Unitarity, Dispersion Relations, Cutkosky's Cutting Rules

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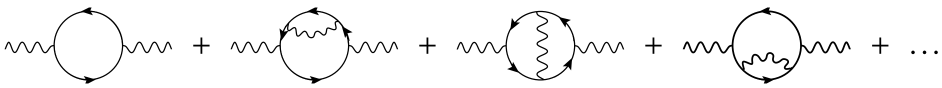
For more information about unitarity, dispersion relations, and Cutkosky's cutting rules, consult Peskin& Schröder, or rather Le Bellac.

1 Motivation: Gauge Dependence of QFT

A lot of quantum field theory is actually gauge dependent, even though physical observables should not be. Consider for example QED: In exercise four, we computed the one-loop fermion self-energy $\Sigma^1(\xi, q, \mu)$. One finds that the self-energy really depends on ξ, a covariant gauge parameter: $\frac{\partial}{\partial\xi}\Sigma^1 \neq 0$. Consequently, Σ^1 cannot be a physical observable! Maybe a pole of the self-energy at some particular mass is, but certainly not the full amplitude, because of the gauge dependence in Feynman rules:

$$P_{\mu\nu}(k,\xi) = \frac{g_{\mu\nu}}{k^2} - (1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}$$
$$k^\mu P_{\mu\nu}(k,0) = 0$$

The second line implies that in the Landau gauge, where $\xi = 0$, the photon propagator is transversal.
If the Feynman rules are gauge dependent, and we use Feynman rules to write down integrals, which then give us amplitudes, whose sum of absolute squares is physically observable, then it is note quite trivial why the result should be gauge independent.
Take, for example, the one-loop graph. This is just a single graph and it is certainly not gauge-invariant. Still, physical observables must be gauge-independent! In QED, merely the full series of graphs,

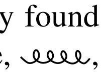


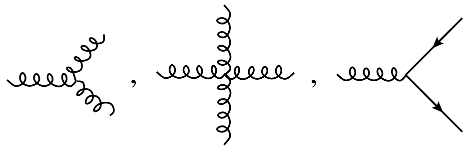
is gauge-independent, even though there are ξ-dependent internal propagators.

- Each graph with ξ in its propagator(s) is gauge dependent.
- The full sum is independent of the gauge.

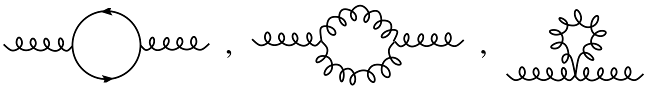
Remark on exerise 4.2: depends on ξ, therefore has a transversal subgraph, so the transversal part vanishes when contracted with the external momentum k_μ, so it kills all ξ dependence.
In QED, ξ drops out "thanks to Ward", but this is more of an accident than a general property of quantum field theories.

1.1 Non-abelian gauge theory: QCD

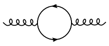
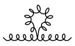
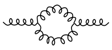
Let's look at quantum chromodynamics, a non-abelian gauge theory. We have already found out before that in a non-abelian gauge theory, self-interaction is possible. If a curly line, , denotes the gauge boson ("gluon"), then the following vertices are allowed:

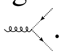
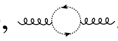


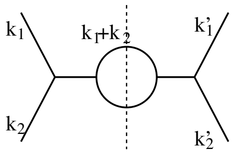
Let us look at QCD from the viewpoint of ξ dependence in a covariant gauge. The possible gauge boson one loop graphs are:



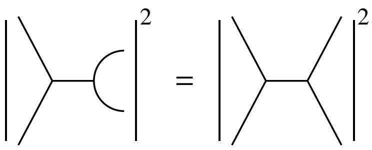
Are they ξ-independent? Are they transversal? We'll find out step by step.

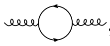
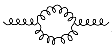
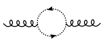
-  $\Rightarrow$ as in QED: ✓
-  $\Rightarrow$ Tadpole graph: The loop is independent of the external momentum, therefore when renormalized, any subtraction removed this graph from the sum. ✓
-  $\Rightarrow$ Two internal propagators with a ξ dependence, no graph to cancel it. Also, it is not transversal.

The incoming and outgoing bosons are transversal. Thus, the graph should also be transversal! However, the internal propagators have transversal and longitudinal degrees of freedom. We need to eliminate the longitudinal contribution of unobserved degrees of freedom somehow.
$\Rightarrow$ We introduce another Feynman rule: The "ghost" particle, denotes by a dotted or dashed line: Ghost propagator , ghost vertex .
The ghost loop, , cancels the extra degrees of freedom.
In 1960, Cutkosky presented a set of rules, which we will get into later on, on how to determine the discontinuity of a Feynman amplitude across a branch cut. These discontinuities are closely connected to the imaginary part of the respective Feynman integral, and to the optical theorem. If we consider a graph where two particles scatter, and the intermediate propagator has a closed loop,



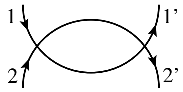
then the imaginary part is somewhat connected to



which is obtained by cutting the original graph in half, indicated by the dashed line.
$\Rightarrow$ The imaginary part of a Feynman integral has something to do with forward on-shell scattering. Still, this does not solve the problem that only certain restricted sets of polarizations for graphs like , ,  are allowed, but internal particles can have any polarization.

1.2 Scalar Field Theory

Let us look at ϕ^4_4 theory, especially one out of the three graphs of the one-loop vertex.



$$\to -i\frac{(ig)^2}{2}\int \mathrm{d}^D q \frac{i^2}{(q^2-m^2+i\epsilon)\big((q-k)^2-m^2+i\epsilon\big)}$$

The two momenta on the left are considered incoming, whereas the two momenta on the right are considered outgoing. Moreover, the momentum k denotes the sum of incoming momenta, which is, of course, equal to the sum of outgoing momenta: $k = k_1 + k_2 = k_1' + k_2'$. Moreover, $k^2 = (k_1 + k_2)^2 \equiv s$ denotes the center of mass energy. If we introduce Feynman parameters, we can basically get down to the integral

$$\int_0^\infty \mathrm{d}A_1 \mathrm{d}A_2 \frac{\exp\left(-\frac{A_1 A_2 s - m^2 (A_1+A_2)^2}{A_1+A_2}\right)}{(A_1+A_2)^2}$$

which will result in the expression

$$\begin{aligned} &-\frac{g^2}{2(4\pi)^2}\int_0^1 \mathrm{d}x \;\ln\left(\frac{m^2 - sx(1-x)}{m^2 - s_0 x(1-x)}\right) \\ =&-\frac{g^2}{16(2\pi)^2}\int_{-1}^{+1} \mathrm{d}y \;\ln\left(\frac{4m^2 - s(1-y^2)}{4m^2 - s_0(1-y)^2}\right) =: I_R \end{aligned}$$

To get from the first to the second line, we substituted $x = \frac{1}{2} - \frac{1}{2}y$. The logarithm in I_R has branch points.

If s is small enough, there is no problem passing the real axis, it is totally regular. For larger s, however, there is a branch cut.
I_R is analytic in the cut s-plane:

$$I_R = \lim_{\epsilon\to 0} I_R(s + i\epsilon, s_0)$$

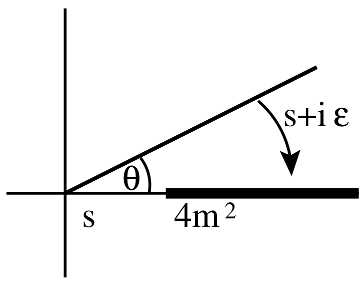


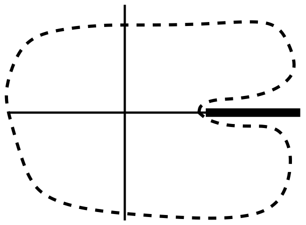
Figure 1: The imaginary part of the logarithm is characterized by the angle θ.

Let us write down the imaginary part:

$$\Im I_R(s,s_0) = \frac{\pi g^2}{16(2\pi)^2}\int_{-\left(1-\frac{4m^2}{s}\right)^{\frac{1}{2}}}^{+\left(1-\frac{4m^2}{s}\right)^{\frac{1}{2}}} \mathrm{d}y = \frac{\pi g^2}{8(2\pi)^2}\sqrt{\frac{s-4m^2}{s}}$$

Of course, this is just one specific example of the computation of an imaginary part of a graph. In general, there are dispersion relations.

1.3 Dispersion Relations



Let $f(s)$ be analytic in the cut-plane $\mathbb{C}/(4m^2\dots)$.

$$f^*(s) = f(s^*)$$

$$f(s) = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \frac{\Im f(s')}{s-s'}\mathrm{d}s'$$

There is a dispersion relation for $f(s)$, namely

$$f(s) - f(s_0) = \frac{s-s_0}{\pi}\int\limits_{4m^2}^{+\infty} \frac{\Im f(s')}{(s'-s)(s'-s_0)}\mathrm{d}s'$$

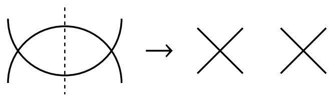
However, anayticity of the S matrix does not tell us enough to not deal with field quantization, etc.

1.4 Cutkosky's cutting rules

Let us look at the imaginary part of I_R again. According to Cutkosky,

$$\Im I_R = \frac{1}{4}\int \frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6 2\omega_k 2\omega_{k'}}(2\pi)^4\delta^{(4)}\left(k_1+k_2-k_1'-k_2'\right)\left|T^{(1)}\right|^2$$

Here, $\left|T^{(1)}\right|^2$ denotes the one-loop matrix element of the decay $\times$, which is just the tree-level vertex and therefore proportional to the coupling g alone.
The imaginary part of a loop graph has something to do with forward scattering:



The internal particles have been put on-shell!
This comes from:

- Unitarity of the S-matrix: $SS^\dagger = \mathbb{I}$
- trivial and nontrivial part of the S-matrix: $S = \mathbb{I} + iT$

$$\Rightarrow \quad (\mathbb{I}+iT)(\mathbb{I}-iT^\dagger) = \mathbb{I} \quad \Rightarrow T - T^\dagger = iTT^\dagger$$
$$\Rightarrow T_{fi} - T_{if}{}^\dagger = i\sum_n (2\pi)^4\delta(k_f-k_i)T_{fn}T_{ni}{}^\dagger$$

This decomposition is a building block of S-matrix theory:

- Analyticity properties of Feynman amplitudes as functions of invariants kinematics $(s,t,u,\dots)$ is assumed.
- Dispersion relations for all amplitudes.

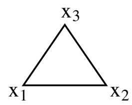
Cutkosky's cutting rules tell us to put internal propagators on the mass-shell in order to determine the imaginary part of a Feynman amplitude.

1.4.1 Derivation of Cutkosky's cutting rules

Before we start, let us briefly note that even though everything that Cutkosky says is there, is actually there, but then again, there might be more that Cutkosky did not discuss.
Here's a reminder of the various scalar propagtors we introduced last term, and their properties.

$$\begin{aligned} \Delta^{\pm}(x) &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \delta(k^2 - m^2)\Theta(\pm k^0)e^{-ikx} \\ \Delta^{\pm}(x) &= \left(\Delta^{\mp}(x)\right)^* \\ \Delta^{\pm}(x) &= \Delta^{\mp}(-x) \\ \Delta^{+} &= \langle 0|\phi(x)\phi(0)|0\rangle \\ \Delta^{-} &= \langle 0|\phi(0)\phi(x)|0\rangle \\ \Delta_F &= \Theta(x^0)\Delta^+(x) + \Theta(-x^0)\Delta^-(x) \end{aligned}$$

The last propagator, Δ_F, is the Feynman propagator.
Let us play a game where we consider the amplitude $F(x_1, x_2, x3)$ assigned to a triangle graph,



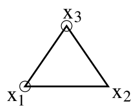
so that

$$F(x_1, x_2, x_3) = (-ig)^3 \Delta_{12}\Delta_{23}\Delta_{31}$$

with the notation $\Delta_{ij} := \Delta_F(x_i - x_j)$.
Next, we enlarge the Feynman rules: We allow to underline some quantities in the amplitude, $F(x_1, \ldots, \underline{x_i}, \ldots, x_N)$, where

Δ_{kl}	$\rightarrow$	Δ_{kl}	if x_k and x_l are both not underlined.
Δ_{kl}	$\rightarrow$	Δ^+_{kl}	if x_k is underlined, but x_l is not underlined.
Δ_{kl}	$\rightarrow$	Δ^-_{kl}	if x_k is not underlined, but x_l is underlined.
Δ_{kl}	$\rightarrow$	Δ^*_{kl}	if x_k and x_l are both underlined.
$(-ig)$	$\rightarrow$	$(+ig)$	if a vertex is underlined.

Let's look at the triangle graph again. Denote "underlining" of spacetime point x by circles around the vertices at x:



So far, this has been pure combinatorics!

Theorem Assume $x_i^0 > x_j^0 \ \forall j \neq i$. ($x_i$ happend later than x_j.) Then:

$$F(x_1, \ldots, \underline{x_i}, \ldots, \underline{x_j}, \ldots, x_N) = -F(x_1, \ldots, x_i, \ldots, \underline{x_j}, \ldots, x_N)$$

Reminder: The sum over all possible underlinings vanishes.

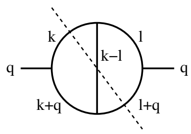
$$\sum_{\substack{\text{all} \\ \text{under-} \\ \text{linings}}} F(x_1, \ldots, x_N) = 0$$

The external particles are plane waves. If the sum is zero, which are the contributions than are not immediately zero?
Using a Fourier transform, we can find that

$$\sum_{\substack{\text{all}\\\text{under-}\\\text{linings}}} F(x_1,\ldots,x_N) = 0 \quad \xrightarrow{\text{FT}} F(k_1,\ldots,k_N) + \bar{F}(k1,\ldots,k_N)$$

$$= -\sum_{\text{cuts}} F_c(k_1,\ldots,k_N)$$

The sum in Fourier space goes over all "cuts" - but what is a cut in a diagram?
We (or rather, Cutkosky did) define a cut such that the diagrams falls apart into two diagrams, and for every line that needs to be removed, or cut, we replace the propagator with a δ function.

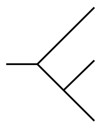


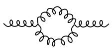
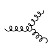
$$\Rightarrow \frac{1}{k^2-m^2} \xrightarrow[\text{}]{\substack{\text{replace}\\\text{by}}} \delta(k^2-m^2)$$

$$\frac{1}{(k-l)^2-m^2} \xrightarrow{\substack{\text{replace}\\\text{by}}} \delta\Big((k-l)^2-m^2\Big)$$

$$\frac{1}{(l-q)^2-m^2} \xrightarrow{\substack{\text{replace}\\\text{by}}} \delta\Big((l+q)^2-m^2\Big)$$

So in this graph, we get three δ functions. With five integrations over scalar variables, this leaves us with two dimensions of spacetime for this amplitude. If we put the cut propagators on the mass shell, we get a scattering amplitude:



where the incoming part is referred to as "sunny" and the outgoing part as "shady".
If we apply the concept of dispersion relations, the optical theorem, and Cutkosky rules on  $\rightarrow \Big|$  $\Big|^2$, we get a contribution $\ln\left(\frac{q^2}{\mu^2}\right)$. In the loop, we sum over four degrees of freedom. In the scattering, there are only two polarization degrees of freedom. Feynman introduced the ghost particle to compensate this mismatch.