Noether-Lefschetz and Gromov-Witten

The purpose of the seminar is to understand [7] and [8]. A slightly wider survey of these two papers can be found in [10]. We also need to understand [3] which plays a crucial role in both papers. For all the three papers mentioned above we need to cover some background material. Some recommended references are: [1] for the general theory of K3 surfaces, [6] for Gromov-Witten theory and [5] for automorphic forms.
The basic objects of study are 1-parameter families of K3 surfaces. To every such (suitably decorated) family we can associate certain intersection numbers on the moduli space of K3 surfaces called Noether-Lefschetz numbers. They are proved to be related to modular forms. On the other hand, for any such family, one "expects" a finite number of curves of prescribed genus and degree. In [7] Maulik and Pandharipande find a relationship between the number of curves on the 3-fold total space on one side and the Noether-Lefschetz numbers (shortly NL) of the family and the number of curves of fixed genus and degree on K3-surfaces on the other.
This relationship can be used in several directions. In a certain example, we can use modular forms in order to compute Noether-Lefschetz numbers and sequently use the (NL-enumerative) relation in order to compute the number of curves on K3-surfaces (see [8]). Another application ([7]) is to use the enumerative geometry in order to compute Noether-Lefschetz numbers for certain special families of K3-surfaces (e.g a general Lefschetz pencil of quartics). The second application is rather surprising: we use the theory of modular forms and "modern enumerative geometry" in order to solve a more classical geometric problem.

1 Overview (20.04)

2 Let's get familiar with K3 surfaces (27.04)

- what is a K3 surface, examples, why would we think that we can "count" curves on K3 surfaces?
- explain the Picard lattice
- the period map

- the surjectivity of the period map
- what is a quasi-polarization
- the definition of the moduli space of quasi-polarized K3 surfaces
- the statement of global Torelli (no proof)

A reference for this could be [1], or maybe [2] or [9]

3 Noether-Lefschetz for K3 surfaces and Noether-Lefschetz numbers (04.05)

- An overview of classical Noether-Lefschetz, Hodge theoretic approach (it should take 20-25 minutes)
- the Noether-Lefschetz locus for these moduli spaces via the period map (this is codimension 1!)
- define the divisors $P_{\Delta,\delta}$ and $D_{h,d}$
- define NL numbers and prove finiteness (section 1.4, 1.5, 1.6 in [7])

4 Modular forms(11.05)

this should give a short introduction to (classical) modular forms.

5 Vector valued modular forms of half integral weight and Heegner divisors(18.05)

- section 4.2 in [7]
- define Heegner divisors (4.3 in [7])

the last two topics can be found in [5], section 2.4

6 Borcherds' theorem for Heegner divisors (25.05)

- define the generating series Φ in [7] section 4.4
- compare Heegner divisors with $D_{h,d}$
- the computation of NL for K3's (section 4.4 in [7])

7 Borcherds theorem (01.06)

- state the theorem at page 30 in [7] and look at the proof in [3] and [4]

8 Gromov-Witten and Gopakumar-Vafa invariants (08.06)

- definitions
- examples of why are the genus zero GV invariants counting "the right thing"

An overview can be found in [6].

9 Introduction to toric varieties (15.06)

- cones and fans
- polytopes and homogeneous coordinate rings
- examples

10 Overview of mirror symmetry for CY complete intersections in toric varieties(22.06)

Ideally, this talk should present the "algorithm" of computing GW invariants of CY complete intersections in toric varieties via hypergeometric equations. More precisely, we will look at chapter 2 in [6] and if time allows, we will look at the rough idea in 5.5 (just for the quintic 3-fold!) .

11 GW of K3 surfaces, some conjectures on GW of K3's (29.06)

- define GW for K3 surfaces (section 2.3)
- conjectures 1 and 2
- maybe discuss the relationship between Conjecture 2 in genus 0 and Yau-Zaslow

12 The relationship between NL and GV invariants (06.07)

prove the main theorem in [7] (sections 3.1, 3.2)

13 NL for pencils of quartics (13.07)

prove Theorem 2 in [7] (sections 5.1-5.5)

References

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[6] D. Cox, S. Katz, Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs 68, American Mathematical Society, 1999.

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[9] D. Morrison, Lecture notes on the geometry of K3 surfaces, available at http://www.cgtp.duke.edu/ITP99/morrison/cortona.pdf

[10] R. Pandharipande, Maps, sheaves and K3 surfaces, available at http://www.math.princeton.edu/ rahulp/clay.ps