Height Bounds in Diophantine Geometry

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GRK Summer School in Alpbach 2.-7. Sept. 2007

The Manin-Mumford Conjecture

Conjecture

Let C be a smooth projective algebraic curve defined over $\mathbb{C}$ of genus at least 2 embedded into its Jacobian J, then $C \cap J_{\mathrm{tors}}$ is finite.

Generalized by Lang to

Conjecture

Let S be a semi-abelian variety defined over $\mathbb{C}$ and let $X \subset S$ be an irreducible closed subvariety. If X is not an irreducible component of an algebraic subgroup of S, then the intersection $X \cap S_{\mathrm{tors}}$ is not Zariski dense in X.

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A simple example

Example

- $\mathbb{G}_m$ is the multiplicative group, i.e. $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$.
- Consider the curve $C \subset \mathbb{G}_m^2$ defined by

$$X + Y = 1.$$

- Then

$$C \cap (\mathbb{G}_m^2)_{\mathrm{tors}} = \{(\zeta, \zeta^{-1}), (\zeta^{-1}, \zeta)\}$$

where ζ is a primitive 6th root of unity.

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Definition

Let S be a semi-abelian variety defined over $\mathbb{C}$. We define

$$S^{[r]} = \bigcup_{\substack{H \text{ alg. subgrp.} \\ \textbf{codim H}\geq\textbf{r}}} H(\mathbb{C})$$

$$= \{x \in S(\mathbb{C});\ x \text{ contained in an algebraic subgroup of codimension } \geq r\}.$$

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$$(\mathbb{G}_m^n)^{[1]} = \{(x_1, \ldots, x_n) \in \mathbb{G}_m^n;\ x_1^{a_1} \cdots x_n^{a_n} = 1,$$
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Intersecting subvarieties with $S^{[r]}$

Question

Let X be a subvariety of a semi-abelian variety S, what can be said about $X \cap S^{[r]}$? We are specially interested in the critical values $r = \dim X$ and $r = 1 + \dim X$.

Remark

Manin-Mumford: $X \cap S^{[r]}$ is non-dense if $r = \dim S$ and if X is not an irreducible component of an algebraic subgroup of S.

Example

If C is a curve in $\mathbb{G}_m^n$, then $C \cap (\mathbb{G}_m^n)^{[1]}$ is always infinite.

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On heights I

Definition (of the height)

The (absolute logarithmic Weil) height $h(x)$ of an **algebraic number** x is defined as follows:

There exists a unique $P = a_d T^d + \cdots + a_0 \in \mathbb{Z}[T]$, the minimal polynomial, with

- $P(x) = 0$,
- P irreducible over $\mathbb{Z}$, $a_d > 0$, and
- $P = a_d(T - \xi_1)\cdots(T - \xi_d)$ with $\xi_i \in \mathbb{C}$.

$$h(x) := \frac{1}{d} \log \left(a_d \prod_{|\xi_i|>1} |\xi_i| \right) = \frac{1}{d} \log(\text{Mahler measure of } P) \geq 0.$$

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Definition (of the height cont.)

If $x = (x_1, \ldots, x_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ we define

$$h(x) = h(x_1) + \cdots + h(x_n).$$

A property

We have $h(x^k) = |k| h(x)$ for $k \in \mathbb{Z}$.

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Two basic facts I

Theorem (Northcott)

Let $C, D \in \mathbb{R}$, then

$$\{x \in \mathbb{G}_m^n(\overline{\mathbb{Q}});\ h(x) \leq C \quad \textit{and} \quad [\mathbb{Q}(x):\mathbb{Q}] \leq D\}$$

is finite.

Proof idea

The constants C and D bound the absolute values of the coefficients and degrees of the minimal polynomials of $x_1, \ldots, x_n$.

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Theorem (Kronecker)

If $x \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$, then

$$h(x) = 0 \quad \textit{if and only if} \quad x \in (\mathbb{G}_m^n)_{\mathrm{tors}}.$$

Proof idea of "$\Rightarrow$"

If $h(x) = 0$ then $h(x^k) = |k|h(x) = 0$ for all $k \in \mathbb{Z}$. The set $\{x^k\}$ has bounded height and degree.
Northcott: $\{x^k\}$ is finite. There exists $k > k'$ with

$$x^k = x^{k'} \quad \text{so} \quad x^{k-k'} = 1.$$

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Let $C \subset \mathbb{G}_m^n$ be an algebraic curve defined over $\overline{\mathbb{Q}}$. Assume that C is not contained in the **translate of a proper algebraic subgroup** *(= proper coset) of $\mathbb{G}_m^n$. Then*

- *$C \cap (\mathbb{G}_m^n)^{[1]}$ has bounded height,*
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A consequence using Northcott's Theorem

Example

If C is a curve as in the Theorem of Bombieri, Masser, and Zannier and K is a number field, then

$$C(K) \cap (\mathbb{G}_m^n)^{[1]}$$

is finite.

Strategy of proof

The finiteness statement in BMZ's Theorem rests on:

Proposition

Assume $C \subset \mathbb{G}_m^n$ is a curve which is not contained in a proper algebraic subgroup. If $B \in \mathbb{R}$, then

$$\{x \in C \cap (\mathbb{G}_m^n)^{[2]};\ h(x) \leq B\} \quad \text{is finite.}$$

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The proof of this result uses

"relative Lehmer-type height lower bounds."

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Let us consider again the curve C defined by $X + Y = 1$ in $\mathbb{G}_m^2$.

- We already know $\# C \cap (\mathbb{G}_m^2)^{[2]} = 2$.
- In 2000, Cohen and Zannier proved that if $(x, y) \in C \cap (\mathbb{G}_m^2)^{[1]}$, in other words if

$$x \neq 0, 1 \quad \text{and} \quad x^a(1-x)^b = 1 \quad (a, b) \in \mathbb{Z}^2 \backslash \{0\}$$

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$$h(x) \leq \log 2 \quad \text{and} \quad h(y) \leq \log 2.$$

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Theorem (Maurin 2007)

Let $C \subset \mathbb{G}_m^n$ be an algebraic curve defined over $\overline{\mathbb{Q}}$. Assume that C is not contained in a **proper algebraic subgroup** *of $\mathbb{G}_m^n$, then*

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Let $C \subset \mathbb{G}_m^n$ be a curve not contained in a proper coset, $n \geq 2$, and let $\gamma_1, \ldots, \gamma_s \in \overline{\mathbb{Q}}^*$ be multiplicatively independent. Then

$$C \times \{(\gamma_1, \ldots, \gamma_s)\} \subset \mathbb{G}_m^{n+s}$$

is a curve but not in proper algebraic subgroup.

If $x \in C(\overline{\mathbb{Q}})$ and $k \in \mathbb{N}$ such that the coordinates of x^k are in $< \gamma_1, \ldots, \gamma_s >$, then

$$(x, \gamma_1, \ldots, \gamma_s) \in C(\overline{\mathbb{Q}}) \times \{(\gamma_1, \ldots, \gamma_s)\} \cap (\mathbb{G}_m^n)^{[2]}.$$

Therefore, $C \cap \Gamma$ is finite for a subgroup $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ of finite rank.

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A result in arbitrary dimension I

Theorem (Bombieri, Zannier 2000)

Let $X \subset \mathbb{G}_m^n$ be irreducible and defined over $\overline{\mathbb{Q}}$ of any dimension. Set

$$X^{\mathrm{o}} = X \setminus \bigcup_{\substack{H \subset X \text{ coset} \\ \dim H \geq 1}} H,$$

then

$$X^{\mathrm{o}} \cap (\mathbb{G}_m^n)^{[n-1]}$$

has bounded height.

Remark (on curves)

The subgroup codim. size $n-1$ is not optimal for curves if $n \geq 3$.

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Remark (on hypersurfaces)

If X is a hypersurface, i.e. $\dim X = n-1$, then the subgroup codim. size $n-1$ is optimal.

Conclusion

Combining BMZ and BZ's Theorem: we have boundedness of height results with optimal subgroup codim. for **curves** and **hypersurfaces**.

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Abelian varieties

What is known for subvarieties of abelian varieties?

Theorem (Viada 2003)

Let E be an elliptic curve and $C \subset E^g$ an irreducible curve both over $\overline{\mathbb{Q}}$. Assume C is not contained in a **proper coset***. Then*

- *any Néron-Tate height is bounded on $C \cap (E^g)^{[1]}$,*
- *if E has complex multiplication, then $C \cap (E^g)^{[2]}$ is finite.*

Theorem (Rémond 2005)

Boundedness of height for curves in any abelian variety.

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Why the complex multiplication hypothesis?

Remark

Finiteness for $C \cap (E^g)^{[2]}$ is deduced as in Bombieri, Masser, and Zannier's Theorem:
One needs a "relative Lehmer-type height lower bound" which is presently only available for elliptic curves with CM.

The obstruction

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Algebraic subgroups of E^g

Remark

If E is an elliptic curve over a field of characteristic 0, then $\mathrm{End}(E)$ is an order in a number field of degree 1 or 2 over $\mathbb{Q}$.

Example

Let H be an algebraic subgroup of E^g of codim. r. There exist $\mathrm{End}(E)$-linearly independent vectors

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A finiteness conjecture in higher dimension

Conjecture (Zilber)

Let S be semi-abelian and X an irreducible closed subvariety of S both over $\mathbb{C}$. There exists a **finite** *set $\mathcal{H}$ of proper algebraic subgroups of S such that*

$$X \cap S^{[1+\dim X]} \subset \bigcup_{H \in \mathcal{H}} X \cap H.$$

Remark

If $X \not\subset$ proper algebraic subgroup of S then Zilber would imply

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Similar conjectures were stated by Pink and by Bombieri, Masser, Zannier for $S = \mathbb{G}_m^n$.

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X^{oa} is a natural subset of X which tries to eliminate trivial counterexamples.

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Anomalous subvarieties I

Remark

Let X, H be subvarieties of A (or $\mathbb{G}_m^n$ or even S). Let

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If X and H are in general position one expects

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An irreducible closed subvariety $Y \subset X$ is called anomalous if there exists a **coset** H such that $Y \subset H$ and

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Any anomalous subvariety of an irreducible curve $C \subset A$ must equal C. The coset H must satisfy $\dim H < \dim A$. Hence,

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We define

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Theorem (Bombieri, Masser, and Zannier 2006)

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The same holds for $\mathbb{G}_m^n$ replaced by an abelian variety.

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Moreover, if E is an elliptic curve and $X \subset E^g$ then

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Height bound and finiteness results are currently more available in the abelian case as opposed to the multiplicative case:

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Height bound or finiteness results for subvarieties of $\mathbb{G}_m^n$ other than curves and hypersurfaces with optimal subgroup codim. are rather sparse.

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