Cycles, Regulators and L^2–cohomology
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Disclaimer: These notes contain a very special selection of the literature and represent my own viewpoint only.

Part I

Algebraic K–theory, Higher Chow Groups and Motivic Cohomology

Literature/History

Milnor (Princeton UP), Weibel (homepage), Srinivas (Birkhäuser), Rosenberg (Springer), K-theory handbook (Springer), Bloch (homepage), Fulton (Springer), Levine Motives (AMS), Voevodsky Orange Book (Princeton UP), Mazza/Weibel (AMS), many articles by Bloch, Suslin, Friedlander, Voevodsky and Levine.

Class groups, Kummer-Vandiver conjecture, Picard group, Legendre/Jacobi/Hilbert Symbols, Dirichlets Unit Theorem, Brauer group, Whitehead group, Brauer group, Grothendieck group K_0 and K_n^{top} (1950's), Quillens K_n^{alg} (1970), Higher Chow groups, motivic cohomology, new Grothendieck topologies.

$\zeta(12) = \frac{691}{6825 \cdot 93555}\pi^{12}$ (Euler, Bernoulli) and $K_{22}(\mathbb{Z}) = \mathbb{Z}/691\mathbb{Z}$ (Soulé).

$K_0(R)$

A finitely generated projective module P is an R-module such that $P \oplus Q \cong R^N$.

Definition

R commutative integral domain with 1. Then

$$K_0(R) = \frac{\mathbb{Z}[\text{Iso classes of f.g. projective modules P}]}{\langle [P \oplus P'] = [P] + [P'] \rangle}.$$

$K_0(R) \to \mathbb{Z}$ Rank. $K_0(R) = \mathbb{Z}$ iff every P is stably trivial, i.e., $P \oplus R^M \cong R^N$.

Examples

- (R, m) local or PID, then $K_0(R) = \mathbb{Z}$ and every P is actually free.
- $R = \mathcal{O}_K$ ring of integers (Dedekind domain), then $K_0(R) = \mathcal{C}\ell(\mathcal{O}_K) \oplus \mathbb{Z}$. Every projective module of rank n can be written as $I \oplus R^{n-1}$ with $I \subset \mathcal{O}_K$ an ideal.
- Projective modules are locally free hence correspond to vector bundles. $K_0(X)$ can be defined for schemes in the same way.

$GL(R)$

Definition

$$GL(R) = \bigcup GL_n(R) = \lim GL_n(R).$$

$$E(R) = \{e_{ij}(a) = 1 + \delta_{ij} \cdot a \mid a \in R,\ i \neq j\}$$

elementary matrices.

Any upper triangular matrix $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in GL_{2n}(R)$ is elementary.

Same for lower triangular matrices.

Lemma (Properties=Axioms for Steinberg group)

- $e_{ij}(a)e_{ij}(b) = e_{ij}(a+b)$.
- $[e_{ij}(a), e_{k\ell}(b)] = 1$ *if* $j \neq k$ *und* $i \neq \ell$.
- $[e_{ij}(a), e_{jk}(b)] = e_{ik}(ab)$ *if* i, j, k *pairwise distinct.*
- $[e_{ij}(a), e_{ki}(b)] = -e_{kj}(-ba)$ *if* i, j, k *pairwise distinct.*

Whitehead trick

Lemma (Whitehead)

$$E(R) = [GL(R), GL(R)] = [E(R), E(R)]$$

is a perfect normal subgroup of $GL(R)$.

Definition (Bass)

Abelian group

$$K_1(R) = GL(R)/E(R) = H_1(GL(R), \mathbb{Z}).$$

Measures properties of $GL(R)$ somehow.

Proof of Whitehead Lemma

Proof.
(a) If $A \in GL_n(R)$ then

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in E(R).$$

(b)

$$\begin{pmatrix} [A,B] & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

(c) $e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$ if i, j, k pairwise distinct. □

Relative K_0 and K_1

$I \subset R$ ideal. For $* = 0, 1$ define

$$K_*(R, I) = \mathrm{Ker}(p_{1*} : K_*(D(R, I)) \to K_*(R)),$$

where $D(R, I) = \{(x, y) \in R \times R \mid x - y \in I\}$ (double ring). It induces long exact sequence

$$K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(R, I) \to K_0(R) \to K_0(R/I)$$

(not in general surjective)

Examples

- (R, m) local or euclidean, then $K_1(R) = R_{\mathrm{ab}}^{\times}$.
- In general $SK_1(R) := \mathrm{Ker}(\det : K_1(R) \to R^{\times})$.
- $R = \mathcal{O}_K$ ring of integers, then $SK_1(R) = 1$ trivial.
- There are PID with $SK_1(R) \neq 1$ (Bass).
- $SK_1(\mathbb{R}[x, y]/x^2 + y^2 - 1) = \mathbb{Z}/2\mathbb{Z}$.

Dirichlet's Unit Theorem

$$\mathcal{O}_K^\times/\mathrm{Torsion} \hookrightarrow \prod_\sigma \mathbb{R}, \quad r \mapsto \log|\sigma(r)|$$

is a lattice in the hyperplane $H = \{y_\sigma = y_{\bar\sigma}, \sum y_\sigma = 0\}$.

Class number formula:

$$\zeta_K^*(0) = -\frac{h}{w} \cdot R_K.$$

Order of vanishing $= r_1 + r_2 - 1$.

Example: $K = \mathbb{Q}$, $\zeta(0) = -\frac{1}{2}$, $h = 1$, $w = 2$, $R = 1$.

Milnor's K_2

Definition

$$K_2(R) := \mathrm{Ker}(St(R) \to E(R)),$$

where $St(R)$ is freely generated by $x_{ij}(a)$ with Whitehead's four relations imposed.

Theorem

$$0 \to K_2(R) \to St(R) \to E(R) \to 0$$

universal central extension of perfect group $E(R)$, i.e., $St(R)$ and $St_n(R)$, $n \geq 3$ are perfect and any central extension of $St(R)$ splits.

Examples and Computations

- $K_2(R) = H_2(E(R), \mathbb{Z})$.
- (Matsumoto, van der Kallen, Kolster) F field or local ring with residue field with ≥ 4 elements, then

$$K_2(F) = F^\times \otimes_\mathbb{Z} F^\times / \langle a \otimes (1-a),\ a \otimes -a,\ a \neq 0, 1 \rangle.$$

- F finite field, then $K_2(F) = 0$.
- $K_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $K_2(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \oplus \text{uniquely divisible}$, $K_2(\mathbb{C}) = \text{uniquely divisible}$.
- $K_2(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_p \mathbb{F}_p^\times$.
- $K_2(\mathcal{O}_K)$ is finite (Garland).

Milnor K-theory

Definition
R field or local ring with sufficiently large residue field,

$$K_n^M(R) = R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times / \langle a \otimes (1-a) \otimes a_3 \otimes \cdots , \; a \otimes -a \otimes a_3 \otimes \cdots , \; a \neq 0, 1$$

Bloch-Kato Conjecture: $(\ell \neq \mathrm{char}(R))$

$$K_n^M(R)/\ell = H^n_{\mathrm{et}}(R, \mu_\ell^{\otimes n})$$

for R a field or a smooth, local k-algebra (Rost, Voevodsky).

$K_n^M(\mathcal{O}_K) = (\mathbb{Z}/2\mathbb{Z})^{r_1}$ for $n \geq 3$ (Bass/Tate).

Singular Topological Homology

$X \in \mathrm{Top}.$

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \ \sum x_i = 1\} \in \mathrm{Top}.$$

$$C_n(X, \mathbb{Z}) = \mathbb{Z}[f : \Delta_n \to X \text{ continuous}].$$

$$H_n^{\mathrm{sing}}(X, \mathbb{Z}) = \frac{\mathrm{Ker}(C_n(X, \mathbb{Z}) \xrightarrow{\partial} C_{n-1}(X, \mathbb{Z}))}{\mathrm{Im}(C_{n+1}(X, \mathbb{Z}) \xrightarrow{\partial} C_n(X, \mathbb{Z}))}.$$

Similar for cubes $\square^n = [0, 1]^n$.

CW complexes

$$X = \bigcup_{n=0}^{\infty} X_k \in \mathrm{Top}$$

Hausdorff, compactly generated. Each X_n is obtained from X_{n-1} by successively adjoining n–cells Δ_n (colimit=pushout):

$$X_n = X_{n-1} \coprod_{\partial \Delta_n} \Delta_n.$$

X has weak topology: $A \subset X$ closed iff $A \cap X_n$ closed for all n.

Homotopy groups

$(X, *) \in \mathrm{Top}_*$. Abelian group (for $n \geq 2$)

$\pi_n(X, *) = \text{Homotopy classes of } f : (S^n = \Delta_n \coprod_{\text{equator}} \Delta_n, *) \to (X, *).$

Hurewicz Map:

$$\pi_n(X, *) \longrightarrow H_n^{\text{sing}}(X, \mathbb{Z}).$$

Classifying Space BG and EG

G=group (discrete topology). There are CW complexes BG and EG together with a topological fibration $\pi : EG \to BG$ with fiber G and a cellular (and nice) G-action on EG with quotient BG. BG=Eilenberg-MacLane space for G, i.e. $\pi_1(BG, *) = G$ and $\pi_i(BG, *) = 0$ for $i \geq 2$.

Construction of BG: $X_0 = G$, $X_n = X_{n-1} * G$ (join by lines), $EG = \lim X_n$, $BG = EG/G$.

+-Construction of Quillen

Theorem (Quillen)

$(X, *)$ connected CW complex. $N \subset \pi_1(X, *)$ perfect, normal subgroup. Then there is a map of CW complexes $f : X \to X^+$ such that

1. *f_* is quotient map $\pi_1(X, *) \to \pi_1(X, *)/N$.*
2. *For all local systems L on X^+ one has $H_i(X, f^* L) = H_i(X^+, L)$.*

Proof.

Attach 2-cells to kill N, hence (1). Then attach 3-cells to correct homology in (2). □

Theorem (Kan/Thurston)

X connected CW-complex. Then there is a group $T = T(X)$ and a perfect subgroup $N \subset T$ such that X is homotopy equivalent to BT^+.

Higher K-groups of Quillen

Definition

$$K_n(R) = \pi_n(K_0(R) \times BGL(R)^+, *) \text{ for } n \geq 0.$$

Theorem

This is old definition for $n = 1, 2$.

Proof.

$n = 0$ clear by definition. $n = 1$ by property (1). For $n = 2$ note that $K_2(R) = H_2(E(R), \mathbb{Z})$ and $BE(R)^+$ is simply connected. Hence $BE(R)^+$ is a universal covering of $BGL(R)^+$. Since $BE(R)^+$ is 1–connected we have $\pi_2(BGL(R)^+) = \pi_2(BE(R)^+) = H_2(BE(R)^+) = H_2(BE(R), \mathbb{Z}) = H_2(E(R), \mathbb{Z}) = K_2(R)$. $\square$

Other models and properties

Other models for K-theory spaces:

- (Volodin) $BGL(R)^+ = BGL(R)/X(R)$, where $X(R)$ is acyclic subcomplex with $\pi_1(X(R),*) = St(R)$. Concretely $X(R) = \bigcup_{n,\sigma} BT_n^\sigma(R)$, $\sigma \in \Sigma_n$.
- (Karoubi-Villamayor) R regular ring, $BGL^+(R) = BGL(R[\Delta_\bullet])$ topological realization of simplicial ring $R[\Delta_\bullet] : \cdots \to R[t_0,t_1] \to R[t_0] \to R \to 0$.
- (Quillen) $\Omega BQ\mathcal{A}$, $\mathcal{A} =$ category of projective R-modules.

Further properties: $BGL(R)^+$ is a commutative H–group and $K_*(R)$ is a graded commutative $K_0(R)$-algebra (Loday). Hurewicz map $K_n(R) \otimes \mathbb{Q} \hookrightarrow H_n(GL(R),\mathbb{Q})$ has as image the primitive elements (Milnor/Moore theorem). $GL(R)$ can be replaced by GL_m for m large (Suslin stability).

A Filtration

$K_*(R) \otimes \mathbb{Q}$ is a special λ–ring. This induces Adams operations and a γ-filtration with graded pieces

$$K_n(R) \otimes \mathbb{Q} = \bigoplus_p \mathrm{Gr}^p_\gamma K_n(R) \otimes \mathbb{Q}.$$

Jouanolou's trick: For every X=smooth algebraic variety$/k$ there is an affine $\mathbb{A}^n$–torsor $\mathrm{Spec}(R) \to X$. We can thus define $K_n(X) := K_n(R)$. Example: $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \to \mathbb{P}^n$ (picture). Then

$$K_n(X) \otimes \mathbb{Q} = \bigoplus_p \mathrm{Gr}^p_\gamma K_n(X) \otimes \mathbb{Q}.$$

Motivic cohomology is a way to define graded pieces integrally.

Examples and computations

Theorem (Quillen)

$F = \mathbb{F}_q$ finite field. Then $K_{2n}(F) = 0$ for $n \geq 1$ and $K_{2n-1}(F) = \mathbb{Z}/(q^n - 1)\mathbb{Z}$ for $n \geq 1$.

Theorem (Borel)

K a number field. Let d_n be the vanishing order of $\zeta_K(1-n)$. The rank of $K_{2n}(\mathcal{O}_K)$ is 0 for $n \geq 1$ and the rank of $K_{2n-1}(\mathcal{O}_K)$ is equal to d_n for $n \geq 1$. $d_1 = r_1 + r_2 - 1$ for $n = 1$ and $d_{2k} = r_2$ or $d_{2k+1} = r_1 + r_2$ for $k \geq 1$. Furthermore $\zeta_K^*(1-n) = q_n \cdot R_n^B(K)$, $q_n \in \mathbb{Q}^\times$, R_n^B=Borel regulator (covolume of lattice in R^{d_n}).

Examples: $K_3(\mathbb{Z}) = \mathbb{Z}/48\mathbb{Z}$, $K_4(\mathbb{Z}) = 0$, $K_5(\mathbb{Z}) = \mathbb{Z}$, $K_6(\mathbb{Z}) = 0$, $K_7(\mathbb{Z}) = \mathbb{Z}/240\mathbb{Z}$, $K_8(\mathbb{Z}) = 0$, ..., $K_{22}(\mathbb{Z}) = \mathbb{Z}/691\mathbb{Z}$, etc.

Optimistic Finiteness Conjectures

Conjecture (Bass)

R regular, finitely generated $\mathbb{Z}$-algebra. Then $K_n(R)$ is finitely generated. If R is not regular, one may take $G = K'$-theory instead.

Theorem (Quillen/Grayson)

True for Dedekind rings $R = \mathcal{O}_K$ and K_n for curves over finite fields ($\dim(X) \leq 1$ regular).

Conjecture (Lichtenbaum)

$$\zeta_K^*(1-n) = \pm 2^? \frac{|K_{2n-2}(\mathcal{O}_K)|}{|K_{2n-1}(\mathcal{O}_K)_{\mathrm{tors}}|} \cdot R_n^B.$$

Evidence: For totally real fields K and $n = 2$ (Birch-Tate conjecture) this follows from a result of Mazur and Wiles (Iwasawa main conjecture). Abelian case by Fleckinger/Kolster/Nguyen Quang-Do and Huber/Kings.

Chow Groups

X equidimensional, quasi-projective$/F$.

$$CH^p(X) = \frac{\mathbb{Z}[W \text{ irreducible codim p subvariety}]}{\langle \mathrm{div}(f) \mid f \in k(W)^\times \ \mathrm{codim}(W) = p-1 \rangle}$$

Goal: Extend localization sequence for $U = X \setminus A$, A closed:

$$CH^p(X,1) \to CH^p(U,1) \to CH^{p-r}(A) \to CH^p(X) \to CH^p(U) \to 0.$$

$CH^p(-,n)$ Bloch's higher Chow groups (Borel–Moore theory).

Examples

- X smooth, quasi-projective, $CH^1(X) = \mathrm{Pic}(X)$.
- X compact Riemann surface, Abel-Jacobi map $(X, *) \to \mathrm{Jac}(X)$, $P \mapsto (\int_*^P \omega_1, \ldots, \int_*^P \omega_g)$ where ω_i runs through a basis of 1-forms. Induces an isomorphism $CH^1(X) \to \mathrm{Jac}(X)$.
- X smooth, projective (Kähler) manifold, Albanese map $(X, *) \to \mathrm{Alb}(X)$, $P \mapsto (\int_*^P \omega_1, \ldots, \int_*^P \omega_g)$ as above. $CH_0^n(X) \to \mathrm{Alb}(X)$ is surjective but not injective in general (Mumford $n = \dim(X) = 2$).

Gersten Resolution

Sheafify Quillen K–theory in Zariski topology:

$$0 \to \mathcal{K}_n \to \bigoplus_{x \in X^{(0)}} i_* K_n(k(x)) \to \bigoplus_{x \in X^{(1)}} i_* K_{n-1}(k(x)) \to \dots$$

$$\dots \to \bigoplus_{x \in X^{(n-1)}} i_* k(x)^\times \xrightarrow{\mathrm{div}} \bigoplus_{x \in X^{(n)}} i_* \mathbb{Z} \to 0.$$

Flasque resolution of $\mathcal{K}_n$ in Zariski topology (Bloch/Ogus).

Theorem (Quillen, Bloch's formula)

X smooth, quasi-projective/F, then $CH^n(X) = H^n(X, \mathcal{K}_n)$.

Same for Milnor K-theory sheaf (Moritz Kerz, thesis 06/2008).

Bloch's Higher Chow Groups

$\Delta_F^n = \mathrm{Spec}(F[t_0, \dots, t_n]/\sum t_i - 1)$ algebraic simplex with $n+1$ codimension 1 faces $\{t_i = 0\}$. X quasi-projective, equidimensional variety$/F$.

$Z^p(X, n) = \mathbb{Z}[W \subset X \times \Delta^n \text{ irred. subvariety of codim p, admissible}]$.

$$\partial : Z^p(X, n) \to Z^p(X, n-1), W \mapsto \sum_{i=0}^{n} (-1)^i W \cap \{t_i = 0\}.$$

$$CH^p(X, n) = H_n(Z^p(X, \bullet), \partial).$$

Cubical version $H_n(C^p(X, \bullet), \partial)$ with $W \subset X \times \square_F^n = X \times (\mathbb{P}^1 \setminus \{1\})^n$ up to degenerate cycles.

$$\partial W := \sum_{i=1}^{n} (-1)^{i-1} \left(W \cap \{z_i = 0\} - W \cap \{z_i = \infty\} \right).$$

Properties (proved by Bloch)

- Covariant for proper maps and contravariant for flat maps.
- $CH^*(X,*)$ has product structure (add p and n) for smooth X.
- Homotopy invariance $CH^p(X \times \mathbb{A}^m, n) = CH^p(X, n)$.
- Localization sequence as above for $U = X \setminus A$.
- $K_n^M(R) = CH^n(R, n)$ for R is a field (Nesterenko/Suslin, Totaro) or local and smooth with sufficiently large or infinite residue fields (Elbaz-Vincent/SMS, Kerz).
- Beilinson/Soulé Vanishing: $CH^p(F, n) \otimes \mathbb{Q} = 0$ for $n \geq 2p \geq 1$. True for $p = 0$ and $p = 1$.
- There are HCG over Dedekind domains (Levine).
- Bloch's formula generalized: $CH^p(X, 1) = H^{p-1}(X, \mathcal{K}_p)$.

BLLFS Spectral Sequence

There exists a spectral sequence for X smooth

$$CH^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X).$$

(Bloch/Lichtenbaum, Levine Friedlander/Suslin).

It degenerates over $\mathbb{Q}$ and we get a Riemann-Roch statement (Bloch, Levine)

$$\mathrm{gr}^p_\gamma K_n(X) \otimes \mathbb{Q} = CH^p(X, n) \otimes \mathbb{Q},$$

Computations/Exercises

Theorem (Bloch)

X smooth, quasi-projective. Then $CH^1(X,n) = 0$ for $n \geq 2$ and $CH^1(X,1) = H^0_{\mathrm{Zar}}(X, \mathcal{O}_X^\times)$.

Exercise (1): Prove this!
Hint: Use cubical coordinates and localization to reduce to a field F. Then, if $W = \mathrm{div}(F(x_1, \ldots, x_n)) \in C^1(F,n)$ satisfies $\partial(W) = 0$, you may assume that the intersection of W with every codim 1 face is empty. Now construct a function $G(x_1, \ldots, x_{n+1})$ such that $\mathrm{div}(G)$ has boundary W.

Exercise (2): Use $CH^*(\mathbb{P}^n_F) = \mathbb{Z}[h]/(h^{n+1})$, $h = c_1(\mathcal{O}(1))$, localization and homotopy invariance to compute $CH^*(\mathbb{P}^n_F, *)$ as an algebra over $CH^*(F,*)$.

Exercise (3): Look at the cubical "Totaro" cycles

$$C_a \in C^2(F,3) : x \mapsto [1 - \frac{a}{x}, 1 - x, x] \in \square^3$$

and compute ∂C_a for $a \in F$.

Part II

Cohomology, Motives and Regulators

Literature

Voisin I/II (Cambridge), Carlson/SMS/Peters (Cambridge), Crashkurs (SMS), K-theory Handbook (Springer), various papers of Bloch/Kato, Deninger/Scholl, Goncharov, Levine (Mixed Motives, AMS), Kerr/Lewis/SMS (Compositio 2006), Nori's unpublished work.

De Rham Cohomology

Definition (De Rham)

X/F smooth algebraic variety. $\Omega^i_{X/F}$ sheaf of algebraic i–forms. $H^i_{dR}(X/F)$ is the F–vector space

$$H^i_{dR}(X/F) = \mathbb{H}^i_{\mathrm{Zar}}(X, \Omega^\bullet_{X/F}).$$

Hodge filtration:

$$F^p H^i_{dR}(X/F) = \mathbb{H}^i_{\mathrm{Zar}}(X, \Omega^{\geq p}_{X/F}).$$

Periods

X/F smooth, projective variety, $\sigma : F \hookrightarrow \mathbb{C}$ an embedding, $X(\mathbb{C})$ associated compact complex manifold. Let $H^i(X,\mathbb{Z})$ and $H_i(X,\mathbb{Z})$ be singular (co)homology with piecewise differentiable chains Γ.

Theorem (Period Isomorphism)

$$H^n_{dR}(X/F) \otimes_F \mathbb{C} = \mathrm{Hom}(H_n(X(\mathbb{C}),\mathbb{Q}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$
$$\omega \mapsto \left(\Gamma \mapsto \int_\Gamma \omega\right).$$

There is a also version for pairs (X,D), D NCD in X. Integrating closed algebraic n–forms $\omega \in \Omega^n_{X/F}$ ($\dim(X) = n$) over n–chains, we get Kontsevich/Zagier type periods $\int_\Gamma \omega$, if everything is defined over $F \subset \overline{\mathbb{Q}}$.

MZV

Multiple zeta values ($n_m \geq 2$)

$$Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m) := \sum_{k_1<k_2<\cdots<k_m} \frac{z_1^{k_1}\cdots z_m^{k_m}}{k_1^{n_1}\cdots k_m^{n_m}}$$

$$\zeta(n_1,\ldots,n_m) := Li_{n_1,\ldots,n_m}(1,\ldots,1) = \int_0^1 \frac{dt}{1-t} \circ \frac{dt}{t} \circ \cdots \frac{dt}{t} \circ \cdots$$

Iterated integral. Take geometric series and integrate (Kontsevich).

Example:

$$\zeta(3) = \int_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz} \notin \mathbb{Q}.$$

Complex Forms

Let $F = \mathbb{C}$. Then there is an inclusion of (double) complexes

$$\mathbb{C} \hookrightarrow (\Omega^\bullet_{X/\mathbb{C}}, d) \hookrightarrow (\mathcal{E}^{\bullet,\bullet}, \partial, \bar{\partial})$$

$\mathcal{E}^{p,q}$ is the sheaf of $\mathbb{C}$–valued differentiable (p,q)–forms

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Total complex: $\mathcal{E}^n = \bigoplus \mathcal{E}^{p,q}$ with differential $d = \partial + \bar{\partial}$.

The resolution induces an isomorphism

$$H^n_{dR}(X/\mathbb{C}) = \frac{\mathrm{Ker}\left(H^0(X, \mathcal{E}^n) \to H^0(X, \mathcal{E}^{n+1})\right)}{\mathrm{Im}\left(H^0(X, \mathcal{E}^{n-1}) \to H^0(X, \mathcal{E}^n)\right)}.$$

Hodge decomposition

Theorem (Hodge decomposition)

$X(\mathbb{C})$ compact Kähler manifold, e.g. projective. Then every class $\alpha \in H^m(X(\mathbb{C}), \mathbb{C})$ has a harmonic representative α_0 with $\Delta\alpha_0 = 0$. If $\alpha = \sum \alpha^{r,s}$ then every $\alpha^{r,s}$ has a harmonic representative.

Corollary

$$H^m(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{r+s=m} H^{r,s}(X),$$

where $H^{p,q}(X) = H^q(X(\mathbb{C}), \Omega^p_{X/\mathbb{C}})$, a complex vector space of harmonic (p,q)–forms.

$$F^p H^m(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{r \geq p} H^{r,m-r}(X).$$

Pure Hodge Structures

A Pure Hodge structure of weight m is a free $\mathbb{Z}$–module $H = H_{\mathbb{Z}}$ together with a descending filtration

$$H_{\mathbb{C}} = H \otimes \mathbb{C} = F^0 \supset F^1 \supset F^2 \supset \cdots \supset$$

such that $H_{\mathbb{C}} = F^p \oplus \overline{F^{m-p+1}}$. Denote $H^{p,q} = F^p / F^{p+1}$. Then $H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$.

Mixed Hodge Structures

A Mixed Hodge structure is a free $\mathbb{Z}$–module $H = H_{\mathbb{Z}}$ together with two filtrations:

- Increasing Weight Filtration $W_\bullet$ of $H_{\mathbb{Q}}$.
- Decreasing Hodge Filtration $F^\bullet$ of $H_{\mathbb{C}}$.

These are compatible: $F^\bullet$ induces on Gr_m^W a pure Hodge structure of weight m.

Example

$X(\mathbb{C})$ compact Kähler manifold, D smooth divisor in X. Gysin sequence

$$\to H^{k-2}(D,\mathbb{Z}) \to H^k(X,\mathbb{Z}) \to H^k(U,\mathbb{Z}) \to H^{k-1}(D,\mathbb{Z}) \to H^{k+1}(X,\mathbb{Z})$$

The cohomology groups of X and D have pure Hodge structures, but the cohomology of U has a mixed Hodge structure with $W_0H^*(U,\mathbb{Z}) = \mathrm{Im}H^*(X,\mathbb{Z})$ and $W_1H^*(U,\mathbb{Z}) = H^*(U,\mathbb{Z})$.

Logarithmic De Rham Complex

(X, D) as above. Then we have exact sequence of complexes of sheaves

$$0 \to \Omega_X^\bullet \to \Omega_X^\bullet(\log D) \xrightarrow{\mathrm{Res}} \Omega_D^{\bullet-1} \to 0,$$

where $\Omega_X^\bullet(\log D)$ is generated by $\frac{dz_1}{z_1}$ and dz_i for $i \geq 2$ if $D = \{z_1 = 0\}$ locally. The filtration $W_0\Omega_X^\bullet(\log D) = \Omega_X^\bullet$ induces the weight filtration on hypercohomology.

More Boundary Divisors

Let D be a NCD in X. Define $\Omega_X^\bullet(\log D)$ by generators $\frac{dz_i}{z_i}$ for $i \leq m$ and dz_j for $j \geq m+1$ if locally $D = \{z_1 \cdots z_m = 0\}$. Then there is again a weight filtration $W_\bullet$ by order of poles on $\Omega_X^\bullet(\log D)$ with $W_0\Omega_X^\bullet(\log D) = \Omega_X^\bullet$ and

$$\mathrm{Gr}_k^W \Omega_X^\bullet(\log D) = j_* \Omega_{D_k}^{\bullet - k},$$

with $j : D_k = \bigcap_{\mathrm{k-fold}} D_i \hookrightarrow X$.

Deligne's Mixed Hodge Structures

Deligne has defined mixed Hodge structures on the cohomology of varieties (possibly singular and not compact), even on simplicial varieties (Hodge III).

This extends to locally constant coefficients $H^n(X, \mathbb{V})$ by the work of M. Saito and S. Zucker.

Motive of a cycle

Assume $W \in \mathrm{Ker}\left(CH^p(X,n) \to H^{2p-n}(X,\mathbb{Q})\right)$. This defines an extension of mixed Hodge structures (Bloch):

$$0 \to H^{2p-n-1}(X) \to \mathbb{E} \to \mathbb{Z}(-p) \to 0.$$

where $\mathbb{Z}(-p)$ is the Tate–Hodge structure of weight $2p$. Extension class:

$$[\mathbb{E}] \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-p), H^{2p-n-1}) = J^{p,n}(X)$$

by Carlson' theory. It is known that this extension class coincides up to a constant with Bloch's Abel–Jacobi map.

Examples

Example I: 2 points $[P]-[Q] \in CH^1(X)$, compact curve X:

$$0 \to H^1(X) \to H^1(X \backslash \{P,Q\}) \to \mathbb{Z}(-1) = \mathrm{Ker}\left(H^0(\{P,Q\}) \to H^2(X)\right) \to 0$$

Extension class: $\alpha \mapsto \int_P^Q \alpha$ for all 1–forms α.

More generally Abel–Jacobi map $CH^p(X) \to J^p(X)$:

$$0 \to H^{2p-1}(X) \to H^{2p-1}(X \setminus |W|) \to \mathbb{Z}(-p) \to 0,$$

$$\mathbb{Z}(-p) \subset \mathrm{Ker}\left(H^{2p}_{|W|}(X) \to H^{2p}(X)\right).$$

Example II: $[a]-[1] \in CH^1(F,1)$:

$\mathrm{Coker}\left(H^0(\mathbb{P}^1 \setminus \{0,\infty\}) \to H^0(\{1,a\})\right) = \mathbb{Z}(0) \to$
$H^1(\mathbb{P}^1 \setminus \{0,\infty\},\{1,a\}) \to H^1(\mathbb{P}^1 \setminus \{0,\infty\}) = \mathbb{Z}(-1) \to 0.$

Extension class: $\log(a) = \int_1^a \frac{dz}{z}$.

General Case

Assume W as above, $U = \Delta^n_X \setminus |W|$.
$\partial U = \partial\Delta^n_X \setminus |\partial W|$, hence by weak purity

$$H^{2p-2}(X) \to H^{2p-2}(\partial\Delta^n_X) \to H^{2p-1}(U, \partial U) \to H^{2p-1}(U) \to H^{2p-1}(\partial U$$

We need: $H^i(\partial\Delta^n_X) = H^i(X) \oplus H^{i-n+1}(X)$, i.e., $\partial\Delta^n$ is like a real $(n-1)$-sphere. But by a diagram chase

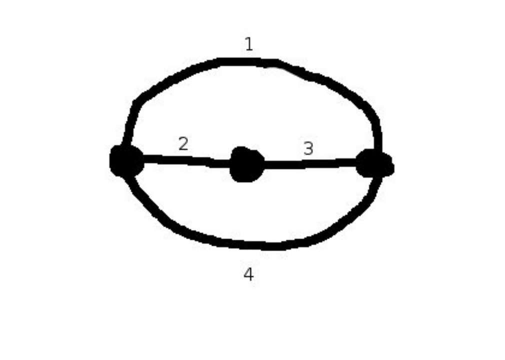
$$\mathrm{Ker}\left(H^{2p-1}(U) \to H^{2p-1}(\partial U)\right) \subseteq \mathrm{Ker}\left(H^{2p}_{|W|}(\Delta^n_X)^o \xrightarrow{\beta} H^{2p}_{|\partial W|}(\partial\Delta^n_X)^o\right),$$

($\circ$=forgetting supports) hence

$$0 \to H^{2p-n-1}(X) \to \mathbb{E} \to \mathbb{Z}(-p) \to 0.$$

Graph Hypersurfaces

A graph Γ

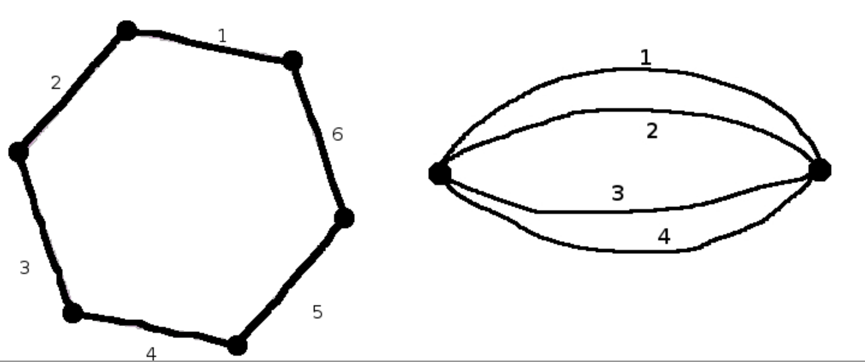


defines a polynomial equation

$$\Psi_\Gamma = \sum_T \prod_{e \notin T} x_e = 0.$$

T runs through all spanning trees (no loops).

Examples



$$\Psi = x_1 + x_2 + x_3 + x_4 + x_5 + x_6, \quad \Psi = x_1 x_2 x_3 x_4 \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right)$$

Motive of a Feynman graph

Log–divergent case: $2n$ edges, n loops: Motive is

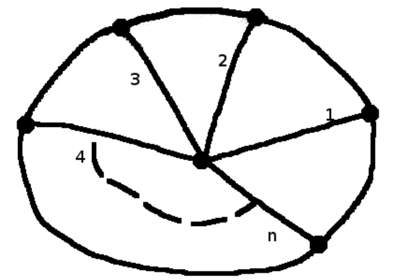
$$H^{2n-1}(\mathbb{P}^{2n-1} \backslash \{\Psi_\Gamma = 0\}, \bigcup_{i=0}^{2n-1} \{x_i = 0\})$$

This defines a period:

$$P(\Gamma) := \int_{\sigma^{2n-1}} \frac{\Omega}{\Psi_\Gamma^2}.$$

σ^{2n-1}=topological simplex.

Log-Divergent Feynman Motives



Periods (Broadhurst-Kreimer, Bloch-Esnault-Kreimer):

$$P(\Gamma) := \int_{\sigma^{2n-1}} \frac{\Omega}{\Psi_\Gamma^2} = const \cdot \zeta(2n-3)!$$

Mixed Tate-Motives ? Hopf-Algebra !

Nori's Abelian Category of Mixed Motives

Abelian category $NMM(k) = \mathrm{Rep}(G_{\mathrm{mot}})$ ($k \subset \mathbb{C}$) with objects (X, Y, i) and morphisms of triples. "Good objects" are such that $H_j(X(\mathbb{C}), Y(\mathbb{C})) = 0$ for $j \neq i$. $\mathbb{Z}(1) = H_1(\mathbb{G}_m)$ inverted.

Lemma (Basic Lemma)

$X(\mathbb{C})$ affine, $\dim(X) = n$, $Z \subset X$ closed, $\dim(Z) \leq n - 1$. Then there is a closed subset $Y \supset Z$ such that

- $\dim(Y) \leq n - 1$.
- *$H_i(X(\mathbb{C}), Y(\mathbb{C})) = 0$ for $i \neq n$.*
- *$H_n(X(\mathbb{C}), Y(\mathbb{C}))$ finitely generated.*

Lemma gives rise to a filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ with $H_j(X_j, X_{j-1})$ finitely generated.

Deligne Cohomology

X compact Kähler manifold (e.g. projective). Deligne cohomology:

$$H^i_{\mathcal{D}}(X,\mathbb{Z}(p)) = \mathbb{H}^i(X_{\mathrm{an}}, (2\pi i)^p\mathbb{Z} \to \mathcal{O}_X \to \cdots \to \Omega_X^{p-1}).$$

For X smooth, quasi-projective/$\mathbb{C}$ define:

$$H^i_{\mathcal{D}}(X,\mathbb{Z}(p)) = \mathbb{H}^i(\overline{X}_{\mathrm{an}}, \mathrm{Cone}(Rj_*\mathbb{Z}(p) \oplus F^p \hookrightarrow \Omega^\bullet_{\overline{X}}(\log D)[-1])).$$

Examples:

$$H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) = H^1(\mathcal{O}^*_{X,\mathrm{alg}}) = \mathrm{Pic}(X), \quad H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) = H^0(\mathcal{O}^*_{X,\mathrm{alg}}).$$

Intermediate Jacobians

Albanese: $(d = \dim_{\mathbb{C}}(X))$

$$0 \to \mathrm{Alb}(X) = \frac{H^0(X, \Omega^1_X)^*}{H_1(X, \mathbb{Z})} \to H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d)) \to H^{d,d}(X, \mathbb{Z}) = \mathbb{Z} \to 0.$$

Intermediate Jacobian of Griffiths:

$$0 \to J^p(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^p + H^{2p-1}(X, \mathbb{Z})} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to F^p H^{2p}(X, \mathbb{Z}) \to 0.$$

Generalized Intermediate Jacobian:

$$0 \to J^{p,n}(X) = \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p + H^{2p-n-1}(X, \mathbb{Z}(p))} \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to F^p H^{2p-n}(X, \mathbb{Z}(p)) \to 0$$

Deligne Class

We want to construct a map

$$cl^{p,n} : CH^p(X,n) \to H_{\mathcal{D}}^{2p-n}(X,\mathbb{Z}(p)),$$

which restricts to $CH^p_{\mathrm{hom}}(X,n) \to J^{p,n}(X)$.

Examples:

$$CH^d(X) \to H_{\mathcal{D}}^{2d}(X,\mathbb{Z}(d)), \quad CH^1(X,1) \xrightarrow{\mathrm{id}} H_{\mathcal{D}}^1(X,\mathbb{Z}(1)),$$

$$CH^2(X) \to H_{\mathcal{D}}^4(X,\mathbb{Z}(2)).$$

(generalization of: Albanese map, algebraic invertible functions (GAGA), Griffiths Abel–Jacobi map)

Behaviour

Theorem (Green/Voisin)

$X \subset \mathbb{P}^4$ very general hypersurface of degree ≥ 6, then

$$cl^2 : CH^2(X)_{\mathrm{hom}} \to J^2(X)$$

has torsion image.

Theorem (SMS,JAG1997)

$X \subset \mathbb{P}^3$ hypersurface of degree $d \geq 1$, then

$$cl^{2,1} : CH^2(X,1)_{\mathrm{hom}} \to J^{2,1}(X) = \frac{H^2(X,\mathbb{C})}{F^2 + H^2(X,\mathbb{Z}(2))}$$

has countable image modulo $\mathrm{NS}(X) \otimes \mathbb{C}^*$. If $d \geq 5$ and X very general, then image is equal to $\mathrm{NS}(X) \otimes \mathbb{C}^*$ modulo torsion.

Quartic K3 surfaces have in general large image (SMS, Voisin-Oliva). Cycles in families give rise to inhomogenous Picard-Fuchs equations (del Angel/SMS).

KLM formula

X smooth, projective/$\mathbb{C}$. Then

Theorem

If $Z = \sum a_i W_i \in CH^p(X, n)$ is a cycle homologous to zero, such that each irreducible components intersects all real faces properly, then the Abel–Jacobi image of Z is given by the following current:

$$\begin{aligned}
\alpha \mapsto \frac{1}{(2\pi i)^{d-p+n}} \Big[& \sum a_i \int_{W_i \setminus \pi_2^{-1}[-\infty,0]\times\square^{n-1}} \pi_2^*\big(\log z_1 d\log z_2 \wedge \ldots \wedge d\log z_n\big) \wedge \pi_1^*\alpha \\
& - (2\pi i) \sum a_i \int_{W_i \cap \pi_2^{-1}[-\infty,0]\times\square^{n-1} \setminus \pi_2^{-1}[-\infty,0]^2\times\square^{n-2}} \pi_2^*\big(\log z_2 d\log z_3 \wedge \ldots\big) \wedge \pi_1^*\alpha \\
& + \cdots + (-2\pi i)^{n-1} \sum a_i \int_{W_i \cap \pi_2^{-1}([-\infty,0]^{n-1}\times\square^1) \setminus W_i \cap \pi_2^{-1}[-\infty,0]^n} \pi_2^*(\log z_n) \wedge \pi_1^*\alpha \\
& + (-1)^n (2\pi i)^n \int_\Gamma \pi_1^*\alpha \Big],
\end{aligned}$$

where $\partial\Gamma = Z \cap \pi_2^{-1}[-\infty, 0]^n$.

The existence of Γ follows from Z being homologous to zero.

Bloch-Beilinson Type Conjectures

Conjecture

X smooth, projective/$\mathbb{C}$. Then there is a finite filtration

$$CH^p(X,n)\otimes\mathbb{Q} = F^0 \supset F^1 \supset F^2 \supset \cdots \supset 0$$

which is compatible with products and satisfies $CH^p_{\mathrm{hom}}(X,n) = F^1$. If X is defined over a number field, then $F^2 = 0$ and

$$CH^p(X,n)\otimes\mathbb{Q} \hookrightarrow H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Beilinson's conjectures (refined by Bloch/Kato) give a precise description of the image in terms of special values of L–series $L(H^i(X))$.

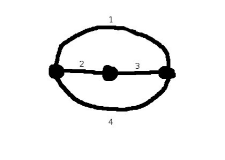
Beilinson's formula

$$\mathrm{Gr}_F^{\nu} CH^p(X,n) \otimes \mathbb{Q} = \mathrm{Ext}^{\nu}_{NMM(k)}(\mathbb{Q}(-p), H^{2p-n-\nu}(X)).$$

Controls structure of higher Chow groups and computes extension groups of abelian category of mixed motives.

Computations/Exercise

Exercise (1): Compute the graph polynomial of



Exercise (2): Prove that $CH^1(X,1) \to H^1_{\mathcal{D}}(X,\mathbb{Z}(1))$ is an isomorphism and both groups are the algebraic invertible functions.

Exercise (3): Use the KLM-formula to show that $cl^{2,3}$ of $C_1 : x \mapsto [1-\frac{1}{x}, 1-x, x]$ in $\mathbb{C}/\mathbb{Z}(2)$ is $Li_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, i.e., C_1 is 24–torsion in $CH^2(\mathbb{Q},3) = \mathbb{Z}/24\mathbb{Z}$.

Hint: The KLM-formula in this case looks like

$$cl^{2,3}(C_1) = -\int_{C_1 \cap \{z_1 \in \mathbb{R}^-\}} \log(z_2)\, d\log(z_3) - (2\pi i) \sum_{p \in C_1 \cap \{z_1, z_2 \in \mathbb{R}^-\}} \log(z_3(p)).$$

Part III

VHS, Higgs bundles, Shimura Varieties and L^2–cohomology

Literature

Carlson/SMS/Peters (Cambridge), articles by Viehweg/Zuo (2000-2007), Möller/Viehweg/Zuo (2007) and SMS/Viehweg/Zuo (2008).

Local Systems

$f : A \to X$ smooth, projective morphism between quasi-projective varieties$/\mathbb{C}$. $X \subset \bar{X}$ smooth compactification with NCD $D = \bar{X} \setminus X$.

Local system The m–the cohomology groups $H^m(A_t, \mathbb{C})$ form a local system $\mathbb{V} = R^m f_* \mathbb{C}$. It corresponds to a monodromy representation $\rho : \pi_1(X, *) \to GL_n(\mathbb{C})$, where $n = \dim_{\mathbb{C}} H^m(A_0, \mathbb{C})$. This gives rise to a vector bundle $V = \mathbb{V} \otimes \mathcal{O}_X$ on X. There is a Hodge filtration $V = F^0 \supset F^1 \supset \cdots$ by vector bundles.

Gauß–Manin connection:

$$\nabla : V \to V \otimes \Omega^1_X, \ \nabla^2 = 0$$

is $\mathbb{C}$–linear. By Griffiths transversality we have $\mathcal{O}_X$–linear

$$\mathrm{Gr}^p \nabla : F^p / F^{p+1} \to F^{p-1} / F^p \otimes \Omega^1_X.$$

Unipotency, Deligne extension

Theorem (Borel, Landman)

The local monodromies T around each component of D are quasi-unipotent:

$$(T^\nu - 1)^{n+1} = 0.$$

We will always assume that $\nu = 1$, hence monodromy is unipotent.

Theorem (Deligne)

Assume monodromy is unipotent. Then V and the Hodge bundles F^p have extensions as vector bundles to $\bar{X}$ such that

$$\mathrm{Gr}^p\nabla : F^p/F^{p+1} \to F^{p-1}/F^p \otimes \Omega^1_{\bar{X}}(\log D).$$

are still maps of vector bundles.

Polarization

Cohomology groups $H^m(A_0, \mathbb{C})$ have a decomposition into primitive parts:

$$H^m(A_0, \mathbb{C}) = H^m_{\mathrm{pr}}(A_0, \mathbb{C}) \oplus LH^{m-2}(A_0, \mathbb{C})$$

L=Lefschetz operator.

Primitive cohomology comes with a polarization Q. It satisfies $Q(H^{p,q}, H^{r,s}) = 0$ if $(r, s) \neq (q, p)$ and $Q(i^{p-q}u, \bar{u}) > 0$.

Modular Curves

Upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Homogenous space for $G = SL_2(\mathbb{R})$, $z \mapsto \frac{az+b}{cz+d}$.

Stabilizer of i is $K = SO(2) = U(1)$ (maximal compact).

$G/K = \mathbb{H}$ Hermitian symmetric domain.

Modular Curves

A Modular Curve is a quotient $X = \Gamma \backslash \mathbb{H}$, where

$$\Gamma \subset SL_2(\mathbb{Z})$$

is an discrete, torsion–free, "arithmetic" subgroup.

X is a Riemann surface (not compact in general). Can be compactified using "cusps" at infinity.

Examples: Congruence subgroups

$$\Gamma_0(N) = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N\}$$

$$\supset \Gamma_1(N) = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N\}$$

$$\supset \Gamma(N) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$$

Good Modular Curves, Families of Elliptic Curves

$$X(N) = \Gamma(N)\backslash\mathbb{H}, \quad X_1(N) = \Gamma_1(N)\backslash\mathbb{H}, \quad X_0(N) = \Gamma_0(N)\backslash\mathbb{H},$$

parametrize elliptic curves with additional structure on N–torsion points:

$$X(N) = \{(E,\varphi) \mid \varphi : E_{\mathrm{N-tor}} \cong (\mathbb{Z}/N\mathbb{Z})^2\},$$

$$X_1(N) = \{(E,P) \mid N\cdot P = 0\},\ X_0(N) = \{(E,C) \mid C \cong \mathbb{Z}/N\mathbb{Z}\}.$$

For $N \geq 3$ they form a good quotient and there is a universal family of elliptic curves over $X(N)$.

Uniformization

$f : E \to X$ family of curves E_λ for $\lambda \in X$, e.g. Legendre family $y^2 = x(x-1)(x-\lambda)$.

Let $\omega(\lambda) = \frac{dx}{y}$ be "the" holomorphic 1–form on E_λ.

Periods are elliptic integrals $\int_\gamma \omega$ over loops γ in $\pi_1(E_\lambda)$.

They form hypergeometric functions in λ.

Then the period map

$$X \to \mathbb{H}, \lambda \mapsto \frac{\int_{\gamma_1} \omega}{\int_{\gamma_2} \omega}$$

is multivalued, but locally biholomorphic.

Explicit Modular Curves

j–line: $X(1) = \Gamma(1)\backslash\mathbb{H} = \mathbb{C}$ (affine line, no good family).

Klein Quartic $\overline{X}(7) = \{x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0\} \subset \mathbb{P}^2$ with 24 cusps.

$\overline{X}_0(11) = \{y^2 + y = x^3 - x^2 - 10x - 20\} \subset \mathbb{P}^2$, elliptic.

Connected Shimura varieties

G reductive (e.g. semisimple) algebraic group$/\mathbb{Q}$ such that $G^{ad} = G/Z(G)$ is of Hermitian type, i.e., $X^+ = G^{ad}/K$ Hermitian symmetric domain.

$\Gamma \subset G^{ad}(\mathbb{Q})$ arithmetic subgroup, i.e., commensurable to $G_{\mathbb{Z}}(\mathbb{Z})$ for some embedding $G \hookrightarrow GL_r$.

Congruence subgroups Γ contain $\mathrm{Ker}\,(G_{\mathbb{Z}}(\mathbb{Z}) \to G_{\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}))$.

Essential types: $SU(p,q)$, Sp_{2g}, $SO(2,n)$, $SO^*(2n)$, E_6, E_7.

Locally symmetric variety $\mathcal{M} = \Gamma \backslash X^+$.

$\mathcal{M} \subset \mathcal{M}^*$ Baily–Borel compactification using sections of $\omega_{\mathcal{M}}^{\otimes M}$ (automorphic forms).

Example: $X(N) \subset \overline{X}(N)$. It is a quotient of $X \cup \mathbb{P}^1(\mathbb{Q})$.

Hecke correspondences

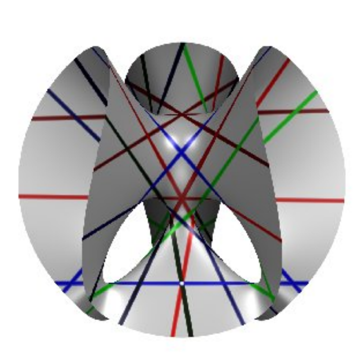
Assume Γ arithmetic. Any $q \in G(\mathbb{Q})$ induces Hecke correspondence T_q

$$\left(\Gamma \cap q^{-1}\Gamma q\right) \backslash X^+ \hookrightarrow \left(\Gamma \backslash X^+\right)^2 \rightrightarrows \Gamma \backslash X^+.$$

These operate on forms, i.e., on cohomology groups and on

Shimura subvarieties (Hecke translates).

Hilbert modular varieties



Clebsch diagonal cubic surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_0 + x_1 + x_2 + x_3 + x_4 = 0$$

Model of a Hilbert modular surface of level $N = 2$ for $\mathbb{Q}(\sqrt{5})$
(Hirzebruch 1976)

Hilbert modular varieties

F totally real number field of degree d.

$X = \Gamma \backslash \mathbb{H} \times \cdots \times \mathbb{H}$.

$\Gamma \subset SL_2(\mathcal{O}_F)$ arithmetic subgroup.

X carries a family of d–dim. abelian varieties with extra endomorphisms.

Siegel space

$$Sp(2g,\mathbb{R}) = \{\Omega \mid \Omega^T I_g \Omega = I_g\},\ I_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

$\mathbb{H}_g = Sp(2g,\mathbb{R})/U(g)$ Siegel upper half space, i.e.,

$$\mathbb{H}_g = \{\tau \in \mathbb{C}^{g\times g} \mid \mathrm{Im}(\tau) > 0,\ \tau^T = \tau\}.$$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{R})$ acts on $\mathbb{H}_g$ via fractional linear transformations like in the case $\mathbb{H}_1$ where $Sp(2,\mathbb{R}) = SL_2(\mathbb{R})$.

$U(g)$ is embedded via $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, if $M = A + Bi \in U(g)$.

$\mathbb{H}_g$ parametrizes (polarized) Hodge structures of weight $m = 1$.

$\mathcal{A}_g$

Assume we have a family $f : A \to X$ of g–dimensional abelian varieties. Then $\mathbb{V} = R^1 f_* \mathbb{C}$ has the extended Hodge bundles $F^1 = \bar{f}_* \Omega^1_{\bar{A}/\bar{X}}(\log \bar{f}^{-1} D)$ and $F^0/F^1 = R^1 \bar{f}_* \mathcal{O}_{\bar{X}}$ where $\bar{f}$ is a compactification of f.

Period map: $X \to \mathcal{A}_g = \Gamma \backslash \mathbb{H}_g$, $\Gamma \subset Sp(2g, \mathbb{Z})$, where an abelian variety A_t, $t \in X$, gets sent to its $g \times 2g$ (normalized) period matrix $\int_\gamma \omega \mathbb{H}_g$ (Riemann bilinear relations).

Example: Burkhardt Quartic

$$\{x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4 = 0\} \subset \mathbb{P}^4(\mathbb{C}).$$

Orthogonal Shimura varieties

$G = SO(2, n)$ orthogonal for form $x_1^2 + x_2^2 - x_3^2 - ... - x_{n+2}^2$.
$K = SO(2) \times SO(n)$.

$SO(2, 1)$, $SO(2, 2)$: modular curves and (Hilbert) modular surfaces.

$SO(2, 3)$: Siegel $\mathcal{A}_2$, i.e., $Sp(4, \mathbb{R})$.

$SO(2, n)$, $n \leq 19$: Moduli space of polarized K3 surfaces.

Ball quotients/Picard modular surfaces

$\mathcal{O}$ ring of integers for imaginary quadratic number field, e.g. $\mathcal{O} = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}] \subset \mathbb{Q}(\sqrt{-3})$.

Picard modular surfaces: $\overline{X}$, a smooth, projective compactification of $X = \Gamma\backslash\mathbb{B}_2$, where $\Gamma \subset U(2,1;\mathcal{O})$ arithmetic subgroup.

(2–dim. Shimura subvariety in $\overline{\mathcal{A}}_3$)

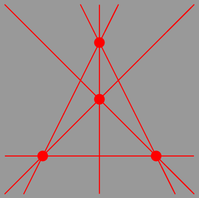
Picard curves (1880)

$$C_{s,t} : y^3 = x(x-1)(x-s)(x-t), \quad (s,t) \in \mathbb{C}^2 \subset \mathbb{P}^2(\mathbb{C}).$$

Genus 3 curves with extra $\mathbb{Z}/3\mathbb{Z}$ automorphism.

Discriminant locus Δ: 6 lines, 4 cusps:

$\Delta = \cup \Delta_{i,j}$:



Uniformization

The family has 6–dimensional periods (Euler PDE), 3 of which define a multivalued map

$$\mathbb{P}^2 \setminus \Delta \to \mathbb{B}_2 = \{|z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{C}^2 \subset \mathbb{P}^2(\mathbb{C}).$$

$\mathbb{B}_2$ is a homogenous space for $U(2,1)$.

Picard 1880

$\mathbb{P}^2 \setminus \mathrm{cusps} \cong \Gamma \backslash \mathbb{B}_2$, where

$$\Gamma = \{\gamma \in U(2,1)(\mathcal{O}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mod (1-\omega)\}.$$

($\mathcal{O} = \mathbb{Z}[\omega]$ Eisenstein numbers, $\omega^3 = 1$)

Hirzebruch–Holzapfel surfaces

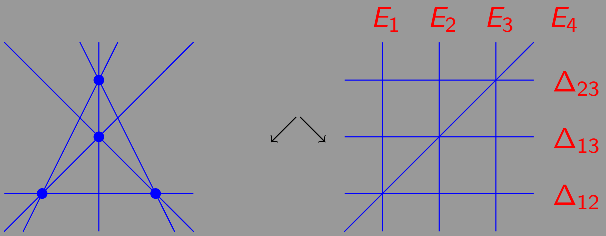
Hirzebruch (Math. Annalen 1984) constructed a series of surfaces with $c_1^2/c_2 \to 3$ for $n \to \infty$ and conjectured that they are compactified ball quotients $\overline{X_n}$ with $X_n = \Gamma_n \backslash \mathbb{B}_2$, $\Gamma_n \subset U(2,1;\mathcal{O})$.

Later Holzapfel proved that and constructed more examples. See his many publications on the subject.

Holzapfel's surface $\widetilde{E \times E}$

Main Example: $\overline{X}$ = blow–up of $E \times E$ in 3 points (P_i, P_i), where $E = \{y^2 z = x^3 - z^3\}$ CM elliptic curve with automorphism $x \mapsto \omega x$ of order 3 and fixed points P_1, P_2, P_3.

This is a (birational) covering of $\mathbb{P}^2$: Blow up 4 cusps and blow down all 3 strict transforms of lines $\Delta_{i,4}$.

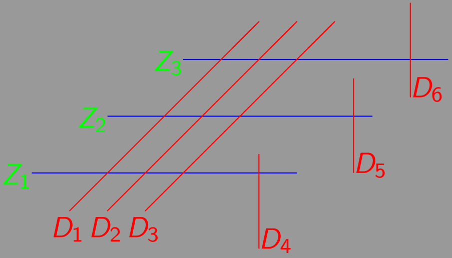


Holzapfel's surface $\widetilde{E \times E}$

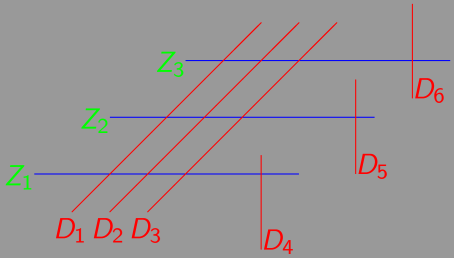
Then take branched cover along horizontal and vertical lines. Diagonal curve splits into 3 elliptic curves which correspond to strict transforms of the 3 diagonals

$$(z, w), \quad (z, \omega w), \quad (z, \omega^2 w) \subset E \times E.$$

Together we get 6 elliptic cusp curves $D_1, \ldots, D_6$:



Holzapfel's surface $\widetilde{E \times E}$



Z_1, Z_2, Z_3 are rational modular curves with 4 cusps. They carry a special family of Jacobians of type $E \times S^2(E_\lambda)$. There are 3 more modular elliptic curves $\tilde{\Delta}_{ij}$. Proof by proportionality equality below.

Log–Higgs bundles

A Higgs bundle on a compact Kähler manifold X is a pair (E, θ) with E a vector bundle on X and a holomorphic section $\theta \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X)$ with $\theta \wedge \theta = 0 \in H^0(X, \mathrm{End}(E) \otimes \Omega^2_X)$.

A (log–)Higgs bundle: on a compactifiable (e.g. quasi–projective) Kähler manifold $X = \overline{X} \backslash D$ with normal crossing boundary D, same definition, but $\theta \in H^0(\overline{X}, \mathrm{End}(E) \otimes \Omega^1_{\overline{X}}(\log D))$ such that $\theta \wedge \theta = 0 \in H^0(\overline{X}, \mathrm{End}(E) \otimes \Omega^2_{\overline{X}}(\log D))$.

Main Class: Variations of Hodge structures

Assume: $\mathbb{V}$ a $\mathbb{C}$–VHS of weight w on $X = \overline{X} \setminus D$, D=NCD, unipotent local monodromy, $\mathbb{V} \otimes \mathcal{O}_X = F^0 \supseteq F^1 \supseteq \ldots$.

Deligne extension: $\overline{\mathbb{V} \otimes \mathcal{O}_X} = F^0 \supseteq F^1 \supseteq \ldots$ over $\overline{X}$.

The Higgs bundle corresponding to $\mathbb{V}$ is $E = \left(\oplus_{a+b=w} E^{a,b}, \theta\right)$
$E^{a,b} = F^a / F^{a+1}$ and

$$\theta : E^{a,b} \to E^{a-1,b+1} \otimes \Omega^1_{\overline{X}}(\log D)$$

the (extended) graded part of Gauß–Manin connection.

Uniformized examples in weight one

Modular curves: $X = \Gamma\backslash\mathbb{H}$, $\mathbb{V} = R^1 f_*\mathbb{C}$, Higgs bundle $E = L \oplus L^{-1}$, $\Omega^1_X(\log D) = L^2$, $L = \bar{f}_*\omega_{\bar{C}/\bar{X}}(\log D)$, $\theta = id : L \to L^{-1} \otimes \Omega^1_X(\log D)$ tautological.

Hilbert modular surfaces: $X = \Gamma\backslash\mathbb{H} \times \mathbb{H}$, Higgs bundle

$$E = L_1 \oplus L_1^{-1} \oplus L_2 \oplus L_2^{-1}, \quad \Omega^1_X(\log D) = L_1^{\otimes 2} \oplus L_2^{\otimes 2}$$

with $\theta = id : L_i \to L_i^{-1} \otimes L_i^{\otimes 2}, \quad i = 1, 2$ tautological.

Picard modular surfaces: $X = \Gamma\backslash\mathbb{B}_2$, $R^1 f_*\mathbb{C} = \mathbb{V}_1 \oplus \mathbb{V}_2$ (see below) Higgs bundle for $\mathbb{V}_1$ looks like

$$E_1 = \left(\Omega^1_X(\log D) \otimes L^{-1}\right) \oplus L^{-1}.$$

Again θ tautological.

Example Holzapfel's surface

Theorem (Holzapfel, Picard, Simpson)

(a) Holzapfel's surface $\overline{X} = \widetilde{E \times E}$ is a compactified ball quotient with $X = \mathbb{B}_2/\Gamma$ with $\Gamma \subseteq SU(2,1;\mathcal{O})$.
(b) The universal family $f : A \to X$ of Jacobians has an eigenspace decomposition

$$R^1 f_* \mathbb{C} = \mathbb{V}_1 \oplus \mathbb{V}_2.$$

(c) Family has unipotent local monodromies and the Higgs bundle (E, θ) associated to the Deligne extension of $\mathbb{V}_1$ is of type

$$E = E^{1,0} \oplus E^{0,1}, \quad E^{1,0} = \Omega^1_{\overline{X}}(\log D) \otimes L^{-1}, \quad E^{0,1} = L^{-1},$$

$$L^{\otimes 3} = K_{\overline{X}} + D, \quad \theta = \mathrm{id} : E^{1,0} \to E^{0,1} \otimes \Omega^1_{\overline{X}}(\log D).$$

L^2–Higgs complex: Jost/Yang/Zuo 2003

E Higgs bundle of $\mathbb{C}$–VHS on X. After extending to $\overline{X}$ there is an algebraically defined subcomplex

$$0 \to \Omega^0_{(2)}(E) \to \Omega^1_{(2)}(E) \to \Omega^2_{(2)}(E) \to \dots$$

of the full algebraic Higgs complex

$$E \xrightarrow{\theta} E \otimes \Omega^1_{\overline{X}}(\log D) \to \cdots$$

L^2–conditions: harmonic metric on bundle = Hodge metric $\langle i^{p-q} -, -\rangle$, background metric on X = Poincaré metric at infinity $\sim \frac{i}{2} \frac{dz \wedge d\overline{z}}{|z|^2 \log^2 |z|^2}$ around each divisor $\{z = 0\}$. Depends essentially only on $N = \mathrm{Res}(\theta) : E \to E$.

L^2–Higgs complex

Case of Curves (Zucker):

$$\Omega^0_{(2)}(E) = W_0 + tE, \quad \Omega^1_{(2)} = W_{-2} \otimes \Omega^1_{\bar{X}}(\log D) + E \otimes \Omega^1_{\bar{X}}.$$

where $W_\bullet$=monodromy weight filtration on E and $D = \{t = 0\}$.

Surface Case: if $D = V(z_1)$ is smooth and weight $m = 1$, then

$$\Omega^0_{(2)}(E) = \mathrm{Ker}(\mathrm{Res}(\theta)) \subseteq E,$$

$$\Omega^1_{(2)}(E) = dz_1 \otimes E + dz_2 \otimes \mathrm{Ker}(\mathrm{Res}(\theta)),$$

$$\Omega^2_{(2)}(E) = \frac{dz_1}{z_1} \wedge dz_2 \otimes z_1 E = \Omega^2_{\bar{X}} \otimes E.$$

Monodromy Weight Filtration

Let V be a complex vector space with a nilpotent endomorphism N with $N^{m+1} = 0$ and $N^m \neq 0$. One always has $m + 1 \leq \dim_{\mathbb{C}}(V)$. N=nilpotent logarithm of monodromy near boundary D. There is a filtration

$$0 \subset W_{-m} \subset W_{-m+1} \subset \cdots \subset W_0 \subset W_1 \subset \cdots \subset W_m = V.$$

This is defined as follows: First set

$$W_{m-1} = \mathrm{Ker}(N^m), \quad W_{-m} = \mathrm{Im}(N^m).$$

Then inductively W_k is constructed in such a way that $N(W_k) = \mathrm{Im}(N) \cap W_{k-2} \subset W_{k-2}$ and

$$N^k : \mathrm{Gr}_{m+k}(V) \to \mathrm{Gr}_{m-k}(V)$$

are isomorphisms.

Cohomological correspondence

Theorem (Simpson, Jost/Yang/Zuo)

In this situation ($\mathbb{C}$–VHS, NCD, unipotent)

$$IH^*(\overline{X}, \mathbb{V}) = H^*_{L^2}(X, \mathbb{V}) = H^*_{L^2-\mathrm{Higgs}}(\overline{X}, (E, \theta))$$

$$:= \mathbb{H}^*(\overline{X}, \Omega^0_{(2)}(E) \xrightarrow{\theta} \Omega^1_{(2)}(E) \to \cdots).$$

Eichler–Shimura

$f : E \to X$ modular curve, $X = \mathbb{H}/\Gamma$ "universal" family.
$\mathbb{V} := R^1 f_* \mathbb{C}$ local system, representation of Γ (small enough).

Parabolic (=L^2) cohomology

$$H^1(\overline{X}, j_* \mathrm{Sym}^{\mathrm{k}} \mathbb{V}) = S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)}$$

Cusp forms, Hodge decomposition of L^2–cohomology.

Vanishing for Hilbert modular case

Theorem (Matsushima/Shimura)

$X = \Gamma\backslash\mathbb{H} \times \cdots \times \mathbb{H}$, Γ torsion free arithmetic subgroup and ρ a complex, irreducible, non–trivial representation of Γ. Then

$$H^i_{L^2}(X, \mathbb{V}) = 0 \text{ for } i \neq \dim(X) = n,$$

and $H^{\dim(X)}_{L^2}(X, \mathbb{V})$ is a space of automorphic forms.

Proof using Higgs bundles

Case of modular curves: $\mathbb{V} = R^1 f_* \mathbb{C}$, Higgs bundle $E = L \oplus L^{-1}$, $\Omega^1_{\overline{X}}(\log D) = L^2$: Eichler–Shimura

$$H^1_{L^2}(\overline{X}, \mathrm{Sym}^k \mathbb{V}) = H^1(\overline{X}, L^{-k}) \oplus H^0(\overline{X}, L^{k+2}).$$

Proof for $k = 1$: Higgs complex quasi–isomorphic to $L^{-1} \xrightarrow{0} L \otimes \Omega^1_{\overline{X}}(\log D) = L^3$.

Case of Hilbert modular surfaces:

$$H^2_{L^2}(\overline{X}, \mathbb{V}^{(m_1,m_2)}) = H^0(\overline{X}, L_1^{m_1} \otimes L_2^{m_2} \otimes K_{\overline{X}}) \oplus H^1(\overline{X}, L_1^{m_1+2} \otimes L_2^{-m_2}(-D))$$

$$\oplus H^1(\overline{X}, L_1^{-m_1} \otimes L_2^{m_2+2}(-D)) \oplus H^2(\overline{X}, L_1^{-m_1} \otimes L_2^{-m_2}).$$

Some computations with Zuo

Theorem (Ragunathan, Li–Schwermer, Saper)

Let $\mathbb{W}$ be an irreducible representation of Γ, i.e. a local system on X. If the highest weight of $\mathbb{W}$ is regular, then one has $H^1_{L^2}(X, \mathbb{W}) = 0$.

Example: $\mathbb{W}_{a,b}$ kernel of the natural maps

$$S^a\mathbb{V}_1 \otimes S^b\mathbb{V}_2 \longrightarrow S^{a-1}\mathbb{V}_1 \otimes S^{b-1}\mathbb{V}_2.$$

$\mathbb{W}_{a,b}$ has regular highest weight if $a, b > 0$.

Theorem (Zuo/SMS)

One has $H^0(\overline{X}, S^n\Omega^1_{\overline{X}}(\log D)(-D) \otimes L^{-m}) = 0$ for all $m \geq n \geq 3$.

Proof

Consider $\mathbb{W}_{a,b}$ for $a, b > 0$. The corresponding Higgs bundle $E_{a,b}$ is a subbundle of $S^a E_1 \otimes S^b E_2$. $E_{a,b}$ contains the vector bundle $S^{a+b}\Omega^1_{\overline{X}}(\log D) \otimes L^{-a-2b}$. If we compute H^1 of the corresponding Higgs complex, then in degree one there is a term $S^{a+b+1}\Omega^1_{\overline{X}}(\log D) \otimes L^{-a-2b}$, i.e. a symmetric $(a+b+1)$–tensor which is neither in the kernel of θ nor killed by the differential θ from degree zero. It therefore survives in $H^1(\overline{X}, E_{a,b})$. For $a, b > 0$ we have however $H^1(\overline{X}, E_{a,b}) = 0$ and hence we have $H^0(\overline{X}, S^{a+b+1}\Omega^1_{\overline{X}}(\log D)(-D) \otimes L^{-a-2b}) = 0$. Setting $n = a + b + 1 \geq 3$ and $m = a + 2b \geq a + b + 1 = n$ we obtain the assertion. $\square$

Miyaoka's Result

Miyaoka: If $X = \overline{X}$ compact 2–dim. ball quotient, then

$$H^0(X, S^N\Omega^1_X \otimes L^{-N}) = 0 \quad \forall N \geq 1 \quad (L = K_X^{1/3}).$$

For $N = 3$ this is related to our method, since $H^1(X, \mathrm{End}(\mathbb{V}_1)) = 0$ by Ragunathan's theorem and the Higgs complex for $\mathrm{End}(\mathbb{V}_1)$ contains $H^0(X, S^3\Omega^1_X \otimes L^{-3})$. With L^2–conditions Miyaoka's theorem is not known (our method below gives only a cuspidal version twisted by $-D$).

For D smooth, using Biquard's work, we can however use Hermitean–Yang–Mills techniques to imitate Miyaoka without twist (work in progress Yang/Zuo/SMS).

Some applications to algebraic cycles

Theorem (Schoen)

A multiple of the normal function $AJ(C_t - C_t^-)$ associated to the Ceresa cycle is contained in the maximal abelian subvariety $J^2_{\mathrm{ab}}(JC_t)$ of the intermediate Jacobian $J^2(JC_t)$ for every t.

Zuo/SMS

Let X be a Picard modular 3–fold ($g = 4$). Then a general fiber of $f : A \to X$ has non–trivial $CH^3_{(2)}(A_t)$, even modulo algebraic equivalence.

Sketch of Proof

Only $g = 3$: We compute cohomology group $H^1_{L^2}(X, R^3 f_* \mathbb{C}_{\mathrm{pr}})$.

This means we have to compute the primitive part of the Higgs bundle $\Lambda^3 E$, where E is the uniformizing Higgs bundle $E = E^{1,0} \oplus E^{0,1} = \left(\Omega^1_X(\log D) \otimes L^{-1}\right) \oplus L^{-1}$.

One computes

$$E^{3,0}_{\mathrm{pr}} = L^2,\ E^{2,1}_{\mathrm{pr}} = \mathcal{O}_{\overline{X}} \oplus \left(\Omega^1_{\overline{X}}(\log D) \otimes L^{-1}\right) \oplus \left(S^2\Omega^1_{\overline{X}}(\log D) \otimes L^{-2}\right),$$

$$E^{1,2}_{\mathrm{pr}} = \mathcal{O}_{\overline{X}} \oplus \left(\Omega^1_{\overline{X}}(\log D) \otimes L^{-2}\right) \oplus \left(S^2\Omega^1_{\overline{X}}(\log D) \otimes L^{-4}\right),\ E^{0,3}_{\mathrm{pr}} = L^{-2}.$$

Therefore the complex of which we want to compute H^1 is

$$E^{2,1}_{\mathrm{pr}} \to E^{1,2}_{\mathrm{pr}} \otimes \Omega^1_{\overline{X}}(\log D) \to E^{0,3}_{\mathrm{pr}} \otimes \Omega^1_{\overline{X}}(\log D)$$

is quasi-isomorphic to

$$\mathcal{O}_{\overline{X}} \xrightarrow{0} \left(S^3\Omega^1_{\overline{X}}(\log D) \otimes L^{-4}\right) \oplus \Omega^1_{\overline{X}}(\log D) \to 0.$$

Sketch of Proof

The abelian part of the intermediate Jacobian corresponds to a saturated sub Higgs bundle which is contained in $\mathrm{Ker}\left(E^{1,2}_{\mathrm{pr}} \to E^{0,3}_{\mathrm{pr}} \otimes \Omega^1_X(\log D)\right)$, hence

$$E_{\mathrm{ab}} = E^{2,1}_{\mathrm{ab}} \oplus E^{1,2}_{\mathrm{ab}} = \mathcal{O}_X^{\oplus 2} \subset E^3_{\mathrm{pr}} = \bigoplus_{a+b=3} E^{a,b}.$$

Let the quotient bundle be $F = E^3_{\mathrm{pr}}/E_{\mathrm{ab}}$. Then the complex

$$F^{2,1} \to F^{1,2} \otimes \Omega^1_X(\log D) \to F^{0,3} \otimes L^3$$

is quasi–isomorphic to $S^3\Omega^1_X(\log D) \otimes L^{-4}$ in degree 1, hence has no H^0. $\square$

Cohomology of Picard modular surfaces

Let us consider the surface X of Holzapfel again. We will show a method to prove:

Theorem [MMWYZ 2005]

The intersection cohomology $IH^q(X, \mathbb{V}_1)$ vanishes for $q \neq 2$.
X general $\Longrightarrow IH^1(X, \mathbb{V}_1) \subseteq H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D) \otimes \Omega^1_{\overline{X}} \otimes L^{-1})$.

Note: Since $G = SU(2,1)$ we cannot expect vanishing for arbitrary Γ, hence this is a coincidence.

Proof

Without L^2–conditions:

$$
\begin{array}{ccccc}
 & & \left(\Omega^1_X(\log D)\otimes L^{-1}\right) & \oplus & L^{-1} \\
 & & \downarrow\cong & & \downarrow \\
 & \left(\Omega^1_X(\log D)^{\otimes 2}\otimes L^{-1}\right) \quad \oplus & \left(L^{-1}\otimes\Omega^1_X(\log D)\right) & & 0 \\
 & \downarrow & & & \\
\left(\Omega^1_X(\log D)^{\otimes 2}\otimes L^{-1}\otimes\Omega^2_X(\log D)\right) \quad \oplus & \left(L^{-1}\otimes\Omega^2_X(\log D)\right). & & &
\end{array}
$$

Therefore it is quasi–isomorphic to a complex

$$L^{-1}\xrightarrow{0}S^2\Omega^1_X(\log D)\otimes L^{-1}\xrightarrow{0}\Omega^1_X(\log D)\otimes\Omega^2_X(\log D)\otimes L^{-1}$$

with trivial differentials.

Proof

- As L is nef and big, we have

$$H^0(L^{-1}) = H^1(L^{-1}) = 0.$$

- Hence we get

$$\mathbb{H}^1(\overline{X}, (E^\bullet, \vartheta)) \cong H^0(\overline{X}, S^2\Omega^1_{\overline{X}}(\log D) \otimes L^{-1}).$$

- Impose L^2–conditions: Since

$$\Omega^1(E)_{(2)} \subseteq \Omega^1_{\overline{X}} \otimes E$$

we conclude that

$$IH^1(X, \mathbb{V}_1) \subseteq H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D) \otimes \Omega^1_{\overline{X}} \otimes L^{-1}).$$

Proof

- Now restrict to union $Z = \coprod \mathbb{P}^1$ of 3 modular curves. We get $(\Omega^1_{\overline{X}}(\log D) \otimes L^{-1})^{\oplus 2} \to \Omega^1_{\overline{X}}(\log D) \otimes \Omega^1_{\overline{X}} \otimes L^{-1} \to \Omega^1_Z \otimes \Omega^1_{\overline{X}}(\log D) \otimes L^{-1}$.
- By Bogomolov–Sommese vanishing
$$H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D) \otimes L^{-1}) = 0,$$
since L is nef and big.
- In order to prove the vanishing, it is hence sufficient to show that
$$H^0(Z, \Omega^1_{\overline{X}}(\log D) \otimes \Omega^1_Z \otimes L^{-1}) = 0.$$

Proof

But Z is a disjoint union of $\mathbb{P}^1$'s and one easily computes that

$$0 \to \mathcal{O}_Z(-2) \to \Omega^1_X(\log D) \otimes \Omega^1_Z \otimes L^{-1} \to \mathcal{O}_Z(-1) \to 0.$$

On global sections this proves the assertion. $\square$

Arakelov inequalities

Theorem (Faltings 83 et.al.)

$\overline{f} : \overline{Y} \to \overline{X}$ family of abelian varieties of dim $= g$ over a curve $\overline{X}$, semistable in codimension one ($\Rightarrow$ unipotent), $E = E^{1,0} \oplus E^{0,1}$ associated Higgs bundle, then

$$\deg(E^{1,0}) \leq \frac{g}{2} \deg \Omega^1_{\overline{X}}(\log D) = \frac{g}{2}(2g(\overline{X}) - 2 + \sharp D).$$

Corollary

$\overline{X} = \mathbb{P}^1$, $g = 1$, f not isotrivial, then $\sharp D \geq 4$.

Proof

$A := F^{1,0} \subseteq E^{1,0}$ non–flat part (split off unitary local system).

May assume wlog $A = E^{1,0}$ and $\theta : A \to B \otimes \Omega^1_X(\log D)$ isomorphism, $B \subseteq E^{0,1}$.

$A \oplus B \subseteq E$ sub Higgs bundle $\Rightarrow \deg(A \oplus B) \leq 0$.

$$\begin{aligned} \Rightarrow \deg(A) &= \deg(B) + \mathrm{rk}(B) \cdot \deg \Omega^1_X(\log D) \\ &\leq -\deg(A) + g \cdot \deg \Omega^1_X(\log D). \end{aligned}$$

□

Equality

Theorem (Viehweg, Zuo 2004)

Equality in the theorem holds, iff θ is an isomorphism (maximal Higgs field). This implies up to an étale cover that $f : Y \to X$ is a product $A \times_X E \times_X E \times_X \cdots \times_X E$, where $E \to X$ is a modular family of elliptic curves.

Sketch of proof: Equality $\Rightarrow$ local system is $\mathbb{L} \otimes \mathbb{U}_1 \oplus \mathbb{U}_2$ with $\mathbb{U}_i$ unitary. $\mathbb{L}$ Higgs bundle rank two, is uniformizing: $\tilde{\varphi} : X \to \mathcal{D} = \mathbb{H}$ period map. θ maximal $\Rightarrow$ $\tilde{\varphi}$ locally biholomorphic, hence isomorphism and $X = \mathbb{H}/\Gamma$. $\square$

Upshot: Extremal cases in Arakelov inequalities lead to special subvarieties = (translates of) Shimura varieties.

Surface Case: Viehweg/Zuo 2005

$f : X \to Y$ semistable family of abelian varieties of dim $= g$ over a surface Y, smooth over $U = Y \setminus S$, and with period map $\varphi : U \to A_g$ finite. Then:

$$c_1(f_*\omega_{X/Y}) \cdot c_1(\omega_Y(S)) \leq \frac{g}{4} c_1^2(\omega_Y(S)).$$

If one has equality and Griffiths–Yukawa Coupling

$$\tau^g : \wedge^g F^{1,0} \to \wedge^{g-1} F^{1,0} \otimes F^{0,1} \otimes \Omega^1_Y(\log S) \to \cdots \to \wedge^g F^{0,1} \otimes S^g \Omega^1_Y(\log S)$$

does not vanish, then X is a generalized Hilbert modular surface.

If as above and $g = 3$ and Griffiths-Yukawa Coupling does vanish, then

$$c_1(f_*\omega_{X/Y}) \cdot c_1(\omega_Y(S)) \leq \frac{2}{3} c_1^2(\omega_Y(S))$$

and X is a generalized Picard modular surface.

Hirzebruch-Höfer

Give a non–singular, compact curve $\bar{C} \subset \bar{Y}$ such that $\bar{C}$ intersects the boundary S of $\bar{Y}$ transversal for simplicity. Then the relative proportionality inequality saying that

$$2 \cdot \bar{C}.\bar{C} \geq -(K_{\bar{Y}} + S).\bar{C},$$

if Y is a Hilbert modular surface,

$$3 \cdot \bar{C}.\bar{C} \geq -(K_{\bar{Y}} + S).\bar{C},$$

if Y is a ball quotient. If the compactification $\bar{Y}$ is a Mumford compactification, or more generally if $\Omega^1_{\bar{Y}}(\log S_{\bar{Y}})$ is numerically effective (nef) and if $\omega_{\bar{Y}}(S_{\bar{Y}})$ is ample with respect to Y, then equality implies that $\tilde{C}$ is a complex subball of $\tilde{Y}$.

Relative Proportionality in $SO(2,n)$, $SU(n,1)$ type

joint work with Kang Zuo (Mainz), Eckart Viehweg (Essen)

Theorem i) If $\mathcal{M}$ is Shimura of $SO(n,2)$-type, $Z \subset \mathcal{M}$ arbitrary subvariety of dimension $d \geq 1$, and if the Griffiths-Yukawa coupling $\theta^2_{\bar{Z}} \neq 0$ then

$$d \cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(N_{\bar{Z}/\bar{\mathcal{M}}}) + (n-d) \cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(\Omega^1_{\bar{Z}}(\log S_{\bar{Z}})) =$$
$$n \cdot \big(\deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(\Omega^1_{\bar{Z}}(\log S_{\bar{Z}})) - d \cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(E^{2,0}_{\bar{Z}})) \geq 0.$$

The equality implies that Z is a Shimura subvariety of $\mathcal{M}$ of Hodge type for $SO(d,2)$.

Relative Proportionality in $SO(2,n)$, $SU(n,1)$ type

Theorem ii) If $\mathcal{M}$ is Shimura of type $SO(n,2)$, $Z \subset \mathcal{M}$ arbitrary subvariety of dimension $d \geq 1$, and if the Griffiths-Yukawa coupling $\theta^2_{\bar{Z}}$ is zero then

$$(d+1) \cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(N_{\bar{Z}/\bar{\mathcal{M}}}) + (n-d-1) \cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(\Omega^1_{\bar{Z}}(\log S_{\bar{Z}})) =$$
$$n \cdot \big(\deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(\Omega^1_{\bar{Z}}(\log S_{\bar{Z}})) - (d+1) \cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(E^{2,0}_{\bar{Z}})\big) \geq 0.$$

The equality implies that Z is either the translate of a Shimura curve in $\mathcal{M}$ or, if $\dim(Z) > 1$, that Z is a Shimura subvariety of $\mathcal{M}$ of Hodge type for $SU(d,1)$.

Relative Proportionality in $SO(2,n)$, $SU(n,1)$ type

Theorem iii) If $\mathcal{M}$ is Shimura of type $SU(n,1)$, $Z \subset \mathcal{M}$ arbitrary subvariety of dimension $d \geq 1$,then the Griffiths-Yukawa coupling $\theta^2_{\bar{Z}}$ is zero and

$$(d+1)\cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(N_{\bar{Z}/\bar{\mathcal{M}}}) + (n-d)\cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(\Omega^1_{\bar{Z}}(\log S_{\bar{Z}})) =$$
$$(n+1)\cdot \big(\deg_{\omega_{Z}(S)}(\Omega^1_{\bar{Z}}(\log S_{\bar{Z}})) - (d+1)\cdot \deg_{\omega_{\bar{Z}}(S_{\bar{Z}})}(E^{2,0}_{\bar{Z}})\big) \geq 0.$$

Again the equality implies that Z is either the translate of a Shimura curve in $\mathcal{M}$ or, if $\dim(Z) > 1$, that Z is a Shimura subvariety of $\mathcal{M}$ of Hodge type for $SU(d,1)$.

Sketch of Proof of i)

Look at variation of Hodge structures $\mathbb{V}$ on $\mathcal{M}$. Associated Higgs bundle is

$$E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}.$$

$E^{2,0}$ generates a saturated Higgs bundle $F \subset E$ with $F^{2,0} = E^{2,0}$, $F^{0,2} = E^{0,2}$ and

$$E^{2,0} \otimes T_Z(-\log S) \to F^{1,1}.$$

By Simpson one has $\deg(F) \leq \deg(E)$. On the other hand by duality $\deg(F^{2,0}) + \deg(F^{0,2}) = 0$. Hence

$$0 \geq \deg(F^{1,1}) \geq \deg(E^{2,0} \otimes T_Z(-\log S))$$

whence the second inequality follows.

Sketch of Proof of i)

The first equality follows since $c_1(\omega_Z(S)) = n \cdot \deg(E^{2,0})$.

Equality holds if and only if Z is a totally geodesic subvariety. By the assumption on Griffiths–Yukawa coupling Z is rigid. Hence Z is Shimura of type $SO(2,d)$ by Mumford et al..

Inverse problem

Question: For $i \in I$ Shimura varieties $W_i \subset Z \subset \mathcal{M}$ intersecting boundary transversal (for simplicity here) and satisfying HHP. Is then Z Shimura ?

Theorem i) If $\sigma_i : W_i \to \mathcal{M}$ are of type $SO(d-1,2)$ for all $i \in I$ and satisfy the HHP equality

$$\mu_{\omega_{\bar{W}_i}(S_{\bar{W}_i})}(N_{\bar{W}_i/\bar{Z}}) = \mu_{\omega_{\bar{W}_i}(S_{\bar{W}_i})}(T_{\bar{W}_i}(-\log S_{\bar{W}_i})),$$

and if $\#I \geq \rho^2 + \rho + 1$, then $Z \subset \mathcal{M}$ is a Shimura subvariety of Hodge type for $SO(d,2)$.

Inverse problem

Theorem ii) Assume that the Griffiths-Yukawa coupling vanishes on $\bar{Z}$. If $\sigma_i : W_i \to \mathcal{M}$ are Shimura varieties of type $SU(d-1,1)$, if

$$\frac{\deg_{\omega_{\bar{W}_i}(S_{\bar{W}_i})}(N_{\bar{W}_i/\bar{Z}})}{\mathrm{rk} N_{\bar{W}_i/\bar{Z}}} = \frac{\deg_{\omega_{\bar{W}_i}(S_{\bar{W}_i})}(T_{\bar{W}_i}(-\log S_{\bar{W}_i}))}{d+1},$$

and if $\# I \geq \rho^2 + \rho + 1$, then $Z \subset \mathcal{M}$ is a Shimura subvariety of Hodge type for $SU(d,1)$.

Sketch of Proof

Some power of $\omega_Z(S)$ has sections. We may assume that Z is a surface. There is a linear combination $D = \sum n_i W_i$ of W_i with $D^2 > 0$. We have a saturated sub Higgs sheaf F as above. One checks that $c_1(F)^2 \geq 0$, $c_1(F) \cdot D = 0$. By Hodge index theorem one has $c_1(F)^2 = 0$. Again this implies that Z is Shimura.

Exercises

Exercise (1): Prove the Eichler-Shimura isomorphism, i.e., the Hodge decomposition of $H^1_{L^2}(X, \mathrm{Sym}^k \mathbb{V})$ for a family of elliptic curves $f : E \to X$ over a modular curve X, where $\mathbb{V} = R^1 f_* \mathbb{C}$. Hint: Use Higgs bundle $E = \mathrm{Sym}^k(L \oplus L^{-1})$ and $L^2 = \Omega^1_{\bar{X}}(\log D)$.

Exercise (2): Let V be a complex vector space of dimension n and N a nilpotent operator with $N^{m+1} = 0$ and $N^m \neq 0$. Show by induction that the monodromy weight filtration exists.

Exercise (3): Let $f : A \to X$ be a family of surfaces with (primitive) Hodge numbers $h^{2,0} = h^{1,1} = h^{0,2} = 1$ over a curve X. Compute $H^1_{L^2}(X, \mathbb{V})$, $\mathbb{V} = R^2 f_* \mathbb{C}$. Hint: Classify all possibilities for the nilpotent monodromy operator N in terms of the Jordan normal form and compute the L^2–Higgs complex $\Omega^*_{(2)}(E)$.