Embeddings of local fields in simple algebras and simplicial structures on the Bruhat-Tits building

Daniel Skodlerack

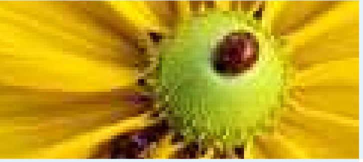
20. Juni 2008

Notation

Notation

- $\mathbb{N} = \{1, 2, \ldots\}$ $\mathbb{N}_r := \{1, \ldots, r\}$.
- (F, ν) non-archimedean local field, $D|F$ a central skewfield, $d := \sqrt{[D : F]} < \infty$. $L|F$ max. unramified field in D, $[L : F] = d$
$$D \supseteq L \supseteq F$$
- Assume that π_D normalizes L.
$$D = L \oplus L\pi_D \oplus L\pi_D^2 \ldots \oplus L\pi_D^{d-1}$$
- $A := M_m(D)$ the and $V := D^m$, right D vector space, $m \in \mathbb{N}$ fixed.

The outlook



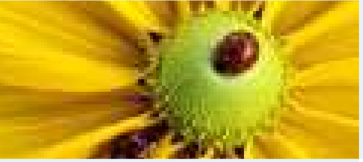
The outlook

Martin Grabitz and Paul Broussous have classified embeddings

$$E^\times \subseteq \text{ compact modulo center group } \subseteq M_m(D)$$

and introduced invariants. The question of E.W. Zink was:
Is there a geometric way to find the invariants using euclidean Bruhat Tits buildngs as geometrical object together with an affine map.

Embeddings



hereditary order

Definition 1 A *hereditary order* $\mathfrak{a} \subseteq M_m(D)$ is a subring of $M_m(D)$, s.t. there is a $g \in GL_m(D)$ s.t. $g\mathfrak{a}g^{-1}$ is of the form

$$\begin{pmatrix} D^\circ & D^{\circ\circ} & \dots & D^{\circ\circ} \\ \vdots & \dots & \dots & \vdots \\ D^\circ & \dots & D^\circ & D^{\circ\circ} \\ D^\circ & \dots & & D^\circ \end{pmatrix}^{n_1,n_2,\dots,n_r}$$

where $\sum n_i = m$.

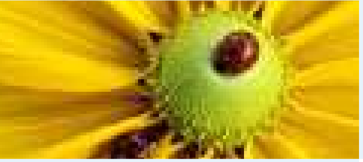
Embedding

Definition 4 An *embedding* is a pair $(E, \mathfrak{a})$ satisfying

1. E is a field extension of F in A,
2. $\mathfrak{a} \in \text{Her}(A)$ is normalised by $E^\times$.

$(E, \mathfrak{a}) \backsim (E', \mathfrak{a}')$ if there is a $g \in A^\times$, such that $gE_Dg^{-1} = E'_D$ and $g\mathfrak{a}g^{-1} = \mathfrak{a}'$.

An example for embeddings are pearl embeddings. (soon)



Pearl embedding

Definition 6 Let $f|d$ and $r \leq m$. An *embedding datum* is a $f \times r$-matrix λ of non-negative integer entries s.t. in every column is non-zero, and the sum of all entries is m.The *pearl embedding* of λ is the embedding $(E, \mathfrak{a})$, s.t.

1. $[E : F] = f$ and E is in the image of

$$x \in L \mapsto \mathrm{diag}(M_1(x), M_2(x), \ldots, M_r(x)) \text{ where}$$

$$M_j(x) = \mathrm{diag}(\sigma^0(x)\,\mathrm{I}_{\lambda_{1,j}}, \sigma^1(x)\,\mathrm{I}_{\lambda_{2,j}}, \ldots, \sigma^{f-1}(x)\,\mathrm{I}_{\lambda_{f,j}})$$

2. $\mathfrak{a} \in \mathrm{Her}(A)$ in standard form according to $m = n_1 + \ldots + n_r$ where $n_j := \sum_{i=1}^{f} \lambda_{i,j}$.

Equivalent vectors

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \end{pmatrix}^T.$$

Definition 7

1. $w = (w_1, \ldots, w_t) \sim w' = (w'_1, \ldots, w'_t)$ (real entries) if there is a k, s.t.

   $$w = (w'_k, \ldots, w'_t, w'_1, \ldots, w'_{k-1}).$$

   We write $< w >$ for the equivalence class.

2. For a $t \times s$-matrix M we put $\mathrm{row}(M) := (m_{1,1}, \ldots, m_{1,s}, m_{2,1}, \ldots, m_{2,s}, \ldots, m_{t,s})$.

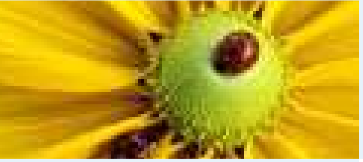
3. $M \sim N$ if $\mathrm{row}(M) \sim \mathrm{row}(N)$.

Theorem 1 *[BG00, 2.3.3 and 2.3.10]*

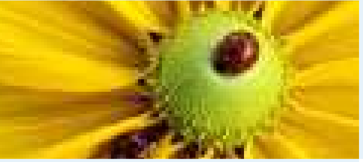
1. *Two pearl embeddings are equivalent if and only if the embedding datas are.*

2. *In any class of embeddings lies a pearl embedding.*

Definition 8 By the theorem to an embedding corresponds one class of embedding datas, called *embedding type* (notion from V. Secherre).



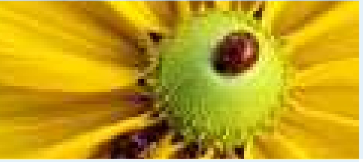
The euclidean building of $GL_m(D)$



building

A building of rank $m-1$ is a poset $(\Omega, \leq)$ s.t.

- $\bar{S} := \{S' \in \Omega |\ S' \leq S\}$ is poset isom. to a simplex, $S \in \Omega$ (faces).
- Every face has not more then $m-1$ *vertices* ($=$ minimal elements).
- Every face lies in a face with $m-1$ vertices ($=$ maximal elements $=$*chambers*).
- $\Omega = \bigcup \mathcal{A}$, where $\mathcal{A}$ is a set of chamber subcomplexes of rank $m-1$, *apartments*.
- There are poset isomorphisms between $\Sigma, \Sigma' \in {}'\mathcal{A}$.

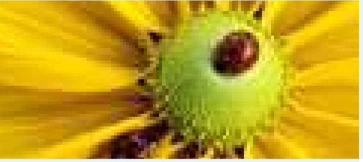


Euclidean building

A building is called euclidean if every apartment is a isomorphic to a cell decomposition of an f.d. euclidean space with an infinite affine reflection group.

$|S| := \{\sum_{\text{v vertex of } S} \lambda_v v| \ \sum \lambda_v = 1 \ \lambda_v > 0\}$ *geometric realisation g.r. of* S

$$|\Omega| := \cup\{|S|| \ S \in \Omega\}.$$



Lattice functions

With $\mathrm{Latt}_{D^\circ}^{m,V}$ we denote the set of full D°- lattices in V. The word full will be omitted. Definitions:

- A left continuous monoton decreasing (all w.r.t. $\subseteq$) function $r \in \mathbb{R} \rightarrow \Lambda(r) \in \mathrm{Latt}_{D^\circ}^{m,V}$ is called D°-*lattice function* of V, if $\forall r \in \mathbb{R}: \ \Lambda(r)\pi_D = \Lambda(r + \frac{1}{d})$.
- The set of D° lattice functions is denoted by $\mathrm{Latt}_{D^\circ}^1 V$.
- $\Lambda_1 \backsim \Lambda_2$ iff $\exists s \in \mathbb{R} : \forall r \in \mathbb{R} : \Lambda_1(r) = \Lambda_2(r + s)$.
- $\mathrm{Latt}_{D^\circ} V := \mathrm{Latt}_{D^\circ}^1 V / \backsim$

Affine Structure

Definition 10 A D-basis (v_i) of V is called splitting basis of a lattice function $[\Lambda]$, if

$$\forall r \in \mathbb{R}: \ \Lambda(r) = \oplus_{i=1}^{m} (\Lambda(r) \cap R_i).$$

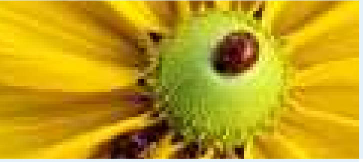
Affine structure: For $[\Lambda]$ and $[\Lambda']$ we can find a splitting basis (v_i), thus

$$\Lambda(r) = \oplus_{i=1}^{m} v_i D^{\circ\circ [r-\alpha_i]+} \text{ and } \Lambda'(r) = \oplus_{i=1}^{m} v_i D^{\circ\circ [r-\alpha_i']+}.$$

For $\lambda \in [0,1]$ one defines

$$\lambda[\Lambda] + (1-\lambda)[\Lambda'] := [\Lambda''] \text{ with}$$

$$\Lambda''(r) := \oplus_{i=1}^{m} v_i D^{\circ\circ [r-\lambda\alpha_i-(1-\lambda)\alpha_i']+}.$$



The description of the building with lattice function

The g.r. of the eucl. building of $GL_m(D)$ we denote by $\mathcal{I}$.

Theorem 5 ([BL02] section I (2.5)) $\mathcal{I} \cong \mathrm{Latt}_{D^\circ} V$ *$GL(D)^\times$-equivariant, affine.*

Apartments: A *frame* $R = \{R_i | 1 \leq i \leq m\}$ is a set of m linearely independent 1-dim. D-subspaces of V.

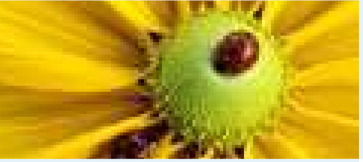
$$\mathrm{Latt}_R V := \{[\Lambda] |\ \Lambda \text{ is splitt by } R\}.$$

$$\text{Apartments} \ = \ \{\mathrm{Latt}_R V |\ R \text{ frame}\}.$$

Faces: They are given by the hereditary orders of A,

$$\text{Her}(A) := \{\mathfrak{a} |\ \mathfrak{a} \text{ is a hereditary order}\}$$

$$\text{Def.: } \mathfrak{a} \leq \mathfrak{a}' \quad \text{if } \mathfrak{a} \supseteq \mathfrak{a}'$$

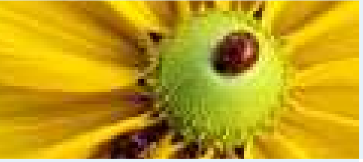


Simplicial structure

- A lattice function $[\Lambda]$ lies on the face $\mathfrak{a}_\Lambda = \{a \in A|\ a\Lambda(r) \subseteq \Lambda(r)\ \forall r \in \mathbb{R}\}$.
- The range of a lattice function is a lattices chain. This lattice chain represents the face $\tilde{F}$ of the simplicial building s.t. $p \in |\tilde{F}|$.
- Lattice chains are in 1-1 correspondence to hereditary orders.

Theorem 6 (P.Broussous, B.Lemaire)

1. *The simplicial complex of $\mathcal{I}$ is isomorphic to $(\text{Her}(A), \supseteq)$.*
2. *The hereditary order of rank k correspond to the faces of rank k, i.e. of dimension $k-1$.*
3. *Maximal her. orders, correspond to the vertices and minimal her. orders to the chambers.*



The affine map j_E

Notation

$$A = M_m(D) \supseteq B = C_A(E) \supseteq E \supseteq F$$

- $E|F$ is a unram. field extension of degree $[E:F]|d$ in A.
- B is the centraliser of E in A.
- It is $\mathcal{I}_E$ the g.r. of the eucl. building of B.

Existence and Uniqueness of j_E

Theorem 8 *[BL02, part of Thm 1.1.] There exists a unique application* $j_E : \mathcal{I}^{E^\times} \rightarrow \mathcal{I}_E$ *such that*

1. j_E *is* $B^\times$*-equivariant.*
2. j_E *is affine.*

Moreover j_E^{-1} *can be caracterised as the unique* $B^\times$*-equivariant affine map* $\mathcal{I}_E \rightarrow \mathcal{I}$.

j_E in terms of lattice functions 1

This is due to Broussous and Lemaire [BL02] II 3.1.
We have $E \cong i(E) \subseteq L$ (F-Algebrahomomorphism).
$E \otimes_F i(E) \cong \bigoplus_{k=0}^{[E:F]-1} i(E)$ with the decomposition
$1 = \sum_{k=0}^{[E:F]-1} 1^k$
So we get $V = \bigoplus_k V^k$, $V^k := 1^k V$, w.l.o.g. s.t. $V^{k+1} = V^k \pi_D$ and $V^{[E:F]-1}\pi_D = V^0$.

Remark 3 The skewfield $\Delta := \mathrm{C}_D(i(E))$ is central over $i(E)$ of index $\frac{d}{[E:F]}$.

1. $B \cong \mathrm{End}_\Delta(V^0)$.
2. $B \cong M_m(\Delta)$.

Theorem 9 *[BL02, II 3.1.] In terms of lattice functions j_E has the form*

$$j_E^{-1}([\Theta]) = [\Lambda],$$

with

$$\Lambda(s) := \bigoplus_{k=0}^{f-1} \Theta(s - \frac{k}{d})\pi_D^k, \; s \in \mathbb{R}.$$

Barycentric coordinates

Orientation

For the simplicial complexes of $\mathcal{I}$, $\mathcal{I}_E$ we write $(\Omega, \leq)$, $(\Omega_E, \leq)$. For the lattices corresponding to a face H or point x we write lattices(H), lattices(x). We define an orientation on Ω_E.

Definition 11 An edge $H = \{e, e'\} \in \Omega_E$ is said *to be oriented towards* e' if there are $\Gamma \in$ lattices(e) and $\Gamma' \in$ lattices(e'), such that $\dim_{\kappa_D}(\Gamma/\Gamma') = 1$. (write $e_1 \rightarrow e_2$) *An oriented chamber* is a tupel $(e_1, \ldots, e_m)$ of m different vertices which lie in a common chamber s.t. $e_i \rightarrow e_{i+1}$ and $e_m \rightarrow e_1$.

Oriented barycentric coordinates type

Definition 12 Assume $x \in \mathcal{I}_E$. An equivalence class of a tuple $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}_+^m$ is called *the local type of* x, if there is an oriented chamber $(e_1, \ldots, e_m)$ of Ω_E such that $x = \sum_{i=1}^m \mu_i e_i$.

Proposition 1 *For* $x \in \mathcal{I}_E$ *there is only one local type.*

The theorem

Vector of pairs

Definition 13 $m', t \in \mathbb{N}$. Take

$$w \in \mathrm{Row}(m', t) := \{w \in \mathbb{N}_0^{m'} \,|\, \sum_i w_i = t\}, \text{ i.e.}$$

$$w = (0, \ldots, 0, w_{i_0}, 0, \ldots, 0, w_{i_1}, 0, \ldots, 0, w_{i_k}, 0, \ldots, 0)$$

with $w_{i_j} > 0$, and we can represent $< w >$ by a $(k+1)$-tupel of pairs

$$((w_{i_0}, i_1 - i_0), (w_{i_1}, i_2 - i_1), \ldots, (w_{i_{k-1}}, i_k - i_{k-1}), (w_{i_k}, i_0 + m' - 1 - i_k))$$

In this way we can map $< w >$ to a class of a vector of pairs, which we denote:

$$\text{pairs}(< w >) :=< (w_{i_0}, i_1 - i_0), (w_{i_1}, i_2 - i_1), \ldots, (w_{i_k}, i_0 + m' - 1 - i_k)$$

There is a duality map $<>^c$: $\mathrm{Row}(m', t) \rightarrow \mathrm{Row}(t, m')$.

Definition 14 Given w as above and pairs$(< w >) =< (a_0, b_0), \ldots, (a_k, b_k) >$ we define the *complement of* $< w >$, denoted by $< w >^c$ to be the class $< w' >$, such that pairs$(< w' >) =< (b_0, a_1), (b_1, a_2), (b_2, a_3), \ldots, (b_k, a_0) >$.

Theorem 10 (S.) *Given* $\mathfrak{a} \in$ *Her*$(A)^{E^\times}$ *and a matrix* λ *s.t.* $< \lambda >$ *is the embedding type of* $(\mathfrak{a}, E)$ *and assume* $< \mu >$ *to be the local type of* $j_E(M_\mathfrak{a})$, *where* $M_\mathfrak{a}$ *is the barycentre of the face corresponding to* $\mathfrak{a}$. $< \mathrm{row}(\lambda) >$ *is obtained as follows*

1. $rf\mu \in \mathbb{N}_0^m$ *and*
2. $< \mathrm{row}(\lambda) > =< fr\mu >^c$.

Example

For example take $r = 2,\ [E : F] = 6,\ \dim_D V = 7,$

$$j_E(M_{\mathfrak{a}}) = \frac{3}{12}b_0 + \frac{2}{12}b_1 + \frac{1}{12}b_2 + \frac{0}{12}b_3 + \frac{0}{12}b_4 + \frac{4}{12}b_5 + \frac{2}{12}b_6.$$

$< 12\mu > = < 3, 2, 1, 0, 0, 4, 2 >$
$\equiv < (3, 1), (2, 1), (1, 3), (4, 1), (2, 1) >$
$< 12\mu >^c \equiv < (1, 2), (1, 1), (3, 4), (1, 2), (1, 3) >$
$\equiv < 1, 0, 1, 3, 0, 0, 0, 1, 0, 1, 0, 0 >$ .Applying theorem 10 we get the embedding data

$$\begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Bibliography

[BG00] P. Broussous and M. Grabitz. Pure elements and intertwining classes of simple strata in local central simple algebras. *COMMUNICATION IN ALGEBRA*, 28(11):5405–5442, 2000.

[BL02] P. Broussous and B. Lemaire. Buildings of $GL(m, D)$ and centralizers. *Transformation Groups*, 7(1):15–50, 2002.