

Arakelov theory and log-singular hermitian vector bundles

José Ignacio Burgos Gil
Universidad de Barcelona

The spectrum of the ring of integers of a number field $S = \text{Spec}(\mathcal{O}_K)$ is an affine variety. Following Weil, we can “compactify” S by adding the finite set Σ of classes of complex embeddings of K . The elements of Σ can be seen as the “points at infinity” of S . The “compactified” variety $\bar{S} = S \cup \Sigma$ behave like a projective curve defined over a field. For instance the product formula is the analogue of the residue formula, and Minkowski’s theorem on the geometry of numbers can be seen as an analogue of the Riemann-Roch theorem for projective curves.

The basic idea of Arakelov Geometry is that a regular projective flat variety X over S can be “compactified” by adding the complex variety X_Σ , that correspond to the set of complex embeddings Σ . The whole object (X, X_Σ) behaves like a projective variety over a field and many theorem of projective algebraic geometry have their arithmetic analogues. Such theorems will relate algebraic geometry on X with complex analysis on X_Σ .

In this spirit, Gillet and Soulé have developed an arithmetic intersection theory and a theory of arithmetic characteristic classes from hermitian vector bundles. Here the hermitian metric plays the role of the extension of the vector bundle to the compactified variety (X, X_Σ) . Originally the theory has been developed for smooth hermitian metrics. Nevertheless, in the applications, it became apparent that the restriction to smooth metrics was too strong.

In this series of lectures we will review the definition of arithmetic Chow groups and we will show how to extend this theory to a class of singular hermitian metrics, the log-singular hermitian metrics, that appear naturally when considering moduli problems.

List of lectures:

1. Deligne-Beilinson cohomology and characteristic classes.
2. Arithmetic Chow rings and arithmetic characteristic classes.
3. Arithmetic characteristic classes of log-singular hermitian vector bundles.

Abstract of the lectures:

Lecture 1. Deligne-Beilinson cohomology and characteristic classes.

In the first lecture we will review the Hodge theory and the Deligne-Beilinson cohomology of complex algebraic varieties. We will give an introduction to the Chern-Weil theory of characteristic classes of hermitian vector bundles by means of Chern forms. Moreover we will discuss Bott-Chern forms, that measure the lack of additivity of the Chern forms, and iterated Bott-Chern forms, that provide characteristic classes from higher algebraic K -theory.

Lecture 2. Arithmetic Chow rings and arithmetic characteristic classes.

This lecture will be devoted to the definition of arithmetic Chow rings with values in a cohomology theory, and to study its basic properties. As a particular case we will recover the original definition of arithmetic Chow rings due to Gillet and Soulé. We will also review the theory of arithmetic characteristic classes from hermitian vector bundles to the arithmetic Chow rings.

Lecture 3. Arithmetic characteristic classes of log-singular hermitian vector bundles.

In the last lecture we will discuss the extension of the formalism of arithmetic characteristic classes to the log-singular hermitian metrics and we will show examples of application of such formalism.