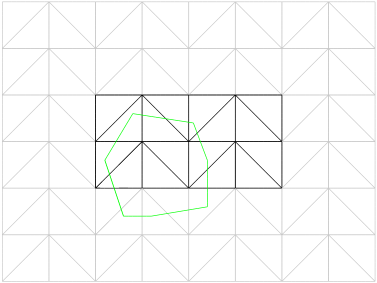

Figure: Ω and $\mathscr{C}_2$

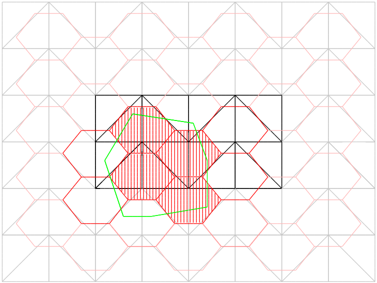

Figure: $\mu_2(\overline{\Omega})$

Sketch of proof VII

- For $m \gg 0$, an easy calculation shows

$$\sum_{\overline{u_m} \in \overline{\Omega}} \mathrm{vol}(\{u_m\}^{f_{\mathscr{L}}}) \sim m^n \ \mathrm{vol}(\overline{\Omega}) \frac{\mathrm{vol}(\Lambda^L)}{\mathrm{vol}(\Lambda)}$$

  and hence

$$\mu(\overline{\Omega}) = n! \ \mathrm{vol}(\overline{\Omega}) \frac{\mathrm{vol}(\Lambda^L)}{\mathrm{vol}(\Lambda)} \ .$$

- By construction, we have $\mathrm{supp}(\mu) = S(\mathscr{A})$. □

Canonical measure for $X = A$

Corollary [Gu3]

If $X = A$ is a d-dimensional abelian variety, then the canonical measure $c_1(L, \|\ \|_{\mathrm{can}})^{\wedge d}$ for an ample line bundle L is equal to the Haar measure μ on the skeleton $\mathbb{R}^n/\Lambda$ of A determined by $\mu(\overline{\Omega}) = \deg_L(A)$.

Proof.

If A is totally degenerate, then the claim follows from the above proof and the fact that Chambert-Loir's measures have total measure equal to the degree. We skip the general case. □

Diophantine geometry

Example

The diophantine equation $x^4 - y^4 = 5$ has only finitely many rational solutions, e.g. $(\frac{3}{2}, \frac{1}{2})$.

In general, we have for any number field K the *Mordell-conjecture*.

Theorem (Faltings 1983)

An algebraic curve of genus $g > 1$ has only finitely many points with coordinates in K.

- A central tool is the *height* of a point.
- The height measures the arithmetic complexity of the point.
- e.g. $h(\frac{3}{2}, \frac{1}{2}) = \log(3)$, as we have the projective solution $(2 : 3 : 1)$.

Product formula [BG,Ch.1]

- Let M_K be the set of absolute values on the number field K which extend the usual absolute value or the p-adic absolute values on $\mathbb{Q}$.
- For $v \in M_K$ extending $q \in M_{\mathbb{Q}}$, let $K_v, \mathbb{Q}_q$ be the completions and let $\mu(v) := \frac{[K_v:\mathbb{Q}_q]}{[K:\mathbb{Q}]}$.
- For non-zero $\alpha \in K$, we have the *product formula*

$$\prod_{v \in M_K} |\alpha|_v^{\mu(v)} = 1.$$

Remark

- If $K = k(B)$ is the function field of a smooth curve B over an algebraically closed field, then every point $v \in B$ induces the discrete absolute value $|f|_v := e^{-\operatorname{ord}(f,v)}$ and we set $M_K := B$.
- The product formula holds here as in the number field case.

In the following, the field K is either a number field or a function field.

Semipositive admissible metrics [Gu4,§3]

- Let L be an ample line bundle on the projective variety X over K.
- If v is non-archimedean, then we are going to apply the theory of semipositive admissible metrics on the Berkovich analytic space X_v^{an}.
- If $v|\infty$, then X_v^{an} is a complex space and there is also a notion of semipositive admissible metric $\| \ \|_v$ on L_v^{an}. For X smooth, this means that $\| \ \|_v$ is a smooth hermitian metric with semipositive curvature.

Example

If $\mathscr{L}$ is an ample O_K-model for $L^{\otimes m}$, then we have seen that $\| \ \|_{\mathscr{L},v}^{1/m}$ defines a semipositive formal metric on L_v^{an} for $v \not| \infty$.

Definition

A *semipositive admissible metric* $\| \ \|$ on L is a family of semipositive admissible metrics $\| \ \|_v$ on L_v^{an}, $v \in M_K$, which are as in the above example up to finitely many $v \in M_K$.

Heights [BG,Ch.2]

Let $\overline{L}$ be the ample line bundle L endowed with a semipositive admissible metric $\| \ \|$.

Definition

The *height* of $P \in X(\overline{K})$ is given by

$$h_{\overline{L}}(P) := -\sum_{w \in M_F} \mu(w) \log \|s(x)\|_w,$$

where F/K is a finite extension with $P \in X(F)$ and $s(x) \in L_x \setminus \{0\}$.

- $\mu(w)$ ensures that the height does not depend on F.
- The product formula shows that the height does not depend on $s(x)$.

Theorem (Weil)

The height does not depend on $\| \ \|$ up to bounded functions.

Néron–Tate–heights [BG,Ch.9]

- Let A be an abelian variety over K with an ample even line bundle L.
- For $v \in M_K$, let $\| \ \|_{\mathrm{can},v}$ be the canoncial metric of L_v^{an} with respect to a fixed rigidification of L.
- This induces a semipositive admissible metric $\| \ \|_{\mathrm{can}}$ on L.

Definition

We call $\hat{h}_L := h_{(L,\| \ \|_{\mathrm{can}})}$ the *Néron–Tate–height* with respect to L.

- By Weil's theorem, $\hat{h}_L(P) = \lim_{m\to\infty} m^{-2} h_{(L,\| \ \|)}(mP)$ for any semipositive admissible metric $\| \ \|$ on L (*Tate's limit formula*).
- $\hat{h}_L$ is a positive semidefinite quadratic form.
- The kernel of the associated bilinear form is the torsion group.
- We get canonical semidistance $d_L(P,Q) := \hat{h}_L(P-Q)$ on A.

The Bogomolov conjecture over the number field K [Zh]

Definition

A *torsion subvariety* of A has the form $B + t$ for an abelian subvariety B and a torsion point t of A.

For a closed subvariety X of A, we have the *Bogomolov conjecture*:

Theorem (Ullmo 1998 for curves, Zhang 1998 in general)

- *There are only finitely many maximal torsion subvarieties in X.*
- *$\hat{h}_L$ has a positive lower bound on their complement in X.*

- This is a statement for points with coordinates in $\overline{K}$.
- The torsion points are dense in every torsion subvariety.
- The statement is independent of the choice of L.

Bogomolov conjecture over the function field $K = k(B)$

Many proofs are easier for function fields:

- Fermat's conjecture: Tschebyscheff, Liouville, Korkine, 19th century
- Mordell conjecture: Manin, Grauert, Samuel, 1963-1966

Theorem (Gu2)

The Bogomolov conjecture holds if A is totally degenerate with respect to some $v \in M_K$.

- The Bogomolov conjecture is wrong if X and A are defined over k.
- It is conjectured only if $\mathrm{Tr}_{L/k}(A) = 0$ for all finite L/K.
- The Bogomolov conjecture was known only for some curves (e.g. $g = 2$) due to Moriwaki, Yamaki.
- Recent work of Zhang and Faber give all curves $g \leq 4$ and more examples.

Setup for equidistribution

The proof of the Bogomolov conjecture relies on the following equidistribution result:

- If X is a closed subvariety of the abelian variety A and L is an even ample line bundle on A.
- We fix a place $v \in M_K$ and an embedding $\overline{K} \hookrightarrow \mathbb{C}_{K_v}$ over K to identify $X(\overline{K})$ with a subset of X_v^{an}.
- Note that the absolute Galois group $G := \mathrm{Gal}(\overline{K}/K)$ acts on $X(\overline{K})$.
- Suppose that (P_n) is a small generic sequence in $X(\overline{K})$:
  - *generic* means $\{n \in \mathbb{N} \mid P_n \in Y\}$ is finite for every closed $Y \subsetneq X$.
  - *small* means that $\lim_{n\to\infty} \hat{h}_L(P_n) = 0$.
- We consider the discrete probability measure μ_n on X_v^{an} which has support GP_n and is equidistributed on this Galois orbit.

Equidistribution theorem [Yu], [Gu4]

Theorem

We have the weak convergence $\mu_n \to (\deg_L(X)^{-1} c_1(L|_X, \|\ \|_{\mathrm{can},v})^{\wedge d}$ *of regular probability measures on* X_v^{an}.

Remark

There is a generalization to arbitrary projective varieties X and any semipositive admissible metric $\|\ \|$ on L, where now small means that $h_{(L,\|\ \|)}(P_n)$ converges to the height $h_{(L,\|\ \|)}(X)$ of X.

If K is a number field, the equidistribution theorem was proved by:

- Szpiro, Ullmo and Zhang for $v|\infty$ and positive curvature at v.
- Chambert-Loir for $v \not| \infty$ if $\|\ \|_v$ is induced by an ample model.
- Yuan in general.

Methods of proof [Yu], [Gu4]

- If the curvature is positive (or the metric is induced by an ample model), then the arithmetic Hilbert-Samuel formula is used to prove the fundamental inequality

$$h_{(L,\| \|)}(X) \leq \liminf_{n\to\infty} h_{(L,\| \|)}(P_n).$$

- A variational principle for metrics on L is used to deduce the equidistribution theorem from the fundamental inequality.
- This is possible as the variational metrics remain semipositive.
- For semipositive admissible metrics, this is no longer true.
- Yuan's idea is to prove a variational form of the fundamental inequality based on Siu's theorem in the theory of big line bundles.
- This is good enough to prove the equidistribution theorem as above.

Yuan's proof may be adapted to function fields. This was done by Faber in the special case $h_{(L,\| \|)}(X) = 0$ and independently by [Gu4] in general.

Tropical equidistribution theorem [Gu2,§5]

- Let A be an abelian variety which is totally degenerate with respect to a fixed $v \in M_K$ and let X be a d-dimensional closed subvariety.
- Let $(P_n)_{n\in\mathbb{N}}$ be a small generic sequence in X as before.
- Let us consider the following discrete probability measure on $\mathrm{Trop}(X)$:

$$\mu_n = \frac{1}{|GP_n|} \sum_{Q\in GP_n} \delta_{\overline{\mathrm{val}}(Q)}.$$

Theorem (Gu2)

Then μ_n converges weakly to a strictly positive volume form μ on $\mathrm{Trop}(X)$, i.e. $\mathrm{Trop}(X)$ is a finite union of d-dimensional polytopes Δ such that $\mu|_\Delta$ is a positive multiple of the Lebesgue measure.

This follows by taking $\overline{\mathrm{val}}_*$ in the previous equidistribution theorem and then using the explicit description of the canonical measures.

Proof of the Bogomolov conjecture I [Gu2,§6]

It is easy to see that the Bogomolov conjecture is equivalent to:

Theorem (Gu2)

Let X be a closed subvariety of the abelian variety A over K. We assume that A is totally degenerate with respect to $v \in M_K$. If X is no torsion subvariety of A, then there is no small generic sequence in $X(\overline{K})$.

- Similarly as in Zhang's proof, we can assume that the morphism

$$\alpha : X^N \longrightarrow A^{N-1}, \quad \mathbf{x} \mapsto (x_2 - x_1, \ldots, x_N - x_{N-1})$$

  is generically finite for N sufficiently large.
- If the Bogomolov conjecture is wrong, then there is a small generic sequence in $X(\overline{K})$.
- Then there is also a small generic sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X^N.
- We conclude that $\alpha(\mathbf{x}_n)$ is a small generic sequence in $Y = \alpha(X^N)$.

Proof of the Bogomolov conjecture II [Gu2,§6]

- We get equidistribution measures μ on $\mathrm{Trop}(X^N)$ and ν on $\mathrm{Trop}(Y)$.
- By construction, we have $\nu = \alpha_{\mathrm{aff}}(\mu)$ for the canonical α_{aff}:

$$\begin{array}{ccc} X^N & \xrightarrow{\alpha} & Y \\ \overline{\mathrm{val}}\downarrow & \circlearrowleft & \downarrow\overline{\mathrm{val}} \\ \mathrm{Trop}(X^N) & \xrightarrow{\alpha_{\mathrm{aff}}} & \mathrm{Trop}(Y) \end{array}$$

- The diagonal X in X^N satisfies $\alpha(X) = 0$.
- The same holds for the diagonal $\mathrm{Trop}(X)$ in $\mathrm{Trop}(X^N) = \mathrm{Trop}(X)^N$.
- There is an Nd-dimensional simplex Δ in $\mathrm{Trop}(X^N)$ with d-dimensional face in $\mathrm{Trop}(X)$.
- $\dim(\alpha_{\mathrm{aff}}(\Delta)) < \dim(\Delta)$ and hence $\nu(\alpha_{\mathrm{aff}}(\Delta)) = 0$.
- This proves $\mu(\Delta) = 0$ which contradicts the strict positivity of μ. □

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