

Analytic Torsion

(1)

Motivation: Let $0 \rightarrow V_0 \xrightarrow{d_0} \dots \xrightarrow{d_n} V_n \rightarrow 0$ be a complex

of finite-dim. vector spaces with metrics

$\langle \cdot, \cdot \rangle_{L^2}$ on each V_j . Set $H^q(V, d) := \ker d_q / \text{im } d_{q-1}$,

$d_q^* := \text{adjoint to } d_q \text{ w.r.t. } \langle \cdot, \cdot \rangle_{L^2}$ and

$\Delta_q := (d + d^*)^2|_{V_q}$, then $H^q(V, d) \cong \ker \Delta_q \subseteq V_q$

and

$$V_0 = H^0 \oplus \text{im } d_0^*$$

$$\xrightarrow{d_0 \cong}$$

$$V_1 = H^1 \oplus \text{im } d_1^* \oplus \text{im } d_0$$

$$\xrightarrow{d_1 \cong}$$

$$V_2 = H^2 \oplus \text{im } d_2^* \oplus \text{im } d_1$$

\vdots

Thus $\dim \sum (-1)^q H^q = \dim \sum (-1)^q V_q$. Similarly

$$\det V := \bigotimes_q \left(\Lambda^{\max} V_q \right)^{(-1)^q} \quad (\text{with } \mathcal{Z}^{-1} := \mathcal{Z}^*)$$

$$= \frac{\det(H^0 \oplus \text{im } d_0^*) \otimes \det(H^2 \oplus \text{im } d_2^* \oplus \text{im } d_1) \otimes \dots}{\det(H^1 \oplus \text{im } d_1^* \oplus \text{im } d_0) \otimes \dots}$$

can.

$$= \det H^0.$$

Metric on V .

Metric on $\det V$.

Metric on $H^* \subseteq V$.

(Quillen-) metric $\|\cdot\|_Q^2$ on $\det H \cong \det V$. \neq L^2 -metric $\|\cdot\|_{L^2}^2$ on $\det H^*$

Lemma: For $v \in \det H^*$

$$\log \|v\|_Q^2 = \log \|v\|_{L^2}^2 + \sum_q (-1)^q \notin \log \det [\Delta_q |_{(\ker \Delta_q)^\perp}]$$

Proof: For $w \in \text{im } d_q^*$, $\|dw\|^2 = \langle d^* dw, w \rangle = \langle \Delta w, w \rangle$.

Hence $\frac{\det \text{im } d_q}{\det \text{im } d_q^*} = \frac{\det d(\text{im } d_q^*)}{\det \text{im } d_q^*}$ is isometric to

$(\mathbb{C}, 1 \cdot 1^2 \cdot \det \Delta|_{\text{im } d_q^*})$.

□

Now consider

\hat{E} holomorphic Hermitian vector bdl. (3)
 M^n compact Hermitian manifold

and $\Omega^{0,q}(M, E) := \Gamma^*(M, E \otimes \Lambda^q T^{0,1} M) =$ the Hilbert-space
of antiholomorphic q -forms with coefficients in E .

The de Rham-Op. d on differential forms induces

$$0 \rightarrow \Omega^{0,0}(M, E) \xrightarrow{\bar{\delta}} \dots \xrightarrow{\bar{\delta}} \Omega^{0,n}(M, E) \rightarrow 0.$$

$\bar{\delta} \leftarrow$ Dolbeault-Op.

Let $\square_g := (\bar{\delta} + \bar{\delta}^*)^2 : \Omega^{0,q}(M, E) \rightarrow \Omega^{0,q}(M, E)$

be the Kodaira-Laplacian. Then

$$Z(s) := \sum_{q=0}^n (-1)^{q+1} q \operatorname{Tr} \left[(\square_g|_{H^{q-1}})^{-s} \right] \quad (\operatorname{Re} s > \frac{n+1}{2})$$

has a meromorphic continuation, holom. at $s=0$.

Def.: Ray-Singer torsion $T(M, \hat{E}) := Z'(0) \in \mathbb{R}$.

Th. (Bott-Bott-Gillet-Soulé): If M Kähler fibration, B fiber, \hat{E} vector bdl.

$$\Rightarrow (\det H^*(B, E|_B), \| \cdot \|_{L^2}^2 \cdot e^{T(M, \hat{E})}) \rightarrow B \text{ is Hermitian vector bdl.}$$

Example: i) $M := \text{ell. curve } Y^2 = 4X^3 - g_2 X - g_3 = \frac{\mathbb{C}/\langle z + \tau z \rangle}{z : 1}$

$$\square, (e^{\mu(z)} dz) = \frac{1/\mu'^2}{\ln \tau} e^{\mu(z)} dz \quad \text{for all } \mu \in \Lambda^*$$

$$T(M, G) = \frac{d}{ds} \Big|_{s=0} \sum_{\substack{\mu \in \Lambda^* \\ \mu \neq 0}} \left(\frac{1/\mu'^2}{\ln \tau} \right)^{-s}$$

2. coeff of T_d -class
Discriminant

Kronecker

$$\frac{v_0(M)}{1853} = -\log \ln \tau - \frac{1}{12} \log |g_2^3 - 27g_3^2|^2$$

$$\text{ii) } T(\mathbb{P}^1, G) = \frac{d}{ds} \Big|_{s=0} \sum_{l \geq 1} \frac{2l+1}{l^3(l+1)^3} = \frac{1}{4} - 2\zeta'(-1)$$

$$\text{iii) } T\left(\frac{SO(7)}{SO(5)SO(2)}\right) = -\frac{229403}{120960} - \frac{5}{12} \zeta'(-5) - \zeta'(-3) + \frac{149}{12} \zeta'(-1)$$

$$+ \frac{1139}{7560} \log 2 + \frac{1}{2} \log 60$$

$g : M \rightarrow M$ isometry, $g^E : g^* E \rightarrow E$ fibrewise isometry.

Def.: Eg. tension $T_g(M, \bar{\epsilon}) := \frac{d}{ds} \Big|_{s=0} \sum (-1)^{\frac{g+1}{2}} \text{Tr}(g^* \circ \square_g^{-s})$.

Consider $\pi: M \xrightarrow{\exists} B$ Kähler fibration, $\overset{\bar{E}}{\downarrow}_M$ Herm. holom., (5)

then $T(\bar{E}, \bar{E}_t) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{-1}{\rho(s)} \int_0^\infty t^{s-1} \sum_q (-1)^{q+1} q \operatorname{Tr} e^{-t \square_q} dt \in C^\infty(B)$

Set $\mathfrak{Z}(s) := \frac{-1}{\rho(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_s N_t e^{-A_t^2} dt \in \sum \Omega^{q,q}(B)$.

*"number op."
(tensor)*

Bismut - superconnection
1. order Diff. op. on
 $\Omega^*(B, \Omega^*(\mathfrak{Z}, E_{12}))$

Def. (Bismut, k.'92): Torsion form $T_\pi(\bar{E}) := \mathfrak{Z}'(0) \in \sum_{\text{ind}+\text{ind}} \Omega^{q,q}(B)$

Then $T_\pi(\bar{E})^{[0]} = T(\mathfrak{Z}, E_{12})$.

Ex.: $\overset{\bar{E}}{\downarrow}_B^k$ Herm. holom. vector bundle, $\Lambda \in \mathbb{Z}^{2k}$ lattice bdl.

$\pi: M := E/\Lambda \rightarrow B$ torus bdl.

$$\mathfrak{Z}(s) := \frac{\Gamma(2k-1-s)}{(2\pi)^k (k-1)! \Gamma(s)} \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} (\|\lambda\|^2)^{s+1-2k} (\det \|\lambda\|^2)^{k(k-1)}$$

$$\in \Omega^{k-1, k-1}(B) \quad (\text{for } \operatorname{Re} s < 0).$$

Then $T_\pi(0) = \frac{\mathfrak{Z}'(0)}{Td(\bar{E})}$.

Def.: Set $\tilde{K}(n) := \{\sum a_j \bar{E}_j + \eta \mid a_j \in \mathbb{Z}, \bar{E}_j \rightarrow M \text{ Herm. holom. vector bdl.}\}$, (6)

$$\eta \in \sum \alpha^{2,q}(M) /_{\text{ind}+1 \text{ ind}} \} / \{ \bar{E} \sim \bar{E}' + \bar{E}'' - \text{ch}(\bar{E}) \mid \\ \xi: 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \text{ exakt}\}$$

Th. (Bismut, k. '92): For $\pi: M \xrightarrow{\cong} B$ proper fibration, it

$$\pi_!: \tilde{K}(M) \rightarrow \tilde{K}(B)$$

$$\pi_! \bar{E} := \sum (-1)^q H^q(B, E|_B) + T_\pi(\bar{E})$$

$$\pi_! \eta := \int_B \eta \cdot Td(TB) \quad \text{volldef.}$$

Set $\text{ch}: \tilde{K}(M) \rightarrow \sum \alpha^{2,q}(M)$, $\text{ch}(\eta) := \frac{i}{2\pi} \eta$, then

$$\text{ch}(\pi_! \bar{E}) = \pi_! \text{ch}(\bar{E}).$$

This can be extended to a direct image for general proper maps $f: M \rightarrow N$, which verifies (for suitable metrics on the mfds.) $(f_! \circ f_*)_! = f_{!*} \circ f_!^*$

(Bismut, Gillet, Soulé, k., Leandre, Berthomieu, Ma).

Def.: Gillet-Soulé's Anakarov K₀-theory of an arithm. var. X ⁽²⁾

$\hat{K}(X) := \left\{ \sum a_j \bar{E}_j + \gamma \mid a_j \in \mathbb{Z}, E_j \rightarrow X \text{ vbdl. with Hermitian metric on } \frac{E_j}{X_0} \right\}$

$$\eta \in \Sigma \alpha^{g, \pm}(X_C) /_{(\text{mod } \lambda \omega + \lambda \omega)} \quad (\bar{E} \sim \bar{E}' + \bar{E}'' - \widehat{\text{ch}}(\xi) \mid \xi: 0 \rightarrow E' \rightarrow E'' \rightarrow 0)$$

Th.: For $f: X \rightarrow Y$ proper morphism, torsion forms define a direct image

$$f_! : \hat{K}(X) \rightarrow \hat{K}(Y)$$

$$f_! \bar{E} := \sum (-1)^g R^g f_* E + T_f(E).$$

Th. (Bismut, Gillet, Soulé, ¹⁹⁹², Faltings): For $f: X \rightarrow Y$ proper,

$$\begin{array}{ccc} \hat{K}(X)_Q & \xrightarrow{\tau \cdot \widehat{\text{ch}}} & \widehat{\text{CH}}(X)_Q \\ f_! \downarrow & & \downarrow f^* \\ \hat{K}(Y)_Q & \xrightarrow{\widehat{\text{ch}}} & \widehat{\text{CH}}(Y)_Q \end{array} \quad \text{commutes}$$

with $\tau = \widehat{\text{Td}}(Tf) \cdot (1 - \alpha(R(\overline{Tf})))$, R being the additive char. class with

$$R(L) = \sum_{\substack{l \geq 1 \\ l \text{ odd}}} (2S'(-l) + S(-l) \sum_{j=1}^l \frac{1}{j}) \frac{c_*(L)^l}{l!}$$

for a line bdl. L .

(Comprehensively written proof only for $\dim Y = 1$)

Assume an isometry g acting on M , with fixed pt. submanifold M^g .⁽⁸⁾

Def.: $\text{ch}_g(\bar{E}) := \text{Tr } g^* e^{-\langle \bar{N}^g \rangle^2 / 2\pi i} \in \Sigma \Omega^{2,2}(M^g)$,

$$\text{Td}_g(\bar{E}) := \text{Td}(\bar{E}^g) \text{ch}_g(1 \cdot \bar{E}^g \setminus \bar{E}^{g*})^{-1}$$

Let an arithm. var. X be acted upon by the group scheme of N -th roots of unity $\mu_N := \text{Spec } \mathbb{Z}[X]/X^{N-1}$. Set

$$\hat{K}^{MN}(X) := \left\{ \sum a_j \bar{E}_j + \gamma \mid \begin{array}{l} \bar{E}_j \text{ eq. vbdll., } \gamma \in \Sigma \Omega^{2,2}(X^2) \\ \text{ind} + \text{ind} \end{array} \right\}$$

modulo $\bar{E} \sim \bar{E}' + \bar{E}'' - \text{ch}_g(\bar{E})$.

TL. (K., Roessler): Eq. torsion forms provide direct images in \hat{K}^{MN} .

Th. (K., Roessler, arithm. Lefschetz fixed pt. formula) (2001):

Let N be the normal bdl. to $X^{MN} \rightarrow X$ for an arithm. variety

$\pi: X \rightarrow \text{Spec } \mathbb{Z}$ with μ_N -action. Then

$$\begin{array}{ccc} \hat{K}^{MN}(X) & \xrightarrow{\tau \cdot \rho} & \hat{K}^{MN}(X^{MN}) \\ \pi_! \downarrow & & \downarrow \pi_! \\ \hat{K}^{MN}(\text{Spec } \mathbb{Z}) & \longrightarrow & \hat{K}^{MN}(\text{Spec } \mathbb{Z}) \end{array} \quad \text{commutes}$$

with ρ being restriction to the fixed pt. subscheme and

$$\tau = \left(\sum (-1)^q \lambda^q \bar{N}^* \right)^{-1} \cdot (1 - a(R_g(N_C)))$$

With $\mathfrak{I}_L(e^{iq}, s) := \sum_{k>0} \frac{e^{ikq}}{k^s}$, R_g is the additive (9)

such that for line bdl. \mathcal{I} and

$$\tilde{R}_g(z) := \sum_{\ell \geq 0} [\mathfrak{I}'_L(g^\ell, -\ell) + \frac{1}{2}\mathfrak{I}_L(g^\ell, -\ell)] \sum_j \frac{z^j}{j!} \frac{c(z)^\ell}{\ell!}$$

$$\in \Sigma \mathcal{A}^{2,2}(X_C^{MN}),$$

$$R_g(z) := \tilde{R}_g(z) - \tilde{R}_g(z^*).$$

$$\text{Thus } \pi_!^{MN} \bar{E} = \pi_!^{MN} \frac{\bar{E}|_{X_C^{MN}}}{\sum (-1)^q q^2 N^*} - \int Td_g(N) R_g(N) ch_g(E).$$

Combining the Lefschetz formula with the arithm. GRR

$$\begin{array}{ccc} \hat{K}^{MN}(x)_Q & \xrightarrow{\cong \cdot \rho} & \hat{K}^{MN}(X_C^{MN})_Q \xrightarrow{\text{td} \cdot \hat{ch}} \hat{CH}(X_C^{MN}) \otimes \mathbb{C} \\ \pi_! \downarrow & & \pi_! \downarrow & & \pi_* \downarrow \\ \hat{K}^{MN}(\text{Spec } \mathbb{Z})_Q & \xrightarrow{\cong} & K^{MN}(\text{Spec } \mathbb{Z})_Q & \xrightarrow{\hat{ch}} & \hat{CH}(\text{Spec } \mathbb{Z}) \otimes \mathbb{C} \end{array}$$

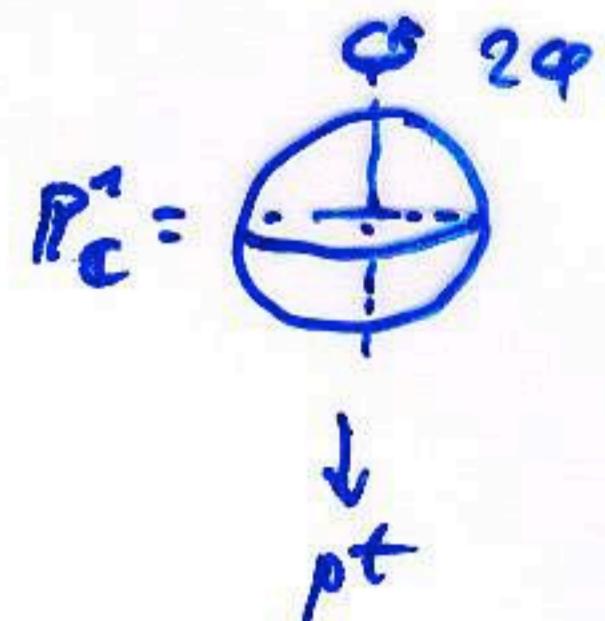
shows

$$\hat{ch}(\pi_!^{MN} \bar{E}) = \pi_*^{MN} (\widehat{Td}_g(\widehat{T\pi}) \widehat{ch}_g(\bar{E})) - \int_{X_C^{MN}} Td_g(T\pi) R_g(T\pi) ch_g(E)$$

Example:

$\mathbb{P}_{\mathbb{Z}}^1 \xleftarrow{\text{G}(k)} (k > 0)$
 $\pi \downarrow R^0 \pi_* \mathcal{O}(k) =: M$ rank $k+1$ \mathbb{Z} -module of
 $\text{Spec } \mathbb{Z} \xleftarrow{\text{homog. pol. of degree } k}$ in 2 variables

$\mathbb{G}_m = \text{Spec } \mathbb{Z}[X, \frac{1}{X}]$ -action on $\mathbb{P}_{\mathbb{C}}^1$



corresponding to rotation of S^2

M decomposes into rank 1 weight spaces w.r.t. this action,

which are generated by sections $s_m : (z_0 : z_1) \mapsto z_0^m z_1^{k-m}$.

$$\text{Thus } \widehat{\deg} \pi_* \mathcal{O}(k) = \widehat{\deg} R^0 \pi_* \mathcal{O}(k) + \frac{1}{2} \text{Tg}(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}(k))$$

$$= \sum_{m=0}^k e^{i(k-2m)\varphi} \log \|s_m\| + \frac{\partial}{\partial s} \Big|_{s=0} \sum_{l \geq 1} \frac{\sin(2l+k+1)\varphi}{\sin \varphi \cdot l^s (l+k+1)^s}$$

$\left(\frac{1}{4\pi} \int_0^{2\pi} d\alpha \int_0^\pi \cos \frac{m}{2}\vartheta \sin^{k-m-2} \frac{\vartheta}{2} \sin \vartheta d\vartheta \right)^{1/2}$

$$= -\frac{1}{2} \sum_{m=0}^k e^{i(k-2m)\varphi} \cancel{\log [(k+1)(m)]} + \frac{1}{2} \sum_{m=1}^{k+1} \frac{\sin(2m-k-1)\varphi}{\sin \varphi} \cancel{\log m}$$

$$+ \frac{\cos(k+1)\varphi}{\sin \varphi} R^{\text{rot}}(2\varphi) = \sum_{\substack{\text{north pole} \\ \text{south pole}}} \text{Td}_g(T\mathbb{P}_{\mathbb{C}}^1) R_g(T\mathbb{P}_{\mathbb{C}}^1) \text{ch}_g(\mathcal{O}(k))$$

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Applications:

i) Eq. torsion for circle actions

$\begin{matrix} \bar{E} \\ \downarrow \\ M \end{matrix}$ Hermitian vbdl., S^1 -action on M, \bar{E} ,

having (some) model over $\text{Spec } \mathbb{Z}$ with corresponding \mathbb{G}_m -action.

Lemma (Roessler, K. 2002): There is a rational function

$Q \in \mathbb{R}(x)$ such that for almost all $g_t = e^{2\pi i t}$

$$T_{g_t}(M, \bar{E}) = \underbrace{\int_M Td_{g_t}(TM) R_{g_t}(TM) ch_{g_t}(\bar{E})}_{\text{topological term, very non-rational in } g_t} + Q(e^{2\pi i t})$$

ii) Arithmetic Bott residue formula

Assume $\begin{matrix} X \\ \downarrow \\ \text{Spec } \mathbb{Z} \end{matrix}$ with \mathbb{G}_m action, S^1 -isometry at x_0 .

Classical Bott-residue formula: Let $\phi: \text{End}(C^n) \rightarrow \mathbb{C}$

be an $\text{Ad } GL(n, \mathbb{C})$ -invariant polynomial \Rightarrow

$$\int_{X_0} \phi(TX) = \sum_{p \text{ fixed pt.}} \frac{P(\Theta)}{\det \Theta} \quad \begin{matrix} P(\Theta) \\ \text{infinitesimal action} \\ \text{on } T_p X \end{matrix}$$

if S^1 acts with isolated fixed pts.

Th. (arithm. Bott residue formula, Roessler, K.) (2002):

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Consider $q_j \in \mathbb{N}$: $\sum_{j=1}^m q_j = n+1$ and e.g. vbdls. $\overset{\widehat{E}_j}{\downarrow}_X$ ($1 \leq j \leq m$).

Let K be the vector field induced by the S^1 -action \Rightarrow

$$\pi_* \prod \widehat{c}_{q_j}(\widehat{E}_j) \cdot (1 - \alpha(S_K(TX))) = \pi_* \overset{G_m}{\underset{\text{current}}{\prod}} \frac{\prod \widehat{c}_{q_j, k}(\widehat{E}_j)}{\widehat{c}_K^{\text{top}}(N)} \cdot (1 - \alpha(r_K(N_C))).$$

\uparrow
topological
Term at fixed pt.

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III) Jantzen sum formula for lattice representations of Chevalley schemes
(Kaiser, K. 2002)

G_c compact Lie group, $T \subset G_c$ maximal torus, λ dominant weight

induces line ball. \downarrow^{L_λ} such that
 G_c/T flag space

$H^*(G_c/T, L_\lambda)$ is the irred. G_c -repr. of highest weight λ .

Consider action of generic $t \in T$ on G/T , then

fixed pts $(G_c/T)^t \cong W_G$ Weyl group.

Atiyah-Bott '68: Classical Lefschetz formula \Rightarrow

$$\text{character } \chi_{P_G + \lambda} = \sum_{\mu \text{ weight}} \mu \cdot \dim H^*(G_c/T, L_\mu) = \sum_{w \in W_G} \frac{e^{2\pi i w \lambda(t)}}{\prod_{\alpha \text{ pos. root}} (1 - e^{-2\pi i w \alpha(t)})}.$$

Now consider this situation in an Arakelov context:

$G \rightarrow \text{Spec } \mathbb{Z}$ Chevalley scheme, P parabolic subgroup,

$X := G/P$ smooth flag variety, $A := P$ -module of weight λ , $A \cong \mathbb{Z}$,

$L_\lambda \rightarrow G/P$ induced by A

Th. (Jantzen sum formula): Let ℓ be the length of $w \in W_G$ such that $w^{-1}(\rho_G + \lambda)$ is dominant. Then

$$\begin{aligned} & -(-1)^\ell \sum_{\mu \text{ weight}} \mu \log \operatorname{covol} \overline{H^\ell(X, \mathbb{Z}_2)}_{\mu, \text{free}} \\ & + \sum_{q=0}^n (-1)^q \sum_{\mu \text{ weight}} \mu \log \# H^q(X, \mathbb{Z}_2)_{\mu, \text{tors}} \\ & = -\frac{1}{2} \sum_{\alpha \text{ root}} \sum_{k=1}^{\pm \langle \alpha^\vee, \rho_G + \lambda \rangle} (\pm \chi_{\rho_G + \lambda \mp k\alpha} \log k) - \frac{1}{2} \sum_{\alpha \text{ root}} \log \alpha^\vee(x_0) \cdot \chi_{\rho_G + \lambda} \\ & \quad x_0 \in \text{fixing metric on } G/p \end{aligned}$$

Method of proof: Arithm. fixed pt. formula \Rightarrow

$$\begin{aligned} & \sum_{\mu \text{ weight}} \mu \log \frac{\operatorname{covol} \overline{H^*(X, \mathbb{Z}_2)}_{\mu, \text{free}}}{\# H^*(X, \mathbb{Z}_2)_{\mu, \text{tors}}} - \frac{1}{2} T_g(X, \mathbb{Z}_2) \\ & = \underbrace{\text{fixed pt. expression}}_{\text{Grothendieck-Demazure}} \end{aligned}$$

Identify Hodge embeddings of $H^*(X, \mathbb{Z}_2) \rightsquigarrow$ description of L^2 -metrics

Computation of $T_g(X, \mathbb{Z}_2)$: Construct a tower of fibrations

$G/B \leftarrow$ Borel subgroup

\downarrow
 G/P_α with fibres being Hermitian symmetric spaces.

\downarrow
 G/G

(i) Value of torsion for symmetric spaces (K. 1995), functoriality of direct image in \widehat{K} (Kaiser, K. for this case, Ma)

\Rightarrow value of $T_g(G/B, \mathbb{Z}_\lambda)$

(ii) Now consider fibration $\begin{array}{c} G/B \\ \downarrow \\ G/P \end{array} \rightarrow P/B$, apply functoriality again

\Rightarrow value of $T_g(G/P, \mathbb{Z}_\lambda)$

Remark: This does not work for $G = G_2, F_4, E_8$; one has to use

quaternionic symm. spaces instead, and quat. torsion.

IV) Height of generalized flag var. G/P (Kaiser, K. 2002)

Def.: Height of arithm. var. X $\xrightarrow{\text{proj.}}$ $h(X) := \pi_X^* \widehat{G}(G(1))^{n+1} \in \mathbb{R}$

Properties: $(k_0 : - : k_n)_{\in \mathbb{Z}} \in \mathbb{P}^n \Rightarrow h((k_0 : - : k_n)) = \log \sqrt{\sum k_j^2}$

$h(X) \geq 0$ (Faltings)

$h(X)$ gives lower bound on height of integral pts. in X
(Bost-Gillet-Soule, Shouwu Zhang)

$$\log \# \{ s \in \Gamma(X, E(m)) \mid \|s\| \leq 1 \} \geq \text{rk } E \cdot h(X) \cdot \frac{m^{n+1}}{(n+1)!} + O(m^n \log m).$$

Let Ht denote the additive char. class which equals for

a line bdl. $Ht(\mathcal{I}) := \sum_{k=0}^{\infty} \frac{(-c_k(\mathcal{I}_\lambda))^k}{2(k+1) (k+1)!}$

(i.e. $Ht(x) = \frac{1}{2x} (\log |x| - \Gamma'(1) - E_7(-x))$).

Th.: For a certain canonical vbdll. $\overset{\mathbb{F}}{\downarrow}_{G/P}$,

$$h(G/P, \mathcal{I}_\lambda) = \sum_{(G/P)_C} Ht(F) ch(\mathcal{I}_\lambda).$$

This corresponds to more explicit formulae in terms of roots.

Ex.: $h(X_0^2 + \dots + X_g^2 = 0) = \frac{7531}{420}$ (Cassaigne-Milnor)

$$h(E_6/\mathrm{SO}(10)S^1) = \frac{9123967}{4620}$$

$$h(G(6, 15)) = 1.364 \times 10^{29} \in \mathbb{Z}$$

For the Lagrangian Grassmannian,

$$h(Sp(d)/U(d)) = \sum_{\varepsilon_1, \dots, \varepsilon_d} \frac{1}{\prod_{j < k} (\varepsilon_j j + \varepsilon_k k)} \sum_{l=1}^{\frac{d(d+1)}{2}} \sum_{j < k} \frac{1}{2 \ell (\varepsilon_j j + \varepsilon_k k)}$$

$$\cdot \left[\left(\sum \varepsilon_m m \right)^{\frac{d(d+1)}{2}} - \left(\sum \varepsilon_m m \right)^{\frac{d(d+1)}{2} - l+1} \left(\sum \varepsilon_m m - (2 - \delta_{jk} (\varepsilon_j j + \varepsilon_k k)) \right)^l \right].$$

Lemma: $h(G/P) = \sum_{k=1}^{\min\{2c(G)-2, n+1\}} \frac{k e}{2k}$ with $k \in \mathbb{Z}$,

$c(G)$:= Coxeter number of G .

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V) Hirzebruch proportionality principle for moduli of ab. var.

Classical Hirzebruch prop. princ.: G/k compact symm. space,

G'/k non-compact dual

(e.g. $G/k = \mathbb{P}^1_{\mathbb{C}} = \frac{\mathrm{SU}(2)}{\mathrm{SU}(1)\mathrm{U}(1)}$, $G'/k = \text{upper half plane} = \frac{\mathrm{SU}(1,1)}{\mathrm{SU}(1)\mathrm{U}(1)}$)

$\Gamma \subset G'$ cocompact, $\Gamma \backslash G'/k$ smooth \Rightarrow

$$\exists \iota: H^*(G/k) \rightarrow H^*(\Gamma \backslash G'/k)$$

such that for $\overset{E}{\downarrow}_{G/k}$, $\overset{E'}{\downarrow}_{G'/k}$ associated to same k -repr.

$$\iota(c_j(E)) = c_j(E').$$

Hirzebruch's proof: comparison of the Chern-Weil forms

representing the char. classes.

Hirzebruch-Mumford $\rightsquigarrow \exists \iota: H^*(\frac{\mathrm{Sp}(d-1)}{\mathrm{U}(d-1)}) \rightarrow H^*(\Omega_d)$

moduli of $\overset{\sim}{\Gamma}$ p.p. abelian vars.

such that

$$\iota(c_j(S)) = c_j(E)$$

tautol. bdl.
on Sp/U Hodge bdl.

Structure of $H^*(Sp(d-1)/U(d-1))$: Graded comm. ring generated by $c_j(s)$ with relations $c_{2j}(s \oplus s^*) = 0$ ($j > 0$), $c_d(s) = 0$.

Proof of prop. princ. for this case using Lefschetz-Riemann-Roch (clss.):

Let π_B^Y be an ab. scheme of rel. dim. d, $E := \text{Lie}(\pi_B)^\ast = (R\pi_X)_\ast^\ast$.

$$\text{Then } (I) R\pi_X \mathcal{O} \cong \Lambda^d E$$

$$(II) T_\pi \cong \pi^\ast E^\ast$$

Lefschetz-RR applied to a fibrewise action of g on π_B^Y

$$\text{states } ch_g(R\pi_X \mathcal{O}) = \pi_X^\ast [Td_g(T_\pi) ch_g(E)]$$

$$(I) // \qquad \qquad \qquad (II) //$$

$$ch_g(\Lambda E^\ast) \qquad \qquad \qquad \pi_X^\ast \pi^{g^\ast} Td_g(E^\ast)$$

$$\frac{c^{\text{top}}(E^\ast)}{Td_g(E)} \qquad \qquad \qquad // \\ Td_g(E^\ast) \cdot \pi_X^\ast \pi^{g^\ast} 1$$

$$\text{or } c^{\text{top}}(E^\ast) = Td_g(E \oplus E^\ast) \cdot \pi_X^\ast \pi^{g^\ast} 1.$$

a) Consider $g = \text{id}$. Then $c^{\text{top}}(E) = 0$.

b) Consider $g = -1$. Then $1 = Td_{-1}(E \oplus E^\ast) \cdot 4^d$

and from this easily $c_{2j}(E \oplus E^\ast) = 0$ ($j > 0$).

(19)

Now do the same proof in Arakelov geometry for

$\pi: Y \rightarrow B$ princ. pol. abelian scheme of rel. dim. d
 B proj. arithm. variety

Scale metrics on fibres such that $\text{vol } Y_B = 1$.

Then $\overline{R\pi_* G} = \Lambda \bar{E}^*$, $\overline{T\pi} = \pi^* \bar{E}^*$ (Berthelot, Breen, Kresing).

a) arithm. Riemann-Roch $\Rightarrow \hat{c}^{\text{top}}(\bar{E}) = a(Y)$

with $\gamma = \gamma'(0)$, $T_x^{(k)} = \frac{\gamma}{T\alpha(E)}$ as in the ex. for torsion forms.

b) fixed pt. formula \Rightarrow

$$\hat{c}_{2k}(E \oplus \bar{E}^*) = \left(\frac{2\gamma'(1-2k)}{3(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{j} \right) \cdot a(\text{ch}(E)^{[2k-1]} \cdot (2k-1)!).$$

Kor. 1: There is a ring homom.

$$h: \widehat{CH}^*(\mathbb{Sp}(d-1)/\mathbb{U}(d-1))_{\mathbb{Q}} \rightarrow \widehat{CH}^*(B)_{\mathbb{Q}} / (a(\gamma))$$

$$\text{given by } h(\hat{c}_k(\tilde{s})) = \hat{c}_k(\bar{E}) \cdot \left(1 + a \left[\sum_1^{d-1} \frac{\gamma'(1-2k)}{3(1-2k)} (2k-1)! \cdot \text{ch}(E)^{[2k-1]} \right] \right).$$

Proof uses Tamvakis' explicit description of $\widehat{CH}^*(\mathbb{Sp}(d-1)/\mathbb{U}(d-1))$.

$$\text{Kor. 2: } \hat{c}_1(E)^{\frac{d(d-1)}{2}+1} = a(r_d \cdot c_1(E)^{\frac{d(d-1)}{2}} + \phi(E) \cdot Y)$$

with $r_d \in \mathbb{R}$ explicit, ϕ a certain Chern-Weil form, namely

$$r_d = \frac{2^{\frac{(d-1)(d-2)}{2}+1} \prod_{k=1}^{d-1} (2k-1)!!}{(\frac{d(d-1)}{2})!} \sum_{k=0}^{d-2} \left(-\frac{2\pi'(-2k-1)}{3(-2k-1)} - \sum_j \frac{2k+1}{j} \right).$$

Conj.: This principle should extend to non-compact moduli

in Bongartz-Kramer-Kühn's framework (possibly up to

multiples of $\log 2$). In particular

$$h(M_d) = \frac{r_d+1}{2} \deg(M_d).$$

moduli of p.p. ab.var.

VI) Gross-Deligne period conj. (Roessler, Maillet)

X
 $\pi \downarrow$
 $\text{Spec } \mathbb{Z}$

apply fixed pt. formula to $\bar{E} := \sum (-1)^q \lambda^q \bar{\tau}_\pi$, then

$$\tau_\pi(\bar{E}) \stackrel{\text{Ray-Singer}}{=} 0, \text{ thus}$$

$$\widehat{ch}_{\mu_N}(\overline{R\pi_* E}) = \underbrace{\pi_*(Td_{\mu_N}(\bar{\tau}_\pi) \widehat{ch}_{\mu_N}(\bar{E}))}_{=0} - \underbrace{\int_{X_{\mathbb{C}}^{\mu_N}} Td_g(TX) ch_g(E) R_g(TX)}_{\int_{X_{\mathbb{C}}^{\mu_N}} c^{\text{top}}(TX^g) R_g(TX)^{[0]}}$$

→ special case of Gross-Deligne period conjecture
 (generalizing Chowla-Selberg)