Computational and Diagrammatic Techniques for Perturbative Quantum Electrodynamics

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Contents

Chapter 1

Introduction

The notion of fields is certainly one of the most successful concepts in physics. A prominent example is the theory of classical Electrodynamics which describes the dynamics of charged bodies in terms of interactions with an electromagnetic field. On one hand, electric charges generate an interacting electromagnetic field; the electromagnetic field, on the other hand, produces forces on the electric charges. A quantitative description of these interactions is provided by Maxwell's equations and the Lorentz force law. These laws successfully explain many phenomena above the atomic length scale. Below this scale, quantum effects can not be neglected and require concepts from quantum theory.

Quantum Field Theory provides a set of methods to quantize classical field theories such as Electrodynamics. In combination with perturbative techniques, it reproduces the classical predictions in the low-energy limit. On the other hand, additional quantum corrections emerge in the high-energy regime. Due to these quantum corrections, Quantum Electrodynamics has been able to explain of new effects such as the Lamb shift, photon-photon scattering, or soft Bremsstrahlung. Further results of Quantum Electrodynamics provided crucial refinements of our understanding of physical quantities. For example, it transpires that the electric charge is not a physical constant, but rather a dynamical parameter whose magnitude increases with increase of energy. Another prominent example is the anomalous magnetic moment of the electron a_e, which describes the difference between Quantum Electrodynamics and the classical field theory in the coupling-strength of a spin-$^1/_2$ electron to a magnetic field. The theoretical prediction [1, 2]

$$a_e^{\text{th}} = 1\ 159\ 652\ 181.664(23)(16)(763) \times 10^{-12} \tag{1.1}$$

agrees with the most recent experimental value of [3]

$$a_e^{\exp} = 1\ 159\ 652\ 180.73(0.28) \times 10^{-12} \tag{1.2}$$

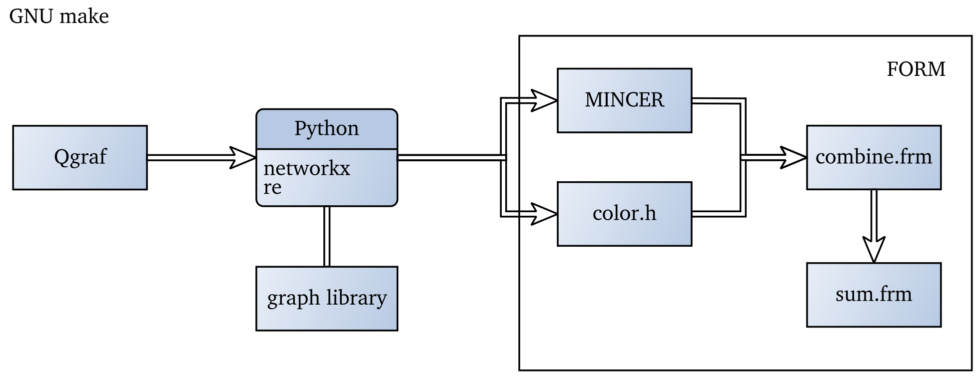
to such an extreme precision that its theoretical estimation was the key to the most precise determination of the mass of the electron [4]. Here, it is important to note that the actual challenge that underlies these predictions is the computation of an enormous number of Feynman graphs.

Despite enormous successes in comparing perturbative Quantum Electrodynamics to experiments, the theory and the quantization procedures are still lacking a rigorous mathematical foundation. Nonetheless, the theory of quantized fields provides heuristic arguments which propose many global properties and restrictions; for instance causality, unitarity, gauge invariance, and renormalizability, to name but a few. Unfortunately, in many cases it is far from obvious how these properties affect Feynman graphs or how they can be exploited in perturbative calculations. This motivates our basic proposal to study the relation between global properties of Quantum Field Theory and its perturbative sector. This thesis intends to conduct such an enquiry for the topic gauge invariance and indeed we conclude that global symmetry properties as Ward identities [5, 6, 7] and the Landau-Khalatnikov transformation [8] are indeed a consequence of structural properties of Feynman graphs.

On a technical level, the computation of Feynman graphs requires involved mathematical methods to handle divergent expressions which emerge in the associated integrals. These techniques are referred to as renormalization and form a core part of our research and this thesis. In [9], Kreimer discovered that the perturbative sector of a Quantum Field Theory possesses an underlying Hopf algebra structure; this Hopf algebra of Feynman graphs provides a mathematically rigorous description of the renormalization process. Furthermore, Kreimer's discovery might be seen as the starting point for a course of research that revealed beautiful connections of Feynman graphs in scalar Quantum Field Theory and mathematical concepts from algebra, algebraic geometry, combinatorics, and number theory. To name only a few, we refer to the work of Bloch and Kreimer on Cutkosky rules [10], Brown's construction of the cosmic Galois group [11], Schnetz's proof of the zigzag conjecture [12], and algebraic lattices [13, 14]. The full extent of these structures and how they can be exploited in perturbative computations is the subject of ongoing research. Further, it is an interesting questions how these findings convey to gauge theories such as Quantum Electrodynamics. In this thesis, we start with exploring the Hopf algebra structure and its applications in perturbative computations for Quantum Electrodynamics and its non-abelian analogue Quantum Chromodynamics.

Computational methods

A major effort that underlies this thesis has been the computation of a large number of Feynman graphs. This paragraph gives a short account on our computational setup and acknowledges the software components and programs that have been used in is course.



We use QGRAF [15] to generate all one-particle irreducible Feynman graphs of a specific external leg type and loop order. As a preparation for the actual computation, the notation of these Feynman graphs must be rearranged in a way compatible with succeeding programs and libraries. This is accomplished by a Python script developed by the author. It utilizes the RE library for convenient support of regular expressions and the NETWORKX library which provides a broad arsenal of objects and algorithms pertaining to graph theory. By searching for graph isomorphisms, every Feynman graph of the generated list is mapped to an archetype graph defined in our external library. This identifies the topology of a Feynman graph and provides an appropriate labelling of its edges. Eventually, the PYTHON script produces the following output: a first list that describes the kinematics of every Feynman graph with appropriate momentum labellings, a second list of expressions to compute the color factors of every Feynman graph, and a visualization of each Feynman graph in form of a png-file generated by the GRAPHVIZ package.

For the consecutive evaluation of these Feynman graphs, we use FORM [16] and its parallel version TFORM [17] which are symbolic manipulation systems tailor-made for this task. The color factors are evaluated with the COLOR package by van Ritbergen, Schellekens, and Vermaseren [18]. On the other hand, the MINCER package [19, 20] evaluates the kinematics in dimensional regularization and provides us with an epsilon expansion at $D = 4 - 2\varepsilon$ dimensions. The computation of a Feynman graph is completed by combining the derived color factors with the epsilon expansion of that graph. Finally, the derivation is completed by summing up the results of every Feynman graph. It worth remarking that we were computing a number of Feynman graph of the magnitude 10.000. Therefore, all described steps have been automatized with usage of GNU MAKE scripts.

In a consecutive computation, we used the PYTHON based computer algebra system SAGEMATH to derive renormalized Green's functions and renormalization group functions from the output of our FORM setup.

Further, we like to acknowledge the usage of the AXODRAW packages [21, 22, 23] to draw and display the Feynman graphs in this thesis and the usage of YED for the creation of the flowchart above.

Outline of the thesis

The succeeding chapter describes computational and diagrammatic techniques for Quantum Electrodynamics.

Section 2.1 provides some introductory material of Quantum Electrodynamics concerning its quantization, symmetries, and anomalies. In following section, we introduce the cancellation identities of Quantum Electrodynamics and demonstrate how cancellations among Feynman graphs imply global properties as Ward identities in paragraph 2.2.2. In addition to that, we show the cancellation identities are useful to determine the dependence on the covariant gauge parameter in paragraph 2.2.3.

Section 2.3 introduces essential methods of renormalization. Followed by a discussion of the Hopf-algebraic renormalization of Quantum Electrodynamics in the linear covariant gauge in section 2.4.

These results are applied to the massless self-energy of the electron in section 2.5. A non-perturbative argument proves the well-known conjecture that the anomalous dimension of the electron only depends on the gauge parameter at first order in perturbation theory.

Chapter 3 reports on our enquiry and applications of the Hopf algebra structure for the renormalization of non-abelian gauge theories; it can be understood as prerequisite for a generalization of our diagrammatic techniques to scrutinize the gauge dependence. Paragraph 3.6 contains the derived renormalization group functions in the $\widetilde{\text{MOM}}$q and $\widetilde{\text{MOM}}$h schemes.

Chapter 2

Quantum Electrodynamics

Quantum Electrodynamics is the Quantum Field Theory which is the main subject of our enquiry. In an introductory section, we start with a discussion of the classical Lagrangian theory, which is referred to as Spin-$\frac{1}{2}$ Electrodynamics, and discuss its symmetry properties and applications of perturbation theory, following the classic approach of [24]. This culminates in a description of the perturbative sector by Feynman rules.

In the subsequent section, the analysis of the Feynman rules yields cancellation identities and it is shown that these diagrammatic identities are the reason for crucial global properties of the Green's functions. Namely, Ward identities and a characterization of the gauge dependence are a consequence of a purely combinatorial analysis of the resulting cancellations among Feynman graphs.

A naive evaluation of a Feynman graph beyond the tree-level results in divergent expressions. Therefore, the consecutive step in our enquiry is the renormalization of Quantum Electrodynamics. For this propose, we follow [9, 25, 26] and generalize their concepts of Hopf-algebraic renormalization to the case of Quantum Electrodynamics with a linear covariant gauge fixing. Remarkably, the Hopf algebra allows for another global result — the closed formula for the coproduct of Green's functions which proves the Callan-Symanzik equation and trivializes the combinatorics of kinematic renormalization schemes.

Finally, the combination of both techniques cancellation identities and Hopf-algebraic renormalization allows for a study of the gauge dependence of renormalized Green's functions. This is demonstrated in an analysis of the self-energy of the electron. A non-perturbative argument proves a well-known conjecture that the anomalous dimension of the electron is gauge dependent only at the first loop order.

2.1 Spin-1/2 Electrodynamics

Spin-$\frac{1}{2}$ Electrodynamics is a field theory consisting of an electromagnetic field with the covariant four-potential A_μ coupled to the spin-$\frac{1}{2}$ Dirac field ψ

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\not{\partial} - e\not{A} - m \right) \psi. \tag{2.1}$$

Here, the Dirac adjoint is denoted by $\overline{\psi} := \psi^\dagger \gamma_0$, m is the mass of the electron, and e is the positive elementary charge, whereas results within the course of this thesis will be mainly expressed in terms of the so-called fine-structure constant

$$\alpha = \frac{e^2}{4\pi}. \tag{2.2}$$

It should be remarked that we are employing natural units ($c = 1$, $\varepsilon_0 = 1$, $\hbar = 1$) throughout this thesis. Further notice that we might use the terms four-potential, electromagnetic field and gauge field synonymously in a slight abuse of language. Appendix A provides the full set of our conventions to an interested reader. Here, we like to conclude with the equations of motion which restricts the Dirac field to satisfy

$$\left(i\not{\partial} - e\not{A} - m \right) \psi = 0 \tag{2.3}$$

and the gauge field obeys

$$\partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu \psi. \tag{2.4}$$

The construction of an exact solution of this coupled system of partial differential equations provides an enormous challenge. Two topics might be useful to approach this kind of task. Symmetries are expected to provide some insight into the properties of solutions and the next paragraph is devoted to this topic. Furthermore, perturbative techniques will be introduced in the sequel.

2.1.1 Symmetries

The action of Spin-$\frac{1}{2}$ Electrodynamics is defined as the spacetime integral of the Lagrangian density (2.1). It is symmetric under the following set of transformations:

1. Spacetime translations. Given a four-vector $a \in \mathbb{R}^{3,1}$ in the Minkowski space, the action is invariant under the transformation

$$\left.\begin{aligned} A_(x) &\mapsto A_\mu(x-a) \\ \psi(x) &\mapsto \psi(x-a) \end{aligned}\right\} \tag{2.5}$$

which can be regarded as a consequence that the Lagrangian density does not explicitly depend on the spacetime

$$\mathscr{L}\left(A(x), \overline{\psi}(x), \psi(x), x\right) = \mathscr{L}\left(A(x), \overline{\psi}(x), \psi(x)\right). \tag{2.6}$$

2. Lorentz invariance. An appropriate representation of the proper orthochronous Lorentz group for the Dirac and gauge field is defined by

$$\left.\begin{aligned} A_\mu(x) &\mapsto \mathcal{M}(\Lambda)_\mu{}^\nu A_\nu(\Lambda^{-1}x) \\ \psi(x) &\mapsto S(\Lambda)\psi(\Lambda^{-1}x) \\ \overline{\psi}(x) &\mapsto \overline{\psi}(\Lambda^{-1}x)S^{-1}(\Lambda) \end{aligned}\right\}. \tag{2.7}$$

This guarantees Lorentz invariance by construction if combined with the compatibility condition

$$S^{-1}(\Lambda)\gamma_\mu S(\Lambda) = \mathcal{M}(\Lambda)_\mu{}^\nu \gamma_\nu. \tag{2.8}$$

3. Gauge invariance. For a local function $\omega : \mathbb{R}^{3,1} \to \mathbb{R}$ with appropriate regularity, the following set of transformations

$$\left.\begin{aligned} A_\mu(x) &\mapsto A_\mu(x) + \partial_\mu \omega(x) \\ \psi(x) &\mapsto e^{-ie\omega(x)}\psi(x) \\ \overline{\psi}(x) &\mapsto \overline{\psi}(x)e^{ie\omega(x)} \end{aligned}\right\} \tag{2.9}$$

leaves the Lagrangian invariant. Further, taking $\omega = \text{const}$ yields the conserved fermion current $j^\mu = e\overline{\psi}\gamma^\mu\psi$.

While the above transformation are symmetries of the full action including the mass term, the following transformations yield symmetries in the limit of a vanishing fermion mass $m = 0$ only.

4. Chiral symmetry. Introduce the right-handed and left-handed fermionic spinors as

$$\psi_R := \tfrac{1}{2}(1+\gamma_5)\psi \quad \text{and} \quad \psi_L := \tfrac{1}{2}(1-\gamma_5)\psi, \tag{2.10}$$

then the massless Lagrangian respectively respects the following vector and axial symmetries

$$\left.\begin{aligned}\psi_L &\mapsto e^{i\theta_V}\psi_L\\ \psi_R &\mapsto e^{i\theta_V}\psi_R\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}\psi_L &\mapsto e^{i\theta_A}\psi_L\\ \psi_R &\mapsto e^{-i\theta_A}\psi_R\end{aligned}\right\}. \tag{2.11}$$

Note that the vector current coincides with j^μ from above, whereas the axial current reads $j_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$.

5. Dilation invariance. If $\alpha \in \mathbb{R}$ is a real scalar, then the massless action remains invariant under the following scale transformations

$$\left.\begin{aligned}A(x) &\mapsto e^{\alpha}A(e^{\alpha}x)\\ \psi(x) &\mapsto e^{3/2\alpha}\psi(e^{\alpha}x)\\ \overline{\psi}(x) &\mapsto \overline{\psi}(e^{\alpha}x)e^{3/2\alpha}\end{aligned}\right\}. \tag{2.12}$$

6. Special conformal transformations. For a four-vector $c \in \mathbb{R}^{3,1}$ in the Minkowski space, define the conformal scale

$$\sigma(x,c) = 1 + 2(c\cdot x) + c^2x^2. \tag{2.13}$$

Then, a finite special conformal transformation reads [27]

$$\left.\begin{aligned}x_\mu &\mapsto x'_\mu = \frac{x_\mu + c_\mu x^2}{\sigma(x,c)}\\ A_\mu(x) &\mapsto \frac{\partial x^\nu}{\partial x'^\mu}A_\nu(x)\\ \psi(x) &\mapsto \sigma(x,c)(1+\not{c}\not{x})\psi(x)\end{aligned}\right\} \tag{2.14}$$

and leaves the massless action invariant.

All these symmetries give raise to conservation laws which are encoded in conserved Noether currents. Appendix B gives a comprehensive account on the infinitesimal generators of the above transformations, their conformal algebra, and the resulting Noether currents.

2.1.2 Perturbation theory and quantization

For the sake of a first perturbative analysis, let us focus on the equations of motion of the Dirac field (2.3) and assume that the gauge field only contributes a static term $\not{A}$. In order to construct solutions, it is convenient to employ the method of Green's functions — that is, instead of directly approaching the homogeneous differential equation, one rather solves the distribution-valued equation

$$\left(i\partial\!\!\!/ - eA\!\!\!/(x) - m\right) S_A(x) = i\delta^{(4)}(x) \tag{2.15}$$

in the unknown distribution S_A. Once one has determined such an distribution S_A, a solution of the homogeneous differential equation is readily provided by the spacial integral of the product of S_A with an initial value ψ_0 at a particular time [24]. It should be remarked that S_A further allows to construct solutions if an inhomogeneous term is added to the differential equation (2.3). Therefore, S_A is called fundamental solution or Green's function in the context of classical field theory. However, the quantization of fields leads to a more complicated notion of Green's functions; hence S_A and suchlike distributions are termed propagator.

The above distribution-valued differential equation can equivalently be formulated as an integral equation

$$S_A(x) = S_F(x) - ie\int d^4y S_F(x-y)A\!\!\!/(y)S_A(y), \tag{2.16}$$

where S_F denotes the Dirac propagator which is a solution of the free Dirac equation

$$\left(i\partial\!\!\!/ - m\right) S_F(x) = i\delta^{(4)}(x). \tag{2.17}$$

Of course, a closed solution of the differential or integral equations are known only for very few occasions of the gauge field $A\!\!\!/$. At this point, it becomes necessary to introduce some kind of physically convenient approximation and it turns out that a large class of problems can be handled accordingly by means of perturbative techniques.

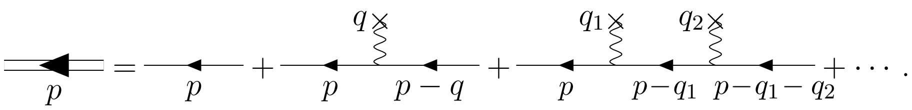
Under the assumption that the electromagnetic field induces only small perturbations to the free propagator, it is possible to solve the integral equation in terms of a formal Neumann series in the electromagnetic charge. Abstaining from a discussion of convergence, the solution reads

$$\begin{aligned} S_A(x) = S_F(x) &- ie\int d^4y S_F(x-y)A\!\!\!/(y)S_F(y) \\ &+ (-ie)^2\int d^4y d^4z S_F(x-y)A\!\!\!/(y)S_F(y-z)A\!\!\!/(z)S_F(z) + \cdots . \end{aligned} \tag{2.18}$$

This series determines the Dirac propagator in presence of the (static) gauge field A_μ as a sequence of position space integrals. In the following, Fourier transformation will be applied in order to substitute the position space integrations by integrations over momentum space variables. This appears to be a convenient procedure as a Fourier transform translates the differential equation (2.17) into an algebraic equation such that the free propagator becomes a rational function in momentum space. In momentum space, the formal Neumann series reads

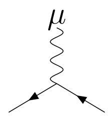
$$\begin{aligned} S_A(p) = S_F(p) - ie \int d^4q S_F(p) \not{A}(q) S_F(p-q) \\ + (-ie)^2 \int d^4q_1 d^4q_2 S_F(p) \not{A}(q_1) S_F(p-q_1) \not{A}(q_2) S_F(p-q_1-q_2) + \cdots . \end{aligned} \tag{2.19}$$

Here, S_A and S_F represent the Fourier transformed propagators, which is indicated by their momentum space arguments $p, q, q_1,$ and q_2; in slight abuse of notation we use the same symbol for the Fourier transformed expressions. As an alternative to the algebraic description of this series, each term might be depicted by a graph which consists of straight and wavy edges to represent the free Dirac propagators and the electromagnetic field, respectively. In this way, the solution of the integral equation is cast into the elegant diagrammatic representation

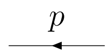


(2.20)

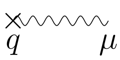
The double-lined propagator represents S_A and the other algebraic expressions (2.19) are reconstructed by employing the following set of rules to the graphs



$$= -ie\gamma^\mu \tag{2.21}$$



$$= i \left(\frac{\not{p} + m}{p^2 - m^2} \right) \tag{2.22}$$



$$= A_\mu(q), \tag{2.23}$$

where the Dirac propagator is obtained by solving the Fourier transformed free Dirac equation (2.17). Further, momentum conservation holds at every vertex and hence restricts the parametrization by momentum variables. Indeed, this course was prominently propagated by Feynman [24] and these graphs and rules are customarily referred to as Feynman graphs and Feynman rules.

It is worth remarking that the distribution-valued equation (2.17) does not give raise to an unique propagator, but rather requires the specification of further boundary conditions. This freedom is fixed by adopting the Feynman prescription to regulate the propagator — a condition necessary for compatibility with quantization. The physical reason for this choice might be due to Stückelberg's interpretation [28] that negative wave frequencies can be understood as antiparticles propagating with reversed time.

Further, the reader should be aware that the dynamical character of the gauge field and its evolution under the equation (2.4) has been neglected in this course. Taking the dynamical character into account leads to the conclusion that the photon source (2.23) must be replace by another perturbation series which is determined by the equations of motion of the gauge field.

Quantization

So far, the perturbation series only contains tree-level Feynman graphs. This underlines its entirely classical character. Quantum Field Theory provides a set of methods to quantize fields such as canonically quantization [29] and path integrals [30]. Unfortunately, none of these formalisms is capable of providing a mathematically rigorous definition of a theory of interacting quantum fields and we will not go into further details of field quantization. Instead, we will view a quantized field theory as determined by its perturbative sector and refer to it as perturbative Quantum Electrodynamics; that is to define the quantized theory by means of Feynman graphs and Feynman rules, which can be deduced in the scope of classical field theory. This point of view is certainly close to the classical approach suggested by Feynman in [31].

As a result of quantization, Feynman graphs are not restricted to tree structures anymore, but rather contain loops which might be interpreted as self-interactions mediated via virtual particles. As outlined above, the tree-level terms originate from perturbation theory of classical field theory. The additional quantum contributions to the propagators and the interaction vertices give rise to the definition of Green's functions which can be reduced to consist of all one-particle irreducible (that is two-edge-connected) Feynman graph of a certain external leg structure. These Green's functions can be thought of as the building blocks for the renormalization process as discussed in paragraph 2.4.2. Beside of that, the discussion of cancellation identities in paragraph 2.2.3 requires us to consider Green's functions which consist of connected Feynman graphs, rather then one-particle irreducible.

2.1.3 Gauge fixing

It is instructive to apply perturbative techniques of the preceding paragraph to the gauge field or, more precisely, to its free equations of motion

$$(g^{\mu\nu}\,\Box - \partial^\mu \partial^\nu)\, A_\nu = 0. \tag{2.24}$$

However, employing a Fourier transform to these equations reveals an obstacle. The momentum space expression of this differential operator is not invertible. Therefore, it is not possible to derive a propagator for these equations of motion. The standard approach to resolve this issue is to add a gauge fixing term $\mathscr{L}_{\mathrm{GF}}$ to the Lagrangian (2.1) that breaks gauge invariance and further modifies the equations of motion for the gauge field in such a way that its Fourier transform allows for an inverse. In this sense, any kind of perturbative approach for a gauge theory requires a gauge fixing.

Obviously, there is not an unique choice for such a gauge fixing. Nonetheless, there is a customary choice for the sake of minimizing the number of terms in the perturbation series. This is the (linear) covariant gauge

$$\mathscr{L}_{\mathrm{GF}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2, \tag{2.25}$$

where ξ denotes the gauge parameter. One can think of ξ as a Lagrangian multiplier, which implies that the gauge field obeys the Lorenz gauge condition

$$\partial_\mu A^\mu = 0. \tag{2.26}$$

This consideration is the reason for identifying $\mathscr{L}_{\mathrm{GF}}$ as a gauge fixing. Further, the reader should be aware that it does not alter the physical electric and magnetic fields, hence it is considered to be a non-physical term — physical results should not depend on the non-physical gauge parameter ξ. This motivates us to ascertain how Green's functions and other quantities depend on the non-physical gauge parameter in the succeeding sections.

Note that within this thesis, statements concerning the gauge dependence will always refer to the class of linear covariant gauges. Therefore, the characterization of the gauge dependence is understood as a characterization of the dependence on the gauge parameter ξ.

Finally, we note that the added gauge fixing term yields the modified equations of motion

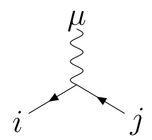
$$\left(g^{\mu\nu}\,\Box + (\frac{1}{\xi} - 1)\partial^\mu \partial^\nu \right) A_\nu = 0, \tag{2.27}$$

$$ \tag{2.28}$$

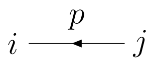
which gives raise to a perfectly invertible operator after a Fourier transformation.

2.1.4 Feynman rules in the covariant gauge

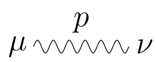
After addition of the covariant gauge fixing, a perturbative analysis of the equations of motion of the gauge field results in an explicit expression for the photon propagator in straight analogy to our discussion of the Dirac field. Conclusively, the Feynman rules of Quantum Electrodynamics in the linear covariant gauge read



$$= ie\gamma^\mu_{ij} \tag{2.29}$$



$$= i\left(\frac{\not{p} + m}{p^2 - m^2}\right)_{ij} \tag{2.30}$$



$$= -i\frac{1}{p^2}\left[g^{\mu\nu} - (1-\xi)\,\frac{p^\mu p^\nu}{p^2} \right]. \tag{2.31}$$

Further, each closed fermion loop contributes a factor of -1 due to the Pauli's exclusion principle which can be shown to be a consequence of the half-integer spin of the Dirac field [32]. It should be noted that ξ is the gauge parameter specified by the linear covariant gauge. Although the gauge parameter will be renormalized and hence change under the flow of the renormalization group, it is sometimes convenient to choose a particular value for the gauge parameter. A commonly used value is the Feynman gauge ($\xi = 1$); this choice minimizes the number of Lorentz tensors in gauge-fixed photon propagator and hence substantially reduces the number of terms generated in the computation of Feynman graphs. Another particular useful choice is the Landau gauge ($\xi = 0$), which reduces the photon propagator to its transversal component. The importance of the

Landau gauge is due to the fact that a Ward identity guarantees this value to be a fixed point under the renormalization flow. For the general covariant gauge fixing, it is convenient to decompose the photon propagator into its transversal and longitudinal components by means of the following tensors

$$T^{\mu\nu}(p) = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \quad \text{and} \quad L^{\mu\nu}(p) = \frac{p^\mu p^\nu}{p^2}. \tag{2.32}$$

They fulfil the following projection rules

$$L^{\mu\sigma} L_{\sigma\nu} = L^\mu{}_\nu, \qquad T^{\mu\sigma} T_{\sigma\nu} = T^\mu{}_\nu, \tag{2.33}$$
$$T^{\mu\sigma} L_{\sigma\nu} = 0, \quad \text{and} \quad L^{\mu\sigma} T_{\sigma\nu} = 0. \tag{2.34}$$

Here, we suppressed the explicit notation of the momentum dependence as all tensor are supposed to depend on the same momentum. The latter line implies that the transversal parts do not interfere with the longitudinal parts in a chain of photon propagators.

2.1.5 Restoring gauge invariance

As the photon propagator clearly depends on the non-physical gauge fixing term, the first question one likes to address is: how does a photon propagator in Quantum Electrodynamics without specification of a gauge fixing relates to the proposed photon propagator? This paragraphs provides a naive attempt to answer this question.

We start with the Lagrangian of Spin-$\frac{1}{2}$ Electrodynamics $\mathscr{L}$ without a gauge fixing. Then, the gauge fixing term $\mathscr{L}_{\text{GF}}$ is added and immediately subtracted such that the theory effectively remains gauge invariant

$$\mathscr{L} = \underbrace{\mathscr{L} + \mathscr{L}_{\text{GF}}}_{(i)} \underbrace{- \mathscr{L}_{\text{GF}}}_{(ii)}. \tag{2.35}$$

The first part of the Lagrangian (i) allows for a perturbative analysis resulting in the Feynman rules as presented above. In particular, one obtains the above expression of the photon propagator which we like to denote as

$$P_{\mu\nu}(p) = \frac{-i}{p^2} \left(T_{\mu\nu}(p) + \xi L_{\mu\nu}(p) \right), \tag{2.36}$$

where T and L respectively denote the transversal and longitudinal tensors with the momentum p. The second part (ii) contributes a new two-photon interaction term which has to be included in the perturbation series. This term gives rise to the expression

$$V_{\mu\nu}(p) = \frac{i}{\xi} p^2 L_{\mu\nu}(p). \tag{2.37}$$

In a Feynman graph, this novel vertex only couples in-between a pair of photon propagators. Interpreting suchlike expressions as contributions to an effective photon propagator, the novel vertex can indeed be absorbed. This effective photon propagator consists of the original photon propagator and all possible ways to build a chain of the novel vertex enclosed by two photon propagators

$$\widetilde{P}_{\mu\nu}(p) = P_{\mu\nu}(p) + P_{\mu\sigma_1} \sum_{n \geq 1} V_{\sigma_1\rho_1} P_{\rho_1\sigma_2} \cdots V_{\sigma_n\rho_n} P_{\rho_n\nu}(p). \tag{2.38}$$

Following this line thoughts, one indeed finds that the effective photon propagator adopts the expected decomposition into a transversal and a longitudinal part

$$\widetilde{P}_{\mu\nu}(p) = \frac{-i}{p^2} \left(T_{\mu\nu} + \widetilde{\xi} L_{\mu\nu} \right), \tag{2.39}$$

provided one introduces the effective gauge parameter $\widetilde{\xi}$. However, this parameter

$$\widetilde{\xi} := \xi \sum_{n \geq 0} 1 \tag{2.40}$$

turns out to diverge and hence incorporates the pathological behaviour the free equations of motion of the gauge field (2.4). This derivation suggests the interpretation that adding the covariant gauge fixing (2.25) squeezes the amplitude of the longitudinal part of the photon propagator to a finite value. On the other hand, the gauge fixing does not affect the transversal part of the propagator, which might therefore be considered as the physical part.

2.1.6 Anomalies

The paragraph 2.1.1 summarized several symmetries of the Spin-$\frac{1}{2}$ Electrodynamics as a classical field theory. This paragraph gives a non-exhaustive report on how these symmetries are modified due to self-interactions in the quantization process. Special emphasis lies on the symmetries of the massless Lagrangian.

Firstly, consider the class special conformal transformations. It can be shown that the linear covariant gauge fixing $\mathscr{L}_{\text{GF}}$ (2.25) breaks the invariance under special conformal transformations and hence causes a non-vanishing term which spoils the conservation of the conformal Noether currents. However, beside of this term, quantum corrections yield further contributions which violate the conformal invariance. More precisely, perturbative terms give raise to a non-vanishing trace in the energy-momentum tensor; the interested reader finds an extensive discussion of the so-called trace anomaly in [33] and references therein. This case illustrates what is termed anomaly — contributions due to quantum corrections that spoil classical conservation laws or affect the classical equations of motion.

In the light of this issue, recall that the literature suggests a customary way to draw conclusions from the gauge invariance of a quantized field theory: first define Green's functions by means of time-ordered vacuum expectation values of operators which evolve in the interaction picture; then, consider the derivative of a Green's function which includes a classically conserved current; apply the current conservation and work out additional Schwinger terms, which are due to the time-ordering. This implies an identity, usually termed Ward identity, that relates a class of different Green's functions [34]. Alternatively, this kind of Ward identity can be derived by exploiting invariance properties of the path integral [30].

Note that the same reasoning is readily applied to the case of dilation invariance [35] and results in the identity

$$\left(\sum_{1\leq i\leq n} x_i^\mu \frac{\partial}{\partial x_i^\mu} - d_G\right) G(x_1, \cdots, x_n) = 0 \tag{2.41}$$

with the scaling dimension

$$d_G = d_\psi N_\psi + d_A N_A - 4 \tag{2.42}$$

of the connected Green's function G, where $d_\psi = {}^3\!/_2$ and $d_A = 1$ are the scaling dimensions of the fermion and photon fields, and the number of external fermion and gauge fields of the Green's function G is denoted by N_ψ and N_A. For the instance of the Fourier transformed electron propagator $S(p)$, Ward identity associated to scale invariance reads

$$\left(p_\mu \frac{\partial}{\partial p_\mu} + 4 - 2d_\psi\right) S(p) = 0, \tag{2.43}$$

which is indeed respected by the tree-level approximation ${}^1\!/_{\not p}$ of the electron propagator. However, it turns out that quantum corrections also spoil the dilation invariance. For a qualitative understanding of this statement, it suffices to consider the first-order correction (in the loop counting parameter α) to the electron propagator

$$S(p) = \frac{1}{\not p}\left[1 + \xi\left(\frac{\alpha}{4\pi}\right)L + \mathrm{O}\left(\alpha^2\right)\right], \quad \text{with} \quad L = \ln\left(\frac{-p^2}{\mu^2}\right). \tag{2.44}$$

Obviously, the kinematic variable L depends on the momentum p and yields a non-vanishing contribution which spoils the dilation Ward identity (2.43). However, at this order, the dilation identity can be restored by replacing the scaling dimension d_ψ of the fermionic field by the coupling dependent series

$$\widetilde{d}_\psi(\alpha) = d_\psi + \xi\frac{\alpha}{4\pi} + \mathrm{O}\left(\alpha^2\right). \tag{2.45}$$

The coupling dependent terms indicate that quantum corrections modify the classical scaling behaviour and are hence called anomalous (scaling) dimension of the electron. However, considering even higher orders of quantum corrections, the scaling Ward identity ceases to persist and is replaced by the renowned Callan-Symanzik equation [36, 37, 38]

$$\left(\partial_L + \beta\alpha\partial_\alpha + \delta\xi\partial_\xi - \gamma\right)\not p S(p) = 0, \tag{2.46}$$

where γ represents the anomalous dimension as introduced above and the other renormalization group functions β and δ respectively describe the scaling behaviour of the coupling parameter α and the gauge parameter ξ. Further details on the Callan-Symanzik equation including a Hopf-algebraic derivation can be found in paragraph 2.4.4.

As one expects the scaling behaviour of Green's functions to be related to observable quantities, it is worth emphasizing that the one-loop anomalous dimension (2.45) immediately reveals a striking point: renormalization group functions are gauge dependent objects. This observation funds our motivation to characterize the dependence of Green's function on this non-physical parameter.

Another crucial remark pertains to the derivation of the Ward identity (2.41). The derivation in the canonical or path integral formalism appears to be insensitive to the possible emergence of anomalies. For this reason, our enquiry aims to completely avoid the application of classical symmetries to Green's functions; this includes gauge invariance and its residual relatives. Instead, we rather commit ourselves to the perturbative regime and study the analytic properties of Feynman rules to derive Ward identities and characterize the gauge dependence.

After, becoming aware of the anomalous behaviour of the special conformal transformations and dilations, it should be mentioned that the chiral symmetry is also violated. The axial current is anomalous. However, due to the Adler-Bardeen theorem [39], this anomaly is restricted to the first loop order and it is possible to define a modified axial current that is indeed conserved [40].

Finally, it is reasonable to remark that despite of their formidable appearance and the related breakdown of symmetries, anomalies should actually be appreciated as they provide excellent opportunities to compare theory with experiments. A prominent instance of massive Quantum Electrodynamics is the anomalous magnetic moment of the electron as discussed in the introduction.

2.2 Cancellation identities

This section constitutes our enquiry of global properties of Green's functions. It is perturbative by nature — that is, it is based on diagrammatic identities which are a consequence of the analytical structure of the Feynman rules. The discussion presented here is an extension of the results of [41].

Firstly, it transpires that tree-level identities imply cancellation between certain Feynman graphs. These cancellation identities apply in two different scenarios. The first pertains to Green's functions with longitudinal photon legs and implies the renowned Ward identities — as an example the Ward-Takahashi identity [5, 6, 7] is derived. The second application pertains to the longitudinal part of the photon propagator where the resulting cancellations allow for a characterization and reconstruction of the gauge-dependent parts of the Green's function. This characterization by means of a Dyson-Schwinger type equation can be thought of as a perturbative derivation of the famous Landau-Khalatnikov formula [8] and is readily generalized to Quantum Electrodynamics at higher dimensions.

2.2.1 Diagrammatic cancellations

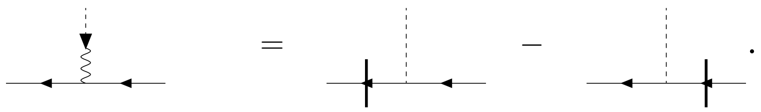
The cornerstone of our enquiry is a remarkably simple tree-level identity, namely

$$\gamma_\nu k^\nu = \left(\not{p} + \not{k} - m\right) - \left(\not{p} - m\right) . \tag{2.47}$$

This algebraic identity is useful when considering two electron propagators that enclose a vertex with a longitudinal contracted photon leg

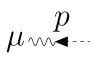
$$\frac{1}{\not p + \not k - m}\gamma_\nu k^\nu \frac{1}{\not p - m} = \frac{1}{\not p - m} - \frac{1}{\not p + \not k - m},$$

the product which contains two fermionic propagators (three gamma matrices) can be written as a sum of two fermionic propagators (one gamma matrix each). The right-hand side appreciates some analytical simplifications; it is therefore reasonable to introduce a diagrammatic representation for these terms

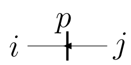


(2.48)

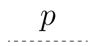
The novel representation includes cancelled electron propagators (visualized by slashed electron lines) and a dashed line that couples to the electron without contributing the usual γ^ν factor. To be more precise, the analytic expressions are visualized by means of the following set of auxiliary Feynman rules.



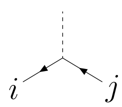
$$= p^\mu \tag{2.49}$$



$$= i\left(\frac{\not p - m}{\not p - m}\right)_{ij} = i\delta_{ij} \tag{2.50}$$



$$= \frac{1}{p^2} \tag{2.51}$$



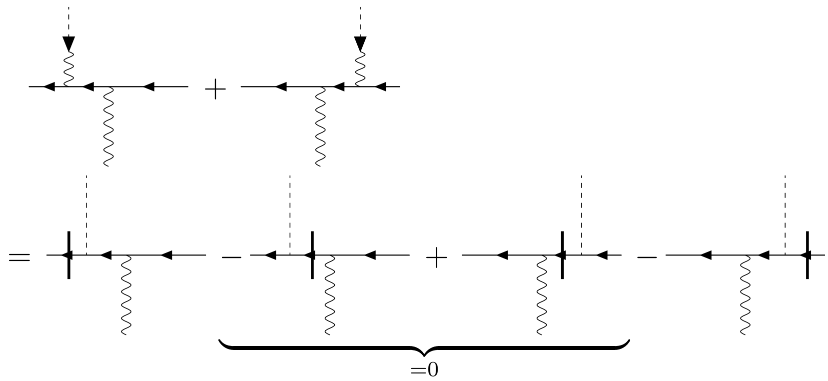
$$= ie\delta_{ij} \tag{2.52}$$

$$= i\xi \tag{2.53}$$

The first equation introduces a convention: the orientation of the momentum p is meant to be defined by the direction of the triangle. Also note that the dashed line is associated with a scalar propagator and we introduced a vertex that connects two dashed lines. These notations are anticipated here; these will prove themselves useful for the description of the longitudinal part of the photon propagator in paragraph 2.2.3.

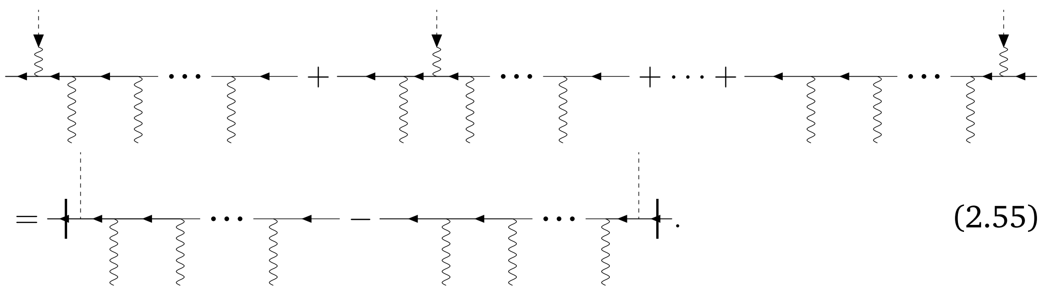
Diagrammatic cancellation identities

The tree-level identity (2.48) replaces a longitudinal contracted Feynman graph by the sum of two Feynman graphs with auxiliary edge and vertex types. The actual simplification of this replacement is that the number of gamma matrices decreases. This decrease is indicated by the slashed electron propagators in the diagrammatic identity. Note that the slashed propagator corresponds to a unit matrix in spinor space and can hence be contracted to a point without changing the analytical structure of the Feynman graph. In combination with the relative minus sign in the identity (2.48), this leads to cancellations of the following type.



(2.54)

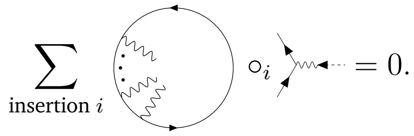
This cancellation readily generalizes to the case of an electron line with an arbitrary number of attached photon legs. Our first cancellation identity reads



(2.55)

When considering all possible insertions of a momentum-contracted photon vertex into an electron line, all terms cancel except for terms from the very first and the very last insertion place. It is worth emphasizing that there is no restriction on the photon lines below the horizontal fermion line. Pairs of them might be connected with the photon propagator or even vacuum polarization graphs. This feature enables the tree-level identity to describe cancellations between Feynman graphs containing loops.

The second cancellation identity is concerned with the insertion of a momentum-contracted photon leg to a closed electron loop. Let us denote the insertion of a graph γ into the graph Γ at an insertion point i by $\Gamma \circ_i \gamma$. Then, the sum over all possible insertions of a momentum-contracted photon vertex into a closed electron loop vanishes



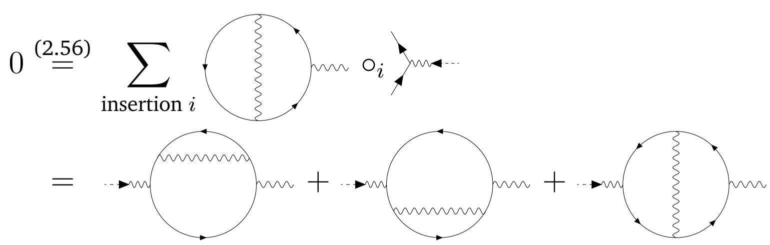
(2.56)

This identity is easy conclusion of the first cancellation identity. To see this, close the horizontal electron line in the first identity to a loop and contract the slashed electron propagators. This yields two equivalent closed electron loops with a relative minus sign. The analytic terms assigned to these graphs differ in the momentum running through the closed electron loop. To cancel both graphs, one needs to shift the loop momentum assigned to the closed electron loop. The reader should be aware that this provides a restriction to the employed renormalization scheme. Within this thesis, we restrict ourselves to dimensional regularization, which is well-known to satisfy this requirement [42].

Note that the second cancellation identity already has strong implications on the global properties of the vacuum polarization $\Pi^{\mu\nu}$. One of its conclusions is the Ward identity

$$p_\mu \Pi^{\mu\nu}(p) = 0, \tag{2.57}$$

which reveals the vacuum polarization to be transversal — that is it vanishes when one of its external photon legs are contracted with the assigned momentum. Here, we visualize the instance of the second loop order.



(2.58)

2.2.2 The Ward-Takahashi identity

The preceding graphical cancellation identities are useful to derive relations between different types of Green's functions. In this regard, cancellation identities enhance perturbative properties from Feynman graphs to the global notion of Green's functions and hence are capable to provide information beyond perturbation theory. Indeed, we will now demonstrate that the Ward-Takahashi identity can be derived solely from the cancellation identity (2.48).

In the following, the sum over all connected Feynman graphs of a particular external leg type is referred to as connected Green's function and will be denoted by attaching its external legs to a rectangular box. In a similar way, circular boxes represent one-particle irreducible, or in other words two-edge-connected, Green's functions.

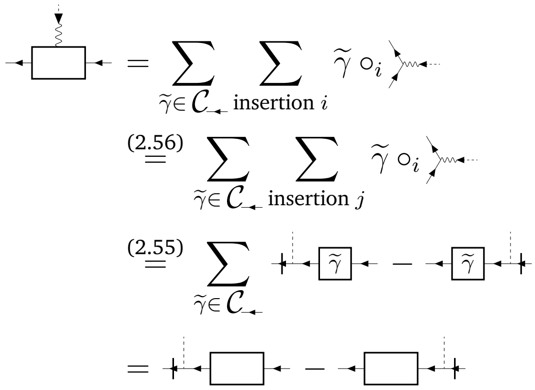
It is worth noticing that the cancellation identities interrelate certain subclasses of Feynman graphs rather than entire Green's functions. These subclasses can be characterized by defining the following equivalence relation. Let γ be a connected Feynman graph of vertex type, we write $\mathrm{res}(\gamma) =$ to depict the external legs of the Feynman graph γ. For every such a vertex graph γ, construct an electron propagator graph $\widetilde{\gamma}$ which is obtained by deleting both, the external photon leg and the adjacent vertex, and connecting the electron edges which have been adjacent to the deleted vertex. Now, a connected vertex Feynman graphs γ_1 is $\sim$-related to another connected vertex Feynman graph γ_2 iff the deletion of their external photon legs results in the same electron propagator graph, that is

$$\gamma_1 \sim \gamma_2 :\Leftrightarrow \widetilde{\gamma_1} = \widetilde{\gamma_2}. \tag{2.59}$$

The relation $\sim$ clearly inherits the properties reflexivity, symmetry, and transitivity from the graph equality relation $=$ in its definition and hence is an equivalence relation. As a result, the set of connected vertex Feynman graphs partitions into equivalence classes. Moreover, this set of equivalence classes is bijective to the set of connected electron propagator graphs. To prove this statement, map every equivalence class $[\gamma]$ to the graph $\widetilde{\gamma}$. This mapping is injective (equivalence classes are disjoint) and surjective (given an arbitrary target $\widetilde{\gamma}$, insertion of an electron-photon vertex defines an appropriate preimage). Denoting the set of connected Feynman graphs with the external leg structure r by $\mathcal{C}_r$, the above equivalence reads

$$\mathcal{C}_{\,}/\!\sim\; \cong \mathcal{C}_{\,}. \tag{2.60}$$

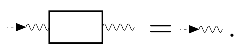
Now, given an electron propagator graph $\widetilde{\gamma}$, this set equivalence assigns an equivalence class of vertex type graphs to the graph $\widetilde{\gamma}$. By definition, this equivalence class consists of all graphs which reduce to $\widetilde{\gamma}$ after removal of their external photon legs. In other words, all graphs of the equivalence class can be constructed by inserting an electron-photon vertex into an electron edge of $\widetilde{\gamma}$ in all possible ways. In this way, the set equivalence (2.60) allows us to rewrite the sum over all connected vertex type Feynman graphs as the summation over all connected propagator graphs in combination with insertion of an electron-photon vertex in all possible ways.



(2.61)

Here, the summation over insertions i refers to all insertions of the electron-photon vertex into an arbitrary electron propagator of $\widetilde{\gamma}$, whereas the insertions j are restricted to insertions into the external electron line — all insertions into closed electron loops vanish as a consequence of the cancellation identity (2.56). The remaining sum over insertions into the external electron line allows to employ the cancellation identity (2.55). This results in a sum over electron propagator graphs which indeed yields the entire connected electron propagator Green's function due to the equivalence (2.60).

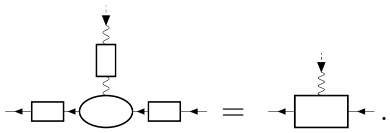
A similar reasoning restricts the connected photon Green's function. For a Feynman graph of loop order one or higher, the longitudinally contracted external photon leg is attached to a closed electron loop. Further, all graphs which are generated by moving the longitudinal photon leg along the closed electron loop are part of the connected Green's function. The sum over these classes of graphs vanishes due to the cancellation identity (2.56). Therefore, the only remaining term of the longitudinally contracted photon Green's function is the tree-level contribution



(2.62)

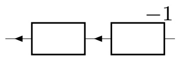
Note that this argument generalizes to an arbitrary number of external photon legs.

Finally, it is the Ward identity for the connected vertex function and the Ward identity for the connected photon propagator function that restrict the one-particle irreducible vertex function in the following way. Recall that the one-particle irreducible vertex function satisfies by definition



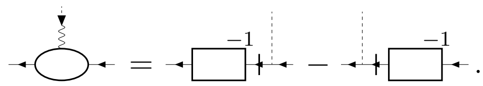
(2.63)

Due to the Ward identity (2.62), the longitudinally contracted photon Green's function on the left-hand side vanishes up to its tree-level contribution. Further, we amputate the connected electron propagator functions by multiplication with their inverses. In our diagrammatic language, a superscript minus one is assigned to the rectangular boxes to denote the inverse propagator functions which satisfy the diagrammatic condition



$$= S(p)S^{-1}(p) = \mathbb{1}_{4\times 4}. \tag{2.64}$$

Now, due to the Ward identity for the connected vertex function (2.61), the one-particle irreducible vertex function with a longitudinally contracted photon obeys the diagrammatic identity



(2.65)

Denoting the momentum of the outgoing electron line by p and the momentum of the incoming electron line by q, this diagrammatic identity translates into the renowned Ward-Takahashi identity

$$(p_\mu - q_\mu)\Gamma^\mu(p,q) = \frac{1}{S(p)} - \frac{1}{S(q)}. \tag{2.66}$$

2.2.3 Reconstruction of the gauge dependence

The preceding section gave an account of how cancellation identities apply to Feynman graphs with a longitudinally contracted external photon leg and imply Ward identities. The essential point of this section is the observation that the gauge dependent terms of the photon propagator give rise to a Lorentz tensor with a similar longitudinal structure. Therefore, cancellation identities are also useful to study the gauge dependence of Green's functions. The highlights of this enquiry are closed formulas for the gauge dependent terms of the electron and vertex Green's function. As the photon Green's function is gauge independent, this provides a complete characterization of how generic Green's functions depend on the gauge parameter.

Consider the tree-level photon propagator of Quantum Electrodynamics with a linear covariant gauge fixing

$$P^{\mu\nu}(p) = \frac{-i}{p^2}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right), \tag{2.67}$$

where ξ denotes the gauge parameter.For the renormalization process, it will be crucial to treat the gauge parameter as a renormalized variable. However, at an intermediate stage, it is possible to evaluate the gauge parameter at a particular value. Here, we denote this value by the constant ξ^* such that the gauge-fixed version of the photon propagator reads

$$P^{\mu\nu}_{\xi^*}(p) = \frac{-i}{p^2}\left(g^{\mu\nu} - (1-\xi^*)\,\frac{p^\mu p^\nu}{p^2}\right). \tag{2.68}$$

It is worth remarking that the Feynman gauge $\xi^* = 1$ is particularly convenient since this choice minimizes the number of Lorentz tensors in gauge-fixed propagator and hence substantially reduces the number of terms generated by each Feynman graph.

Now, assume that we have evaluated the gauge parameter at ξ^* and have performed the Feynman graph computations in this particular gauge fixing. Then we need to reconstruct the actual dependence on the gauge parameter ξ in order to renormalize the divergent Green's functions. This is due to the fact that the gauge parameter needs to be understood as a renormalized parameter which can be utilized to absorb divergences; for a detailed discussion of the renormalization process, the reader is referred to section 2.3.

Notice that the photon propagator in the general linear gauge can be decomposed into the gauge fixed propagator P_{ξ^*} and a gauge parameter dependent tensor

$$P^{\mu\nu}(p) = P^{\mu\nu}_{\xi^*}(p) - i\,(\xi-\xi^*)\,\frac{p^\mu p^\nu}{(p^2)^2}. \tag{2.69}$$

In the general linear gauge, evaluating a Feynman graph results in a polynomial in the gauge parameter ξ and its degree is determined by the number of photon propagators which is bounded by the number of loops of the considered Feynman graph. In the following, the result of a Feynman graph is considered as an expansion in powers of the gauge parameter ξ at ξ^*. The constant part of this expansion is obtained by replacing every photon propagator with the gauge-fixed propagator P_{ξ^*}. Starting from the constant term, terms of higher order in the gauge parameter are constructed by inserting a gauge parameter dependent tensor

$$- p^{\mu} \, \frac{1}{p^2} \, \left[i \, (\xi - \xi^*) \right] \, \frac{1}{p^2} \, p^{\nu} \tag{2.70}$$

instead of a gauge-fixed propagator P_{ξ^*}. Obviously, the number of these gauge dependent tensors determines the order in the (shifted) gauge parameter — for instance the quadratic term in $(\xi - \xi^*)$ is constructed by considering all ways to choose two gauge-fixed propagators and replace them with this longitudinal tensor. As indicated above, the gauge dependent tensor can be represented by means of the auxiliary Feynman rules (2.49-2.53) as two longitudinal photon legs which are connected to a two-valent vertex by two scalar propagators. Rephrasing the previous example in a diagrammatic language, the quadratic term in $(\xi - \xi^*)$ of a Green's function is constructed by evaluating all Feynman graphs with exactly two longitudinal photon propagators (2.70) (and an arbitrary number of the gauge-fixed photon propagator P_{ξ^*}).

Quenched Quantum Electrodynamics

At this point of the discussion, we will restrict ourselves to Feynman graphs that contain no closed electron lines — this is usually referred to as the quenched sector of Quantum Electrodynamics. The influence of closed electron lines will be enquired in the next paragraph.

In the following, we define an equivalence relation that relates certain classes of Feynman graphs containing the gauge dependent tensor (2.70) such that cancellation identities apply to these classes of Feynman graphs in a similar fashion as in the preceding discussion of the Ward-Takahashi identity in paragraph 2.2.2.

Let r denote an external leg structure consisting of exactly one external electron line and an arbitrary number of photon legs. Consider the set $\mathcal{C}^r_m$ of all connected quenched Feynman graphs of the external leg structure r which possess exactly m longitudinal photon edges

$$\mathcal{C}^r_m = \{\gamma : \gamma \text{ connected quenched Feynman graph with } m \text{ longitudinal photons and } \operatorname{res}\gamma = r\} . \tag{2.71}$$

Further, it is useful to consider the set of these kind of Feynman graphs with an additional marking on one of the longitudinal photon edges

$$\widetilde{\mathcal{C}^r_m} = \{(\gamma, e) : \gamma \in \mathcal{C}^r_m, e \text{ longitudinal photon edge of } \gamma\} . \tag{2.72}$$

There is an equivalence relation on $\widetilde{\mathcal{C}^r_m}$ by defining (γ_1, e_1) and (γ_2, e_2) to be $\sim$-related iff the considered Feynman graphs γ_1 and γ_2 are equal up to their marked longitudinal photon edges

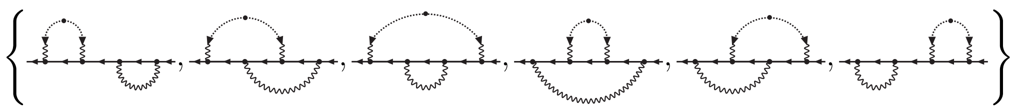
$$(\gamma_1, e_1) \sim (\gamma_2, e_2) :\Leftrightarrow (\gamma_1 - e_1) = (\gamma_2 - e_2) \tag{2.73}$$

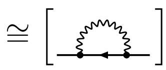
where the $-$ denotes the deletion operation in the sense of deleting the photon edge and forgetting its adjacent vertices to avoid the emergence of two-valent electron vertices. As in the preceding discussion of the Ward-Takahashi identity, the relation $\sim$ inherits all properties of an equivalence relation due to the fact that equality between graphs is an equivalence relation in its own right. Consequently, the relation $\sim$ decomposes the set $\widetilde{\mathcal{C}^r_m}$ into equivalence classes and by definition each equivalence class $[(\gamma, e)]$ can be represented by the connected quenched Feynman graph with $(m-1)$ longitudinal photon edges $(\gamma - e)$. Mapping each equivalence class onto its representative defines a bijective map and proves the equivalence

$$\widetilde{\mathcal{C}^r_m}/\!\sim\; \cong \mathcal{C}^r_{m-1}. \tag{2.74}$$

Firstly, it should be remarked that by considering only quenched Feynman graphs any kind of difficulty originating from tadpole subgraphs is avoided and the considered Feynman graphs remain connected even after the deletion of the marked longitudinal photon edges. Secondly, the number of gauge-fixed photon edges is an invariant in the above definitions and hence the equivalence classes can be refined by considering a specific number gauge-fixed photon edges in $\widetilde{\mathcal{C}^r_m}$.

The following example illustrates these circumstances: consider the set of connected quenched electron propagator graphs with exactly one longitudinal and exactly one gauge-fixed photon propagator. In this case, the marking is already determined since every Feynman graph possesses only a single longitudinal photon edge. Deleting this marked longitudinal edge yields a one-loop connected electron propagator graph in $\mathcal{C}_0^{\leftarrow}$. As there is only one such a graph, the considered subset of $\widetilde{\mathcal{C}_1^{\leftarrow}}$ decomposes into a single equivalence class



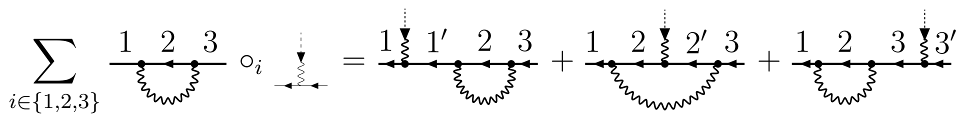


(2.75)

where we used the one-loop graph of $\mathcal{C}_0^{\leftarrow}$ as the representative of the equivalence class in a slight abuse of notation.

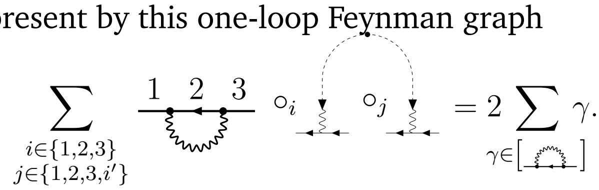
As seen from this example, the elements of a equivalence class are generated by considering all possible ways to attach a longitudinal photon edge (and assigning the marking to this attached longitudinal edge) into the representative graph of the equivalence class. By definition of the equivalence relation $\sim$ and the representative of an equivalence class, this observation is not a particular feature to this specific example, but remains true for a generic equivalence class. However, instead of attaching a longitudinal photon edge it is useful to rephrase this modification in terms of two iterated insertions of a longitudinal contracted electron-photon vertex and connect the inserted vertices such that the longitudinal tensor (2.70) is constructed.

Let us illustrate the iterated insertions in the context of the above example. The one-loop representative graph possesses three places $i \in \{1,2,3\}$ for the first insertion of a longitudinal contracted electron-photon vertex. This generates three Feynman graphs



(2.76)

and each of them possesses four successive insertion places $j \in \{1,2,3,i'\}$ for the second insertion of a longitudinally contracted electron-photon vertex. All these insertion places are used for the two successive insertions into the one-loop Feynman graph. This generates all graphs of the equivalence class that is represent by this one-loop Feynman graph

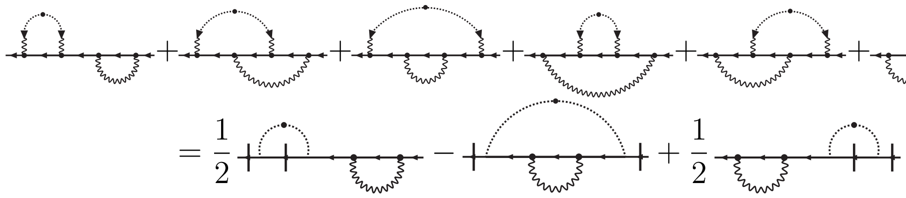


(2.77)

Here the longitudinally contracted electron-photon legs are connected by two scalar propagators and a two-valent vertex in convenience with the auxiliary Feynman rules (2.49-2.53) such that they constitute the gauge dependent tensor (2.70). The factor of 2 is due to the fact that permuting both longitudinally contracted electron-photon vertices corresponds to a different pair of insertions (i, j) that yields the same graph. To exemplify this, consider the first graph of the right-hand side of (2.76). An insertion of the second vertex either into the place 1 or into the place $1'$ yields the first graph of our accumulation of the equivalence class (2.75).

Eventually, up to the technical notion of the marking, a set of quenched connected Feynman graphs with m longitudinal photon propagators decomposes into equivalence classes such that all graphs of an equivalence class are generated by suchlike iterated insertions of two longitudinally contracted electron-photon vertices. This kind of insertion obeys the cancellation identity (2.55) which can be iterated and simplifies the computation of the graphs of the equivalence class.

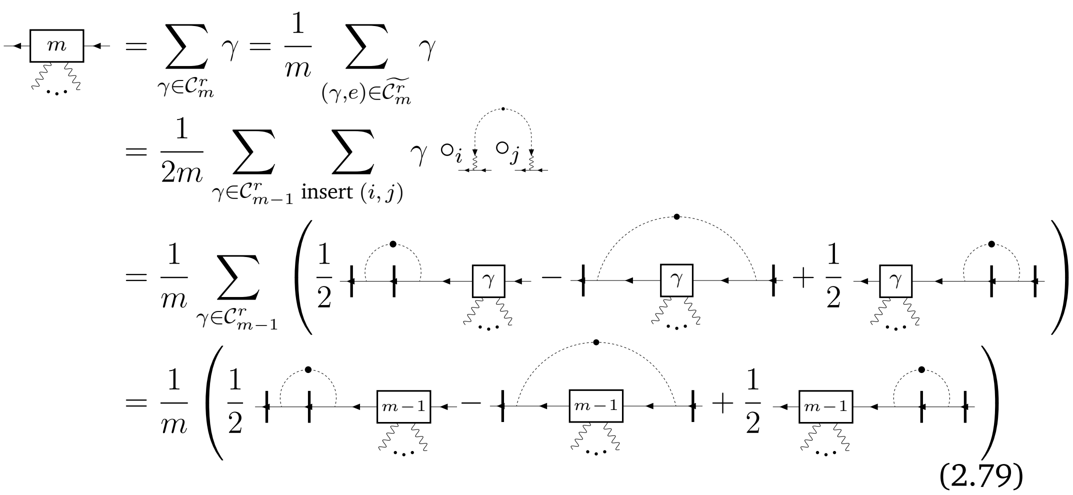
In the case of our example, this reads



(2.78)

where the three remaining graphs result from applying the cancellation identity (2.55) twice to resolve the iterated insertion in the left-hand side of (2.77).

In the general case of a connected quenched Green's function with a single external electron line and an arbitrary number of external photon legs, consider the $(\xi - \xi^*)^m$ term of this Green's function. Rewriting this term by means of an iterated insertion and applying the cancellation identity (2.55) shows that a term of order m in the gauge parameter is determined by the term of order $(m-1)$.



(2.79)

Here, the factor of $1/m$ is introduced in the first line because every graph with m longitudinal photon edges provides m different markings. In the second line, we used that fact that the sum over an equivalence class can be rewritten by means of an iterative insertion into the representative graph of this equivalence class; notice that the set of representative graphs is characterized by (2.74). In the third line, we applied the cancellation identity

(2.55) twice to rewrite the iterated insertion. Finally, we rewrote everything in terms of the connected quenched Green's functions of order $(m-1)$ in the (shifted) gauge parameter.

Closed electron loops

Now, consider the set Π^n_m consisting of all Feynman graphs which are constructed by drawing a closed electron loop and attaching n external photon legs, m longitudinal photons propagator, and an arbitrary number of gauge-fixed propagators to the closed electron loop.

With this definition, $\cup_{m\geq 0}\Pi^2_m$ is the set of Feynman graphs that defines the quenched beta function. However, we will see that all gauge dependent terms cancel such that effectively only Π^2_0 contributes to the quenched beta function.

First notice that every Feynman graph of Π^n_m is one-particle irreducible by construction and moreover each graph of this set remains one-particle irreducible even after the deletion of one of its longitudinal photon propagators

$$\gamma \in \Pi^n_m, e \text{ longitudinal edge of } \gamma \Rightarrow (\gamma - e) \in \Pi^n_{m-1}. \tag{2.80}$$

This property allows us to construct an equivalence relation in a very similar way as in the quenched sector. Firstly, introduce the set of Feynman graphs with a single closed electron loop, n external photon legs, and m longitudinal photon edges where one of the longitudinal edges received a marking

$$\widetilde{\Pi^n_m} = \{(\gamma, e) : \gamma \in \Pi^n_m \text{ and } e \text{ longitudinal edge of } \gamma\}. \tag{2.81}$$

Secondly, an equivalence relation $\sim$ is defined on $\widetilde{\Pi^n_m}$ by setting

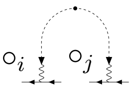
$$(\gamma_1, e_1) \sim (\gamma_2, e_2) :\Leftrightarrow (\gamma_1 - e_1) = (\gamma_2 - e_2). \tag{2.82}$$

As a result, the set $\widetilde{\Pi^n_m}$ decomposes into equivalence classes. Mapping such an equivalence class $[(\gamma, e)]$ onto the Feynman graph $(\gamma - e) \in \Pi^n_{m-1}$ defines a bijection such that every equivalence class possesses a natural representative Feynman graph

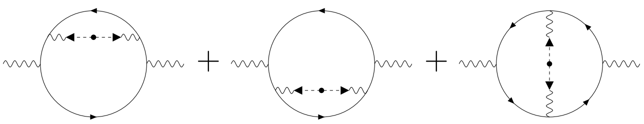
$$\widetilde{\Pi^n_m}/\!\sim\, = \Pi^n_{m-1}. \tag{2.83}$$

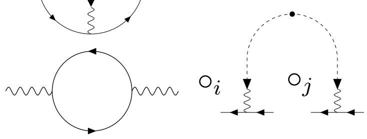
By construction, every Feynman graph of a particular equivalence class can be generated by iterated insertion of two longitudinally contracted electron-photon vertices into the associated representative graph in strict analogy to the quenched sector.

Given such an equivalence class $[\gamma]$ where $\gamma \in \Pi^n_{m-1}$ is the representative graph of the class, the cancellation identity (2.56) for insertions into a closed electron loop applies whenever the sum over all graphs of this equivalence class is considered

$$\sum_{\Gamma \in [\gamma]} \Gamma = \frac{1}{2} \sum_{\text{insertions } (i,j)} \gamma \circ_i \; \circ_j$$  $$\stackrel{(2.56)}{=} 0. \quad (2.84)$$

Again, the number of gauge-fixed propagators is an invariant of the equivalence relation and allows further decomposition of the equivalence classes such that the cancellation remains valid. For instance, consider the two-loop vacuum polarization with one longitudinal photon propagator. This subset of Π^2_1 is characterized by restricting to Feynman graphs without gauge-fixed propagators; it consists of three Feynman graphs



$$= \frac{1}{2} \sum_{\text{insert. } (i,j)}$$  $$= 0. \quad (2.85)$$

which can be rewritten as iterated insertion into the one-loop vacuum polarization. Due to the cancellation identity (2.56), the iterated insertion vanishes. The three two-loop gauge parameter dependent terms cancel and the two-loop vacuum polarization is independent of the (shifted) gauge parameter.

As a conclusion of the equation (2.84), the sum over all Feynman graphs of the type closed electron loop possessing at least one longitudinal tensor vanishes

$$\sum_{\gamma \in \Pi^n_m} \gamma = 0 \quad \text{for } m \geq 1. \quad (2.86)$$

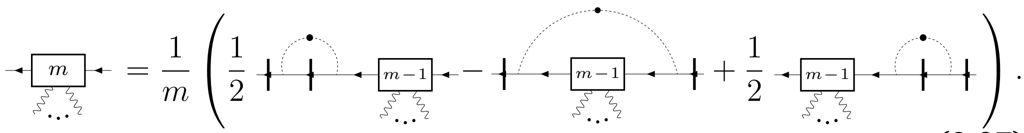
Or in other words, the gauge-dependent longitudinal tensor (2.70) is effectively not inserted into closed electron loops.

Unquenched Green's functions

In paragraph 2.2.2, we demonstrated that the photon Green's function is transversal at an arbitrary loop order. The argument presented there is not restricted to the case of two external photon legs, but naturally generalizes to Green's functions with an arbitrary number of external photon legs as long as there are no external electron legs. As a result, every Green's function of this type can be considered to be transversal in the sense that it vanishes whenever one of its external photon legs is contracted with the associated external momentum. Consequently, the gauge-dependent tensor does not couple to the external photon legs of a Green's function of this kind.

Combining this statement with the conclusion of the preceding paragraph, the gauge-dependent tensor effectively neither couples to internal edges nor to the external photon legs of closed electron loops. By effectively we mean that an insertion into an individual Feynman graph might yield a non-vanishing result, but these contributions cancel once all graphs are considered.

Effectively, the gauge-dependent tensor couples only to external electron lines and hence the cancellation identities of the quenched sector apply for these insertions. Therefore, the characterization of the gauge parameter dependence in the quenched sector (2.79) generalizes to unquenched Quantum Electrodynamics



(2.87)

This formula relates the $(\xi - \xi^*)^m$ term of a connected Green's function to the term proportional to $(\xi - \xi^*)^{m-1}$ of the same Green's function. Given a Green's function with its gauge parameter evaluated at the particular value ξ^*, the formula reconstructs the linear term of the expansion in the shifted gauge parameter and iterative application of the formula eventually determines the full dependence on the general covariant gauge parameter ξ of that Green's function. Therefore, the above formula provides a full characterization of how a Green's function in the linear covariant gauge depends on gauge parameter.

The Landau-Khalatnikov formula revisited

It is instructive to employ the auxiliary Feynman rules (2.49-2.53) on the diagrammatic expressions of the last paragraph and to convey the characterization of the gauge dependence into a momentum space Dyson-Schwinger equation. Recall that due to our conventions for the longitudinal tensor (2.70), the two-valent vertex converts into a factor of the shifted gauge parameter $(\xi - \xi^*)$. We divide this factor together with the factor m to the left-hand side of the equation and rewrite this factor as derivative with respect to the gauge parameter $^m/_{(\xi-\xi^*)} \cong {}^\partial/_{\partial\xi}$. Then, summation over all powers in the (shifted) gauge parameter yields the following characterization for a connected unrenormalized Green's function

$$\frac{\partial G^r}{\partial \xi}(q) = ie^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2} \left[G^r(q+p) - G^r(q) \right]. \tag{2.88}$$

Here, the index r refers to a single external electron line and an arbitrary number of external photon legs. Further, it is crucial to remark that the integration is understood in terms of dimensional regularization as this implies that the (actually divergent) tadpole terms of the diagrammatic formula (2.87) vanish.

This formula can be perceived to be a natural momentum space version of the Landau-Khalatnikov formula which is discussed now. In [8], Landau and Khalatnikov enquired how general gauge transformations affect the Green's functions of Quantum Electrodynamics. The prevailing assumption underlying their work is the absence of anomalies in Green's functions under a general gauge transformation of the kind

$$\left.\begin{array}{rrr} A_\mu \mapsto & A'_\mu = & A_\mu + \partial_\mu \omega \\ \psi \mapsto & \psi' = & e^{ie\omega}\psi \end{array}\right\}. \tag{2.89}$$

More precisely, they claim that the change of a canonically quantized Green's function in a certain gauge is described by employing the gauge transformation on the fields in the correlator of that Green's function

$$\left\langle A'_\mu \cdots \psi' \cdots \overline{\psi}' \right\rangle = \left\langle (A_\mu + \partial_\mu \omega) \cdots e^{ie\omega}\psi \cdots e^{-ie\omega}\overline{\psi} \right\rangle. \tag{2.90}$$

Considering Green's functions as a series of Feynman graphs, this statement restricts the effect of a general gauge transformation to external legs only; internal edges in a Feynman graph are unaffected.

As a result of quantizing the free scalar field ω and identifying its propagator with the longitudinal part of the photon propagator, Landau and Khalatnikov proposed a position space formula for the gauge dependence of the electron propagator. This result was extended by Zumino [43] with respect to the discussion of several gauge-fixings, including the linear covariant gauge. Further, Zumino clarified that the Landau-Khalatnikov formula refers to unrenormalized Green's functions. For the electron propagator function, the Landau-Khalatnikov formula reads

$$S'_F(x-y) = \exp\left\{ ie^2 \left[M(x-y) - M(0) \right] \right\} S_F(x-y), \tag{2.91}$$

where the function M within the exponential denotes the propagator of the scalar field ω. This function is determined by the choice of a gauge-fixing. In the setting from above (the transformation from a specific value ξ^* to another value ξ in the gauge parameter of the the linear covariant gauge), it is given by

$$M(x) = (\xi - \xi^*) \int \frac{d^D p}{(2\pi)^D} \frac{e^{-i(p\cdot x)}}{(p^2)^2}. \tag{2.92}$$

Note that the Landau-Khalatnikov formula (2.91) can be written as differential equation with respect to the gauge parameter

$$\frac{\partial S'_F}{\partial \xi}(x-y) = ie^2 \int \frac{d^D p}{(2\pi)^D} \frac{e^{-ip\cdot(x-y)} - 1}{(p^2)^2} S'_F(x-y). \tag{2.93}$$

Finally, replacing the electron propagator S'_F by its Fourier transform $\hat{S}'_F$ translates the Landau-Khalatnikov formula into the position space

$$\frac{\partial \hat{S}'_F}{\partial \xi}(q) = ie^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2} \left[\hat{S}'_F(q+p) - \hat{S}'_F(q) \right], \tag{2.94}$$

which indeed coincide with our formula for the characterization of the gauge dependence (2.88).

It should be remarked that for additional external photon legs, the Landau-Khalatnikov formula involves additional expressions when compared to our formula (2.88). This is because we have amputated the photon propagator of the external photon leg in our Green's functions. Adding the gauge-dependent longitudinal tensor (2.70) onto an external photon leg yields the additional terms and is in convenience with the global picture of (2.90) where the external photon legs are modified by longitudinal expressions originating from the scalar field ω.

Higher dimensions: QED_6 and QED_8

It is a remarkable fact that a class of scalar Quantum Field Theories in different dimensions share the same behaviour at their fixed points under the renormalization group flow. This property is called universality and was first established for the non-linear sigma model in two dimensions and φ^4 theory in four dimensions [44]. This universality class was extended by further $O(N)$ scalar theories in six [45, 46, 47] and eight dimensions [48].

A similar construction applies to (non-abelian) gauge theories [49, 48]. For the abelian case, theories in six and eight dimensions were constructed and shown to accompany the universality class of customary four-dimensional Quantum Electrodynamics [50, 48].

Notably, the theories within a universality class share a common symmetry — which is the $U(1)$ gauge symmetry in the case of Quantum Electrodynamics. Another shared property is the requirement of perturbative renormalizability. That is the kinematic terms of the Lagrangian have a classical scaling dimension that equals the considered space-time dimension and (coupling) parameters have vanishing scaling dimensions. These properties constitute the prescription to construct higher dimensional theories within a universality class. For further details on this construction, the reader is referred to [48].

Following this prescription, we construct the Lagrangians of Quantum Electrodynamics in $d = 6$ and $d = 8$ dimensions

$$\mathscr{L}_{d=6} = -\frac{1}{4}(\partial_\mu F_{\nu\rho})(\partial^\mu F^{\nu\rho}) - \frac{1}{2\xi}(\partial_\mu \partial^\nu A_\nu)(\partial^\mu \partial^\rho A_\rho) + i\overline{\psi}\not{D}\psi, \tag{2.95}$$

$$\begin{aligned}\mathscr{L}_{d=8} = &-\frac{1}{4}(\partial_\mu \partial_\nu F_{\rho\sigma})(\partial^\mu \partial^\nu F^{\rho\sigma}) - \frac{1}{2\xi}(\partial_\mu \partial_\nu \partial^\rho A_\rho)(\partial^\mu \partial^\nu \partial^\sigma A_\sigma) + i\overline{\psi}\not{D}\psi \\ &+ \frac{g_2^2}{32}(F_{\mu\nu}F^{\mu\nu})(F_{\rho\sigma}F^{\rho\sigma}) + \frac{g_3^2}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}.\end{aligned} \tag{2.96}$$

At eight dimensions, two quartic tensors emerge which are respectively multiplied by the coupling parameters g_2 and g_3. These tensors contribute a four-photon interaction vertex that complicates the discussion of cancellations and the gauge dependence. However, in the following we will show that they do not couple to the gauge-dependent parts of the photon propagator.

The higher derivative terms in the Lagrangians (2.95) and (2.96) require a slightly generalized version of Lagrange equations

$$0 = \frac{\partial \mathscr{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathscr{L}}{\partial\, \partial_\nu A_\mu} + \partial_\rho \partial_\nu \frac{\partial \mathscr{L}}{\partial\, \partial_\rho \partial_\nu A_\mu} - \partial_\sigma \partial_\rho \partial_\nu \frac{\partial \mathscr{L}}{\partial\, \partial_\sigma \partial_\rho \partial_\nu A_\mu}. \tag{2.97}$$

Discarding the interaction terms in the Lagrangians yields the free equations of motion at the varying dimensions.

$$d = 4: \qquad 0 = \left(\partial^\nu F_{\nu\mu} + \frac{1}{\alpha}\partial_\mu \partial^\nu A_\nu\right) \tag{2.98}$$

$$d = 6: \qquad 0 = \Box \left(\partial^\nu F_{\nu\mu} + \frac{1}{\alpha}\partial_\mu \partial^\nu A_\nu\right) \tag{2.99}$$

$$d = 8: \qquad 0 = \Box\Box \left(\partial^\nu F_{\nu\mu} + \frac{1}{\alpha}\partial_\mu \partial^\nu A_\nu\right) \tag{2.100}$$

Structurally the equation of motion for the free gauge field match the four-dimensional case. Therefore, the decompositions of the photon propagator into a transversal and a longitudinal part remains valid at six and eight dimensions. However, an increase of the dimension by two causes an additional factor of the D'Alembert operator $\Box = \partial_\mu \partial^\mu$. The appearance of these D'Alembert operators translates into higher pole behaviour of the photon propagator for increasing dimension. To conclude, the photon propagator in $d = 4, 6,$ and 8 dimensions reads

$$P_d^{\mu\nu}(p, \xi) = \frac{-i}{(p^2)^{(d-2)/2}} \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right). \tag{2.101}$$

The six-dimensional Lagrangian $\mathscr{L}_{d=6}$ does not introduce new interactions and hence it satisfies the same cancellation identities (2.55) and (2.56) as the four-dimensional version. Therefore, the diagrammatic formula (2.87) which characterizes the gauge dependence remains valid. However, the higher pole structure in the Feynman rule for the photon propagator alters the pole structure in the auxiliary Feynman rules for the scalar edges (2.51) and the gauge-dependent tensor (2.70). As a result, the differential equation for the connected unrenormalized Green's function in six dimension features a higher order infrared divergence

$$\frac{\partial G^r}{\partial \xi}(q) = ie^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^{d/2}} \left[G^r(q + p) - G^r(q) \right]. \tag{2.102}$$

Note that this result is in convenience with the description of Landau and Khalatnikov [8] as discussed above.

The eight-dimensional Lagrangian $\mathscr{L}_{d=8}$, on the other hand, introduces a four-photon interaction vertex. As this vertex connects to the transversal and the longitudinal parts of the photon propagator, new cancellation identities are necessary to guarantee the validity of the diagrammatic formula (2.87).

In the following, we demonstrate that the emerging four-photon vertex is transversal; that is it vanishes whenever one of its photon legs is contracted with the associated momentum. In other words the four-photon vertex can be thought of as a modification of the transversal Green's function that describes light-by-light scattering. Consequently, the gauge-dependent part of the photon propagator effectively does not couple to this four-photon vertex and, in close analogy to the discussion of unquenched Green's functions in paragraph 2.2.3, the diagrammatic formula (2.87) remains valid despite the additional interaction vertices.

To prove the transversality, both emerging interaction terms of the Lagrangian $\mathscr{L}_{d=8}$ are analysed separately. This will determine each of them to be transversal. Therefore, each interaction term separately obeys gauge-invariance.

First, the interaction term proportional to the coupling g_2^2 is discussed. This tensor is the square of a pair of contracted Electromagnetic field tensors. Due to the square, the Feynman rule of this tensor factors into two Lorentz tensors. That Lorentz tensor is determined by the expression

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\,. \tag{2.103}$$

Every Electromagnetic potential A contributes a photon leg which carries a Lorentz index μ and a momentum p. Labelling them by 1 and 2, the above tensor is proportional to the expression

$$T^{\mu_1\mu_2}(p_1,p_2) = [(p_1\cdot p_2)g^{\mu_1\mu_2} - p_1^{\mu_2}p_2^{\mu_1}]\,. \tag{2.104}$$

This tensor vanishes when contracted with $p_1^{\mu_1}$ or $p_2^{\mu_2}$; i.e. it is transversal. The Feynman rule of the full vertex is given as a product of two of these tensors with summation over all ways to associate a label to each Electromagnetic potential, which we denote as the sum over all permutations of the four indices $\sigma \in S_4$. This yields

$$\begin{aligned} V_2^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) &= \frac{g_2^2}{32}\sum_{\sigma\in S_4} 4T^{\mu_{\sigma(1)}\mu_{\sigma(2)}}(p_{\sigma(1)}p_{\sigma(2)})T^{\mu_{\sigma(3)}\mu_{\sigma(4)}}(p_{\sigma(3)}p_{\sigma(4)}) \\ &= g_2^2\,[\quad T^{\mu_1\mu_2}(p_1,p_2)T^{\mu_3\mu_4}(p_3,p_4) \\ &\qquad + T^{\mu_1\mu_3}(p_1,p_3)T^{\mu_2\mu_4}(p_2,p_4) \\ &\qquad + T^{\mu_1\mu_4}(p_1,p_4)T^{\mu_2\mu_3}(p_2,p_3)]\,, \end{aligned} \tag{2.105}$$

which is clearly transversal in every Lorentz index

$$(p_i)_{\mu_i}V_2^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = 0 \quad \text{for } i = 1,2,3,4. \tag{2.106}$$

A prove of the transversality of the second interaction term

$$F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \tag{2.107}$$

is more involved and requires us to introduce some notation. We decompose every Electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \tag{2.108}$$
$$\cong \;\nwarrow \;-\; \nearrow \tag{2.109}$$

and represent the first term by an arrow pointing northwest; the term with a minus sign is represented by an arrowing pointing northeast. In this way, the chain of four contracted Electromagnetic field tensors decomposes into a sum of $2^4 = 16$ terms and each of them is represented by a cyclic sequence of 4 arrows either pointing northwest or northeast

$$\nwarrow\nwarrow\nwarrow\nwarrow, \quad \nwarrow\nwarrow\nwarrow\nearrow, \quad \cdots . \tag{2.110}$$

Again, the construction of the full contribution to the four-photon vertex requires the summation over all labellings which is denoted as sum over all permutations of the symmetric group of order four S_4.

$$V_3^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) \cong \frac{g_3^2}{8} \sum_{\sigma\in S_4} \sum_{s_i\in\mathbb{A}(i)} (-1)^{\#\nearrow} \, s_{\sigma(1)} s_{\sigma(2)} s_{\sigma(3)} s_{\sigma(4)} \tag{2.111}$$

Here, the second sum denotes all labelled arrows s_i which are either pointing northwest or northeast and carry a momentum p_i and Lorentz index μ_i

$$\mathbb{A}(i) = \left\{ {}_{p_i}\!\nwarrow^{\mu_i} , \quad {}^{\mu_i}\!\nearrow_{p_i} \right\} \tag{2.112}$$

and $\#\nearrow$ is the number of arrows that point northeast. Note that a Lorentz index is always set over the arrow and the associated momentum is always placed under the arrow. In this arrow notation, the Feynman rules come in a very natural form. Given a sequence of labelled arrows, a gap between two arrows pairs the kinematic data of the arrows. A pair given of a momentum and a Lorentz index translates into a four-momentum by assigning the Lorentz index to the momentum

$$\nwarrow^{\mu_1}\nwarrow_{p_2} \cong p_2^{\mu_1} \quad \text{and} \quad \nearrow_{p_1}^{\mu_2}\nearrow \cong p_1^{\mu_2}, \tag{2.113}$$

while two enclosed momenta become a contracted scalar

$$\nearrow_{p_1\, p_2}\nwarrow \cong (p_1 \cdot p_2), \tag{2.114}$$

and two enclosed Lorentz indices translate into a metric tensor

$$\nwarrow^{\mu_1\mu_2}\nearrow \cong g^{\mu_1\mu_2}. \tag{2.115}$$

In a second step, we need to remember the minus signs from the Electromagnetic field tensor (2.108). Therefore, we associate a minus sign to each arrow that points northeast.

Note that the cyclic contracted tensor (2.107) yields a cyclic sequence of arrows and hence kinematic data of the last arrows is contracted with data from the first arrow. The following examples might be helpful for the reader.

$$\overset{\mu_1}{\underset{p_1}{\nearrow}}\overset{\mu_2}{\underset{p_2}{\nwarrow}}\overset{\mu_3}{\underset{p_3}{\nearrow}}\overset{\mu_4}{\underset{p_4}{\nearrow}} \cong (p_1 \cdot p_2) g^{\mu_2\mu_3} p_3^{\mu_4} p_4^{\mu_1} \tag{2.116}$$

$$\overset{\mu_1}{\underset{p_1}{\nwarrow}}\overset{\mu_2}{\underset{p_2}{\nwarrow}}\overset{\mu_3}{\underset{p_3}{\nearrow}}\overset{\mu_4}{\underset{p_4}{\nearrow}} \cong (p_1 \cdot p_4) g^{\mu_2\mu_3} p_2^{\mu_1} p_3^{\mu_4} \tag{2.117}$$

These specific examples are useful for another reason: for both examples the sequences of arrows exactly matches except for the first arrow. Therefore, the expression from the first example carries a relative minus sign in comparison to the expression from the second example. Further, contracting the expression with the momentum $p_1^{\mu_1}$ associated to the varying arrow yields the same result.

$$(p_1)_{\mu_1}(p_1 \cdot p_2) g^{\mu_2\mu_3} p_3^{\mu_4} p_4^{\mu_1} = (p_1 \cdot p_2)(p_1 \cdot p_4) g^{\mu_2\mu_3} p_3^{\mu_4} \tag{2.118}$$

$$(p_1)_{\mu_1}(p_1 \cdot p_4) g^{\mu_2\mu_3} p_2^{\mu_1} p_3^{\mu_4} = (p_1 \cdot p_2)(p_1 \cdot p_4) g^{\mu_2\mu_3} p_3^{\mu_4} \tag{2.119}$$

Due to the relative sign, both terms eventually cancel in the sum (2.111) when the corresponding Lorentz index is longitudinally contracted. By exploiting the cyclicity of the sequence of arrows and possibly by renaming the external photon edges, this cancellation generalizes to the every Lorentz index of the four-photon vertex proportional to g_3^2

$$(p_i)_{\mu_i} V_3^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) = 0 \quad \text{for } i = 1, 2, 3, 4. \tag{2.120}$$

Eventually, both contributions to the four-photon vertex are transversal and the diagrammatic cancellations and the formula for gauge dependence (2.87) remain valid. Therefore, Green's functions in eight-dimensional Quantum Electrodynamics $\mathscr{L}_{d=8}$ also satisfy the differential equation (2.102).

It should be noted that the emerging interactions terms in the Lagrangian $\mathscr{L}_{d=8}$ are build from products of a pairwise-contracted Electromagnetic field tensor and a chain of cyclic-contracted Electromagnetic field tensors. These might be considered as the archetypes of interactions for Quantum Electrodynamics at even higher dimension. The arguments presented here easily generalize to longer chains of cyclic-contracted tensors and products of chains and pairwise-contracted tensors.

2.3 Renormalization

Recall that only dimensional regularization is considered in the scope of this thesis. In addition to that we will restrict our discussion to the massless limit of Quantum Electrodynamics from here on. It is well-known that a wide class of Feynman graphs gives raise to divergent integrals. A new set of tools is necessary to deal with these divergences and to construct finite and physical meaningful quantities — this the purpose of renormalization.

Within this thesis, we are mainly interested in the Hopf-algebraic description of renormalization. However, in order to provide a broad overview and relate this rather novel approach to the established methods, we also include a discussion of the BPHZ prescription and the renormalization by Z-factors. Parts of this section are based on the work [51].

2.3.1 Divergent Feynman graphs

The Feynman rules (2.29-2.31) map every graph to an analytic expression in momentum space. Due to momentum conservation, there is exactly one independent momentum for each loop of the graph and integration over all loop momenta is understood. This paragraph provides a rudimentary discussion of the convergence behaviour of these integrals. Hereby, we focus on so-called ultraviolet divergences, that is to say divergences originating from a high modulus of the loop momentum.

For an one-particle irreducible subgraph $\gamma \subseteq \Gamma$ of a Feynman graph Γ, we define the superficial degree of divergence

$$\omega(\gamma) = 4|\gamma| - I_\psi(\gamma) - 2I_A(\gamma) \tag{2.121}$$

where $|\gamma|$ denotes the loop number, $I_\psi(\gamma)$ counts the number of internal fermionic edges, and $I_A(\gamma)$ the number of internal photon edges of the subgraph γ. The superficial degree can be interpreted as leading power when all momenta of the subgraph γ are rescaled by a common large factor and is hence often referred to as power counting as originally proposed by Dyson in [52]. This point of view suggests the postulate that a subgraph $\gamma \subseteq \Gamma$ gives raise to a divergent integral if

$$\omega(\gamma) \geq 0. \tag{2.122}$$

A proof of this statement is due to the work of Weinberg [53] as well as Hahn and Zimmermann [54]. We hence will refer to this result as the power counting theorem. A one-particle irreducible subgraph γ with $\omega(\gamma) \geq 0$ is referred to as subdivergence and a Feynman graph Γ with $\omega(\Gamma)$ is called (superficially) divergent.

So far, the definition of the superficial degree of divergence does not account for the specific edges and interactions of Quantum Electrodynamics. In Quantum Electrodynamics, there are fermion and photon edges and one type of vertex that connects two fermion edges with one photon edge. Using these properties in combination with Euler's characteristic $|\gamma| = I_\psi(\gamma) + I_A(\gamma) - V(\gamma) + 1$, the superficial degree of a Quantum Electrodynamics Feynman graph can be written as

$$\omega(\gamma) = 4 - \frac{3}{2}E_\psi(\gamma) - E_A(\gamma), \tag{2.123}$$

where $E_\psi(\gamma)$ and $E_A(\gamma)$ count the number of external fermion and photon edges, respectively. To conclude, the superficial degree of divergence of a Quantum Electrodynamics Feynman graph is characterized by the external leg structure of that graph.

This fact allows for a classification of divergent Feynman graphs. A superficially divergent graph is of the following external leg type: fermion propagator, photon propagator, electron-photon-vertex, three-photon interaction, or four-photon interaction. Note that the sum over all graphs with three external photons vanishes due to Furry's theorem [55]. The four-photon interaction is called light-by-light scattering and supposed to be finite by renormalizability — in other words there is no interaction vertex in Quantum Electrodynamics that could serve to absorb the divergences of the four-photon type. Unfortunately, the author is not aware of a general argument beyond the first-loop order that proves this statement. In [56, 57], a basis of Lorentz tensors was construct for the class four-photon graphs. An alternative basis of Lorentz tensors that includes the anti-symmetric tensor can be found in [58]. Abstaining from the anti-symmetric tensor, a basis consists of 138 Lorentz tensors and the challenge is to show that each coefficient becomes finite when all graphs at a certain loop order are considered. Our analysis in the preceding section shows that the sum over all one-particle irreducible four-photon graphs is transversal (it vanishes whenever one of its external legs is contracted with the associated momentum). This global property restricts the possible occurrence of divergences to 43 Lorentz tensors whose coefficients remain to be proven finite. Nonetheless, we will assume that light-by-light scattering is indeed finite and can be excluded (beside the three-photon interactions) from the renormalization process in the proceeding discussion.

2.3.2 The BPHZ renormalization prescription

After the characterization of divergent Feynman graphs, it is a natural step to provide a method to renormalize the divergences of a Feynman graph — that is to construct a physically reasonable and finite expression from a divergent Feynman integral. This paragraphs gives a short overview of the renormalization prescription which due to Bogoliubov, Parasiuk, Hepp, and Zimmermann [59, 60, 61].

First, let Γ be a Feynman graph that is superficially divergent but has no proper subdivergences. Under these circumstances, the renormalized value $\phi_{\mathrm{R}}(\Gamma)$ is constructed by subtracting a counterterm with similar divergent behaviour such that the superficial divergence is cancelled. To formalize this procedure, define maps ϕ to map a Feynman graph to its integral expression by means of (2.29-2.31) and a renormalization scheme T that maps a Feynman graph to an appropriate counterterm — one specific way to define such a T is to employ the regular Feynman rules but change the values of the external momenta. Then, the renormalized value reads

$$\phi_{\mathrm{R}}(\Gamma) = \phi(\Gamma) - T(\Gamma). \tag{2.124}$$

Now, if Γ does contain proper subdivergences, the above prescription still applies provided that there is a procedure which subtracts all proper subdivergences from Γ. This procedure is denoted by $\overline{\phi}(\Gamma)$, then the renormalized Feynman rules read

$$\phi_{\mathrm{R}}(\Gamma) = \overline{\phi}(\Gamma) + C(\Gamma), \quad \text{where} \quad C(\Gamma) = -T \circ \overline{\phi}(\Gamma) \tag{2.125}$$

denotes an appropriate counterterm in analogy to the preceding case. The superficial degree of divergence provides a characterization of the divergent domains in the momentum integration. Note that the momentum integrations associated to disjoint subgraphs might diverge simultaneously. To account for these divergent domains, it is necessary to consider all products of disjoint subdivergences rather than restricting to individual subdivergences

$$\overline{\phi}(\Gamma) = \sum_{\gamma \triangleleft \Gamma} C(\gamma)\, \Gamma/\gamma, \quad \text{with} \quad C(\gamma) = \prod_{i=1,\ldots,n} C(\gamma_i) \tag{2.126}$$

the sum goes over all subgraphs $\gamma = \gamma_1 \ldots \gamma_n$ consisting of pairwise disjoint subdivergences γ_i, Γ/γ is the graph obtained by shrinking each component of γ to a point, and the counterterm of γ is determined by the counterterms of its factors. Note that the counterterm is meant to subtract only the superficial divergence of a graph in order to avoid a multiple subtraction of nested subdivergences. Therefore, one has to remove all subdivergences before extracting the divergence

$$C(\gamma_i) = -T \circ \overline{\phi}(\gamma_i) \tag{2.127}$$

in convenience with (2.125). This finishes the definition of the renormalization prescription. Notice that the formula that determines $\overline{\phi}(\Gamma)$ implicitly depends on its value of proper subdivergences $\overline{\phi}(\gamma)$ and hence defines a iterative procedure.

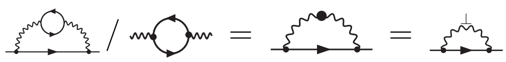
Zimmermann showed that this iteration can be solved by means of the forest formula

$$\phi_{\mathrm{R}}[\Gamma] = \sum_{f \in \mathcal{F}_\Gamma} (-1)^{|f|} \left(\prod_{\gamma \in f} T(\gamma) \right) \phi(\Gamma / f). \tag{2.128}$$

Here, the sum runs over all forest f (a set of divergent one-particle irreducible subgraphs of Γ which are either nested or disjoint) and Γ/f is obtained by contracting all components of the forest f in Γ. The Feynman rules are denoted by ϕ and T is a renormalization scheme as above.

The forest formula allows to lift the renormalization procedure of Feynman integrals to the level of Feynman graphs. In this sense the whole renormalization process is understood as a combinatorial manipulation of Feynman graphs. For every individual Feynman graph Γ, this prescription guarantees a finite renormalized expression $\phi_{\mathrm{R}}[\Gamma]$.

It is worth noting that contraction is sensible to the Lorentz structure of a subgraph. The following example illustrates this issue.


(2.129)

This example demonstrates the contraction of the first order photon subdivergence, which is inserted into the first order electron self-energy graph. In the first equation, the photon subgraph is contracted to a point. That point represents the remaining Lorentz structure of the subdivergence which the transversal tensor $T_{\mu\nu}(k) = g_{\mu\nu}k^2 - k^\mu k^\nu$ — recall the vacuum polarization is transversal (2.57). This Lorentz structure projects both photon propagators on their transversal parts and cancels one of them in the second equation. It remains the transversal part (i.e. Laudau gauge term) of the first order electron self-energy. This result might be anticipated from a calculation of the second order graph. The evaluation of its photon subdivergence yields a purely transversal result. Hence, in an evaluation of the second order graph, only the transversal parts of the photon propagators contribute, therefore the contraction should also yield a purely transversal result.

2.3.3 Z-factor renormalization

The renormalization by means of Z-factors provides an alternative to the renormalization of individual Feynman graphs. Opposed to the BPHZ renormalization prescription, it allows for a direct construction of the renormalized Green's function rather than separately renormalizing each Feynman graph and summing up the renormalized graphs.

As discussed in the paragraph 2.3.1, the set of superficially divergent Green's functions of Quantum Electrodynamics restricts to the electron propagator, the photon propagator, and the electron-photon vertex. Further, one has to assume that the divergences of these Green's functions only emerge in front of Lorentz tensors which are part of expressions provided by the Feynman rules (2.29-2.31). In this case, the divergences can be absorbed by rescaling the vertex and edges with appropriate factors; these factors are usually termed Z-factors.

This line of thought can also be pursued in the Lagrangian formalism. In order to cancel the divergences of Feynman graphs described by the massless Lagrangian (2.1), we add a counterterm Lagrangian $\mathscr{L}_{\mathrm{CT}}$. This defines the renormalized Lagrangian

$$\mathscr{L}_{\mathrm{R}} = \mathscr{L} + \mathscr{L}_{\mathrm{CT}}. \tag{2.130}$$

Every superficially divergent Green's function requires us to introduce a term in the counterterm Lagrangian and the external leg structure of such a Green's function determines the monomial in the fields. Hence, the characterization of superficially divergent Green's function implies that the monomials of the counterterm Lagrangian agree with which of the original Lagrangian (up to constants which absorb the divergences). Therefore it is possible to combine the expressions of both Lagrangians; this procedure introduces now coefficients for the monomials which are the above mentioned Z-factors. As a result, the renormalized Lagrangian is represented as

$$\mathscr{L}_{\mathrm{R}} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + i Z_2 \bar{\psi} \partial\!\!\!/ \psi - e Z_1 \bar{\psi} A\!\!\!/ \psi - \frac{1}{2\xi} Z_4 (\partial_\mu A^\mu)^2, \tag{2.131}$$

where the Z-factors are utilized to cancel the divergences, therefore the fields are referred to as renormalized fields and similarly e and ξ are called renormalized parameters. Note that the gauge fixing renormalization constant is termed Z_4; the other renormalization constants are labelled by the traditional convention [52]. In a consecutive step, the renormalized Lagrangian $\mathscr{L}_{\mathrm{R}}$ is expressed in bare fields. Here, it is assumed that the renormalized Lagrangian matches the original Lagrangian $\mathscr{L}(\psi_B, A_B, e_B, \xi_B)$ when all fields and parameters are replaced by their bare equivalents. This determines the relationship between the bare and the renormalized fields and parameters

$$e_0 = \frac{Z_1}{Z_2 Z_3^{1/2}} e, \quad \psi_0 = Z_2^{1/2} \psi, \quad A_0^\mu = Z_3^{1/2} A^\mu, \quad \xi_0 = \frac{Z_3}{Z_4} \xi. \tag{2.132}$$

Bare fields and parameters are indicated by assigning the subscript zero. In the following, these Z-factors are related to the superficial divergent Green's functions to obtain conditions which described how to determine the counterterms and Z-factors. We will refer to these conditions as renormalization conditions. Moreover, it is worth remarking that the global properties of Green's functions, as originating from Ward identities, impose further relations on the Z-factors. This issue is best exemplified or the instance of the vacuum polarization $\Pi^{\mu\nu}$; that is the one-particle irreducible Green's function with two external photon legs.

First note that the renormalized Lagrangian when expressed in terms of bare fields and parameters gives raise to Feynman rules similar to (2.29-2.31) whereas all parameters are replaced by their bare equivalents. Therefore, our discussion of Ward identities in section 2.2.2 immediately transverse to the bare Green's functions. For the vacuum polarization, the Ward identity reads

$$k_\mu \Pi_0^{\mu\nu}(k) = 0 \quad \text{and} \quad k_\nu \Pi_0^{\mu\nu}(k) = 0. \tag{2.133}$$

In other words, there is no longitudinal part in the vacuum polarization and we can define the tensor reduced vacuum polarization function

$$\Pi_0^{\mu\nu}(k) = \left(g^{\mu\nu} k^2 - k^\mu k^\nu\right) \Pi_0(k^2). \tag{2.134}$$

All divergences of the vacuum polarization graphs are proportional to the transversal Lorentz tensor. This observation generalizes to the bare photon propagator

$$D_0^{\mu\nu}(k, \xi_0) = \frac{1}{k^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \frac{1}{1 - \Pi_0(k^2)} + \xi_0 \frac{k^\mu k^\nu}{k^4}, \tag{2.135}$$

where the longitudinal component is determined by the tree-level photon propagator. Now , the bare photon propagator can be related to its renormalized equivalent by using equation (2.132) and substituting the bare by renormalized fields in the time-ordered product of the field operators or the path integral expression which implies

$$D^{\mu\nu}(k, \alpha, \xi) = \frac{1}{Z_3} D_0^{\mu\nu}(k, \alpha_0, \xi_0) \tag{2.136}$$

This equation indicates that the renormalization constants introduce a scaling between bare and renormalized Green's functions, but do not influence the Lorentz structure; also the renormalized Green's function satisfies the Ward identities and obeys the same equation (2.135) as the bare Green's function. A comparison of their Lorentz structures implies the following renormalization conditions:

$$\Pi(k^2,\alpha,\xi) = \Pi_0(k^2,\alpha_0,\xi_0) + C_3 - C_3\Pi_0(k^2,\alpha_0,\xi_0) \quad (2.137)$$

$$Z_4 = 1 \quad (2.138)$$

The first equation explains the renormalization of the photon Green's function by introduction of the counterterm C_3, for counterterms the convention $Z_i = 1 - C_i,\ i = 1,2,3,4$ is used. The latter condition derives from the longitudinal part of the photon propagator and characterizes the renormalization of the gauge parameter. Eventually, the gauge parameter renormalization constant is solely determined by the renormalization of the vacuum polarization, i.e. by the renormalization constant of the vacuum polarization

$$\xi_0 = Z_3\xi. \quad (2.139)$$

From the perspective of Z-factors, this is the reason for dynamical character of the gauge parameter under renormalization flow and its contributions to the renormalization group equation. Another explanation is to due to the structure of subdivergences of Green's functions. This will be discussed in detail in paragraph .

Renormalization conditions

As demonstrated for the vacuum polarization above, the transition from bare to renormalized Green's functions implies a set of renormalization conditions. To summarize, the full set of conditions for the Green's functions reads

vertex $$\Lambda^\nu(q_1,q_2,\alpha,\xi,\mu) = \Lambda_0^\nu(q_1,q_2,\alpha_0,\xi_0) - C_1\Lambda_0^\nu(q_1,q_2,\alpha_0,\xi_0) \quad (2.140)$$

electron $$\Sigma(q,\alpha,\xi,\mu) = \Sigma_0(q,\alpha_0,\xi_0) + C_2 - C_2\Sigma_0(q,\alpha_0,\xi_0) \quad (2.141)$$

photon $$\Pi(q,\alpha,\xi,\mu) = \Pi_0(q,\alpha_0,\xi_0) + C_3 - C_3\Pi_0(q,\alpha_0,\xi_0) \quad (2.142)$$

Primary, these conditions expose that renormalization is carried out in two steps. First, a substitution of bare to renormalized variables is performed.

Second, the remaining divergences are absorbed by defining appropriate counterterms. Bare and renormalized parameters are connected by a product of renormalization constants, which also might depend on the coupling constant or gauge parameter. Therefore, the substitution replaces bare parameters by series of renormalized parameters. In case of the bare coupling constant and its product with the gauge parameter, the expansion in the renormalized coupling constant up to the second order reads

$$\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3}\alpha = \alpha + (-2C_1 + 2C_2 + C_3)|_1\alpha + O(\alpha^3) \tag{2.143}$$

$$\xi_0\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3} Z_3\xi\alpha = \xi\alpha + (-2C_1 + 2C_2)|_1\xi\alpha + O(\alpha^3). \tag{2.144}$$

At first order, bare parameters are directly rewritten in terms of renormalized parameters, that is $\alpha_0 = \alpha + O(\alpha^2)$ for the coupling constant. This observation simplifies the derivation of one loop counterterms. However, at two loops, the coupling renormalization requires all one loop counterterms.

The Callan-Symanzik equation

As demonstrated above, the Z-factors relate bare and renormalized Green's functions. In our conventions, an one-particle irreducible Green's functions of external leg type r satisfies the equation

$$G^r(p^2, \alpha, \xi) = Z_r G_0^r(p^2, \alpha_0, \xi_0), \tag{2.145}$$

where Lorentz tensors have been amputated such that the Green's function only depends on square of the external momentum (the vertex function is considered with zero momentum transfer at the photon leg). The Z-factors are utilized to absorb the divergences of the bare Green's function and ensure finiteness of the renormalized qualities. Further, one expects the renormalized Green's function to be a finite function on the external kinematic data that shows singular behaviour for some particular kinematic values or limits. Here, the external kinematics restricts to a single momentum p whose Lorentz square parametrizes the singular behaviour. However, the Lorentz invariant p^2 is obviously a dimensional quantity, but arguments of these singular functions should be dimensionless. Therefore, we expect another scale μ that adjusts for the dimensionality of the external data in the renormalized Green's function. Depending on the renormalization scheme, the scale μ is introduced due to the Z-factors or factorized from the the coupling parameter. In either case, renormalized Green's functions and renormalized parameters dependent on this scale while bare Green's functions remain independent of it

$$-Z\mu^2 \frac{d}{d\mu^2} G_B^r = 0. \tag{2.146}$$

Permutation of the derivative with the Z-factor derives the Callan-Symanzik equation [36, 37, 38]

$$\left(\mu^2 \partial_{\mu^2} + \beta\alpha\partial_\alpha + \delta\alpha\partial_\alpha + \gamma\right) G^r = 0, \tag{2.147}$$

where we introduced the anomalous (scaling) dimension

$$\gamma^r = -\mu^2 \frac{d \ln Z_r}{d\mu^2}, \tag{2.148}$$

as known from the introductory discussion of anomalies 2.1.6 and two further renormalization group functions

$$\beta = -\mu^2 \frac{d \ln Z_\alpha}{d\mu^2} = 2\gamma^1 - 2\gamma^2 - \gamma^3 \quad \text{and} \tag{2.149}$$

$$\delta = -\mu^2 \frac{d \ln Z_3}{d\mu^2} = \gamma^3. \tag{2.150}$$

Note that these conventions determine the dynamical evolution laws of the coupling parameter α and the gauge parameter ξ to read

$$\mu^2 \frac{d\alpha}{d\mu^2} = \alpha\beta(\alpha, \xi) \qquad \mu^2 \frac{d\xi}{d\mu^2} = \xi\delta(\alpha, \xi). \tag{2.151}$$

In paragraph 2.4.4, we provide another derivation of the Callan-Symanzik equation which is based on studying the subdivergences of combinatorial Green's functions in the Hopf algebra.

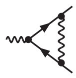
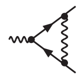
2.3.4 The BPHZ prescription in relation to Z-factors

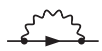
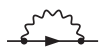
The introduction of Z-factors yielded renormalization conditions for Green's functions which can be interpreted as renormalization of the residue of that Green's function and a renormalization of its parameters. The BPHZ prescription, on the other hand, introduces non of these concepts but provides a procedure to construct counterterms for each Feynman graph individually. To compare both approaches, we employ the BPHZ prescription to each Feynman graph of the self-energy of the electron and compare this to Z-factor renormalization conditions. In this way, it becomes clear how each

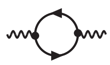
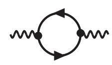
Feynman graph contributes to a particular Z-factor and whether the concept of parameter renormalization can be applied to the renormalization of individual Feynman graphs. We restrict ourselves to the second loop order of the renormalized self-energy. This is the lowest order which involves effects from the renormalization of the gauge parameter.

One-loop counterterms and renormalized parameters

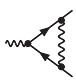
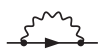
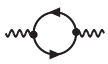
At first loop order, there are only two forests in Zimmermann's forest formula (2.128): the empty graph and the full graph. Further, at this order, bare parameter can simply be replaced by their renormalized equivalents. A comparison with the renormalization conditions (2.140-2.142) determines the counterterms C_i for $i = 1, 2, 3$ at first order.

$\Lambda_0 =$  $C_1 = T\,[$  $]$ (2.152)

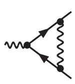
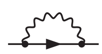
$\Sigma_0 =$  $C_2 = -T\,[$  $]$ (2.153)

$\Pi_0 =$  $C_3 = -T\,[$  $]$ (2.154)

Recall that the operator T originates from the forest formula. It extracts the divergent part of a Feynman graph and represents a particular renormalization scheme. For the next order renormalization, it is crucial to work out the influence of these counterterms on the renormalization of the coupling parameter

$\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3}\alpha = \alpha\Big(1 + T\Big[-2$  $-\,2$  $-$  $\Big] + O(\alpha^2)\Big)\,.$ (2.155)

Notice that the bare coupling parameter which occurs in the Feynman graphs is tacitly substituted by the renormalized coupling, this introduces higher order terms which however do not contribute at second order. Beside the coupling constant, a Feynman graph might also include factors of the gauge parameter ξ_0. The product of both contributes the following renormalization terms

$\xi_0\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3} Z_3 \xi\alpha = \xi\alpha\Big(1 + T\Big[-2$  $-\,2$  $\Big] + O(\alpha^2)\Big)\,.$ (2.156)

It should be remarked that Ward's identity $Z_1 = Z_2$ simplifies the renormalization of the gauge and coupling parameters — the product $\xi\alpha = \xi_0\alpha_0$

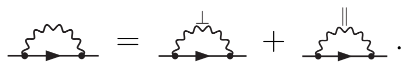
is not renormalized. However, these simplifications might be misleading in the comparison with subdivergences of the forest formula and are therefore not taken into account at this point.

The different substitution rules for the parameters α and $\xi\alpha$ make it necessary to distinguish between longitudinal and transversal photons. Therefore, the photon propagator is divided into the sum of its transversal and longitudinal parts, which are denoted by the labels $\perp$ and $\|$, respectively.

$$P^{\mu\nu}(k,\xi_0) \quad = \quad \frac{1}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \quad + \quad \xi_0 \frac{k^\mu k^\nu}{k^4} \tag{2.157}$$

In this notation, Feynman graphs are built from purely transversal and purely longitudinal photon propagators — all possible combinations need to be taken into account; for instance, the bare self-energy at first loop order becomes a sum of two graphs

$$\Sigma_0|_1 = \tag{2.158}$$



Two-loop renormalization of the self-energy of the electron

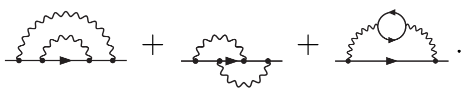
Now the results of the previous paragraph are extend to the second loop order. From the renormalization condition (2.141) and the previous discussion, five term are expected to contribution to the renormalized self-energy

$$\Sigma(\alpha,\xi)|_2 = \Sigma_0|_2 + \Sigma_0|_1(\alpha,\xi) + C_2|_1(\alpha,\xi) - C_2|_1\Sigma_0|_1 + C_2|_2. \tag{2.159}$$

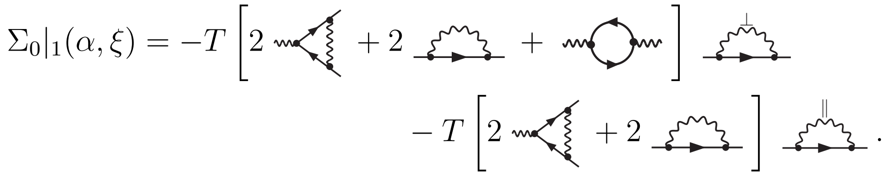
The first term contains all two-loop graphs from the bare self-energy, the second and third term arise from the substitution of bare by renormalized parameters in the first order bare self-energy and its counterterm, the fourth term is determined by the one-loop bare self-energy and its counterterm, the fifth term is the two-loop counterterm, which is determined by comparison to the BPHZ ing.

The bare self-energy of the electron at two loops reads

$$\Sigma_0|_2 = \tag{2.160}$$



In these two-loop graphs, all parameters are tacitly substituted by renormalized parameters, this causes contributions at three and higher loops, but no contribution to the second loop order. However, the parameter substitution in the one-loop graphs contribute at second order

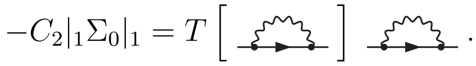


(2.161)

Here, the renormalization of the gauge parameter removes the pathologic divergence in the longitudinal part of the self-energy that comes from the renormalization of the transversal part of the photon self-energy. Note that this is in accordance with the contraction of subdivergences in the forest formula, as exemplified in (2.129). The same applies to the first order counterterm

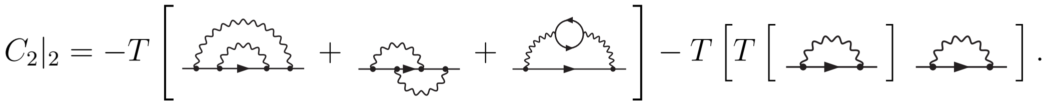
$$C_2|_1(\alpha, \xi) = -T\left[\Sigma_0|_1(\alpha, \xi)\right]. \tag{2.162}$$

The fourth contribution is a product of the one-loop self-energy and its counterterm, again tacitly rewritten in terms of renormalized parameters



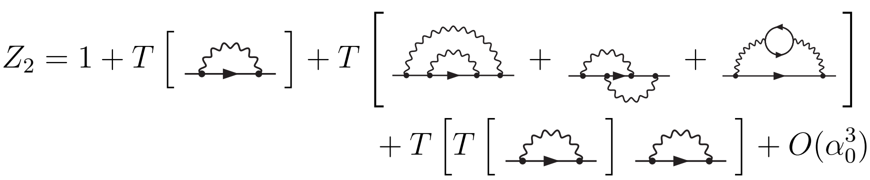
(2.163)

Finally, a comparison between the derived terms and forests from Zimmermann's formula determines the counterterm of the electron self-energy at two loops



(2.164)

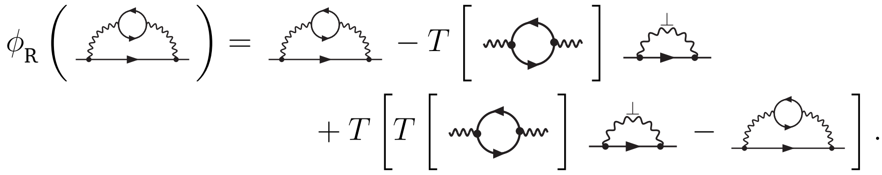
Notice that a composed term of the first order graph contributes. This term arises due to the fact that the counterterm is defined as the difference of terms generated by the forest formula (2.128) and terms resulting from the renormalization of parameters (as the coupling or gauge parameter). For the sake of completeness, we also provide the renormalization constant at two loops



(2.165)

It is worth remarking that we substituted the graphs by their dimensional regularized result $4-2\varepsilon$ dimensions and reproduced the well-known result of Z_2 [62].

The formula (2.161) demonstrates that the counterterms generated by the BPHZ recursion can be partitioned in such a way that the concepts of wave-function renormalization and parameter renormalization apply when the sum of all Feynman graphs are considered. However, this does not apply to individual Feynman graphs. To see this, consider the two-loop contribution with a vaccum polarization subdivergence. The BPHZ forest formula yields



(2.166)

An interpretation in terms of renormalization of the coupling parameter requires the subdivergence of the second term to be compatible with the linear combination of graphs specified by the renormalized coupling (2.155). However, the vertex and self-energy subdivergences are missing. This linear combination is only accomplished when all graphs are taken into account.

Further, recall that the different renormalization constants for the coupling and the gauge parameter appears as an asymmetry (2.161) in structure of subdivergences of longitudinal and transversal photon propagator. This will be of further interest in the succeeding discussion of the Hopf-algebraic approach to renormalization.

2.4 Hopf-algebraic renormalization of QED

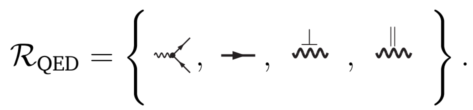
This section introduces all requirements for the Hopf-algebraic renormalization of Quantum Electrodynamics. This involves the definition of the Hopf algebra of QED Feynman graphs, the construction of Green's functions by combinatorial Dyson-Schwinger equations (DSE), and the derivation of a coproduct formula for the Green's functions. Moreover, the derivation of the renormalization group equation by means of the Dynkin operator $S\star Y$ is discussed. Special emphasis is put on the effect of the linear covariant gauge fixing to the Hopf algebra of Quantum Electrodynamics.

2.4.1 Hopf algebra structure

This paragraph introduces the Hopf algebra structure of massless Quantum Electrodynamics in the linear covariant gauge. This construction was first performed for Landau gauge [9, 26] and later extended to a general covariant gauge setting [51].

In the previous section, it was observed that the transversal and the longitudinal part of the photon self-energy give rise to different divergences in the renormalization process. Therefore, both Lorentz structures need to be distinguished and we assign the labellings $\perp$ and $||$ to the different components of the photon propagator. The labelled propagators are understood as different edge types in a Feynman graph.

Let H be the free commutative algebra over $\mathbb{R}$ generated by the set of all divergent one-particle irreducible Feynman graphs together with the product $m : H \otimes H \to H$. The unit $\mathbb{1} \in H$ is identified with the empty graph and the homomorphism $u : \mathbb{R} \to H$ with $u(1) = \mathbb{1}$ denotes the unit map. A Feynman graph is build from propagators and vertices of the set


(2.167)

Further, recall that a Feynman graph is superficially divergent if its external leg structure matches an element of $\mathcal{R}_{\text{QED}}$. Note that according to equation (2.157), the photon propagator is rewritten in terms of the transversal and the longitudinal propagator. A generic photon self-energy graph contributes to both of these Lorentz structures, the projection onto one of these Lorentz structures is denoted by assigning either the $\perp$ or the $||$ label to the external legs of the Feynman graph. In case of a vertex and an electron self-energy graph, the tacit projection onto their divergent Lorentz structures (γ_μ and $\not{q}$) is always understood.

In [9] Kreimer showed that a coproduct $\Delta : H \to H \otimes H$ can be defined by

$$\Delta\Gamma = \sum_{\gamma \trianglelefteq \Gamma} \gamma \otimes \Gamma/\gamma, \quad \Gamma \in H. \tag{2.168}$$

The summation restricts to all subgraphs $\gamma \subseteq \Gamma$ which consist of a product of disjoint subdivergences and includes also the empty and the full subgraph. Further, Γ/γ denotes the Feynman graph obtained by replacing all components of γ by their external structures.

The counit $\varepsilon : H \to \mathbb{R}$ is the homomorphism which satisfies $\varepsilon(\mathbb{1}) = 1$ and vanishes on the complement of $\mathbb{R}\mathbb{1}$. These definitions yield a bialgebra

$(H, m, \Delta, u, \varepsilon)$, which possesses a grading induced by the loop number of a Feynman graph. Hence, this bialgebra is indeed a Hopf algebra [63], its antipode $S : H \to H$ is recursively defined by

$$S(\Gamma) = -\Gamma - \sum_{\substack{\gamma \triangleleft \Gamma \\ \gamma \neq \mathbb{1}}} S(\gamma)\Gamma/\gamma, \quad \Gamma \in H. \tag{2.169}$$

As pointed out in [64], this recursion is solved by a sum over all forests which exclude the full graph Γ

$$S(\Gamma) = -\Gamma - \sum_{f} (-1)^{|f|} \gamma_f \Gamma/\gamma_f, \quad \gamma_f = \prod_{\gamma \in f} \gamma. \tag{2.170}$$

This formula reveals a striking similarity to Zimmermann's forest formula (2.128). Indeed, Zimmermann's forest formula is reproduced by the convolution of the Feynman rules ϕ and a twisted version of the antipode.

$$\phi_{\mathrm{R}}(\Gamma) = m \circ (\phi \circ S_T \otimes \phi) \circ \Delta(\Gamma) \tag{2.171}$$

Here, S_T is the antipode twisted with the renormalization scheme operator T

$$S_T(\Gamma) = -T \circ \left(\Gamma + \sum_{\substack{\gamma \triangleleft \Gamma \\ \gamma \neq \mathbb{1}}} S_T(\gamma)\Gamma/\gamma \right), \quad \Gamma \in H. \tag{2.172}$$

In this formulation, the finiteness of the renormalized expression ϕ_{R} is understood in terms of an algebraic Birkhoff decomposition [65, 66]. It is worth emphasising that the proof of this theorem clarifies why the renormalization scheme operator T has to satisfy the Rota Baxter equation

$$T(\gamma_1)T(\gamma_2) = -T(\gamma_1\gamma_2) + T \circ (T(\gamma_1)\gamma_2 + \gamma_1 T(\gamma_2)), \tag{2.173}$$

a necessary condition of a well-defined renormalization scheme that was previously anticipated by practitioners [42].

Equation (3.70) provides a prescription to renormalize an individual Feynman graph. However, the renormalization conditions (2.140-2.142) and the corresponding counterterms refer to one particle irreducible Green's functions. Therefore, the next topic in our discussion is the relation between the coproduct and one particle irreducible Green's functions.

2.4.2 Green's functions from combinatorial DSE

The coproduct extracts by definition subdivergences of individual Feynman graphs, its application on one-particle irreducible Green's function is best understood in the language of combinatorial DSE.

In [67] and [68] Broadhurst and Kreimer demonstrated how one-particle irreducible Green's functions are constructed as solutions of combinatorial DSE. In this language, a Green's function is built by insertion of subdivergences into skeleton graphs (Feynman graphs which are free of subdivergences). More precisely, for a given skeleton graph γ they defined an insertion operator B_+^γ, which takes a product of Feynman graphs as argument and maps it to the sum of all possible insertions of these Feynman graphs into the skeleton γ multiplied by some combinatorial factor, defined in [68].

To describe Quantum Electrodynamics in the linear covariant gauge, the vertex Green's function is denoted by $X^{\prec}$ and the electron self-energy by $X^{\rightharpoonup}$. The photon self-energy is represented by the Green's functions $X^{\perp}$ and $X^{\parallel}$ which respectively describe the contributions to the transversal and the longitudinal Lorentz structure. With this definition, $X^{\perp}$ is only inserted into the transversal photon propagators and $X^{\parallel}$ only into the longitudinal photon propagators. The Quantum Electrodynamics system of Dyson-Schwinger equations reads as follows.

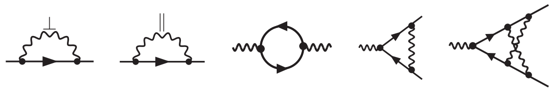
$$X^r = \mathbb{1} \pm \sum_{\substack{\gamma \text{ skeleton} \\ \mathrm{res}(\gamma)=r}} B_+^\gamma \left(X^r Q_\perp^{n_\perp(\gamma)+1} Q_\parallel^{n_\parallel(\gamma)} \right) \quad \text{for } r \in \left\{ \sim\!\!\prec, \rightharpoonup, \perp \right\} \tag{2.174}$$

$$X^{\parallel} = \mathbb{1} - \sum_{\substack{\gamma \text{ skeleton} \\ \mathrm{res}(\gamma)=\parallel}} B_+^\gamma \left(X^{\parallel} Q_\perp^{n_\perp(\gamma)} Q_\parallel^{n_\parallel(\gamma)+1} \right) \tag{2.175}$$

Each of these sums go over all one particle irreducible skeleton graphs of the external leg structure $\mathrm{res}(\gamma)$; the vertex function contains an infinite number of skeleton graphs — some examples are provided in figure 2.1. The number of transversal and longitudinal photon propagators of a Feynman graph γ is denoted by $n_\perp(\gamma)$ and $n_\parallel(\gamma)$. In (2.174), all propagators receive the negative sign, the positive sign only applies for the series vertex graphs. The input of these insertion operators is written in terms of the invariant charges

$$Q_\perp = \frac{(X^{\prec})^2}{(X^{\rightharpoonup})^2 X^{\perp}} \quad \text{and} \quad Q_\parallel = \frac{(X^{\prec})^2}{(X^{\rightharpoonup})^2 X^{\parallel}}. \tag{2.176}$$

Figure 2.1: Low order examples of skeleton graphs. As indicated by the first two graphs, each labelling of photon propagators induces a skeleton graph, these labels are understood in the subsequent graphs.



Notice that the argument of each insertion operator is defined such that every vertex of the skeleton is dressed by a factor $X^{\prec}$, every electron propagator by a factor $1/X^{\rightarrow}$, every transversal and every longitudinal photon propagator by a factor $1/X^{\perp}$ and $1/X^{\parallel}$, respectively. The inverse of a one particle irreducible propagator series corresponds to the series of connected propagator graphs; a one particle irreducible Green's function is build from one particle irreducible vertex insertions, but requires insertions of connected propagator graphs.

The advantage of the combinatorial Dyson-Schwinger equations approach is that a properly defined insertion operator B_+ yields a well-behaved compatibility relation with the coproduct [68, 69],

$$\Delta \circ B_+ = B_+ \otimes \mathbb{1} + (\mathrm{id} \otimes B_+) \circ \Delta. \tag{2.177}$$

This relation allows inductive proofs by induction on the number of subdivergences of a Feynman graph and implies a coproduct formula for Green's functions.

2.4.3 The coproduct of QED Green's functions

A closed formula for the coproduct on Green's functions has first been provided by Yeats [70]. By usage of the compatibility relation (2.177) Yeats derived a formula for the coproduct of one particle irreducible Green's functions in the case of a single invariant charge. However, it should be noted that all her proofs canonically generalize to systems of DSE involving multiple invariant charges by promoting the exponent of the single invariant charge to a multi-index; e.g. in case of the electron self-energy $n \equiv (n_\perp, n_\parallel)$ and

$$Q^n \equiv Q_\perp^{n_\perp} Q_\parallel^{n_\parallel}. \tag{2.178}$$

In addition to that, direct proofs of a coproduct formula for one particle irreducible Green's functions has been provided in [71, 72]. All these results imply the following coproduct formula for an one particle irreducible

Green's function X^r.

$$\Delta X^r = \sum_{0 \le n_\parallel \le n} X^r Q_\perp^{n-n_\parallel} Q_\parallel^{n_\parallel} \otimes X^r_{n;n_\parallel} \quad \text{for } r \in \left\{ \sim\!\!< ,\ \rightarrow ,\ \perp \right\} \tag{2.179}$$

$$\Delta X^\parallel = \sum_{0 \le n_\perp \le n} X^\parallel Q_\perp^{n_\perp} Q_\parallel^{n-n_\perp} \otimes X^\parallel_{n;n_\perp} \tag{2.180}$$

In this formula, $X^r_{n;n_\perp}$ denotes all Feynman graphs of the Green's function X^r which have n loops and $n_\perp$ transversal photon propagators; and $X^r_{n;n_\parallel}$ analogously with restriction to $n_\parallel$ photon propagators. In this formulation, the exponent shift of $n_\perp$ in (2.174) and of $n_\parallel$ in (2.175) was absorbed into the loop number n. Also note that the sum of both exponents of the invariant charges equals the loop number of a cograph, which appear on the right side of the tensor product.

2.4.4 Callan-Symanzik equation

This paragraph provides an Hopf-algebraic discussion of the renormalization group for the class of so-called one-scale Green's function, that is to say, a Green's function of this class depends only on a single external momentum. This obviously applies to propagator functions. In the case of the electron-photon vertex, one requires one external momentum to vanish. In the scope of this thesis, this condition does not provide any serious restrictions as we always consider one-scale renormalization schemes for computational convenience. However, it should be remarked that the techniques introduced here are actually rather general and generalize to the case of an arbitrary number of scales [64].

In the one-scale case, a renormalized Green's function only depends on a single kinematic variable L and allows for an expansion

$$G^r(L, \alpha, \xi) = \phi_{\text{R}}(X^r) = 1 + \sum_{n \ge 1} \gamma^r_n(\alpha, \xi) L^n. \tag{2.181}$$

The coefficients of this expansion are called anomalous dimensions; their relation to the scaling anomalous dimension γ as introduced in 2.1.6 becomes evident at the end of this analysis.

It is well-known [73, 74, 75] that in the case of one-scale Feynman graphs and a proper renormalization scheme (e.g. the $\widetilde{\text{MOM}}$ scheme), the Birkhoff decomposition implies the renormalized Feynman rules ϕ_{R} to be a morphism of bialgebras. As a conclusion the renormalized Feynman rules, or more precisely the anomalous dimensions, can be constructed by means of a combinatorial operation on the Hopf algebra of Feynman graphs

$$\gamma_n^r = \frac{1}{n!}\sigma^{\star n}(X^r) \quad \text{with} \quad \sigma = \phi_{\mathrm{R}} \circ Y^{-1} \circ (S \star Y). \tag{2.182}$$

The product $S \star Y$ is defined as the convolution $m \circ (S \otimes Y) \circ \Delta$ and $\sigma^{\star n}$ denotes the convolution of n maps σ. For further results and properties of σ and the Dynkin operator $S \star Y$, the reader is referred to [69, 76] and references therein. Here, we only exploit that the linear map $S \star Y$ is a projection and vanishes on products of Feynman graphs, as these properties simplify the evaluation of σ on products of combinatorial Green's functions in the succeeding derivations.

Note that a decomposition of the iterations of convolution product

$$(n+1)\gamma_{n+1}^{\leftarrow} = \left(\sigma \star {}^{\sigma^{\star n}}/_{n!}\right)\left(X^{\leftarrow}\right) \tag{2.183}$$

is capable to relate coefficient of different Green's functions by exploiting the coproduct formula (2.179)

$$(n+1)\gamma_{n+1}^{\leftarrow} = \sum_{0 \le n_\parallel \le n} \left[\sigma\left(X^{\leftarrow}\right) + (n - n_\parallel)\sigma\left(Q_\perp\right) + n_\parallel \sigma\left(Q_\parallel\right)\right] \frac{\sigma^{\star n}}{n!}\left(X^{\leftarrow}_{n;n_\parallel}\right). \tag{2.184}$$

In another step, the coefficients n and $n_\parallel$ can be replaced by means of operators which respectively count the power of the coupling parameter α and the power of the gauge parameter. In addition to that the shift in the power of the kinematic variable on the left-hand side of the equation is rewritten in terms of the derivative ∂_L. After this, summation over n reproduces an identity for the one-particle irreducible Green's function

$$\left[-\partial_L + \sigma\left(Q_\perp\right)\alpha\partial_\alpha + \sigma\left(Q_\parallel - Q_\perp\right)\xi\partial_\xi + \sigma\left(X^{\leftarrow}\right)\right] G^{\leftarrow} = 0. \tag{2.185}$$

This equation is readily recognized as the renormalization group equation by defining the anomalous dimension γ^r and the renormalization group functions β and δ as

$$\gamma^r = \sigma(X^r), \tag{2.186}$$

$$\beta = \sigma(Q_\perp) = 2\sigma(X^{\prec}) - 2\sigma(X^{\leftarrow}) - \sigma(X^\perp) = 2\gamma^{\prec} - 2\gamma^{\leftarrow} - \gamma^\perp, \tag{2.187}$$

$$\delta = \sigma(Q_\parallel) - \sigma(Q_\perp) = \gamma^\perp - \gamma^\parallel. \tag{2.188}$$

In these conventions, a renormalized Green's function satisfies the renormalization group equation

$$(-\partial_L + \beta\alpha\partial_\alpha + \delta\xi\partial_\xi + \gamma^{\leftarrow})G^{\leftarrow} = 0. \tag{2.189}$$

It is worth emphasising that the appearance of the two renormalization group functions β and δ is directly linked to the presence of the two invariant charges in the coproduct formula of one particle irreducible Green's function (2.179) and (2.180). This observations complements the above result. Due to the introduction of the covariant gauge parameter ξ, the photon propagator acquires a longitudinal component and further terms emerge in Green's functions. This phenomena is explained by the Hopf algebra. The coproduct of Green's functions contains the novel invariant charge $Q_{\|}$. On the other hand, the gauge parameter requires renormalization and its dynamics is described by means of the renormalization group function δ. As analysed in the above derivation, the appearance of this renormalization group function can be seen as a consequence of the novel invariant charge in the coproduct formula.

2.5 The massless self-energy of the electron

This section deals with the electron propagator and its one-particle irreducible part, the self-energy of the electron, in the massless limit. The combination of diagrammatic techniques developed in the above sections allow us to analyse the gauge dependence for both the bare and the renormalized propagator. At the end of this section, a non-perturbative argument is developed which demonstrates that the anomalous dimension of the electron depends on the gauge parameter only at the first order of perturbation theory.

2.5.1 Gauge dependence of the bare propagator

In the discussion of paragraph 2.2.3, fermions have always been considered to be massive. However, it should be noted that the cancellation identities (2.55), (2.56), and consequently the characterization of the gauge dependence (2.87) remain valid when massless fermions are considered. In the following, the characterization of the gauge dependence is applied to the massless electron propagator.

First, recall that the photon propagator is decomposed into a gauge-fixed term which is determined by the value ξ_0^* and a longitudinal term that is proportional to the shifted gauge parameter $\xi_0 - \xi_0^*$ (2.69), where the subscript zero represents the bare nature of these parameters. Now, the electron propagator is expanded into coefficients $c_{n;m}$ of order n in the bare coupling and order m in the shifted bare gauge parameter

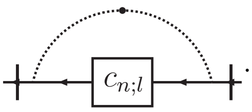
$$S_0(q) = \frac{i}{\not{q}} + \sum_{1\leq n} \sum_{0\leq m\leq n} c_{n;m} \, (\xi_0 - \xi_0^*)^m \alpha_0^n. \tag{2.190}$$

The coefficients $c_{n;m}$ depend on the external momentum q and the special value of the gauge parameter ξ_0^*. A Feynman graph that contributes to this coefficient consists of m gauge dependent tensors (2.70) and $(n-m)$ gauge-fixed photon propagators (2.68).

Now, assume that all Feynman graphs of the electron propagator were computed with the specific value ξ_0^* for the gauge parameter to some loop order N — that corresponds to the coefficients linear in the shifted gauge parameter $c_{n;0}$ with $n \leq N$. Then, the unknown coefficients can be recursively computed by exploiting the characterization of the dependence on the gauge parameter (2.87).

$$c_{n+1;m+1}(q) = \frac{ie_0^2}{m+1} \int \frac{d^D p}{(2\pi)^4} \frac{1}{[-p^2]^2} c_{n;m} \, (q+p) \tag{2.191}$$

$$= -\frac{1}{m+1} \tag{2.192}$$



Note that this reduction can be iterated and is valid at an arbitrary loop order. Therefore, this diagrammatic formula can be thought of as a Dyson-Schwinger equation where subgraphs are linearly inserted into a divergent skeleton graph. The skeleton graph of this Dyson-Schwinger equation can be written in terms of the dimensional regularized one-loop master integral of two scalar propagators with weights x and y

$$G(x,y) = -i \int \frac{d^D k}{(2\pi)^D} \frac{(-q^2)^{x+y-D/2}}{[-k^2]^x \, [-(k+q)^2]^y} \tag{2.193}$$

$$= \frac{\Gamma(D/2 - x)\Gamma(D/2 - y)\Gamma(x+y-D/2)}{(4\pi)^{D/2}\Gamma(x)\Gamma(y)\Gamma(D-x-y)} \tag{2.194}$$

[77], where we have amputated the dependence on the external momentum for notational convenience. By Weinberg's power counting theorem [53, 54], the momentum dependence of the n-loop bare electron propagator $c_{n;m}(q)$ is given by $\not{q}/(q^2)^{1+n\varepsilon}$. Insertion of this momentum dependence into the Dyson-Schwinger equation determines the contribution of the skeleton graph to the electron propagator. In $D = d - 2\varepsilon$ dimensions, it

contributes the factor

$$F(d,\varepsilon,n) = \frac{1}{2}e_0^2\xi_0\left(-q^2\right)^{-\varepsilon}\exp\left[\varepsilon\left(\gamma_E + \frac{\zeta(2)\varepsilon}{2}\right)\right] \times \left[G(d/2-1, 1+n\varepsilon) - G(d/2, 1+n\varepsilon) - G(d/2, n\varepsilon)\right]. \quad (2.195)$$

Here, e denotes the bare coupling parameter, q the external momentum, γ_E the Euler–Mascheroni constant, and $\zeta(z)$ is the Riemann zeta function. The exponential factor does not originate from the Dyson-Schwinger equation, but is implemented for straightforward comparison with results obtained with the MINCER package. In this paragraph, we are only concerned with the four dimensional Quantum Field Theories $(d = 4)$ and hence define $F(\varepsilon, n) := F(4, \varepsilon, n)$ and set $c_{0;0} := i/\not{q}$. Now, an iterative use of the formula (2.192) determines all coefficients

$$c_{n;m} = \frac{1}{m!}c_{n-m;0}\prod_{1\leq j\leq m} F(\varepsilon, n-j) \text{ for } m \geq 1, \quad (2.196)$$

in terms of the coefficient $c_{n;0}$ that are constant in the shifted gauge parameter and the one-loop skeleton function F. For instance, the ε-expansions of all gauge dependent terms at 4 loops

$$c_{4;1}(\varepsilon) = c_{3;0}(\varepsilon)F(\varepsilon, 3), \quad (2.197)$$

$$c_{4;2}(\varepsilon) = \frac{1}{2!}c_{2;0}(\varepsilon)F(\varepsilon, 3)F(\varepsilon, 2), \quad (2.198)$$

$$c_{4;3}(\varepsilon) = \frac{1}{3!}c_{1;0}(\varepsilon)F(\varepsilon, 3)F(\varepsilon, 2)F(\varepsilon, 1), \quad (2.199)$$

$$c_{4;4}(\varepsilon) = \frac{1}{4!}c_{0;0}F(\varepsilon, 3)F(\varepsilon, 2)F(\varepsilon, 1)F(\varepsilon, 0) \quad (2.200)$$

are determined once the ε-expansion is known of the specific gauge parameter ξ_0^* at 3 loops. For analytic results of the electron propagator to three loops the reader is referred to [78, 79, 80, 81].

Recall that these coefficients refer to the connected electron propagator (2.190). However, for the sake of computational convenience, it is necessary to relate these coefficients to the bare self-energy of the electron Σ_0. Define coefficients $p_{n;m}$ to decompose the self-energy

$$\Sigma_0(q) = \not{q}\sum_{1\leq n}\sum_{0\leq m\leq n} p_{n;m}(\xi_0 - \xi_0^*)^m\alpha_0^n. \quad (2.201)$$

Then, the relation between the connected and the one-particle irreducible propagator

$$S_0(q) = \frac{i}{\not{q} - \Sigma_0(q)} \quad (2.202)$$

implies the conversion formulas for the coefficients

$$\tilde{c}_{n;m} := -i\not{q}\, c_{n;m} = \sum_{k\geq 1} \sum_{\substack{n_1+\cdots+n_k=n \\ m_1+\cdots+m_k=m}} p_{n_1;m_1}\cdots p_{n_k;m_k}, \tag{2.203}$$

$$p_{n;m} = \sum_{k\geq 1} (-1)^{k+1} \sum_{\substack{n_1+\cdots+n_k=n \\ m_1+\cdots+m_k=m}} \tilde{c}_{n_1;m_1}\cdots \tilde{c}_{n_k;m_k}. \tag{2.204}$$

In this way, we are able to compare the formula for the coefficient with results from actual perturbative computations. We generate the Feynman graphs of the electron self-energy Σ with QGRAF [15] and perform the computations in FORM [16] and its parallel version TFORM [17] in combination with the MINCER package [19, 20]. For a more detailed discussion of our computational setup the reader is referred to the introduction of this thesis. From these perturbative results, we extract the coefficients $p_{n;m}$ as a series in ε to three loops $n \leq 3$ with an arbitrary power in the shifted gauge parameter $0 \leq m \leq n$.

After the conversion into the connected coefficient (2.203), the MINCER expansions exactly match the formula (2.196) derived by means of the linear Dyson-Schwinger equation. This reconstruction of the gauge dependent terms has been checked for both the Feynman gauge ($\xi_0^* = 1$) and the Landau gauge ($\xi_0^* = 0$).

2.5.2 Hopf-algebraic renormalization

As mentioned in the preceding paragraph, the epsilon expansion was derived for all superficially divergent Feynman graphs up to third loop order. In this paragraph, Hopf-algebraic technique are applied to renormalize the self-energy of the electron. Recall that the renormalized Feynman rules are expressed as a convolution

$$\phi_{\mathrm{R}}\left(X^{\rightarrow}\right) = m \circ (S_T \otimes \phi) \circ \Delta\left(X^{\rightarrow}\right) \tag{2.205}$$

of the twisted antipode with the unrenormalized Feynman rules ϕ. Here, the $\widetilde{\text{MOM}}$ renormalization scheme is employed; that is to say, vertex subdivergences are derived with zero momentum transfer at the photon leg and the square of the remaining external momentum is evaluated at the renormalization point μ^2. This defines the mapping of the operator T.

The renormalized Feynman rules require us to derive the coproduct and the twisted antipode of all Feynman graphs contributing to the combinatorial Green's function $X^{\rightarrow}$. Note that the recursion from the twisted antipode (2.172) is readily rewritten

$$S_T = -T - T \circ m\,(S_T \otimes \mathrm{id})\,\widetilde{\Delta}, \tag{2.206}$$

in terms of the reduced coproduct

$$\widetilde{\Delta} = \Delta - \mathbb{1} \otimes \mathrm{id} - \mathrm{id} \otimes \mathbb{1}. \tag{2.207}$$

This observation together in combination with the closed formulas for the coproduct of a Green's function (2.179) and (2.180) allow us to derive the renormalized Green's function as a recursion over coefficients of bare Green's functions, rather than to perform the recursion for each Feynman graph separately. Finally, these coefficients are replaced by their corresponding epsilon expansion and yield the renormalized self-energy of the electron up to three loops

$$\begin{aligned}\Sigma(\alpha,\xi)/\not{p} &= \xi L\left(\frac{\alpha}{4\pi}\right) + \left(-\frac{1}{2}\xi^2 L^2 - (\frac{3}{2} + 2n_f)L\right)\left(\frac{\alpha}{4\pi}\right)^2 \\ &+\left(\frac{1}{6}\xi^3 L^3 + \left((\frac{3}{2} + 2n_f)\xi - 2n_f - \frac{8}{3}n_f^2\right)L^2 + (\frac{3}{2} - 2n_f + \frac{8}{3}n_f^2)L\right)\left(\frac{\alpha}{4\pi}\right)^3.\end{aligned} \tag{2.208}$$

In this result, the parameter n_f has been introduced. It denotes the number of different fermion generations and hence provides a slight generalization of the original setting where the electron is considered as the only fermion. The advantage of this approach is that the quenched limit is easily accessible by evaluating $n_f = 0$. Further, it is worth noting that the cancellation of poles in the epsilon expansion provides a non-trivial check of the coproduct formula. As another check, the renormalized vacuum polarization was derived in the $\overline{\text{MS}}$ scheme and shown to agree with the result of [82]. A reader interested in these quantities is referred to our detailed discussion of non-abelian gauge theories in section 3.6.

Here, we continue to extract the renormalization group function from the renormalized self-energy above. Recall that the renormalization group equation reads

$$(-\partial_L + \beta\alpha\partial_\alpha + \delta\xi\partial_\xi + \gamma^{\leftarrow})G^{\leftarrow} = 0. \tag{2.209}$$

Further, the renormalization group functions are expanded in the renormalized coupling α and define the following coefficients

$$\beta(\alpha,\xi) = -\sum_{n\geq 1} \beta_n \left(\frac{\alpha}{4\pi}\right)^n, \tag{2.210}$$

$$\gamma^{\rightarrow}(\alpha,\xi) = -\sum_{n\geq 1} \gamma_n \left(\frac{\alpha}{4\pi}\right)^n, \tag{2.211}$$

$$\delta(\alpha,\xi) = -\sum_{n\geq 1} \delta_n \left(\frac{\alpha}{4\pi}\right)^n. \tag{2.212}$$

Now, the renormalized self-energy, or more precisely the renormalized Green's function $G^{\rightarrow} = \not{p} - \Sigma$, is inserted into the renormalization group equation and expanded in the coupling α and the kinetic parameter L. This provides systems of equations that determine the coefficients of the renormalization group functions.

The first loop order of the self-energy Σ determines the anomalous dimension of the electron to satisfy

$$\gamma_1 = \xi, \tag{2.213}$$

which the familiar value known from the introductory discussion of anomalies 2.1.6. The two-loop result of the self-energy relates the first coefficients of the β and δ function and yields the second coefficient of the anomalous dimension

$$\beta_1 = -\delta_1 \qquad \text{and} \qquad \gamma_2 = -\frac{3}{2} - 2n_f. \tag{2.214}$$

At third loop order, the self-energy determines the third order of the anomalous dimension, the first coefficient of the β function (and hence δ function), and relates the second order of the β and δ functions

$$\beta_1 = -\frac{4}{3}n_f, \qquad \beta_2 = -\delta_2, \quad \text{and} \tag{2.215}$$

$$\gamma_3 = \frac{3}{2} - 2n_f + \frac{8}{3}n_f^2. \tag{2.216}$$

It should be remarked that the derived relation between the β and the δ function is in convenience with the Ward identities $\gamma^{\prec} = \gamma^{\rightarrow}$ and $\gamma^{\parallel} = 0$. Further, note that the derived coefficients of the anomalous dimension support the aforementioned conjecture — only γ_1 depends on the gauge parameter ξ.

2.5.3 Higher order gauge parameters

This paragraphs reports on an attempt to find convenient choices for gauge parameter such that the structure of subdivergences and hence the renormalization process simplifies. It is shown that suchlike techniques imply mild restrictions on the gauge-dependent terms of the self-energy. The discussion presented here does not exceed the results published in [51]. The reader who is mainly interested in a characterization of the gauge dependence of the renormalized electron propagator might want to skip this paragraph.

In [83], Johnson and Zumino asserted that all divergences in the self-energy of the electron can be eliminated by a suitable choice of gauge, resulting in a finite wave-function renormalization constant Z_2 and, by Ward's identity, a finite vertex renormalization constant Z_1. The crucial ingredient is the exact knowledge of the behaviour of the electron self-energy under gauge transformations. This has been studied by Landau and Khalatnikov [8] and Zumino [43]. For quenched Quantum Electrodynamics (QED without insertions of photon self-energy graphs), such a gauge has been explicitly constructed by Baker, Johnson, and Willey in [78], where finite solutions of the self-energy were derived by solving a system of truncated Dyson-Schwinger equations. To achieve this, they had to allow for a coupling dependent gauge parameter:

$$\xi_0(\alpha_0) = \frac{3}{2}\frac{\alpha_0}{4\pi} \tag{2.217}$$

Their technique of introducing gauge parameters of higher orders in the coupling parameter was also applied in quenched Quantum Electrodynamics in the first evaluation of the three-loop beta function by Rosner [84] and in Broadhurst's calculation of the anomalous dimensions of the quenched theory to four loops [85]. These results motivate us to examine the behaviour of these coupling dependent gauge parameters under renormalization and which kind of divergences can be cancelled by a suitable choice of gauge. We promote the bare gauge parameter to a series in the bare coupling parameter.

$$\xi_0(\alpha_0) = \sum_{n\geq 0} \xi_0^{(n)} \alpha_0^n \tag{2.218}$$

A gauge fixing corresponds to a specific choice of the parameters $\xi_0^{(n)}$, which are required to be free parameters. However, as we have seen in the previous paragraphs, after renormalization, the gauge parameter becomes a power series in the coupling parameter and its coefficients are determined through the renormalization condition $\xi_0 = Z_3\,\xi$. To avoid this kind of contradiction, we also introduce renormalized higher order gauge parameters $\xi_0^{(n)} = Z_3^{n+1}\xi^{(n)}$ and replace the renormalized gauge parameter by a series of renormalized gauge parameters.

$$\xi(\alpha) = \sum_{n\geq 0} \xi^{(n)}\alpha^n \tag{2.219}$$

This definition maintains the renormalization condition $\xi_0(\alpha_0) = Z_3\xi(\alpha)$, which is necessary due to the fact that the self-energy of the photon is transversal. The photon propagator is modified by a series of longitudinal parts of higher order in the gauge parameter.

$$P^{\mu\nu}(k,\xi_0) = \frac{1}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \xi_0^{(0)}\frac{k^\mu k^\nu}{k^4} + \xi_0^{(1)}\alpha_0\frac{k^\mu k^\nu}{k^4} + \cdots \tag{2.220}$$

In this generalized linear covariant gauge, Feynman graphs are build from vertices and edges of the infinite set

$$\mathcal{R} \in \left\{ \sim\!\!\prec,\ \rightarrow,\ \overset{\perp}{\sim},\ \overset{\|_0}{\sim},\ \overset{\|_1}{\sim},\ \ldots \right\}, \tag{2.221}$$

where the transversal part of the photon propagator is denoted by the $\perp$ label and the longitudinal term which is proportional to ξ_0^j is denoted by the label $\|_j$.

First, the quenched sector of Quantum Electrodynamics in this gauge is discussed. The vertex and electron one particle irreducible Green's functions are solutions of the following system of DSE.

$$\Delta X^r = \mathbb{1} + \sum_{\substack{\gamma \text{ skeleton} \\ \text{res}(\gamma)=r}} B_+^{\gamma}(X^r Q^{|\gamma|}), \quad r \in \left\{ \sim\!\!\prec,\ \rightarrow \right\} \tag{2.222}$$

Where the invariant charge of the quenched theory is defined as

$$Q = \frac{(X^{\prec})^2}{(X^{\rightarrow})^2}. \tag{2.223}$$

Note that these DSE only differ from (2.174) of the full theory by the property that no photon self-energy graphs are inserted into the skeletons. Again, the application of (2.177) implies a closed formula for the coproduct of the one particle irreducible Green's function in the quenched sector.

$$X^{\rightarrow} = \sum_{n\geq 0} X^{\rightarrow} Q^n \otimes X_n^{\rightarrow} \tag{2.224}$$

Here, $X_n^{\rightarrow}$ denotes all n-loop Feynman graphs of the quenched Green's function $X^{\rightarrow}$. It should be remarked that the same formula follows from the coproduct formula of the full theory by dividing out the Hopf-ideals generated by $X^{\perp}_{n_\perp, n_\parallel}$ and $X^{\parallel}_{n_\perp, n_\parallel}$. As demonstrated in the previous paragraph, this coproduct formula in combination with (2.182) restricts the L expansion of the electron self-energy. The absence of transversal photon subdivergences implies that the quenched invariant charge vanishes under $\sigma = \phi_{\mathrm{R}} \circ Y^{-1} \circ (S \star Y)$.

$$\sigma(Q) = 2\gamma^{\prec} - 2\gamma^{\rightarrow} = 0 \tag{2.225}$$

This corresponds to the fact that coupling parameter α and the gauge parameter ξ are not renormalized in the vertex and electron Green's function of the quenched theory and yields the following simplified renormalization group equation.

$$(\partial_L - \gamma^{\rightarrow})(1 - \Sigma_{\text{quenched}}) = 0, \quad \text{with } \gamma^{\rightarrow} = -\partial_L \Sigma_{\text{quenched}}(0) \tag{2.226}$$

This ordinary differential equation determines the self-energy of the electron by means of its anomalous dimension, which is up to a sign the linear log term of the self-energy. Hence, the unique solution reads

$$\Sigma_{\text{quenched}} = 1 - \exp\left(L\gamma^{\rightarrow}\right). \tag{2.227}$$

In this way, the question of a vanishing quenched self-energy of the electron is converted to a vanishing anomalous dimension. Indeed, there is a choice of higher order gauge parameters such that the anomalous dimension vanishes. Recall that the one-loop self-energy graph is only sensitive to the longitudinal part of the photon propagator (see (2.208)). Therefore, the self-energy of the electron vanishes in the Landau gauge $\xi^{(0)} = 0$ at first loop order. Now, observe that the one-loop graph contributes at higher loop orders by the coupling dependent parts of the photon propagator (2.220). Moreover, this shift of the first order graph yields a linear log term which is proportional to a unspecified gauge parameter. In other words, the linear log term of the electron self-energy at loop order n can be cancelled by fixing the value of the gauge parameter $\xi^{(n-1)}$. In this gauge fixing, the anomalous dimension vanishes.

$$\Sigma_{\text{quenched}}(\alpha, \tilde{\xi}(\alpha)) = 0 \quad \text{for some} \quad \tilde{\xi}(\alpha) = \sum_{j \geq 0} \tilde{\xi}^{(j)} \alpha^j \tag{2.228}$$

A vanishing renormalized self-energy implies a finite bare self-energy. In other words, the constructed gauge fixing cancels all divergences and allows a finite renormalization constant Z_2, which is the original statement of Baker, Johnson, and Willey.

Finally, the unquenched case of Quantum Electrodynamics is discussed. As demonstrated in the analysis of the quenched sector, the cancellation of the linear log terms of the self-energy already determines the full set of higher order gauge parameters $\tilde{\xi}(\alpha)$. This in combination with the three-loop result (2.208) determines the gauge to second order.

$$\tilde{\xi}(\alpha) = 0 + \left(\frac{3}{2} + 2n_f\right)\alpha + \left(-\frac{3}{2} + 2n_f - \frac{8}{3}n_f^2\right)\alpha^2 \tag{2.229}$$

Note that the quenched limit $n_f = 0$ coincides with the results of Baker, Johnson, and Willey (2.217) at two loops and Broadhurst at three loops [85]. However, in the full theory, the cancellation of the anomalous dimension of the electron does not imply a vanishing self-energy and at third order a quadratic log term remains:

$$\Sigma(\alpha, \tilde{\xi}(\alpha)) = \left(-2n_f - \frac{8}{3}n_f^2\right) L^2 \left(\frac{\alpha}{4\pi}\right)^3. \tag{2.230}$$

A L dependent term arises from a divergent subgraph in Zimmermann's forest formula. Therefore, this nonvanishing L term corresponds to a remaining subdivergence in the self-energy of the electron. This subdivergence is necessarily cancelled by a divergent counterterm. Hence, the higher order gauge parameters of the generalized linear covariant gauge fixing are not sufficient to remove all divergences from the self-energy of the electron, or equivalently to provide a finite renormalization constant Z_2.

It is interesting to note that the pure existence of this gauge technique restricts the next-to-leading log term in the self-energy of the electron within the original linear covariant gauge. More precisely, a next-to-leading log term proportional to

$$\xi^0 n_f^0 L^{n-1} \left(\frac{\alpha}{4\pi}\right)^n, \quad n \geq 3 \tag{2.231}$$

is forbidden.

In other words, a Feynman graph which contributes to the next-to-leading log term possesses

- (at least) a longitudinal photon propagator
- or (at least) a closed fermion loop.

This follows from the fact that such a term contradicts the existence of higher order gauge parameters which induce the vanishing of the quenched self-energy: Notice that a term of form (2.231) is a quenched contribution and recall that the leading-log term allows a direct evaluation via truncated DSE and reads

$$(-1)^{n+1}\frac{\xi^n L^n}{n!}\left(\frac{\alpha}{4\pi}\right)^n, \quad n \geq 1. \tag{2.232}$$

Now, replace the gauge parameter ξ by the coupling dependent series $\xi(\alpha)$, whose coefficients are fixed by the requirement that the electron anomalous dimension vanishes. As argued before, this guaranties a vanishing electron self-energy. Due to an expansion of the gauge parameter series, a log term of a particular order contributes also at higher loop orders. As a result, the next-to-leading log term at $n+1$ loops receives a contribution from the leading-log term at n loops (2.232). Further, all terms which depend on the gauge parameter vanish because of the vanishing coefficient $\xi^{(0)} = 0$ of (2.229). The only remaining term is of the form (2.231). However, by construction of the series $\xi(\alpha)$, all log terms (including the next-to-leading log term) of the quenched sector vanish. Hence, the next-to-leading log term of the electron self-energy vanishes in the Landau gauge (ξ has non-zero exponent) or is not within the quenched sector (n_f has non-zero exponent).

2.5.4 Gauge dependence of the renormalized propagator

This paragraph contains a non-perturbative enquiry of the gauge dependence of the renormalized electron propagator. The characterization (2.87) of the gauge dependence for the bare Green's function is promoted to a Dyson-Schwinger equation for the renormalized electron propagator by introducing an appropriate counterterm that cancels the inherent infrared divergence. This results in a characterization of the gauge dependence of the renormalized electron propagator.

The starting point of our consideration is the integral equation (2.88), in which the bare parameters and Green's functions are replaced by their renormalized equivalents. Further, we define the reduced Green's function $\widetilde{S} = \not{q} S$ to strip off the Lorentz factor for notational convenience. In these conventions, the characterization of the gauge dependence reads

$$\frac{\partial \widetilde{S}}{\partial \xi}\left(\ln\frac{q^2}{\mu^2}, \xi\right) = ie^2 \int \frac{d^D p}{(2\pi)^D}\, \widetilde{S}\left(\ln\frac{(q+p)^2}{\mu^2}, \xi\right) \frac{\mathrm{Tr}\left[\not{q}(\not{q}+\not{p})\right]}{\left[p^2\right]^2 (q+p)^2}. \tag{2.233}$$

As usual, the Green's function is assumed to be analytic in the kinematic variable. Therefore, it might be expressed in terms of the substitution

$$\widetilde{S}\left(\ln\frac{(q+p)^2}{\mu^2}, \xi\right) = \widetilde{S}\left(-\partial_\rho, \xi\right)\left(\frac{\mu^2}{(q+p)^2}\right)^\rho \Bigg|_{\rho=0}. \tag{2.234}$$

The consecutive step is to insert the substitution and interchange the derivatives with respect to ρ and the dimensionally regularized integration. This produces an infrared divergent integral

$$\frac{\partial\widetilde{S}}{\partial\xi}\left(\ln\frac{q^2}{\mu^2},\xi\right) = ie^2\widetilde{S}\left(-\partial_\rho\right)\left(\mu^2\right)^\rho\int\frac{d^Dp}{(2\pi)^D}\frac{\frac{1}{2}\left[(q+p)^2+q^2-p^2\right]}{\left[p^2\right]^2\left[(q+p)^2\right]^{1+\rho}}. \quad (2.235)$$

Here, the limit $\rho \to 0$ is understood after the dimensionally regularized integration. Employing the power counting theorem to the above integrand reveals the infrared divergence for vanishing p^2. To resolve the infrared divergence, the integrand is modified by subtracting the term

$$\frac{\frac{1}{2}\left[(\mu+p)^2+\mu^2-p^2\frac{\mu^2}{q^2}\right]}{\left[p^2\right]^2\left[(\mu+p)^2\right]^{1+\rho}}\left(\frac{\mu^2}{q^2}\right)^\rho. \quad (2.236)$$

Two conditions are crucial for the construction of this counterterm. First, note that the original integrand and the counterterm coincide at $q^2 = \mu^2$. In other words, quantum correction vanish if the external momentum equals the renormalization point μ. This realizes a kinematic renormalization condition which is known as the $\widetilde{\text{MOM}}$ scheme. Second, both expressions have the same scaling factors in the limit $p \to 0$. This condition ensures that subtration of the counterterm improves the superficial degree of divergence and is the reason for the additional factors μ^2/q^2. It worth remarking that a $\overline{\text{MS}}$ counterterm is readily constructed in a similar fashion.

Now, we employ dimensional regularization in $D = 4 - 2\varepsilon$ dimensions and perform a Wick rotation to simplify expressions

$$\frac{\partial\widetilde{S}}{\partial\xi}\left(\ln\frac{-q^2}{\mu^2}\right) = \frac{\alpha}{4\pi}\widetilde{S}\left(-\partial_\rho\right)\left(\frac{-q^2}{\mu^2}\right)^\rho\left[\left(\frac{-q^2}{4\pi}\right)^{-\varepsilon}-\left(\frac{\mu^2}{4\pi}\right)^{-\varepsilon}\right] \times \underbrace{\left[\frac{1}{2}G(2,\rho)+\frac{1}{2}G(2,1+\rho)-\frac{1}{2}G(1,1+\rho)\right]}_{=-1/\varepsilon+\mathrm{O}(\varepsilon^0)}. \quad (2.237)$$

The function G is defined in (2.193); it depends on the dimensional regulator ε. Notably, the specific combination of these functions has a pole in the dimensional regulator ε and its residue is ρ-independent. Thanks to the subtracted counterterm, the pole terms in ε cancel and we can safely take the limit $\varepsilon \to 0$. After that, the substitution (2.234) is reversed and ρ is evaluated to zero. This finally yields the simple ordinary differential equation

$$\frac{\partial \widetilde{S}}{\partial \xi}\left(\ln \frac{-q^2}{\mu^2}, \xi\right) = \ln\left(\frac{-q^2}{\mu^2}\right)\frac{\alpha}{4\pi}\,\widetilde{S}\left(\ln \frac{-q^2}{\mu^2}, \xi\right). \tag{2.238}$$

This differential equation is easily solved by the ansatz

$$\widetilde{S}(L, \xi) = \exp\left[(\xi - \xi^*)L\frac{\alpha}{4\pi}\right]\widetilde{S}(L, \xi^*) \tag{2.239}$$

where we introduced the kinematic variable $L = \ln\left({-q^2}/{\mu^2}\right)$ and assumed the electron propagator to be known at the particular value ξ^* in the gauge parameter. This solution allows to reconstruct the electron propagator in the general covariant gauge once it is known at a particular gauge such as the Feynman $\xi^* = 1$ or Landau gauge $\xi^* = 0$.

In the following, we consider the standard decomposition of the photon propagator into a transversal and a longitudinal part and hence set $\xi^* = 0$. Recall that the connected propagator function satisfies the renormalization group function

$$\left(-\partial_L + \beta\alpha\partial_\alpha + \delta\xi\partial_\xi - \gamma\right)\widetilde{S}(L, \xi) = 0. \tag{2.240}$$

Further, define the anomalous dimension in the Landau gauge as $\gamma(\alpha, 0)$, then the Landau gauged propagator function obeys the renormalization group equation

$$\left(-\partial_L + \beta\alpha\partial_\alpha - \gamma(\alpha, 0)\right)\widetilde{S}(L, 0) = 0. \tag{2.241}$$

Inserting the solution (2.239) in the first renormalization group equation and using the latter renormalization group equation to relate to the anomalous dimension in the Landau gauge and exploit the fact that $\beta = -\delta$ implies the identity

$$\gamma(\alpha, \xi) - \gamma(\alpha, 0) = -\xi\frac{\alpha}{4\pi}. \tag{2.242}$$

Which proves the statement that the anomalous dimension depends on the gauge parameter only at first loop order in the cases of the $\widetilde{\text{MOM}}$ and $\overline{\text{MS}}$ renormalization schemes.

It is possible to provide a generalization of the above discussion for higher dimensional Quantum Electrodynamics. In paragraph 2.2.3, it was demonstrated that the formula which characterizes the gauge dependence (2.102) persists in $d = 6$ and $d = 8$ dimensions. The integrand is modified in such a way that the infrared divergence survives despite the increased dimension. The analysis from above similarly applies to this d-dimensional setting. The residue of equation (2.237) is slightly modified

$$\left[\frac{1}{2}G\left({}^{d}\!/\!{}_{2},\rho\right)+\frac{1}{2}G\left({}^{d}\!/\!{}_{2},1+\rho\right)-\frac{1}{2}G\left({}^{d}\!/\!{}_{2}-1,1+\rho\right)\right]=-\frac{1}{\Gamma\left({}^{d}\!/\!{}_{2}\right)}\frac{1}{\varepsilon}+\mathrm{O}\left(\varepsilon^{0}\right). \tag{2.243}$$

This results in the following solution for the electron propagator

$$\widetilde{S}(\xi,L)=\exp\left[\frac{\xi L}{\Gamma({}^{d}\!/\!{}_{2})}\frac{\alpha}{4\pi}\right]\widetilde{S}(0,L), \tag{2.244}$$

and the anomalous dimension

$$\gamma(\alpha,\xi)-\gamma(\alpha,0)=-\frac{\xi}{\Gamma({}^{d}\!/\!{}_{2})}\frac{\alpha}{4\pi}, \tag{2.245}$$

which is in convenience with the results reported in [48].

Chapter 3

Non-Abelian Gauge Theories

This chapter reports on the Hopf-algebraic renormalization of non-abelian gauge theories. On one hand, it is instructive to ask how the structural results from the Hopf-algebraic renormalization of Quantum Electrodynamics convey to non-abelian gauge theories. On the other hand, it is necessary to compute the gauge-dependent terms of Green's functions for a prospective study of cancellation identities in non-abelian gauge theories.

3.1 Model and Feynman rules

Consider the Lagrangian of a general non-Abelian gauge theory with a linear covariant gauge fixing

$$\mathscr{L} = \mathscr{L}_{\mathsf{F}} + \mathscr{L}_{\mathsf{YM}} + \mathscr{L}_{\mathsf{GF}} + \mathscr{L}_{\mathsf{GH}}. \tag{3.1}$$

Four parts constitute this model and describe the fermionic fields, the gauge fields, the gauge fixing, and the unobservable ghost fields.

The fermionic fields are represented by a Dirac spinor ψ. The Lagrangian reads

$$\mathscr{L}_{\mathsf{F}} = \overline{\psi} \not{D} \psi, \quad \text{where } (D_\mu)_{jk} = \delta_{jk}\partial_\mu + gA^a_\mu T^a_{jk} \tag{3.2}$$

denotes the covariant derivative and $\{T^a : a = 1, \ldots, N_A\}$ is a basis of infinitesimal generators of the gauge group G. The infinitesimal generators are represented by finite-dimensional matrices whose entries are denoted by the subscripts $j, k = 1, \ldots, N_c$. This type of subscript is also assigned to the fermionic fields ψ_k and is commonly referred to as the color quantum number. Moreover, there are different species of fermionic fields, which are denoted by the flavor quantum number; the flavor index assumes values in

$1, \ldots, N_f$. As is clearly seen in $\mathscr{L}_{\mathsf{F}}$, the indices of these quantum numbers are mostly suppressed in our notation. The same applies to the indices referring to the spinor components of the fermionic fields.

The kinematics and self-interaction of the gauge fields are described by the Yang-Mills Lagrangian

$$\mathscr{L}_{\mathsf{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}, \quad \text{where } F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{3.3}$$

denotes the field strength tensor with the Yang Mills coupling parameter g. The gauge field is denoted by A^a_μ and the antisymmetric tensor f is defined by the Lie bracket of the infinitesimal generators

$$\left[T^a, T^b\right] =: i f^{abc} T^c. \tag{3.4}$$

The combination of the fermionic and the Yang-Mills Lagrangian describes the physical content of a non-abelian gauge theory; it is invariant under the class of non-abelian gauge transformations

$$\left.\begin{aligned} \psi(x) &\mapsto U(\boldsymbol{\omega}(x))\psi(x) \\ A_\mu(x) &\mapsto U(\boldsymbol{\omega}(x))^{-1} \left(A_\mu(x) - ig\partial_\mu\right) U(\boldsymbol{\omega}(x)) \end{aligned}\right\} \tag{3.5}$$

where the representation of the gauge group

$$U(\boldsymbol{\omega}(x)) = \exp\left(-ig\omega^a(x)T^a\right) \tag{3.6}$$

is parametrized by a set of local coordinate functions $\omega^a : \mathbb{R}^{3,1} \to \mathbb{R}$ with appropriate regularity.

However, the linearised field equations for the gauge field gives rise to a singular differential operator. Hence the gauge boson propagator is ill-defined and a perturbative approach is not possible. This problem can be resolved by exploiting the underlying ambiguity, which arises from the gauge invariance of the model, by adding additional terms to modify the field equations of the gauge fields. It is customary to use the covariant gauge fixing

$$\mathscr{L}_{\mathsf{GF}} = -\frac{1}{2\alpha} \left(\partial^\mu A^a_\mu\right) \left(\partial^\nu A^a_\nu\right). \tag{3.7}$$

As before, the summation over repeated Lorentz and adjoint indices is understood. The quadratic expression in the gauge field yields just a linear contribution to the field equations, therefore this gauge fixing is also called the linear covariant gauge. Following the notational conventions of [86], the gauge parameter is denoted by α.

The gauge fixing term $\mathscr{L}_{\mathsf{GF}}$ breaks the gauge invariance and modifies the gauge boson propagator. In order to ensure that this modification does not change the physical predictions of the model, another type of fields must be introduced. These fields are commonly referred to as ghosts as they are of non-physical nature in the sense that they only contribute as virtual interactions between physical fields. Given a gauge fixing, 't Hooft and Veltman [87] provided a diagrammatic prescription to construct these ghost fields. In addition to that, there is the classic prescription due to Faddeev and Popov [88]. In case of the non-abelian gauge theory and linear covariant gauge from above the prescription yields

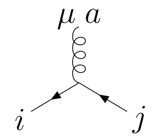
$$\mathscr{L}_{\mathsf{GH}} = -\left(\partial^{\mu}\bar{c}\right) D_{\mu} c. \tag{3.8}$$

This Lagrangian introduces the complex bosonic ghost field, where c and $\bar{c}$ denote the ghost and anti-ghost fields which obey Fermi statistics.

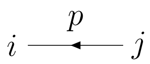
In the framework of perturbation theory, every monomial of the classical Lagrangian is translated into a vertex or an edge; which serve as building blocks to construct graphs in the sense of graph theory. In general, these graphs are mapped to integrals expressions which originate from a perturbative analysis of the fields; this mapping is commonly referred to as Feynman rules. Finally, textbook Quantum Field Theory provides a machinery to quantize the classical Lagrangian, resulting in a perturbation series which represents the sum over all graphs of similar external edges, called legs in the physical literature.

At this point, we like to remark that the specific form of the Lagrangian and hence the form of the Feynman rules depends on various conventions. For instance, ambiguities arise in the Feynman rules from the following operations: a rescaling of fields in the Lagrangian with different constants, choosing different signs or constants in front of each individual Lagrangian, or changing the definition of the covariant derivative. Beside of that, the specific form of the Feynman rules also depends on labelling conventions, such as directions of momenta attached to a vertex. To accommodate these ambiguities, we introduce a constant for each vertex and edge type in the definition of the Feynman rules. In a second step, we will derive restrictions for these constants by examining the renormalization conditions of this model. In this way, it is possible to identify the constants which are fixed by the requirement that the Lagrangian $\mathscr{L}$ is renormalizable and the constants which need to be fitted to experimental data.

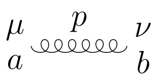
In this course, we define the Feynman rules of the model (3.1) by using the same index conventions as above and introducing the constants $c_1, c_2, c_3, c_{3g}, c_{4g}, \tilde{c}_1,$ and $\tilde{c}_2$. The momentum of an edge is denoted by p with the convention that the direction of the momentum always directs into the vertex; the momentum of a directed edge always points in the same direction.



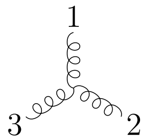
$$= c_1 g \gamma^\mu T^a_{ij} \tag{3.9}$$



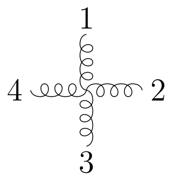
$$= c_2 \delta_{ij} \frac{\not{p} + m}{p^2 - m^2} \tag{3.10}$$



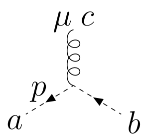
$$= c_3 \delta_{ab} \frac{1}{p^2} \left[g^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right] \tag{3.11}$$



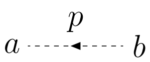
$$= c_{3g} g f_{a_1 a_2 a_3} \begin{bmatrix} g^{\mu_2\mu_3} (p_2 - p_3)^{\mu_1} \\ + g^{\mu_3\mu_1} (p_3 - p_1)^{\mu_2} \\ + g^{\mu_1\mu_2} (p_1 - p_2)^{\mu_3} \end{bmatrix} \tag{3.12}$$



$$= c_{4g} g^2 \begin{bmatrix} f_{a_1 a_2 b} f_{a_3 a_4 b} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right) \\ + f_{a_1 a_3 b} f_{a_4 a_2 b} \left(g^{\mu_1\mu_4} g^{\mu_2\mu_3} - g^{\mu_1\mu_2} g^{\mu_3\mu_4} \right) \\ + f_{a_1 a_4 b} f_{a_2 a_3 b} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_3} g^{\mu_2\mu_4} \right) \end{bmatrix} \tag{3.13}$$



$$= \tilde{c}_1 g f^{abc} p^\mu \tag{3.14}$$



$$= \tilde{c}_2 \delta_{ab} \frac{1}{p^2} \tag{3.15}$$

In order to conclude the discussion of the Feynman rules, we like to anticipate the outcome of a detailed analysis of Ward identities and discuss the implied restrictions for the introduced constants. A detailed discussion of the renormalization conditions can be found in paragraph 3.3.

Ward identities [89, 90, 91] imply that a Green's function with a longitudinally contracted gluon leg vanishes in the on-shell limit for every order in perturbation theory. At the tree-level, this condition relates the three-gluon vertex with the quark-gluon vertex

$$c_{3g} = i \frac{c_1 c_2}{c_3}, \tag{3.16}$$

which is readily deduced from the quark-quark-gluon-gluon amplitude. In a similar way, the four-gluon amplitude relates the four-gluon vertex to the three-gluon vertex

$$c_{4g} = c_3 c_{3g}^2 = -\frac{c_1^2 c_2^2}{c_3}. \tag{3.17}$$

Further, longitudinal gluon terms circulating in loops are required to cancel terms originating from ghost loops. This cancellation determines the ghost-gluon vertex up to an arbitrary constant c_{GH} in the ghost-gluon vertex provided that the ghost propagator is proportional to the inverse of c_{GH}. As a result, all these constants multiply to unity as a closed ghost loop consists of the same number of ghost edges and vertices.

$$\tilde{c}_1 = c_{3g} c_{\mathsf{GH}} = i\frac{c_1 c_2}{c_3} c_{\mathsf{GH}} \tag{3.18}$$

$$\tilde{c}_2 = \frac{c_3}{c_{\mathsf{GH}}} \tag{3.19}$$

On the level of the Lagrangian, the new constant c_{GH} can be interpreted as a rescaling of the ghost fields or as an adjustment in the constant in front of the ghost Lagrangian $\mathscr{L}_{\mathsf{GH}}$

$$\mathscr{L} = \mathscr{L}_{\mathsf{F}} + \mathscr{L}_{\mathsf{YM}} + \mathscr{L}_{\mathsf{GF}} + c_{\mathsf{GH}} \mathscr{L}_{\mathsf{GH}}. \tag{3.20}$$

As indicated before, this constant completely cancels in every Feynman graph and perturbative results are independent of this constant c_{GH}. In particular, the sign in front of the ghost Lagrangian has no physical significance for perturbative methods. This explains the varying sign conventions in the literature. Finally, beside the non-physical ghost constant c_{GH}, the remaining undetermined constants are $c_1, c_2,$ and c_3, which is similar to the situation of an Abelian gauge theory such as Quantum Electrodynamics.

3.2 Color factors

This paragraph gives a short account on color factors and how they are computed in perturbative calculations. Basically, we follow the course of [92, 62, 18].

The gauge group G is a Lie group and hence possesses an associated Lie algebra, which can be identified with the tangent space at the identity of G. As the gauge group is assumed to be finite dimensional, the Lie algebra possesses a basis of matrices T^a with $a \in \{1, \ldots, N_A\}$. The Trace $\mathrm{Tr}(T^a T^b)$

defines a symmetric bilinear form which is non-singular provided that the Lie algebra of G is semi-simple [92]. In this setting, there is a basis of orthonormal infinitesimal generators T^a in the Lie algebra such that

$$\mathrm{Tr}\left(T^a T^b\right) = T_F \delta^{ab} \tag{3.21}$$

for some constant T_F and δ denotes the Kronecker symbol. The commutator of the associated Lie algebra arises from the vector field commutator in the tangent space and defines the structure constants f^{abc} via

$$\left[T^a, T^b\right] =: i f^{abc} T^c. \tag{3.22}$$

As a gauge group is of finite dimension, it is possible to represent the infinitesimal generators T^a by matrices. Further, it is an easy exercise to show that the product $\delta_{ab} T^a T^b$ commutes with every infinitesimal generator. Provided that the matrix representation of the infinitesimal generators T^a is an irreducible representation, it is a consequence of Schur's lemma in its matrix form that the product must be proportional to the unit matrix

$$\delta_{ab}\left(T^a T^b\right)_{ij} =: C_F \delta_{ij} \tag{3.23}$$

with some constant C_F. In the course of classical Lie algebra theory [93], one shows that this object does not depend on the actual choice of the basis elements T^a and hence is called a Casimir invariant or a Casimir operator. With a slight abuse of language, we will call the constant C_F Casimir operator.

Defining the adjoint representation by

$$\left(T_A^a\right)_{bc} := -i f^{abc}, \tag{3.24}$$

the same reasoning applies and gives rise to another Casimir operator, which we denote by

$$f^{acd} f^{bcd} =: C_A \delta^{ab}. \tag{3.25}$$

The structure constants f^{abc} represent a fully antisymmetric tensor and analogously there is a fully symmetric tensor.

$$d^{abc} := \frac{1}{3!} \sum_{\pi \in S_3} \mathrm{Tr}\left(T^{\pi(a)} T^{\pi(b)} T^{\pi(b)}\right) = \frac{1}{2} \mathrm{Tr}\left(T^a \left\{T^b, T^c\right\}\right), \tag{3.26}$$

where S_3 denotes the symmetric group on a set of cardinality three — the summation runs over all permutation of the three indices. Elementary properties of these definitions are the Jakobi identities

1. $f_{abr}d_{cdr} + f_{acr}d_{dbr} + f_{adr}d_{bcr} = 0$

2. $f_{abr}f_{cdr} + f_{acr}f_{dbr} + f_{adr}f_{bcr} = 0.$

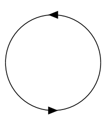
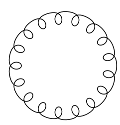
Both identities are proven by replacing the antisymmetric tensors by the commutator (3.22) and making use of the orthogonality (3.21) and the definition of the symmetric tensor (3.26). Then, for instance, the first identity follows from a short calculation:

$$\sum_{\sigma\in A_3} \mathrm{Tr}\left(\left[T^a, T^{\sigma(b)}\right]\left\{T^{\sigma(c)}, T^{\sigma(d)}\right\}\right)$$
$$= \sum_{\sigma\in A_3} \mathrm{Tr}\left(T^a T^{\sigma(b)}\left\{T^{\sigma(c)}, T^{\sigma(d)}\right\} - T^a\left\{T^{\sigma(c)}, T^{\sigma(d)}\right\} T^{\sigma(b)}\right) = 0 \quad (3.27)$$

where we made use of the cyclicity of the trace, $A_3 \subset S_3$ denotes the set of cyclic permutations, and the sum eventually vanishes as (b, c, d) is a cyclic permutation of (c, d, b).

As is clearly seen from the Feynman rules (3.9-3.15), every edge and every vertex of a Feynman graph contributes a Kronecker symbol, basis element of the Lie algebra T^a, or an antisymmetric tensor f^{abc} as a factor, which are commonly referred to as color factors.

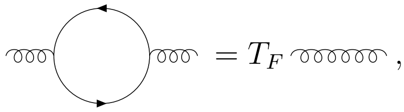
In the following, we introduce some graphical notion of the group-theoretic identities from above; this notion is very convenient to compute the color factors of a Feynman graph. In this paragraph, the edges and vertices of a Feynman graph are meant to represent solely the corresponding color factor. For instance an fermionic propagator just denotes the Kronecker symbol over the color indices of this propagator and the closed graphs

 $= N_c$ and  $= N_A$ (3.28)

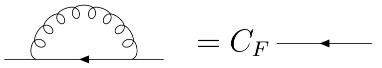
represent self-contracted Kronecker symbols ranging over the fermionic color index $j = 1, \ldots, N_c$ and the adjoint index $a = 1, \ldots, N_A$, respectively. Moreover, if there were different species of fermions involved, then the fermionic loop would also contribute a factor N_f.

Translating the orthogonality relation (3.21) and the Casimir invariants

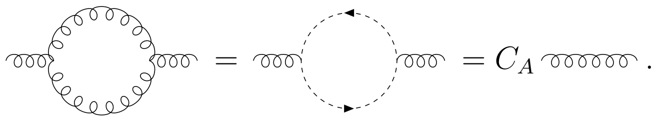
(3.23) and (3.25) into this graphical notion yields



(3.29)



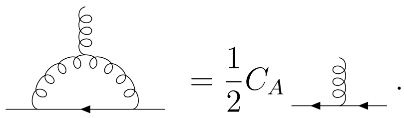
and (3.30)



(3.31)

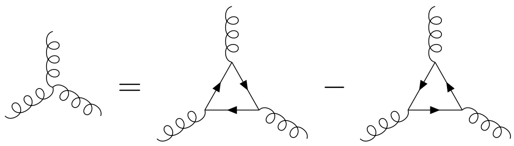
The last equation demonstrates that ghost edges (and vertices) can simply replaced by gluons as they contribute the same color factors.

Another useful identity follows from the antisymmetry of the color factor of the three-gluon vertex

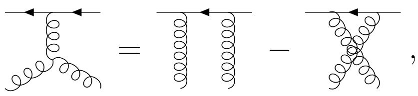


(3.32)

More complicated graphs typically contain more three-gluon vertices; to compute their color factors, the following reduction identities are useful

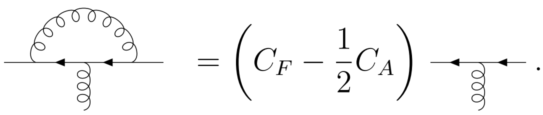


and (3.33)



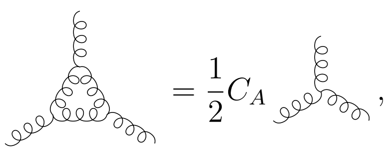
(3.34)

where we implicitly assumed an anticlockwise orientation of the three-gluon vertices. For example, the Gluon crossing relation (3.34) can be used to rewrite the one-loop vertex graph in terms of the preceding examples (3.30) and (3.32)



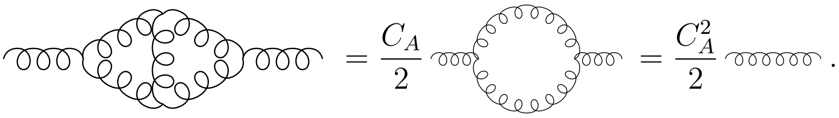
(3.35)

Another non-trivial example is the purely gluonic one-loop three gluon contribution



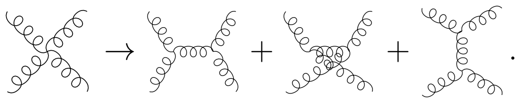
(3.36)

where we used the identity (3.33) to replace one three-gluon vertex by a fermionic loop, which is connect to the other three-gluon vertices such that (3.34) can be applied to remove a second three-gluon vertex which allows the identification of (3.32) subgraph yielding the correct color factor. In general, the identification of known subgraphs simplifies the computation of color factors — the preceding one-loop graph (3.36) allows us to derive the two-loop gluon propagator contribution



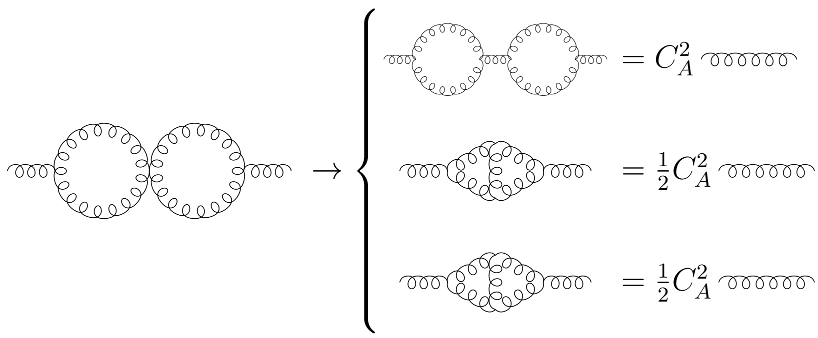
(3.37)

However, it is reasonable to remark that more care is required in replacements of four-gluon vertices. As seen for the Feynman rules of the four gluon vertex (3.13), the individual color factors are intertwined with different Lorentz tensors which influence the result of the momentum dependent factor of a generic Feynman graph. Instead of thinking of a four-gluon term as an individual vertex, we rather have to treat it as a sum of three different expressions representing the individual terms from the Feynman rules. We like to illustrate this in the graphical identity



(3.38)

This issue must be taken into account in the color factor derivation of the following two-loop purely gluonic graph.



(3.39)

Also at higher loop orders, color factors become more complicated and further Casimir invariants are necessary. For a general discussion of these higher Casimir invariants in the context of color factors the reader is referred to [18].

In the physical context of Quantum Chromodynamics, the special case $G = SU(N_c)$ is of particular interest and we will need the special values of the Casimir invariants in this particular gauge group to compare our results of paragraph 3.6 to the literature.

It is well known that the Lie algebra associated to the Lie group $SU(N_c)$ ($N_c \times N_c$ unitary matrices with determinant 1) is spanned by set of the traceless skew-Hermitian $N_c \times N_c$ matrices. However, we will stick to the standard convention of the physics literature and rewrite the Lie algebra exponential such that the infinitesimal generators of the Lie algebra are given by the set of traceless Hermitian $N_c \times N_c$ matrices

$$\mathrm{Tr}\,(T^a) = 0 \quad \text{and} \quad (T^a)^\dagger = T^a, \tag{3.40}$$

which span a $\mathbb{R}$-vector space of dimension $N_A = N_c^2 - 1$.

Due to dimensional reasons, the set $\{I, T^a : a \in N_A\}$ is a basis of the $\mathbb{R}$-vector space of the $N_c \times N_c$ Hermitian matrices. As the anticommutator of two Hermitian matrices is Hermitian, the anticommutator can be expanded in this basis which results in

$$\left\{T^a, T^b\right\} = \frac{2T_F}{N_c}\delta^{ab} I + \frac{2}{T_F} d^{abc} T^c, \tag{3.41}$$

where the coefficients in front of the basis elements were derived by using the trace to define projection operators in combination with (3.40) and (3.21). As a simple conclusion

$$T^a T^b = \frac{1}{2}\left\{T^a, T^b\right\} + \frac{1}{2}\left[T^a, T^b\right] = \frac{T_F}{N_c}\delta^{ab} I + \frac{1}{T_F} d^{abc} T^c + \frac{i}{2} f^{abc} T^c. \tag{3.42}$$

Another consequence of the vanishing trace of the infinitesimal generators T^a is that the symmetric tensor vanishes after an index contraction $d^{abb} = 0$.

In [92], Cvitanović provided an algorithm for the computation of the $SU(N_c)$ Casimir invariants and color factors. The procedure basically relies on the reduction of the number of infinitesimal generators by utilizing the $SU(N_c)$-specific identity

$$\left(T^a\right)_{ij} \left(T^a\right)_{kl} = T_F \left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right). \tag{3.43}$$

For another concise proof of this formula the reader is referred to [94]. This identity allows us to derive the $SU(N_A)$-specific values of the Casimir operators in terms of the dimensions N_c and N_A:

$$C_F = T_F \frac{N_c^2 - 1}{N_c} \tag{3.44}$$

$$C_A = 2T_F N_c \tag{3.45}$$

$$d^{acd} d^{bcd} = \frac{T_F^3}{2} \frac{N_c^2 - 4}{N_c} \delta^{ab}. \tag{3.46}$$

These formulas are useful to compare our results of paragraph 3.6 to the literature which explicitly presumes $SU(N_c)$ as the gauge group of the non-abelian gauge theory.

Finally, Quantum Chromodynamics refers to the gauge group $SU(3)$. Hence $N_c = 3$ and setting $T_F = 1/2$ in convenience with the conventions of [18] implies

$$C_F = \frac{4}{3} \quad \text{and} \quad C_A = 3. \tag{3.47}$$

3.3 Renormalization conditions

In Quantum Field Theory, perturbation series contain Feynman graphs with loops, which might be mapped to divergent integrals by the Feynman rules (3.9-3.15). In order to obtain finite results, all fields and parameters of the model (3.1) must be renormalized. In the sequel, we will discuss appropriate conditions for such a renormalization process.

First, consider the pure Yang-Mill part $\mathscr{L}_{\mathsf{YM}}$ of the model. The Lagrangian consists of a sum of monomials. Monomials quadratic in the fields contribute to the propagator (3.11); the other monomials are associated to exactly one interaction vertex within the graphical notion of the Feynman rules. In the course of perturbation theory, such an interaction vertex receives further quantum corrections from Feynman graphs of the same external legs. In order to deal with divergences of these Feynman graphs, the Lagrangian $\mathscr{L}_{\mathsf{YM}}$ is replaced by its renormalized equivalent

$$\mathscr{L}_{\mathsf{R;YM}} = \mathscr{L}_{\mathsf{YM}} + \mathscr{L}_{\mathsf{CT}}, \tag{3.48}$$

where the last term is constructed in such a way that it contributes appropriate counterterms, which cancel all the divergences of the Feynman graphs originating from $\mathscr{L}_{\mathsf{YM}}$.

In order to understand the structure of the renormalized Lagrangian $\mathscr{L}_{\mathsf{R;YM}}$, we have to discuss some details of the renormalization process.

A first assumption underlying the renormalization procedure is that the kinematics of the sum of all Feynman graphs at a given order and a particular external leg structure is compatible with the corresponding tree-level expression. In other words, once all Feynman graphs at a particular order are taken into account, the remaining divergences must be contained in an expression proportional to the Lorentz structures which are provided by the tree-level propagators and vertices from the Feynman rules (3.9-3.15). Then, we assign a Z-factor to each of the tree-level propagators and vertices and all divergences can be absorbed by rescaling the tree-level expressions by these Z-factors. Since different Z-factors absorb divergences from different classes of Feynman graphs, there are a priori no relations between these Z-factors. However, the Z-factor associated to a particular interaction vertex also affects the renormalization of the coupling parameter which is assigned to this vertex by the Feynman rules. As a result, each vertex and hence each monomial of the renormalized Lagrangian a priori carries an distinct renormalized coupling parameter; even if the unrenormalized coupling parameters used to be the same. Following the conventions of [86] for the Z-factors and coupling parameter, we set

$$\mathscr{L}_{\mathsf{R;YM}} = -\frac{1}{4} Z_3 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2} Z_1 g_1 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \cdot \left(A^\mu \times A^\nu\right) - \frac{1}{4} Z_4 g_4^2 \left(A_\mu \times A_\nu\right) \cdot \left(A^\mu \times A^\nu\right), \quad (3.49)$$

where the distinct renormalized couplings are denoted by g_1 for the three-gluon vertex and g_4 for the four-gluon vertex; further we have used the following abbreviations for the summation over the adjoint indices

$$A_\mu \cdot A_\nu := A_\mu^a A_\nu^a \quad \text{and} \quad (A_\mu \times A_\nu)^a := f^{abc} A_\mu^b A_\nu^c. \quad (3.50)$$

In a similar way, we set up the conventions for the gauge fixing, the fermionic, and ghost parts of the non-abelian gauge theory.

$$\mathscr{L}_{\mathsf{R;GF}} = Z_5 \frac{1}{2\alpha} \left(\partial_\mu A^\mu\right) \cdot \left(\partial_\nu A^\nu\right) \quad (3.51)$$

$$\mathscr{L}_{\mathsf{R;F}} = i Z_2 \bar{\psi} \partial\!\!\!/ \psi + g_F Z_F \bar{\psi} A\!\!\!/ \cdot T \psi \quad (3.52)$$

$$\mathscr{L}_{\mathsf{R;GH}} = -\tilde{Z}_3 \partial_\mu \bar{c} \cdot \partial_\mu c - \tilde{Z}_1 \tilde{g}_1 \left(\partial_\mu \bar{c}\right) \cdot \left(A^\mu \times c\right) \quad (3.53)$$

Eventually, the renormalized version of the non-abelian model (3.1) is defined as

$$\mathscr{L}_{\mathsf{R}} = \mathscr{L}_{\mathsf{R;F}} + \mathscr{L}_{\mathsf{R;YM}} + \mathscr{L}_{\mathsf{R;GF}} + \mathscr{L}_{\mathsf{R;GH}}. \quad (3.54)$$

As another assumption, the renormalized Lagrangian $\mathscr{L}_{\mathsf{R}}$ is required to be equal to the original Lagrangian $\mathscr{L}(\psi_B, A_B, g_B, \alpha_B)$ with all fields and parameters replaced by bare equivalents. This basically translates the proceeding assumption about the kinematics of divergences from Feynman graphs to the Lagrangian level such that the renormalization process can be interpreted as a rescaling of the fields in the Lagrangian. A comparison of the quadratic monomials in the bare and the renormalized formulation of the renormalized Lagrangian implies the following rescalings of the fields

$$A_B^\mu = Z_3^{1/2} A^\mu, \quad \psi_B = Z_2^{1/2}\psi, \quad \eta_B = \tilde{Z}_3^{1/2}\eta. \tag{3.55}$$

These relations define renormalization conditions for the two-point Green's function of the perturbation series

$$G(g, \alpha) = Z G_B(g_B, \alpha_B), \tag{3.56}$$

where the G_B denotes the one-particle irreducible bare Green's function, which depends on the bare parameters, G is the one-particle irreducible renormalized Green's function given in terms of the renormalized parameters, and Z denotes the Z-factor from the rescaling of the corresponding fields (3.55). Extending this convention to one-particle irreducible n-point Green's functions where the associated Z-factor is determined from the associated monomial of the renormalized Lagrangian $\mathscr{L}_{\mathsf{R}}$ implies the renormalization conditions of the coupling parameters:

$$g_B = Z_1 Z_3^{-3/2} g_1, \qquad g_B = Z_4^{1/2} Z_3^{-1} g_4, \tag{3.57}$$

$$g_B = Z_F Z_1^{-1} Z_3^{-1/2} g_F, \qquad g_B = \tilde{Z}_1 \tilde{Z}_3^{-1} Z_3^{-1/2} \tilde{g}_1. \tag{3.58}$$

The renormalized couplings might differ as the various counterterms are determined by different classes of Feynman graphs which are unrelated so far. On the other hand, the bare formulation of the model only has a single coupling parameter, so it is natural to require that this property is maintained during the renormalization process. Hence, we require the equality of the renormalized couplings

$$g_1 = g_4 = g_F = \tilde{g}_1. \tag{3.59}$$

At this point, it is worth to emphasize that the non-abelian model could be renormalized without this requirement at the cost of renormalizing each coupling individually. As a result of these differing couplings, (residual) gauge invariance is broken by quantum corrections. Some authors rather prefer to reverse the logic of this line of thoughts and assert that renormalizability is a conclusion of gauge invariance [95, 96, 97].

The equality of the renormalized couplings in combination with the renormalization conditions for the couplings impose the following restrictions on the Z-factors:

$$\frac{Z_1}{Z_3} = \frac{Z_F}{Z_2} = \frac{\tilde{Z}_1}{\tilde{Z}_3} \quad \text{and} \quad \frac{Z_4}{Z_3} = \left(\frac{Z_1}{Z_3}\right)^2 . \tag{3.60}$$

These are the renowned Slavnov-Taylor identities, they originated from a series of papers by 't Hooft [98], 't Hooft and Veltman [99], and Taylor [89], where it was shown that the non-abelian Green's functions obey some kind of Ward identity, which restricts the structure of divergences in such a way that the Slavnov-Taylor identities can be realized.The work of Slavnov [95] is acknowledged for a concise derivation of the Slavnov-Taylor identities from the invariance of the path integral under infinitesimal gauge transformations and moreover for the first transition from non-abelian Ward identities to the notion of Z-factors — an alternative approach which recognizes the necessity of defining a renormalization scheme was given in [100]. In [86], Celmaster and Gonsalves clarified the significance of the definition of a renormalization scheme for the correctness of the Slavnov-Taylor identities. Given some kind of regularization, the divergences of Feynman graphs are mapped to poles in the regulator and the Z-factors must be defined such that all poles are absorbed. Non-abelian Ward identities provide relations between bare Green's functions which translate into relations for the pole terms in the regulator. However, the Z-factors are not entirely determined by the poles. There is a freedom to include arbitrary finite regulator terms in the definition of the Z-factors. It is the purpose of a renormalization scheme to fix this freedom. In conclusion, if the non-abelian model is meant to be renormalized with a single renormalized coupling, then the Slavnov-Taylor identities must be understood as a renormalization condition which restricts the freedom in defining a renormalization scheme. In other words, once one has defined renormalization conditions for the Z-factors of all propagators and and exactly one vertex function, then the Slavnov-Taylor identities determine the renormalization conditions for all remaining divergent Green's functions and hence a complete renormalization scheme.

3.4 Renormalization group

The preceding paragraph introduced Z-factors which absorb divergences and allow to compute renormalized Green's functions. A discussion of the scaling behavior of these Z-factors reveals that a renormalized Green's function is adequately described by its anomalous dimension and its beta function via renormalization group techniques.

Recall that the explicit definition of a counterterm presumes some kind of method to regularize divergent integrals. For explicit computations, we are solely using dimensional regularization in this thesis. The following paragraph basically presumes this kind of regularization, although the techniques of this paragraph are actually not restricted to this particular method of regularization. The Z-factors are defined in such a way that they cancel the poles of the divergent Feynman graphs. However, the external legs of a generic Feynman graph provide additional kinematic data (such as position or momentum variables) which are dimensional physical quantities and the assigned regularized Feynman integral yields a result which depends on these dimensional parameters. As a result, a counterterm and the corresponding Z-factors must mimic the dimensional parameters of the corresponding propagator or vertex graphs. On the other hand, Z-factors are supposed to be independent of the kinematic data from the external momenta. Therefore, it is necessary to introduce at least one dimensional parameter and replace the kinematic variables of the various vertices and propagators by this or these new parameters. Further, we like to define the Z-factors in convenience with the Slavnov-Taylor identities (3.60) and these are most easily respected by restricting to a single dimensional parameter which is denoted μ and commonly referred to as renormalization scale or renormalization point. In the context of dimensional regularization, some authors rather prefer to argue that one needs to define a dimensional-independent coupling parameter and hence introduces a dimensional parameter μ, which eventually leads to similar conclusions.

Clearly, bare Green's functions are meant to be independent of the renormalization point μ, but Z-factors and hence renormalized Green's functions depend on the renormalization point μ. This circumstance is the basic premise underlying any kind of renormalization group analysis. As a preparation, we rewrite the coupling in terms of the loop-counting parameter $a := {}^{g^2}\!/_{4\pi}$, which obeys the renormalization condition

$$a_0 = Z_a a, \quad \text{with} \quad Z_a = \frac{Z_1^2}{Z_3^3} = \frac{Z_F^2}{Z_2^2 Z_3} = \frac{\tilde{Z}_1^2}{\tilde{Z}_3^2 Z_3} = \frac{Z_4}{Z_3^2}. \tag{3.61}$$

Also recall the renormalization conditions for the gauge parameter

$$\alpha_0 = Z_\alpha \alpha, \quad \text{wtih} \quad Z_\alpha = \frac{Z_3}{Z_5}, \tag{3.62}$$

where $Z_5 = 1$ due to a Ward identity for the gluon propagator in straight analogy to the abelian case. Now, enquire the change of a renormalized Green's function under a rescaling the renormalization point μ and use the fact that the bare Green's function G_B is renormalization point independent. This yields the famous Callan-Symanzik equation

$$\left(-\mu^2\partial_{\mu^2} + \beta\, a\partial_a + \delta\, \alpha\partial_\alpha + \gamma\right) G = 0, \tag{3.63}$$

where we have introduced the following set of functions

$$\beta = \frac{\mu^2}{a}\frac{da}{d\mu^2}, \quad \gamma = -\mu^2\frac{d\ln Z}{d\mu^2}, \quad \text{and} \quad \delta = \frac{\mu^2}{\alpha}\frac{d\alpha}{d\mu^2}. \tag{3.64}$$

While the beta and delta function are universal quantities which describe the scaling of the coupling and gauge parameter and hence a common property of all Green's function, the γ function is called anomalous dimension and explicitly depends on the actual Green's function G. To denote this dependence, we assign an index $r \in \{1, 2, 3, 4, F\}$ to the anomalous dimension

$$\gamma_r = -\mu^2\frac{d\ln Z_r}{d\mu^2}. \tag{3.65}$$

and add a tilde above the γ to refer to the Green's functions which involve ghosts in convenience with our conventions for the Z-factors in the preceding paragraph. In a way similar to the Callan-Symanzik equation, one exploits the invariance of the bare coupling and gauge parameter and derives the so-called renormalization group equations

$$\mu^2\frac{da}{d\mu^2} = -a\mu^2\frac{d\ln Z_a}{d\mu^2} \quad \text{and} \quad \mu^2\frac{d\alpha}{d\mu^2} = -\alpha\mu^2\frac{d\ln Z_\alpha}{d\mu^2}. \tag{3.66}$$

These equations relate the universal functions to the Z-factors of the coupling and gauge parameter renormalization, which we also related to the anomalous dimensions. As a result, the beta and delta functions can be rewritten in terms of the anomalous dimensions.

$$\beta = -\frac{\mu^2}{a}\frac{d\ln Z_a}{d\mu^2} = -\frac{\mu^2}{a}\frac{d}{d\mu^2}\left[2\ln Z_F - 2\ln Z_2 - \ln Z_3\right] = 2\gamma_F - 2\gamma_2 - \gamma_3, \tag{3.67}$$

$$\delta = -\frac{\mu^2}{\alpha}\frac{d\alpha}{d\mu^2} = -\mu^2\frac{d\ln Z_3}{d\mu^2} = \gamma_3 \tag{3.68}$$

Note that the beta and delta function are renormalization scheme dependent as the Z-factors are. Further, the beta function can be expressed in several equivalent ways

$$\beta = 2\gamma_1 - 3\gamma_3 = 2\gamma_F - 2\gamma_2 - \gamma_3 = 2\tilde{\gamma}_1 - 2\tilde{\gamma}_3 - \gamma_3 = \gamma_4 - 2\gamma_3 \qquad (3.69)$$

by implicitly using the Slavnov-Taylor identities (3.61) in the renormalization conditions for the coupling parameter a. Once a relation between the universal functions and the anomalous dimensions has been established, the explicit computation of the Z-factors is not necessary anymore if one has a method to compute the renormalized Green's function G. From their coefficients, the anomalous dimensions and hence the beta function is determined order by order from the Callan-Symanzik equation (3.63) as has been demonstrated for the self-energy of the electron in paragraph 2.5.2. The next paragraph provides a method to directly compute the renormalized Green's functions without causing additional computational expenses.

3.5 Hopf-algebraic renormalization of QCD

The Hopf-algebraic approach to renormalization [9, 67, 26, 51, 71, 101] allows us to renormalize individual Feynman graphs in a similar fashion as the BPHZ procedure [59, 60, 61]. In addition to that, there is a combinatorial description for the coproduct of Green's functions [68, 69, 101] which provides a closed formula for the computation of renormalized Green's functions. This paragraph gives an overview of the basic formulas which are essential for the Hopf-algebraic renormalization of Quantum Chromodynamics.

Recall that the set of Feynman graphs of Quantum Chromodynamics generates a free commutative algebra which possesses a Hopf algebra structure [68, 71]. On this Hopf algebra, the Feynman rules (3.9-3.15) define the map ϕ into the target space of regularized integrals. Then the renormalized Feynman rules, which map a (divergent) Feynman graph Γ onto its renormalized value, can be defined as

$$\phi_{\mathrm{R}}(\Gamma) = m \circ \left(S_T^{\phi} \otimes \phi\right) \circ \Delta(\Gamma), \qquad (3.70)$$

where m and Δ respectively denote the product and the coproduct of the underlying Hopf algebra. The mapping S_T^{ϕ} constructs appropriate counterterms for a given renormalization scheme T and the basic ingredients for its definition shall be discussed now.

A renormalization scheme T is a linear operator which acts on the target space of the Feynman rules with the following constrains. Recall that the Feynman rules ϕ map a Feynman graph Γ to a regularized expression, which factors into a Laurent series in the regulator and a factor which contains the kinematic data of the integral. A renormalization scheme T must maintain the poles of the Laurent series (e.g. minimal subtraction corresponds to the projection onto the pole terms in the regulator) and evaluates the kinematic data at the renormalization scale μ. The evaluation of the kinematic data at the renormalization scale is necessary to describe kinematic renormalization schemes. On the other hand it is a convenient way to introduce the renormalization scale and provides an alternative to the popular assertion that in the course of dimensional regularization the dimensionality of the coupling parameter enforces the introduction of a new dimensional scale. Finally, the requirement, that the renormalized Feynman rules ϕ_{R} shall yield finite expressions, adds a crucial restriction on the renormalization scheme T. It has to satisfy the Rota-Baxter identity

$$T(\Gamma_1)T(\Gamma_2) = -T(\Gamma_1\,\Gamma_2) + T\left(T(\Gamma_1)\Gamma_2 + \Gamma_1 T(\Gamma_2)\right). \tag{3.71}$$

Given a renormalization scheme operator T, counterterms are constructed with the map

$$S_T^{\phi}(\Gamma) = -(T\circ\phi)\,(\Gamma) - \sum_{\mathbb{1}\triangleleft\gamma\triangleleft\Gamma} S_T^{\phi}\,(\gamma)\,(T\circ\phi)\,(\Gamma/\gamma)\,, \tag{3.72}$$

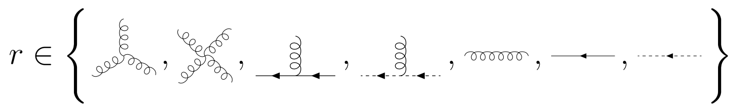
where Γ/γ denotes the graph which is obtained by shrinking every component of γ in Γ to a point and the sum includes all products of disjoint one-particle irreducible divergent subgraphs γ of the graph Γ except for the empty subgraph $\mathbb{1}$ and the full subgraph Γ itself. On the right-hand side of the equation the arguments of S_T^{ϕ} are Feynman graphs with less subdivergences then the left-hand side; further the sum vanishes for graphs Γ which are free of subdivergences, therefore the equation can be understood as an iterative definition of the map S_T^{ϕ}.

From the combinatorial point of view, the Hopf-algebraic formula for the renormalized Feynman rules (3.70) and (3.72) is of a similar complexity as Zimmermann's forest formula in the BPHZ prescription: the renormalization of a single Feynman graph requires to sum over all its products of disjoint subdivergences and to construct counterterms for each subdivergence in an iterative manner. However, the advantage of the Hopf-algebraic description becomes eminent in the discussion of Green's functions. Instead of solving the iteration of subdivergences for each Feynman graph individually, it is possible to provide a prescription which allows the renormalization of entire Green's functions without enquiring the subdivergences of individual Feynman graphs.

A combinatorial Green's function is a formal series in the coupling parameter a with coefficient in the Hopf algebra of Feynman graphs

$$X^r = \mathbb{1} \pm \sum_{\mathrm{res}(\Gamma)=r} a^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)}. \tag{3.73}$$

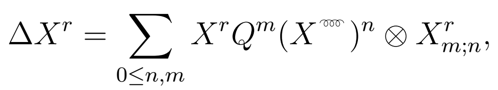
Here, the index r denotes a particular external leg type



(3.74)

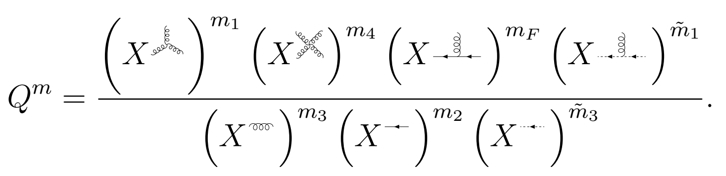
and the sum is over all one-particle irreducible Feynman graphs of the external leg structure r. The symmetry factor of a Feynman graph Γ is denoted by $\mathrm{sym}(\Gamma)$ and $|\Gamma|$ is the number of loops of the graph Γ. Further, the plus sign is used for all combinatorial Green's functions with an external leg structure of vertex type and the minus sign applies to all propagator types.

The coproduct of a combinatorial Green's function for Quantum Chromodynamics in the linear covariant gauge is



(3.75)

where the cograph elements $X^r_{m;n}$ consist of all graphs of X^r which obey the constrains encoded in the indices n and m: the index n counts the power of the gauge parameter α and $m = (m_1, m_4, m_F, \tilde{m}_1, m_3, m_2, \tilde{m}_3)$ is a multi-index whose entries are defined by means of a grading of H with respect to the vertex types and edge types as introduced in [73] and [101, 102]. For example, the entry m_1 equals the number of three-gluon vertices of the cograph minus its number of connected components with three external gluon legs. The factors on the left-hand side of the coproduct are referred to as subdivergences and the invariant charge Q raised by the multi-index m is defined as



(3.76)

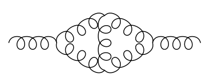
The structure of subdivergences in the coproduct formula can be qualitatively understood from the concept of graph insertions. In a Feynman graph, each vertex and each propagator constitutes an insertion place for Feynman graphs with a matching external leg structure. Further the actual set of Feynman graphs which can be inserted into an insertion place of type r is exactly given by the respective Green's function X^r. In the QCD coproduct formula (3.75), the subdivergences on the left-hand side of the tensor product exactly represent all the (products of) Feynman graphs which can be inserted into the corresponding cographs. Notice that the different treatment of vertices and propagators in (3.76) is due to the fact that an one-particle irreducible Green's function is constructed from one-particle irreducible vertex Green's functions and connected propagator Green's functions.

A combinatorial proof of this intuitive statement is based on the fact that a combinatorial Green's function of external leg structure r can be constructed from a set of skeleton graphs (Feynman graphs of the external leg structure r which have no subdivergences) and considering insertions of subdivergences in all possible ways. Kreimer and Yeats [68, 69, 25] provided such a construction by defining insertion operators B_+^γ where certain sums of these operators commute with the coproduct in a controlled way and fulfil the Hochschild-1-cocycle property

$$\Delta \circ B_+ = \mathbb{1} \otimes B_+ + (B_+ \otimes \mathbb{1}) \circ \Delta. \tag{3.77}$$

This eventually allows for an inductive proof of the coproduct formula (3.75).

It should be remarked that the QCD coproduct formula (3.75) slightly differs from the formula proposed in [51]; the concept of the parallel invariant charge is replaced by the second renormalization factor X^{ooo}. This is due to the fact that in Quantum Chromodynamics the power of the gauge parameter is not bounded by the power of the coupling parameter. For example, the following two-loop graph yields a term of third order in the coupling constant.


(3.78)

However, the proposed interpretation remains the same: every renormalized parameter induces a factor in the left tensor component of the coproduct. Further, the factors are composed of combinatorial Green's functions whose ratio exactly matches the ratio of the renormalization constants in the parameter renormalization, i.e. (3.61) and (3.62). With this interpretation, the coproduct formula basically demonstrates that the structure of subdivergences of the Green's functions is compatible with the concept of wave function and parameter renormalization as described by the introduction of Z-factors in paragraph 3.3.

Once one adapts to the interpretation that the invariant charge describes the renormalization of the coupling parameter, the question naturally arises how the renormalization conditions of the coupling parameter interfere with the invariant charge Q. The renormalization conditions are encoded in the definition of the renormalization scheme T. Notice that in the formula for the renormalized Feynman rules (3.70) and (3.72), the renormalization scheme T acts on the left-hand side of the coproduct. Hence, renormalization conditions as the Slavnov-Taylor identities can be implemented via equivalence relations on left-hand tensor component of the coproduct which effectively turns the underlying coalgebra into a comodule. The compatibility of the Slavnov-Taylor identities and the equivalence relations on the Hopf algebra of QCD Feynman graphs has been extensively discussed in [71]. Further, we refer to paragraph 3.3 where it was argued that a renormalization scheme T is entirely determined by providing renormalization conditions for all propagator Green's functions and a single vertex Green's function. Therefore the invariant charge in the coproduct formula Q^m can be effectively rewritten in terms of the propagator Green's functions and a single vertex Green's function. This notion will be very convenient for the renormalization of the non-abelian gauge theory in a class momentum subtraction schemes in the next paragraph.

3.6 $\widetilde{\text{MOM}}$q and $\widetilde{\text{MOM}}$h renormalization group functions

This paragraph concludes our discussion of the Hopf-algebraic renormalization of Quantum Chromodynamics and summarizes the renormalization group functions which we have derived in momentum subtraction schemes at an asymmetric point.

Whereas minimal subtraction type schemes are designed to minimize the number of terms in the Z-factors, momentum subtraction schemes impose physical boundary conditions on Green's functions at some kind of kinematic configurations. For this reason, momentum subtraction schemes are referred to as physical renormalization schemes. For the Green's functions of propagator type, this kind of boundary conditions usually reads that all loop corrections of this Green's function vanish if its external momentum is evaluated at the kinematic renormalization point μ. The term momentum subtraction scheme applies to all renormalization schemes which provide suchlike boundary conditions for some kind of momentum configuration of the external legs. The original motivation to study these kind of renormalization schemes was for the hope that the perturbative expansion around a coupling parameter which is renormalized in a physical scheme might improve the convergence behavior of the perturbation series [86]. From a modern perspective, the physical boundary conditions of a momentum subtraction scheme can be realized in lattice simulations and hence allows comparison to non-perturbative studies for instance see [103, 104] and references therein.

In order to specify a complete renormalization scheme, it is necessary to provide another boundary condition for one of the vertex Green's functions. Here, we will restrict ourselves to the case of vertices with three external legs and choose one external leg whose momentum is evaluated at zero. As a result, the vertex Green's function depends only on a single momentum scale and can be renormalized in the same way as the propagator Green's function which also reduces the computational complexity of the vertex Green's functions. This kinematic configuration is referred to as a renormalization condition at an asymmetric point and was introduced by Braaten and Leveille in [105].

In the scope of this thesis, two different asymmetric momentum subtraction schemes are analysed. Firstly, the $\widetilde{\text{MOM}}$q scheme is defined by using momentum subtraction at the renormalization point μ for all propagator Green's functions and in addition to that also for the quark-gluon vertex where the momentum of the external gluon is nullified. These renormalization conditions specify the renormalization of the coupling parameter (3.61) and, by usage of the Slavnov-Taylor identities, define the Z-factors for the other vertex type Green's functions. Secondly, the coupling parameter renormalization is determined by the ghost-gluon vertex instead of the quark-gluon vertex and again the momentum of the external gluon is set to zero. This renormalization scheme is referred to as the $\widetilde{\text{MOM}}$h scheme.

On the level of the Hopf algebra, these conditions are implemented by dividing out equivalence classes defined by an ideal which is basically defined by the Slavnov-Taylor identities. This consideration effectively replaces the invariant charge Q depending on the $\widetilde{\text{MOM}}$ scheme by

$$Q_q = \frac{\left(X^{\bar{q}gq}\right)^2}{\left(X^{\bar{q}q}\right)^2 X^{gg}} \quad \text{and} \quad Q_h = \frac{\left(X^{\bar{c}gc}\right)^2}{\left(X^{\bar{c}c}\right)^2 X^{gg}}. \tag{3.79}$$

In the coproduct formula (3.75), the multi-index degenerates to a single index which counts the number of loops of the cographs such that

$$\Delta X^r = \sum_{0 \leq n,m} X^r Q_\cdot^m (X^{\text{gluon}})^n \otimes X^r_{m;n}, \tag{3.80}$$

where $Q_\cdot$ is replaced by Q_q for the $\widetilde{\text{MOMq}}$ or by Q_h for the $\widetilde{\text{MOMh}}$ scheme. With this kind of notion, the renormalization scheme T corresponds to a simple evaluation of the external momentum of the regularized results at the renormalization point μ and the Laurent series in the regulator remains unchanged.

The combination of this closed formula for the coproduct and the formulas for the renormalized Feynman rules (3.70) and the counterterm (3.72) imply closed formulas for the renormalized Green's function at any order in perturbation series. Here, we like to demonstrate this computation at first and second loop order in the Landau gauge $\alpha = 0$. In this gauge, the coproduct formula simplifies to

$$\Delta X^r = \sum_{0 \leq m} X^r Q_\cdot^m \otimes X^r_m. \tag{3.81}$$

Note that this determines the coproduct of the invariance charge

$$\Delta Q = Q \otimes Q. \tag{3.82}$$

In the following, the n-loop coefficient of the Green's function X^r is abbreviated by $x_n := [a^n]X^r$ and $q_n := [a^n]Q_\cdot$. Further, it is convenient to introduce a reduced version of the coproduct

$$\widetilde{\Delta} x := \Delta x - \mathbb{1} \otimes x - x \otimes \mathbb{1}. \tag{3.83}$$

An expansion of the formulas (3.81) and (3.82) yields the following expressions for the coproduct of the Green's function and invariant charge at first and second loop order

$$\widetilde{\Delta} x_1 = 0, \quad \widetilde{\Delta} q_1 = 0, \quad \text{and} \quad \widetilde{\Delta} x_2 = (x_1 + q_1) \otimes x_1. \tag{3.84}$$

From these expressions, the counterterms can be iteratively constructed with the map S_T^ϕ and its recursion (3.72).

$$S_T^\phi(x_1) = -T\left[x_1\right] \tag{3.85}$$

$$S_T^\phi(q_1) = -T\left[q_1\right] \tag{3.86}$$

$$\begin{aligned} S_T^\phi(x_2) &= -T\left[x_2\right] - T\left[S_T^\phi(x_1 + q_1) x_1\right] \\ &= -T\left[x_2\right] + T\left[T\left[x_1 + q_1\right] x_1\right] \end{aligned} \tag{3.87}$$

Finally, renormalized Green's functions can be computed from the following closed formulas.

$$\phi_{\mathrm{R}}(x_1) = x_1 + S_T^{\phi}(x_1) = x_1 - T\,[x_1] \tag{3.88}$$

$$\begin{aligned}\phi_{\mathrm{R}}(x_2) &= x_2 + S_T^{\phi}(x_1 + q_1)x_1 + S_T^{\phi}(x_2) \\ &= x_2 - T\,[x_1 + q_1]\,x_1 - T\,[x_2] + T\,[T\,[x_1 + q_1]\,x_1]\end{aligned} \tag{3.89}$$

The invariant charge is replaced by a sum of Green's functions depending on the renormalization scheme.

$$q_1 = \begin{cases} 2x_1 - 2x_1 - x_1 & \text{for } \widetilde{\text{MOMq}} \\ 2x_1 - 2x_1 - x_1 & \text{for } \widetilde{\text{MOMh}} \end{cases} \tag{3.90}$$

It is worth to emphasize that this approach does not rely on the notion of Z-factors. We rather use FORM im combination with the MINCER and COLOR packages [18] to compute the epsilon expansions of the considered Green's functions in $d = 4 - 2\varepsilon$ dimensions. These results replace the respective coefficients in the closed formulas (3.88) and (3.89) where the external momentum of the Green's function p is evaluated by the renormalization scheme T at the renormalization point μ. This procedure cancels all pole terms in the regulator such that the limit $\varepsilon \to 0$ is well-defined and yields a polynomial in the kinematic parameter $L := \ln(-p^2/\mu^2)$ and its degree is bounded by the loop number. Obviously, the variation with respect to the renormalization point μ can be rewritten in terms of the kinematic parameter

$$\mu^2 \frac{\partial}{\partial \mu^2} = -\frac{\partial}{\partial L} \tag{3.91}$$

and by the Callan-Symanzik equation (3.63), the terms linear L in combination with the renormalization group functions determine the terms of higher order in L. On the other hand, the Callan-Symanzik equation is useful to read off the anomalous dimension of a certain Green's function at a particular order from the renormalized Green's function of that order and the other renormalization group functions at lower orders. As discussed in paragraph 3.3, the beta function is given in terms of the anomalous dimensions in the respective renormalization schemes.

$$\beta^{\widetilde{\text{MOMq}}} = 2\gamma_F - 2\gamma_2 - \gamma_3 \quad \text{and} \quad \beta^{\widetilde{\text{MOMh}}} = 2\widetilde{\gamma}_1 - 2\widetilde{\gamma}_2 - \gamma_3 \tag{3.92}$$

Chetyrkin and Rétey reported on the Quantum Chromodynamics three-loop renormalization group functions in the momentum subtraction schemes at the asymmetric point in [106]. However, their implicit derivation basically rewrites the momentum subtraction renormalization group functions in terms of the minimal subtraction functions. As a result, their $\widetilde{\text{MOM}}$ renormalization group functions are expressed in terms of the minimal subtraction coupling and gauge parameter, while our prescription expresses everything in terms of the respective momentum subtraction parameters. Hence a direct comparison to their results is not possible. Nonetheless, there is a series of non-trivial checks for our results. First of all, we convinced ourselves that our computational setup reproduces the renormalization group functions in the minimal subtraction scheme as recorded by Larin and Vermaseren in [80]. Another check is that all pole terms in the renormalized Green's function actually cancel and the loop corrections for the longitudinal part of the gluon self-energy vanish. Further, we notice that our non-abelian results reproduce the renormalization group of Quantum Electrodynamics in the limit $T_F = 1$, $C_F = 1$, and $C_A = 0$. Here, it should be remarked that the recorded non-abelian functions depend on all Lie group invariants and in particular on T_F which is evaluated at $^1/_2$ by the COLOR package. As recorded in equation (3.44) and (3.45), the Casimir invariants C_F and C_A implicitly contain the T_F dependence for the color factor identities (3.30) and (3.31). However, closed quark loops generate additional T_F terms which require to identify $^{N_f}/_2$ with $T_F N_f$ in the renormalization group functions below.

$$
\begin{aligned}
\gamma_F^{\widetilde{\mathrm{MOMq}}} = & -\left(\frac{a}{4\pi}\right)\left(\frac{3}{4}C_A + \frac{1}{4}\alpha C_A + \alpha C_F\right) \\
& -\left(\frac{a}{4\pi}\right)^2\left(\frac{19}{12}C_A^2 + \frac{25}{4}C_F C_A - \frac{3}{2}C_F^2 - \frac{1}{16}\alpha C_A^2\right. \\
& \qquad \left. +2\alpha C_F C_A - \frac{1}{16}\alpha^2 C_A^2 + \frac{1}{4}\alpha^2 C_F C_A - \frac{1}{12}N_f C_A - N_f C_F\right) \\
& -\left(\frac{a}{4\pi}\right)^3\left(\frac{19889}{576}C_A^3 - \frac{2837}{96}C_A^3\zeta(3) + \frac{10469}{144}C_F C_A^2 - \frac{175}{24}C_F C_A^2\zeta(3)\right. \\
& \qquad -\frac{111}{4}C_F^2 C_A + 12C_F^2 C_A\zeta(3) + \frac{3}{2}C_F^3 - \frac{625}{288}\alpha C_A^3 \\
& \qquad -\frac{5}{24}\alpha C_A^3\zeta(3) + \frac{35}{12}\alpha C_F C_A^2 - \frac{23}{4}\alpha C_F C_A^2\zeta(3) + \frac{23}{3}\alpha C_F^2 C_A \\
& \qquad -8\alpha C_F^2 C_A\zeta(3) - \frac{113}{192}\alpha^2 C_A^3 + \frac{3}{32}\alpha^2 C_A^3\zeta(3) - \frac{79}{48}\alpha^2 C_F C_A^2 \\
& \qquad -\frac{1}{8}\alpha^2 C_F C_A^2\zeta(3) - \frac{5}{4}\alpha^2 C_F^2 C_A + \frac{1}{16}\alpha^3 C_A^3 - \frac{155}{16}N_f C_A^2 \\
& \qquad +\frac{35}{6}N_f C_A^2\zeta(3) - \frac{623}{36}N_f C_F C_A - \frac{4}{3}N_f C_F C_A\zeta(3) - N_f C_F^2 \\
& \qquad +\frac{5}{144}N_f\alpha C_A^2 + \frac{1}{3}N_f\alpha C_A^2\zeta(3) - \frac{1}{6}N_f\alpha C_F C_A + 2N_f\alpha C_F C_A\zeta(3) \\
& \qquad \left. +\frac{1}{24}N_f\alpha^2 C_A^2 + \frac{1}{2}N_f\alpha^2 C_F C_A + \frac{11}{18}N_f^2 C_A + \frac{2}{3}N_f^2 C_F\right)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2^{\widetilde{\mathrm{MOMq}}} = & -\left(\frac{a}{4\pi}\right)(\alpha C_F) \\
& -\left(\frac{a}{4\pi}\right)^2\left(\frac{25}{4}C_F C_A - \frac{3}{2}C_F^2 + 2\alpha C_F C_A + \frac{1}{4}\alpha^2 C_F C_A - N_f C_F\right) \\
& -\left(\frac{a}{4\pi}\right)^3\left(\frac{1381}{16}C_F C_A^2 - \frac{245}{8}C_F C_A^2\zeta(3) - \frac{111}{4}C_F^2 C_A + 12C_F^2 C_A\zeta(3)\right. \\
& \qquad +\frac{3}{2}C_F^3 + \frac{5}{6}\alpha C_F C_A^2 - \frac{15}{4}\alpha C_F C_A^2\zeta(3) + \frac{23}{3}\alpha C_F^2 C_A \\
& \qquad -8\alpha C_F^2 C_A\zeta(3) - \frac{61}{48}\alpha^2 C_F C_A^2 - \frac{1}{8}\alpha^2 C_F C_A^2\zeta(3) - \frac{5}{4}\alpha^2 C_F^2 C_A \\
& \qquad +\frac{1}{8}\alpha^3 C_F C_A^2 - \frac{121}{6}N_f C_F C_A + 4N_f C_F C_A\zeta(3) - N_f C_F^2 \\
& \qquad \left. +2N_f\alpha C_F C_A\zeta(3) + \frac{1}{2}N_f\alpha^2 C_F C_A + \frac{2}{3}N_f^2 C_F\right)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3^{\widetilde{\text{MOMq}}} &= -\left(\frac{a}{4\pi}\right)\left(-\frac{13}{6}C_A+\frac{1}{2}\alpha C_A+\frac{2}{3}N_f\right)\\
&\quad -\left(\frac{a}{4\pi}\right)^2\left(-\frac{49}{6}C_A^2+\frac{49}{24}\alpha C_A^2+\frac{11}{24}\alpha^2C_A^2-\frac{1}{4}\alpha^3C_A^2\right.\\
&\qquad\left.+\frac{19}{6}N_fC_A+2N_fC_F-\frac{2}{3}N_f\alpha C_A-\frac{1}{3}N_f\alpha^2C_A\right)\\
&\quad -\left(\frac{a}{4\pi}\right)^3\left(-\frac{7807}{96}C_A^3-\frac{373}{48}C_A^3\zeta(3)-\frac{91}{9}C_FC_A^2+\frac{52}{3}C_FC_A^2\zeta(3)\right.\\
&\qquad+\frac{3889}{288}\alpha C_A^3-\frac{23}{12}\alpha C_A^3\zeta(3)+\frac{41}{12}\alpha C_FC_A^2-4\alpha C_FC_A^2\zeta(3)\\
&\qquad+\frac{187}{48}\alpha^2C_A^3-\frac{5}{16}\alpha^2C_A^3\zeta(3)-\frac{4}{3}\alpha^2C_FC_A^2-\frac{119}{96}\alpha^3C_A^3\\
&\qquad+\frac{1}{4}\alpha^3C_FC_A^2-\frac{31}{96}\alpha^4C_A^3+\frac{1397}{36}N_fC_A^2+17N_fC_A^2\zeta(3)\\
&\qquad+\frac{353}{9}N_fC_FC_A-\frac{104}{3}N_fC_FC_A\zeta(3)-N_fC_F^2-\frac{463}{72}N_f\alpha C_A^2\\
&\qquad+\frac{2}{3}N_f\alpha C_A^2\zeta(3)-\frac{13}{3}N_f\alpha C_FC_A-\frac{29}{12}N_f\alpha^2C_A^2-\frac{7}{6}N_f\alpha^2C_FC_A\\
&\qquad+\frac{1}{12}N_f\alpha^3C_A^2+\frac{1}{24}N_f\alpha^4C_A^2-\frac{43}{18}N_f^2C_A-\frac{8}{3}N_f^2C_A\zeta(3)\\
&\qquad\left.-\frac{46}{9}N_f^2C_F+\frac{16}{3}N_f^2C_F\zeta(3)\right)
\end{aligned}
$$

$$\begin{aligned}
\beta^{\widetilde{\mathrm{MOMq}}} &= -\left(\frac{a}{4\pi}\right)\left(\frac{11}{3}C_A - \frac{2}{3}N_f\right) \\
&- \left(\frac{a}{4\pi}\right)^2 \left(\frac{34}{3}C_A^2 - \frac{13}{6}\alpha C_A^2 - \frac{7}{12}\alpha^2 C_A^2 + \frac{1}{4}\alpha^3 C_A^2 \right. \\
&\quad \left. - \frac{10}{3}N_f C_A - 2N_f C_F + \frac{2}{3}N_f \alpha C_A + \frac{1}{3}N_f \alpha^2 C_A\right) \\
&- \left(\frac{a}{4\pi}\right)^3 \left(\frac{21655}{144}C_A^3 - \frac{154}{3}C_A^3\zeta(3) - \frac{154}{9}C_F C_A^2 + \frac{88}{3}C_F C_A^2\zeta(3) \right. \\
&\quad - \frac{571}{32}\alpha C_A^3 + \frac{3}{2}\alpha C_A^3\zeta(3) + \frac{3}{4}\alpha C_F C_A^2 - \frac{487}{96}\alpha^2 C_A^3 \\
&\quad + \frac{1}{2}\alpha^2 C_A^3\zeta(3) + \frac{7}{12}\alpha^2 C_F C_A^2 + \frac{131}{96}\alpha^3 C_A^3 - \frac{1}{2}\alpha^3 C_F C_A^2 \\
&\quad + \frac{31}{96}\alpha^4 C_A^3 - \frac{4189}{72}N_f C_A^2 - \frac{16}{3}N_f C_A^2\zeta(3) - \frac{67}{2}N_f C_F C_A \\
&\quad + 24 N_f C_F C_A\zeta(3) + N_f C_F^2 + \frac{13}{2}N_f \alpha C_A^2 + 4 N_f \alpha C_F C_A \\
&\quad + \frac{5}{2}N_f \alpha^2 C_A^2 + \frac{7}{6}N_f \alpha^2 C_F C_A - \frac{1}{12}N_f \alpha^3 C_A^2 - \frac{1}{24}N_f \alpha^4 C_A^2 \\
&\quad \left. + \frac{65}{18}N_f^2 C_A + \frac{8}{3}N_f^2 C_A\zeta(3) + \frac{46}{9}N_f^2 C_F - \frac{16}{3}N_f^2 C_F\zeta(3)\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{\gamma}_1^{\widetilde{\mathrm{MOMh}}} &= -\left(\frac{a}{4\pi}\right)\left(\frac{1}{2}\alpha C_A\right) \\
&- \left(\frac{a}{4\pi}\right)^2 \left(\frac{11}{4}C_A^2 - \frac{3}{4}\alpha C_A^2 - \frac{1}{2}N_f C_A\right) \\
&- \left(\frac{a}{4\pi}\right)^3 \left(\frac{145}{8}C_A^3 + \frac{125}{64}\alpha C_A^3 - \frac{63}{32}\alpha C_A^3\zeta(3) - \frac{53}{64}\alpha^2 C_A^3 \right. \\
&\quad + \frac{15}{32}\alpha^2 C_A^3\zeta(3) - \frac{1}{32}\alpha^3 C_A^3 + \frac{3}{16}\alpha^3 C_A^3\zeta(3) - 7N_f C_A^2 \\
&\quad - \frac{3}{2}N_f C_F C_A + \frac{5}{16}N_f \alpha C_A^2 + \frac{3}{8}N_f \alpha C_A^2\zeta(3) \\
&\quad \left. + \frac{1}{4}N_f \alpha^2 C_A^2 + \frac{1}{2}N_f^2 C_A\right)
\end{aligned}$$

$$
\begin{aligned}
\tilde{\gamma}_2^{\widetilde{\text{MOMh}}} &= -\left(\frac{a}{4\pi}\right)\left(-\frac{3}{4}C_A + \frac{1}{4}\alpha C_A\right) \\
&\quad - \left(\frac{a}{4\pi}\right)^2\left(-C_A^2 - \frac{3}{16}\alpha C_A^2 + \frac{1}{16}\alpha^2 C_A^2 + \frac{1}{4}N_f C_A\right) \\
&\quad - \left(\frac{a}{4\pi}\right)^3\left(-\frac{313}{16}C_A^3 + \frac{47}{16}C_A^3\zeta(3) + \frac{199}{48}\alpha C_A^3 - \frac{5}{8}\alpha C_A^3\zeta(3)\right. \\
&\qquad + \frac{3}{32}\alpha^2 C_A^3 + \frac{3}{8}\alpha^2 C_A^3\zeta(3) - \frac{11}{32}\alpha^3 C_A^3 + \frac{3}{16}\alpha^3 C_A^3\zeta(3) \\
&\qquad + \frac{97}{12}N_f C_A^2 + \frac{1}{4}N_f C_A^2\zeta(3) + \frac{3}{4}N_f C_F C_A - \frac{11}{24}N_f\alpha C_A^2 \\
&\qquad \left. + \frac{1}{4}N_f\alpha C_A^2\zeta(3) - \frac{1}{8}N_f\alpha^2 C_A^2 - \frac{5}{6}N_f^2 C_A\right)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3^{\widetilde{\text{MOMh}}} &= -\left(\frac{a}{4\pi}\right)\left(-\frac{13}{6}C_A + \frac{1}{2}\alpha C_A + \frac{2}{3}N_f\right) \\
&\quad - \left(\frac{a}{4\pi}\right)^2\left(-\frac{23}{6}C_A^2 + \frac{25}{24}\alpha C_A^2 + \frac{11}{24}\alpha^2 C_A^2 - \frac{1}{4}\alpha^3 C_A^2\right. \\
&\qquad \left. + \frac{11}{6}N_f C_A + 2N_f C_F - \frac{2}{3}N_f\alpha C_A - \frac{1}{3}N_f\alpha^2 C_A\right) \\
&\quad - \left(\frac{a}{4\pi}\right)^3\left(-\frac{1997}{36}C_A^3 + 12C_A^3\zeta(3) + \frac{263}{32}\alpha C_A^3 - \frac{41}{8}\alpha C_A^3\zeta(3)\right. \\
&\qquad + \frac{169}{96}\alpha^2 C_A^3 - \frac{5}{8}\alpha^2 C_A^3\zeta(3) - \frac{23}{96}\alpha^3 C_A^3 - \frac{31}{96}\alpha^4 C_A^3 \\
&\qquad + \frac{233}{9}N_f C_A^2 + \frac{131}{12}N_f C_A^2\zeta(3) + \frac{253}{9}N_f C_F C_A - \frac{88}{3}N_f C_F C_A\zeta(3) \\
&\qquad - N_f C_F^2 - \frac{29}{8}N_f\alpha C_A^2 + \frac{1}{4}N_f\alpha C_A^2\zeta(3) - 4N_f\alpha C_F C_A \\
&\qquad - \frac{13}{12}N_f\alpha^2 C_A^2 - \frac{3}{2}N_f\alpha^2 C_F C_A + \frac{1}{12}N_f\alpha^3 C_A^2 + \frac{1}{24}N_f\alpha^4 C_A^2 \\
&\qquad \left. - \frac{5}{3}N_f^2 C_A - \frac{8}{3}N_f^2 C_A\zeta(3) - \frac{46}{9}N_f^2 C_F + \frac{16}{3}N_f^2 C_F\zeta(3)\right)
\end{aligned}
$$

$$\begin{aligned}
\beta^{\widetilde{\mathrm{MOMh}}} &= -\left(\frac{a}{4\pi}\right)\left(\frac{11}{3}C_A - \frac{2}{3}N_f\right) \\
&\quad - \left(\frac{a}{4\pi}\right)^2 \left(\frac{34}{3}C_A^2 - \frac{13}{6}\alpha C_A^2 - \frac{7}{12}\alpha^2 C_A^2 + \frac{1}{4}\alpha^3 C_A^2 \right. \\
&\qquad\qquad \left. - \frac{10}{3}N_f C_A - 2N_f C_F + \frac{2}{3}N_f \alpha C_A + \frac{1}{3}N_f \alpha^2 C_A\right) \\
&\quad - \left(\frac{a}{4\pi}\right)^3 \left(\frac{9421}{72}C_A^3 - \frac{143}{8}C_A^3\zeta(3) - \frac{605}{48}\alpha C_A^3 + \frac{39}{16}\alpha C_A^3\zeta(3) \right. \\
&\qquad\qquad - \frac{173}{48}\alpha^2 C_A^3 + \frac{13}{16}\alpha^2 C_A^3\zeta(3) + \frac{83}{96}\alpha^3 C_A^3 + \frac{31}{96}\alpha^4 C_A^3 \\
&\qquad\qquad - \frac{1009}{18}N_f C_A^2 - \frac{137}{12}N_f C_A^2\zeta(3) - \frac{587}{18}N_f C_F C_A + \frac{88}{3}N_f C_F C_A\zeta(3) \\
&\qquad\qquad + N_f C_F^2 + \frac{31}{6}N_f \alpha C_A^2 + 4N_f \alpha C_F C_A + \frac{11}{6}N_f \alpha^2 C_A^2 \\
&\qquad\qquad + \frac{3}{2}N_f \alpha^2 C_F C_A - \frac{1}{12}N_f \alpha^3 C_A^2 - \frac{1}{24}N_f \alpha^4 C_A^2 + \frac{13}{3}N_f^2 C_A \\
&\qquad\qquad \left. + \frac{8}{3}N_f^2 C_A\zeta(3) + \frac{46}{9}N_f^2 C_F - \frac{16}{3}N_f^2 C_F\zeta(3)\right)
\end{aligned}$$

Chapter 4

Conclusion

In the scope of this thesis, two major advances have been accomplished. Firstly, we enhanced Hopf-algebraic methods for the renormalization of gauge theories. Secondly, we provided a purely perturbative approach to scrutinize the gauge dependence of Quantum Electrodynamics with a covariant gauge fixing.

Hopf-algebraic renormalization of gauge theories

For the Hopf-algebraic renormalization of gauge theories, we pointed out that the linear covariant gauge fixing introduces a longitudinal part in the photon propagator. This non-physical component excludes the usual vacuum polarization. Consequently, transversal and longitudinal components are distinguished in the renormalization process and the gauge parameter is renormalized.

The Hopf algebra of Feynman was adjusted for this effect by treating the longitudinal component of the photon propagator as a new type of residue. Combinatorial Dyson-Schwinger equations provide a combinatorial approach to Green's functions and determine the structure of their subdivergences in form of a closed formula for the coproduct.

In Quantum Electrodynamics, we found that three factors constitute the subdivergences (the left-hand component of the tensor product) of the coproduct formula. The first factor shares the external leg structure of the considered Green's function and was linked to its anomalous dimension. This factor is hence interpreted to renormalize the residue of the Green's function. Both the second and the third factor turned out to be invariant charges whose powers respectively count the grade of the cographs with respect to coupling parameter α and its product with the gauge parameter $\alpha\xi$. Further, the renormalization group functions of these parameters were derived from their invariant charges. This implies that an invariant charge describes the parameter renormalization.

In Quantum Chromodynamics, this picture persists to a certain degree. Again, two factors describe the renormalization of the residue and the gauge parameter. However, there are multiple vertices which possibly contribute a factor of the coupling parameter. Therefore, the invariant charge associated to the coupling parameter is determined by a multi-grading with respect to the types of vertices and edges rather than countering the power of the coupling parameter. However, the implementation of Slavnov-Taylor identities corresponds to factor out quotients of Hopf ideals and we demonstrated that the quotient space allows for a simple invariant charge whose power counts the grade of the cographs with respect to the coupling parameter, in analogy to the abelian case.

These results suggest that every renormalized parameter gives rise to an invariant charge in the coproduct of Green's functions. Further, these invariant charges describe the renormalization group functions of the associated parameters and have been shown to match the Z-factors that describe the renormalization of the associated parameter.

This observation allows for an ease anticipation of coproduct formulas and the structure of subdivergences in other renormalizable Quantum Field Theories. On the other hand, these considerations can be useful to identify appropriate ideals and conditions for the renormalization of parameters in non-renormalizable theories. A first step into this direction might be an inquiry of the question whether the observed multi-grading structure in Quantum Chromodynamics allows to renormalize the theory in an exotic scheme that breaks the Slavnov-Taylor identities.

Gauge dependence of Quantum Electrodynamics

Our inquiry of the gauge dependence of Quantum Electrodynamics is based on cancellation identities which directly followed from an analysis of the Feynman rules of Quantum Electrodynamics. These identities apply to Feynman graphs with longitudinally contracted photons and have been exploited in two different scenarios.

In the first scenario, the longitudinal photon is an external leg of a Green's function. Its defining set of Feynman graphs has been partitioned into equivalence classes such that a cancellation applies to each class. We demonstrated that the resulting cancellations among these graphs imply the well-known Ward identities.

In the second scenario, the gauge-dependent part of the photon propagator was identified as two connected longitudinal photons. In a similar fashion to the first scenario, we have been able to construct a partion whose subsets of Feynman graphs allow for the application of appropriate cancellation identities. This partion is much finer than the sum over all Feynman graphs. The resulting Dyson-Schwinger type equation characterizes the gauge dependence of the bare Green's function; that is it enables us to construct all terms depending on the gauge parameter once the bare Green's functions is known in a specific linear covariant gauge (such as the Feynman or the Landau gauge).

Further, this Dyson-Schwinger equation also reveals another crucial property: after all cancellation have been performed, the gauge-dependent terms only couple to external electron propagators. This ensures the gauge-invariance of on-shell quantities and S-matrix elements. Further, it transpired that our approach can be understood as a natural derivation of the Landau-Khalatnikov formula in momentum space. Another advantage of our method is founded on its perturbative character. It refines the result of Landau and Khalatnikov to subclasses of Feynman graphs and provides formulas of the gauge parameter dependent terms which have been confirmed by perturbative computations to the third order in the coupling parameter.

In the massless limit, the Dyson-Schwinger equation for the bare electron propagator was promoted to a non-perturbative description of the renormalized electron propagator. This proved the famous conjecture

$$\gamma(\xi,\alpha)-\gamma(0,\alpha)=\xi\frac{\alpha}{4\pi}$$

that the anomalous dimension of the electron depends on the gauge parameter only at the first loop order. A consecutive topic is to extend our analysis to the one-particle irreducible electron-photon vertex. Further, it is reasonable to expect that our approach has the potential to shed light on the construction of gauge invariant truncation for Dyson-Schwinger equations.

An ongoing project is the enquiry of cancellation identities for Quantum Chromodynamics, for Quantum Electrodynamics in the 't Hooft-Veltman gauge [87], and for massive gauge bosons.

Our results in Quantum Electrodynamics suggest the expectation that a gauge theory, whose observables are gauge invariant in the on-shell limit, shows similar cancellations such that the longitudinal gauge bosons only couple to the external legs of a Green's function. Therefore, there should be a similar characterization of the gauge dependence for the Green's functions of Quantum Chromodynamics. However, the renormalization group functions which we have derived clearly show a more complicated dependence on the gauge parameter. A possible explanation of this behaviour might be the increased number of interactions. As a result, one expects the gauge dependence in Quantum Chromodynamics to be characterized by systems of Dyson-Schwinger equations rather than a single equation. Similar issues emerge in the case of the 't Hooft-Veltman gauge which is a non-linear gauge and enriches Quantum Electrodynamics with three-photon interactions, four-photon interactions, and a non-trivial ghost sector.

Appendix A

Conventions

This first appendix summarizes some definitions and conventions that have been suppressed in the main discussion for the sake of conciseness.

For the notion of spacetime, we follow the customary convention of the high-energy physics literature. We denote the Minkowski space as $\mathbb{R}^{3,1}$; it is defined as the four-dimensional real vector space $\mathbb{R}^4$ endowed with the Minkowski metric

$$g_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)\,. \tag{A.1}$$

A point of Minkowski space is called four-vector. We refer to its components by assigning a superscripted Lorentz index, which is denoted by a Greek letter and ranges in $0, 1, 2, 3, 4$. The 0-component refers to the time direction. Let a and b be two four-vectors, their product is defined by means of the Minkowski inner product

$$a \cdot b = g_{\mu\nu} a^\mu b^\nu. \tag{A.2}$$

QED Lagrangian

Consider the Lagrange density of Quantum Electrodynamics

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\left(i \not{D} - m\right)\psi. \tag{A.3}$$

Its field content consists of the charged Dirac field ψ and the electromagnetic potential A_μ. The first field is a bispinor and represents a fermion of mass m and electromagnetic charge e. The latter field is included in the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.4}$$

and the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu. \tag{A.5}$$

The square of the field strength tensor is an invariant under Lorentz transformations and is readily expressed in terms of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$

$$F_{\mu\nu}F^{\mu\nu} = 2\left(\mathbf{B^2} - \mathbf{E^2}\right). \tag{A.6}$$

As we consider a Lorentz transformation to be proper (has a unit determinant), there is a second invariant. It relates to the dual field tensor

$$\widetilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}\frac{1}{2}F_{\rho\sigma} \tag{A.7}$$

which incorporates the antisymmetric tensor. Here, we followed the convention of [107]: $\varepsilon^{\mu\nu\rho\sigma}$ equals $+1$ (or -1) whenever $(\mu\nu\rho\sigma)$ is an even (or odd) permutation of (1234). Now, the second Lorentz invariant is readily derived in terms of the electric and magnetic fields

$$F_{\mu\nu}\widetilde{F}^{\mu\nu} = 4\left(\mathbf{E}\cdot\mathbf{B}\right). \tag{A.8}$$

In the paragraph/section ... , we encounter a term quartic in the field strength tensor and it is exciting to notice that it includes both invariants

$$F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} = 2\left(\mathbf{E^2} - \mathbf{B^2}\right)^2 + 4\left(\mathbf{E}\cdot\mathbf{B}\right)^2. \tag{A.9}$$

Another omnipresent abbreviation is the slash notation

$$\not{p} = p_\mu\gamma^\mu, \tag{A.10}$$

where the symbols γ^μ represent the Dirac γ-matrices which are traceless and fulfil the anticommutator relation of a Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_4. \tag{A.11}$$

The four-dimensional unit matrix is denoted by $\mathbb{I}_4$. Furthermore, the four-dimensional representation of the Dirac matrices allows for the definition of another gamma matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.12}$$

which is also traceless, satisfies $\gamma^5\gamma^5 = -\mathbb{I}_4$, and respects the Clifford algebra relations

$$\left\{\gamma^5, \gamma^\mu\right\} = 0. \tag{A.13}$$

The matrix γ^5 is crucial for the notion of chiral symmetry. However, it is worth emphasizing that it does not appear within our calculations of renormalized Green's functions. Therefore, we can employ dimensional regularization and extend the metric tensor and the Dirac matrices to D dimensions; they are assumed to satisfy

1. $g_{\mu\nu} g^{\mu\nu} = D,$ (A.14)

2. $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{I}_4,$ (A.15)

3. $\mathrm{Tr}\left[\gamma^{\mu_1} \ldots \gamma^{\mu_{2n+1}}\right] = 0$ for any odd number of γ's. (A.16)

Appendix B

Conformal symmetry and Noether currents

This paragraph summarizes the enhancement of the notion of conformal transformations to the field content of Spin-$1/2$ Electrodynamics and discusses the conserved Noether currents related to that symmetry.

As we are in particular interested in the behaviour under infinitesimal conformal transformations, we require the structure of a Lie group. An conformal transformation is described by the exponential

$$\exp\left(-a^\mu P_\mu - \frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} - \alpha D - b^\mu K_\mu\right) \tag{B.1}$$

of 15 infinitesimal generators ($M_{\mu\nu}$ is antisymmetric) which respectively generate spacetime translations, proper orthochronous Lorentz transformations, dilations, and special conformal transformations. These infinitesimal generators are defined to act on a field $\Phi \in \{A_\mu, \psi\}$ in the following way.

$$P_\mu \Phi(x) = \partial_\mu \Phi(x) \tag{B.2}$$
$$M_{\mu\nu}\Phi(x) = \left(x_\mu\partial_\nu - x_\nu\partial_\mu + S^\Phi_{\mu\nu}\right)\Phi(x) \tag{B.3}$$
$$D\Phi(x) = \left(d_\Phi + x^\mu\partial_\mu\right)\Phi(x) \tag{B.4}$$
$$K_\mu\Phi(x) = \left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu - 2x_\mu d_\Phi - 2x_\nu S^\Phi_{\mu\nu}\right)\Phi(x) \tag{B.5}$$

Here, S^Φ is the spin matrix that is generated by a Lorentz transformation of the field Φ and d_Φ denotes the scaling dimension of the field Φ; in the concrete case of Quantum Electrodynamics with a gauge field A_μ and Dirac fermion ψ, they read

$$\left(S^A_{\mu\nu}\right)_\rho{}^\sigma = \left(\mathcal{M}_{\mu\nu}\right)_\rho{}^\sigma = g_{\mu\rho}\delta^\sigma_\nu - g_{\nu\rho}\delta^\sigma_\mu, \qquad S^\psi_{\mu\nu} = \Sigma_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right], \tag{B.6}$$

$$d_A = 1, \qquad \text{and} \qquad d_\psi = 3/2. \tag{B.7}$$

It is instructive to check that with these definitions the generators obey the usual conformal algebra.

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho} \\
[P_\mu, P_\nu] &= 0 \qquad\qquad\qquad [M_{\mu\nu}, P_\rho] = g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu \\
[P_\mu, D] &= P_\mu \qquad\qquad\qquad [M_{\mu\nu}, D] = 0 \\
[K_\mu, K_\nu] &= 0 \qquad\qquad\qquad [D, K_\mu] = K_\mu \\
[P_\mu, K_\nu] &= 2M_{\mu\nu} - 2g_{\mu\nu} D \qquad [M_{\mu\nu}, K_\rho] = g_{\nu\rho} K_\mu - g_{\mu\rho} K_\nu
\end{aligned}
\tag{B.8}
$$

Where the first and second line constitute the Poincaré subalgebra. Further, it is reasonable to remark that the spin matrices $\mathcal{M}_{\mu\nu}$ and $\Sigma_{\mu\nu}$ satisfy the first commutator relation.

Given the fact the Lagrangian of Spin-1/2 Electrodynamics is invariant under translations and Lorentz transformations and its *massless* limit further respects dilations and special conformal transformations, one likes to ascertain the Noether currents assigned to these symmetries.

In order to obtain a symmetric energy momentum tensor, it is crucial to symmetrize the derivatives with respect to the Dirac field ψ and $\overline{\psi}$. Therefore, the symmetrized Lagrangian

$$
\mathscr{L} = \frac{1}{2}\overline{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}\overline{\psi}(-i\overleftarrow{\not{\partial}} - m)\psi - e\overline{\psi}\not{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
\tag{B.9}
$$

will be used in this paragraph. The left-pointing derivative is defined to act on the fields to its left-hand side, that is $\overline{\psi}\overleftarrow{\not{\partial}} := \left(\partial_\mu \overline{\psi}\right)\gamma^\mu$. Recall that the equations of motion read

$$
\left.
\begin{aligned}
\left(i\not{\partial} - e\not{A} - m\right)\psi &= 0 \\
\overline{\psi}\left(i\overleftarrow{\not{\partial}} + e\not{A} + m\right) &= 0 \\
\partial_\mu F^{\mu\nu} &= j^\nu
\end{aligned}
\right\}
\tag{B.10}
$$

where $j^\nu := e\overline{\psi}\gamma^\nu\psi$ denotes the conserved fermionic current and further recall the Bianchi identity

$$
\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.
\tag{B.11}
$$

For the rest of the discussion, the Lagrangian is split up into an electromagnetic part and the fermionic part as fellows

$$
\mathscr{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad \mathscr{L}_{\text{F}} = \mathscr{L} - \mathscr{L}_{\text{EM}},
\tag{B.12}
$$

This allows for a separate study of the symmetries of the electromagnetic and fermionic sectors of the theory and of course, it is possible to separately construct the corresponding Noether currents for each sector. However, the reader should keep in mind that fields obey the coupled equations of motions (B.10) of the full Lagrangian. Therefore, the separated Lagrangians do not describe closed physical systems and only the sum of both Noether currents can be expected to be conserved.

Now, the invariance under spacetime translations yields the following Noether currents

$$T^{\mu\nu}_{\text{EM}} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} \partial^{\nu} A_{\alpha}, \tag{B.13}$$

$$T^{\mu\nu}_{\text{F}} = \frac{i}{2} \overline{\psi} \gamma^{\mu} \left(\partial^{\nu} \psi \right) - \frac{i}{2} \left(\partial^{\nu} \overline{\psi} \right) \gamma^{\mu} \psi. \tag{B.14}$$

Exploiting the equations of motion in combination with the Bianchi identity determines the divergence of these tensors to be

$$\partial_{\mu} T^{\mu\nu}_{\text{EM}} = -j_{\alpha} \partial^{\mu} A^{\alpha} \quad \text{and} \quad \partial_{\mu} T^{\mu\nu}_{\text{F}} = e \overline{\psi} \gamma_{\mu} \psi \partial^{\nu} A^{\mu} = j_{\alpha} \partial^{\nu} A^{\alpha}, \tag{B.15}$$

such that the sum of both tensors which actually describes a closed physical system

$$T^{\mu\nu}_{\text{total}} = T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{F}} \quad \text{yields} \quad \partial_{\mu} T^{\mu\nu}_{\text{total}} = 0 \tag{B.16}$$

a vanishing divergence. This Noether current is termed the canonical energy-momentum tensor, but obviously, the form of this tensor is by no means unique as terms with vanishing divergence might be added. Belinfante [108] provided a prescription to construct a symmetric energy-momentum tensor provided that the Lagrangian under consideration is invariant under Lorentz transformations. Further, Callan, Coleman, and Jackiw [109, 110] demonstrated that a class of dilation-invariant Lagrangians allows for a symmetric and traceless energy-momentum tensor.

Fortunately, in the case of the Lagrangian (B.9), it is sufficient to consider the Belinfante improvement to obtain a symmetric energy-momentum tensor that also becomes traceless in the massless limit.

The Belinfante improvement mainly relies on a tensor $B^{\alpha\mu\nu}$ which is anti-symmetric with respect to permutations of the indices α and μ. Then, the anti-symmetry guarantees

$$\partial_{\alpha} \partial_{\mu} B^{\alpha\mu\nu} = 0, \tag{B.17}$$

such that the divergence of the tensor $\partial_{\alpha} B^{\alpha\mu\nu}$ can be safely added to the canonical energy-momentum tensor; still yielding a conserved current. An appropriate choice of these tensors might be deduced from the behaviour of the canonical energy-momentum tensor under gauge transformations [107] or from the conservation laws implied by Lorentz invariance [108, 34]. In our case, the improvement is obtained from the following tensors.

$$B_{\text{EM}}^{\alpha\mu\nu} = F^{\mu\alpha} A^{\nu} \tag{B.18}$$

$$B_{\text{F}}^{\alpha\mu\nu} = -\frac{i}{16} \overline{\psi} \left[\{\gamma^{\alpha}, \Sigma^{\mu\nu}\} + \{\gamma^{\mu}, \Sigma^{\nu\alpha}\} - \{\gamma^{\nu}, \Sigma^{\alpha\mu}\} \right] \psi \tag{B.19}$$

Adding their divergence to the canonical energy-momentum tensor yields the following tensors

$$\Theta_{\text{EM}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + \partial^{\alpha} B_{\text{EM}}^{\alpha\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} F^{\nu}{}_{\alpha} - j^{\mu} A^{\nu} \tag{B.20}$$

$$\Theta_{\text{F}}^{\mu\nu} = T_{\text{F}}^{\mu\nu} + \partial^{\alpha} B_{\text{F}}^{\alpha\mu\nu} \tag{B.21}$$

$$= \frac{i}{4} \begin{bmatrix} \overline{\psi}\gamma^{\mu}\big(\partial^{\nu} + ieA^{\nu}\big)\psi + \overline{\psi}\gamma^{\mu}\big(- \overleftarrow{\partial}^{\nu} + ieA^{\nu}\big)\psi \\ -\overline{\psi}\gamma^{\nu}\big(\partial^{\mu} + ieA^{\mu}\big)\psi - \overline{\psi}\gamma^{\nu}\big(- \overleftarrow{\partial}^{\mu} + ieA^{\mu}\big)\psi \end{bmatrix} + j^{\mu} A^{\nu} \tag{B.22}$$

which become symmetric only in the limit of vanishing current $j^{\mu} = 0$, that is under the assumption of decoupled equations of motion. However, introducing the total energy-momentum tensors as the sum of both improved tensors $\Theta_{\text{total}}^{\mu\nu} = \Theta_{\text{EM}}^{\mu\nu} + \Theta_{\text{F}}^{\mu\nu}$ obviously yields and vanishing divergence

$$\partial_{\mu} \Theta_{\text{total}}^{\mu\nu} = 0, \tag{B.23}$$

and moreover, all non-symmetric terms cancel such that the total energy-momentum tensor becomes symmetric

$$\Theta_{\text{total}}^{\mu\nu} = \Theta_{\text{total}}^{\nu\mu}. \tag{B.24}$$

By making use of the equations of motion (B.10), the trace of the total energy-momentum tensor turns out to vanish provided that the fermion has a vanishing mass

$$g_{\mu\nu} \Theta_{\text{total}}^{\mu\nu} = m \overline{\psi} \psi. \tag{B.25}$$

The symmetric total energy-momentum tensor allows for the definition of another conserved current

$$\mathcal{M}^{\mu\nu\rho} = x^{\nu} \Theta_{\text{total}}^{\mu\rho} - x^{\rho} \Theta_{\text{total}}^{\nu\mu} \qquad \text{with} \quad \partial_{\mu} \mathcal{M}^{\mu\nu\rho} = 0 \tag{B.26}$$

which can also be derived as a consequence of Noether's theorem and the invariance under Lorentz transformations of the Lagrangian (B.9). Further, if one assumes massless electrons $m = 0$, then the total energy-momentum tensor becomes traceless which enables us to define two additional conserved currents

$$\mathcal{D}^\mu = x_\nu \Theta^{\mu\nu}_{\text{total}} \qquad \text{such that} \quad \partial_\mu \mathcal{D}^\mu = 0 \tag{B.27}$$

$$\mathcal{C}^{\mu\nu} = 2x^\mu x_\rho \Theta^{\rho\nu}_{\text{total}} - x^\rho x_\rho \Theta^{\mu\nu}_{\text{total}} \qquad \text{and} \quad \partial_\mu \mathcal{C}^{\mu\nu} = 0 \tag{B.28}$$

and indeed an easy but tedious computation shows that these currents are generated by Noether's theorem and the invariance of the Lagrangian under infinitesimal dilations (B.4) and special conformal transformations (B.5), respectively.

It is interesting to study the effect of the gauge fixing on the symmetries and conserved quantities of the Lagrangian (B.9) in the limit of a vanishing electron mass $m = 0$. The linear covariant gauge fixing term

$$\mathscr{L}_{\text{GF}} = \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \tag{B.29}$$

respects the invariance under spacetime translations as well as the Lorentz invariance and the dilation invariance. However, it turns out to break the invariance under special conformal transformations as generated by the infinitesimal generator (B.5). Therefore, the trace of the energy-momentum tensor is expected to be non-vanishing and the above construction of the dilation current does not apply.

Now, following the above discussion for the gauge-fixed Lagrangian yields a symmetric energy-momentum tensor

$$\widetilde{\Theta}^{\mu\nu}_{\text{total}} = \Theta^{\mu\nu}_{\text{EM}} + \Theta^{\mu\nu}_{\text{GF}} + \Theta^{\mu\nu}_{\text{F}}, \tag{B.30}$$

where the contribution due to the added gauge fixing is

$$\Theta^{\mu\nu}_{\text{GF}} = \frac{1}{\xi} \left\{ g^{\mu\nu} \left[\frac{1}{2} \left(\partial \cdot A\right)^2 + A^\alpha \partial_\alpha \left(\partial \cdot A\right) \right] - A^\mu \partial^\nu \left(\partial \cdot A\right) - A^\nu \partial^\mu \left(\partial \cdot A\right) \right\}. \tag{B.31}$$

In straight analogy to the above discussion, this energy-momentum tensor is a conserved current

$$\partial_\mu \widetilde{\Theta}^{\mu\nu}_{\text{total}} = 0 \quad \text{and} \quad \widetilde{\Theta}^{\mu\nu}_{\text{total}} = \widetilde{\Theta}^{\nu\mu}_{\text{total}} \tag{B.32}$$

its symmetry allows for the definition of another conserved current

$$\widetilde{\mathcal{M}}^{\mu\nu\rho} = x^\nu \widetilde{\Theta}^{\mu\rho}_{\text{total}} - x^\rho \widetilde{\Theta}^{\nu\mu}_{\text{total}} \quad \text{with} \quad \partial_\mu \widetilde{\mathcal{M}}^{\mu\nu\rho} = 0. \tag{B.33}$$

However, the breakdown of the conformal invariance in the gauge-fixed Lagrangian is mirrored by the non-vanishing trace which reads

$$g_{\mu\nu}\widetilde{\Theta}^{\mu\nu}_{\text{total}} = \frac{2}{\xi}\partial_\alpha \left[A^\alpha \left(\partial \cdot A\right)\right]. \tag{B.34}$$

Following Callan, Coleman, and Jackiw [109], we define the virial current of the gauge-fixed Lagrangian

$$V^\mu = \frac{2}{\xi}A^\mu \left(\partial \cdot A\right). \tag{B.35}$$

Finally the conserved current of the dilation invariance is given by

$$\widetilde{\mathcal{D}}^\mu = x_\nu \widetilde{\Theta}^{\mu\nu}_{\text{total}} - V^\mu \quad \text{and satisfies} \quad \partial_\mu \widetilde{\mathcal{D}}^\mu. \tag{B.36}$$

To conclude, the gauge fixing term cause a non-vanishing virial current that indicates the non-vanishing trace in the energy-momentum tensor and the related breakdown of conformal invariance. The virial current linearly contributes to the dilation current, whereas the conformal currents $\mathcal{I}^{\mu\nu}$ cease of be conserved.

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