GRAPHICAL FUNCTIONS IN PARAMETRIC SPACE

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Abstract. Graphical functions are single-valued functions on the complex plane which arise in quantum field theory. We generalize a formula by N. Nakanishi for graphical functions in parametric space. With this result we show that graphical functions are real analytic on the punctured complex plane $\mathbb{C}\backslash\{0,1\}$. Moreover we prove a formula that relates graphical functions of planar dual graphs.

1. Introduction

1.1. **Graphical functions.** Graphical functions were introduced in [11] basically as a tool for calculating Feynman periods in ϕ^4 quantum field theory (see also [10]). Some graphical functions also appear as amplitudes and as correlation functions in $N = 4$ Super Yang-Mills Theory [4], [5].

Let G be a graph with three distinguished vertices labeled 0, 1, and z. We call the vertices 0, 1, z 'external' while all other vertices of G are 'internal'. We fix the dimension

$$d = 2\lambda + 2 > 2 \tag{1.1}$$

and associate to every internal vertex v of G a d-dimensional integration variable $x_v \in \mathbb{R}^d$. The external vertices 0 and 1 correspond to the origin in $\mathbb{R}^d$ and a unit vector (say the column vector $(1, 0, \ldots, 0)^t$), respectively. The vertex z is a variable which for now is a vector in $\mathbb{R}^d$ (soon it will become a complex number). An edge e between vertices u and v corresponds to the quadratic form Q_e which is the square of the Euclidean distance between u and v,

$$Q_e = ||u - v||^2. \tag{1.2}$$

Moreover, every edge e has an edge weight $\nu_e \in \mathbb{R}$. For any subgraph g of G with edge set $\mathcal{E}_g$ we define

$$N_g = \sum_{e\in\mathcal{E}_g} \nu_e \tag{1.3}$$

as the sum of edge weights in g.

The graphical function of G is given by the integral

$$f_G^{(\lambda)}(z) = \left(\prod_{v \text{ internal}} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d x_v}{\pi^{d/2}}\right) \frac{1}{\prod_e Q_e^{\lambda\nu_e}}, \tag{1.4}$$

where the first product is over all internal vertices of G and the second product is over all edges of G.

The convergence of the above integral is equivalent to two conditions named 'infrared' and 'ultraviolet' (this is the weighted analog of Lemma 3.4 in [11]). The infrared condition is that any subgraph g with at least one edge and no edges between external vertices fulfills

$$(d-2)N_g > dV_g^{\mathrm{int}}, \tag{1.5}$$

where V_g^{int} is the number of internal vertices v in g with the property that all edges which are adjacent to v in G are also in g.

The ultraviolet condition is that any subgraph g with at least one edge such that at most one of its V_g vertices is external fulfills

$$(d-2)N_g < d(V_g - 1). \tag{1.6}$$

By symmetry, $f_G^{(\lambda)}$ depends only on the modulus of z and the angle between z and the unit vector 1. Without loss of information we can hence restrict $f_G^{(\lambda)}$ to a two-dimensional plane. We identify this plane with the complex numbers $\mathbb{C}$ and choose the complex number 1 for the unit vector with label '1'. Equivalently, we may specify the vectors associated to the external vertices as

$$0:(0,\ldots,0)^t, \quad 1:(1,0,\ldots,0)^t, \quad z:(\mathrm{Re}z,\mathrm{Im}z,0,\ldots,0)^t. \tag{1.7}$$

From now on we consider graphical functions as functions on $\mathbb{C}$.

In [11] 'completions' of graphical functions were defined. In this article, however, we use uncompleted graphs.

Examples of graphs are depicted in Figure 1.

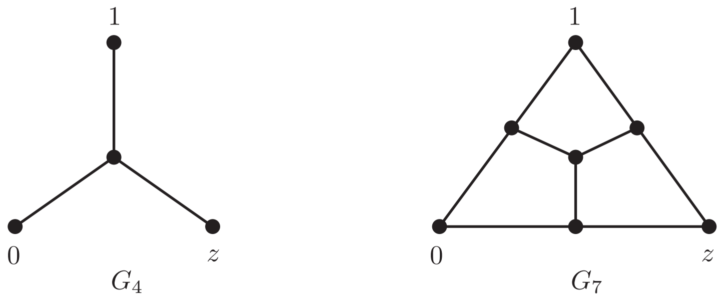

Figure 1: Graphical functions with four and with seven vertices.

In $d = 4$ dimensions G_4 has the graphical function [11]

$$f_{G_4}^{(1)}(z) = \frac{4\mathrm{i}D(z)}{z - \overline{z}},$$

where D is the Bloch-Wigner dilogarithm,

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|).$$

The Bloch-Wigner dilogarithm is a single-valued version of the dilogarithm $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$. It is real analytic on $\mathbb{C}\backslash\{0,1\}$ and antisymmetric under complex conjugation $D(z) = -D(\overline{z})$. These properties of the Bloch-Wigner dilogarithm lift to general properties of graphical functions:

Theorem 1.1. *Let G be a graph which fulfills the infrared and ultraviolet conditions* (1.5) *and* (1.6). *Then the graphical function $f_G^{(\lambda)} : \mathbb{C} \setminus \{0,1\} \longrightarrow \mathbb{R}_+$ has the following general properties:*

(G1)

$$f_G^{(\lambda)}(z) = f_G^{(\lambda)}(\overline{z}). \tag{1.8}$$

(G2) $f_G^{(\lambda)}$ *is single-valued.*

(G3) $f_G^{(\lambda)}$ *is real analytic on* $\mathbb{C}\backslash\{0,1\}$.

It was not possible to prove real analyticity (G3) in full generality with the methods in [11]. In this article we obtain (G3) as a consequence of an alternative integral representation of graphical functions. This integral representation uses parametric space where integration variables are associated to edges of the graph [6], [1].

1.2. **Graph polynomials.** The graph (or Kirchhoff) polynomial of a graph G is defined by associating a variable α_e to every edge e of G and setting

$$\Psi_G(\alpha) = \sum_{T\,\text{span. tree}} \prod_{e\notin T} \alpha_e, \tag{1.9}$$

where the sum is over all spanning trees T of G [7].

Spanning forest polynomials are generalizations of the graph polynomial. They were defined and studied by F. Brown and K. Yeats [3].

Definition 1.2. *Let G be a graph with external vertices $0, 1, z$. Let $p = \{p_1, \ldots, p_n\}$ ($n \leq 3$) be a partition of the set $\{0, 1, z\}$ of external vertices. Let $\mathcal{F}_G^p$ be the set of spanning forests with n trees $T_1 \cup \ldots \cup T_n$ such that the external vertices of p_i are in T_i (and only in T_i). The spanning forest polynomial associated to p is*

$$\Psi_G^p(\alpha) = \sum_{F\in\mathcal{F}_G^p} \prod_{e\notin F} \alpha_e. \tag{1.10}$$

We denote the five partitions of $\{0, 1, z\}$ by $\{01z\}$ if $n = 1$, $\{1z, 0\}$, $\{0z, 1\}$, $\{01, z\}$ if $n = 2$, $\{0, 1, z\}$ if $n = 3$ and drop the wavy brackets in the superscript of Ψ_G^p.

Let $\overline{z}$ be the complex conjugate of $z \in \mathbb{C}$ (which also serves as a label in G). We define

$$\Phi_G(\alpha, z) = \Psi_G^{1z,0}(\alpha)(z-1)(\overline{z}-1) + \Psi_G^{0z,1}(\alpha)z\overline{z} + \Psi_G^{01,z}(\alpha). \tag{1.11}$$

The spanning forest polynomial Ψ_G^{01z} is the graph polynomial Ψ_G while the spanning forest polynomial $\Psi_G^{0,1,z}$ equals the graph polynomial $\Psi_{G/\text{ext}}$ of the graph G/ext that one obtains from G by identifying the three external vertices without changing the edge labels.

Example 1.3. *If we label the three edges adjacent to 0, 1, z in G_4 (see Figure 1) by 1, 2, 3, respectively, then*

$$\begin{aligned}
\Psi_{G_4}^{01z}(\alpha) &= \Psi_{G_4}(\alpha) = 1,\\
\Psi_{G_4}^{1z,0}(\alpha) &= \alpha_1,\\
\Psi_{G_4}^{0z,1}(\alpha) &= \alpha_2,\\
\Psi_{G_4}^{01,z}(\alpha) &= \alpha_3,\\
\Psi_{G_4}^{0,1,z}(\alpha) &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3,\\
\Phi_{G_4}(\alpha, z) &= \alpha_1(z-1)(\overline{z}-1) + \alpha_2 z\overline{z} + \alpha_3.
\end{aligned}$$

Let $\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ be the gamma function. A parametric (i.e. depending on the edge parameters α_e) formula for (massive) position space amplitudes in four-dimensional Minkowski space was given by N. Nakanishi (Equation (8-33) in [8]). In the massless case this formula, translated into Euclidean space, gives a parametric representation for four-dimensional graphical functions. We give an independent proof of the parametric formula in arbitrary dimensions.

Theorem 1.4. *Let G be a non-empty graph with E_G edges, V_G^{int} internal vertices, and three external vertices $0,1,z$. We label the edges of G by $1,2,\ldots,E_G$ and assume that every edge e has an edge weight $\nu_e>0$. We further assume that the graphical function $f_G^{(\lambda)}$ exists. Let*

$$M_G=\lambda N_G-(\lambda+1)V_G^{\mathrm{int}}. \tag{1.12}$$

Then the graphical function is given in parametric space as the projective integral

$$f_G^{(\lambda)}(z)=\frac{\Gamma(M_G)}{\prod_{e=1}^{E_G}\Gamma(\lambda\nu_e)}\int_\Delta\frac{\prod_{e=1}^{E_G}\alpha_e^{\lambda(1-\nu_e)}}{\Phi_G(\alpha,z)^{M_G}\Psi_G^{0,1,z}(\alpha)^{\lambda+1-M_G}}\Omega(\alpha), \tag{1.13}$$

where

$$\Omega(\alpha)=\sum_{e=1}^{E_G}(-1)^{e-1}\alpha_e\mathrm{d}\alpha_1\wedge\ldots\wedge\widehat{\mathrm{d}\alpha_e}\wedge\ldots\wedge\mathrm{d}\alpha_{E_G} \tag{1.14}$$

is the top form in $\mathbb{P}^{E_G-1}\mathbb{R}$ and

$$\Delta=\{(\alpha_1:\alpha_2:\ldots:\alpha_{E_G}),\ \alpha_e>0 \text{ for all } e\in\{1,2,\ldots,E_G\}\}\subset\mathbb{P}^{E_G-1}\mathbb{R} \tag{1.15}$$

is the positive coordinate simplex.

Readers who are not familiar with projective integrals can specialize to an affine integral by setting $\alpha_1=1$ and integrating the α_e, $e>1$ from 0 to ∞.

Theorem 2.1 gives a (Cremona-)dual parametric representation which is valid for any edge weights $\nu_e\in\mathbb{R}$.

Note that M_G is restricted by convergence. From (1.5) with $g=G$ and from (1.6) with $g=G\backslash\{0,1\}$, $g=G\backslash\{0,z\}$, or $g=G\backslash\{1,z\}$ we obtain for a graph G with no edges between external vertices

$$0<M_G<\lambda\min\{N_0+N_1,N_0+N_z,N_1+N_z\}, \tag{1.16}$$

where N_i is the sum of weights of edges adjacent to the external vertex i.

One immediate advantage of the parametric representation is that for many graphs with not more than nine vertices the graphical function can be calculated by parametric integration developed by F. Brown [2] and E. Panzer [9].

1.3. **Planar duals.** An (externally) planar dual $G^\star$ of a graph G with external vertices $0,1,z$ is a planar dual graph which has 'opposite' external labels (see Figure 2, see Definition 4.1 for a precise definition).

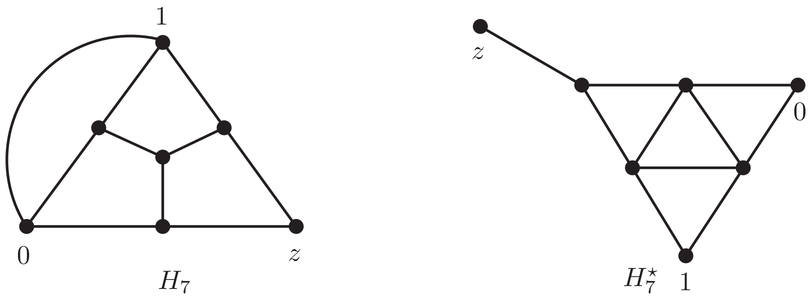

Figure 2: The graphs H_7 and $H_7^\star$ are planar duals.

In the case that $M_G = \lambda + 1$ graphical functions of dual graphs are related:

Theorem 1.5. *Let G be a connected graph with external vertices $0, 1, z$ and edge weights $\nu_e > 0$ such that the graphical function $f_G^{(\lambda)}$ exists and*

$$M_G = \lambda + 1. \tag{1.17}$$

Let G have a dual $G^\star$. The edges $e^\star$ of $G^\star$ are in one to one correspondence to the edges e of G. Let the edge weights $\nu_{e^\star}$ of $G^\star$ be related to the edge weight ν_e of G by

$$\nu_{e^\star} = 1 + \lambda^{-1} - \nu_e. \tag{1.18}$$

Then,

$$f_{G^\star}^{(\lambda)}(z) = \frac{\prod_e \Gamma(\lambda\nu_e)}{\prod_{e^\star} \Gamma(\lambda\nu_{e^\star})} f_G^{(\lambda)}(z), \tag{1.19}$$

where the products are of the edges in G or in $G^\star$, respectively.

Note that ultraviolet convergence (1.6) for a single edge e gives $\lambda\nu_e < \lambda + 1$. Hence $\nu_e^\star > 0$. Similarly, positive edge weights in G ensure that the dual graphical function $f_{G^\star}^{(\lambda)}$ is convergent.

If in four dimensions a graph G has edge weights 1 then a dual graph $G^\star$ has also edge weights 1 and the graphical functions are equal if $M_G = 2$.

One can also use duality for a planar graph G with $M_G \neq \lambda + 1$ if one adds an edge from 0 to 1 of weight $(\lambda+1-M_G)/\lambda$, see the subsequent example and Remark 4.3.

Example 1.6. *We want to calculate the four dimensional graphical function of the graph G_7 in Figure 1 with unit edge weights. We find $M_{G_7} = 1$. To apply Theorem 1.5 we add an edge between 0 and 1 which contributes to the graphical function by a factor of 1 (see Figure 2). Hence $f_{G_7}^{(1)} = f_{H_7}^{(1)}$. Theorem 1.5 gives $f_{H_7}^{(1)} = f_{H_7^\star}^{(1)}$. The graphical function of $H_7^\star$ can be calculated by the techniques completion and appending of an edge* [11]. *We obtain*

$$f_{G_7}^{(1)} = 20\zeta(5)\frac{4\mathrm{i}D(z)}{z-\overline{z}},$$

where $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ is the Riemann zeta function.

One obtains a self dual graph H_4 with $M_{H_4} = 2$ if one adds an edge from 0 to 1 to G_4. In this case planar duality leads to an empty statement.

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2. proof of Theorem 1.4

Although we are mainly interested in the case of three external vertices 0, 1, z the result of this section effortlessly generalizes to an arbitrary number of external vertices $z_1, \ldots, z_{V^{\mathrm{ext}}} \in \mathbb{R}^d$. Definition 1.2 generalizes straighforwardly. The generalization of (1.11) is

$$\Phi_G(\alpha, z) = \sum_{1\leq i<j\leq V^{\mathrm{ext}}} \Psi_G^{z_iz_j,(z_k)_{k\neq i,j}}(\alpha)||z_i - z_j||^2. \tag{2.1}$$

We first prove a (Cremona-)dual version of Theorem 1.4 which has the advantage that it includes the case of negative edge weights. The dual spanning forest polynomials are given by products over edge variables in the spanning forest,

$$\tilde{\Psi}_G^p(\alpha) = \sum_{F\in\mathcal{F}_G^p} \prod_{e\in F} \alpha_e. \tag{2.2}$$

The duality transformation of spanning forest polynomials is given by a coordinate inversion,

$$\begin{aligned} \Psi_G^p(\alpha) &= \Big(\prod_e \alpha_e\Big)\tilde{\Psi}_G^p(\alpha^{-1}), \\ \Phi_G(\alpha, z) &= \Big(\prod_e \alpha_e\Big)\tilde{\Phi}_G(\alpha^{-1}, z). \end{aligned} \tag{2.3}$$

Theorem 2.1. *Let G be a non-empty graph with edge weights $\nu_e \in \mathbb{R}$ and external vertices $z_1, \ldots, z_{V^{\mathrm{ext}}} \in \mathbb{R}^d$ such that the graphical function $f_G^{(\lambda)}$ exists. For any set of non-negative integers n_e such that $n_e + \lambda\nu_e > 0$ we have the following dual parametric representation for $f_G^{(\lambda)}$:*

$$f_G^{(\lambda)}(z) = \frac{(-1)^{\sum_e n_e}\,\Gamma(M_G)}{\prod_e \Gamma(n_e+\lambda\nu_e)} \int_\Delta \left[\Big(\prod_e \alpha_e^{n_e+\lambda\nu_e-1}\partial_{\alpha_e}^{n_e}\Big)\frac{1}{\tilde{\Phi}_G(\alpha,z)^{M_G}\,\tilde{\Psi}(\alpha)^{\lambda+1-M_G}}\right]\Omega(\alpha), \tag{2.4}$$

where M_G is given by (1.12),

$$\tilde{\Psi} = \tilde{\Psi}_G^{z_1,\ldots,z_{V^{\mathrm{ext}}}}, \tag{2.5}$$

the integration cycle Δ is the projective positive coordinate simplex (1.15), and Ω is the projective top form (1.14).

Remark 2.2. *For negative integer $\lambda\nu_e$ one may set $n_e = -\lambda\nu_e + 1$ and trivially perform the α_e integration.*

Proof of the theorem. The proof follows the Schwinger trick (see e.g. [6]). We first assume that G has no edges between external vertices. By convergence G cannot be a single edge, so we may assume that G has at least two edges. From the definition of the gamma function we obtain for $n+\lambda\nu>0$ the formula

$$\frac{1}{A^{\lambda\nu}}=\frac{1}{\Gamma(n+\lambda\nu)}\int_0^\infty \alpha^{n+\lambda\nu-1}(-\partial_\alpha)^n\mathrm{e}^{-\alpha A}\mathrm{d}\alpha. \tag{2.6}$$

We use this formula to replace the product of propagators in the definition (1.4) of the graphical function $f_G^{(\lambda)}$ by an integral over the edge parameters α_e. Because the integrand is positive the integral is absolutely convergent and we can use Fubini's theorem to interchange integrations. By continuity of Gaussian integrals we can also interchange the integration over the vertex variables with the partial derivatives ∂_{α_e} and obtain

$$f_G^{(\lambda)}(z)=\frac{(-1)^{\sum_e n_e}}{\prod_e\Gamma(n_e+\lambda\nu_e)}\int_0^\infty\ldots\int_0^\infty\Big(\prod_e\alpha^{n_e+\lambda\nu_e-1}\partial_{\alpha_e}^{n_e}\Big)\mathcal{I}(\alpha)\prod_e\mathrm{d}\alpha_e, \tag{2.7}$$

where $\mathcal{I}(\alpha)$ is the Gaussian integral

$$\mathcal{I}(\alpha)=\left(\prod_{v\text{ internal}}\int_{\mathbb{R}^d}\frac{\mathrm{d}^dx_v}{\pi^{d/2}}\right)\exp\left(-\sum_e\alpha_eQ_e\right).$$

The quadratic form Q_e is diagonal

$$Q_e=Q_e^1+\ldots+Q_e^d,$$

where the subscript i in Q_e^i indicates the dependence on the ith coordinate of the vertex variables. Hence the integral $\mathcal{I}(\alpha)$ factorizes into d parts, one for each coordinate,

$$\mathcal{I}(\alpha)=\prod_{i=1}^d\mathcal{I}_i(\alpha).$$

The argument in the exponential of $\mathcal{I}_i$ is a quadratic form in the ith coordinate of the vertex variables. The V_G vertex variables of G split into internal and external variables. We arrange the coordinates to the V_G dimensional vector $(x,z)^t$ where $x=(x_v^i)_{v=1,\ldots,V^{\mathrm{int}}}$ and $z=(z_k^i)_{k=1,\ldots,V^{\mathrm{ext}}}$. Then, the quadratic form in the exponential of $\mathcal{I}_i$ has the general structure

$$\sum_e\alpha_eQ_e^i=x^tL^{\mathrm{ii}}(\alpha)x+x^tL^{\mathrm{ie}}(\alpha)z+z^tL^{\mathrm{ei}}(\alpha)x+z^tL^{\mathrm{ee}}(\alpha)z$$

where (by convergence) L^{ii} is positive definite. By symmetry $(L^{\mathrm{ei}})^t=L^{\mathrm{ie}}$ and both L^{ii} and L^{ee} are symmetric. We complete the quadratic form to a perfect square, shift the integration variable to $x+L^{\mathrm{ii}-1}L^{\mathrm{ie}}z$ and obtain by a standard calculation

$$\mathcal{I}_i=\det(L^{\mathrm{ii}})^{-1/2}\exp\Big(z^tL^{\mathrm{ei}}L^{\mathrm{ii}-1}L^{\mathrm{ie}}z-z^tL^{\mathrm{ee}}z\Big). \tag{2.8}$$

From the quadratic forms Q_e the (Laplacian) matrix

$$L=\begin{pmatrix}L^{\mathrm{ii}} & L^{\mathrm{ie}}\\ L^{\mathrm{ei}} & L^{\mathrm{ee}}\end{pmatrix}$$

inherits the structure (with possible multiple edges):

$$L(\alpha)_{uv}=\begin{cases}\sum\limits_{e\text{ incident to }v}\alpha_e & \text{if } u=v,\\ -\sum_{e=\{u,v\}}\alpha_e & \text{otherwise.}\end{cases} \tag{2.9}$$

Now, we orient the edges e in an arbitrary way and define the incidence matrix

$$I(\alpha)_{ev} = \begin{cases} \alpha_e^{1/2} & \text{if } e \text{ begins in } v, \\ -\alpha_e^{1/2} & \text{if } e \text{ ends in } v, \\ 0 & \text{otherwise.} \end{cases}$$

By the above descriptions of I and L it is clear that

$$(I^t I)_{uv} = \sum_e I_{eu} I_{ev} = L_{uv}.$$

With this identity we show that (see (2.5))

$$\det(L^{\mathrm{ii}}) = \tilde{\Psi} :$$

Let $\mathcal{E}_G$ denote the set of edges of G and I_E denote the submatrix of I with rows in E. We use the Binet-Cauchy theorem to calculate the determinant of L^{ii}, yielding

$$\det(L^{\mathrm{ii}}) = \sum_{\substack{E \subseteq \mathcal{E}_G \\ |E| = V^{\mathrm{int}}}} \det(I_E)^2. \tag{2.10}$$

The rows of I_E correspond to edges in G and are of the general form

$$I_{\{e\}} = \alpha_e^{1/2}(0, \ldots, 0, \pm 1, 0, \ldots, 0, \mp 1, 0, \ldots, 0) \tag{2.11}$$

if e connects two internal vertices and

$$I_{\{e\}} = \alpha_e^{1/2}(0, \ldots, 0, \pm 1, 0, \ldots, 0) \tag{2.12}$$

if e connects an internal vertex with an external vertex. Assume E contains a subset E_0 which is either a cycle or a path that connects two external vertices. Choose an orientation on E_0 and set $\mathrm{sgn}(e) = +1$ if an edge $e \in E_0$ is parallel to this orientation; otherwise $\mathrm{sgn}(e) = -1$. Then

$$\sum_{e \in E_0} \mathrm{sgn}(e) \alpha_e^{-1/2} I_{\{e\}} = 0.$$

We conclude that the rows in I_E are linearly dependent and $\det(I_E) = 0$. So, non-zero contributions to (2.10) can only come from forests in $\mathcal{F}_G^{z_1, \ldots, z_{V^{\mathrm{ext}}}}$ (they are spanning because $|E| = V^{\mathrm{int}}$). In this case the matrix I_E is block diagonal (with one block for each tree $T \subset E$) and $\det(I_E)$ factorizes. If we arrange the vertices and edges along T (starting with the external vertex in T) then the block associated to T is triangular with diagonal elements $\pm\alpha_e^{1/2}$, $e \in T$. Altogether

$$\det(I_E) = \begin{cases} \pm \prod_{e \in E} \alpha_e^{1/2} & \text{if } E \in \mathcal{F}_G^{z_1, \ldots, z_{V^{\mathrm{ext}}}}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

and the claim follows.

The next step of the proof is to calculate the inverse of L^{ii}. If $M^{(u,v)}$ is the matrix M with the uth row and the vth column deleted then

$$\left(L^{\mathrm{ii}-1}\right)_{u,v} = \frac{(-1)^{u+v}}{\det(L^{\mathrm{ii}})} \det(L^{\mathrm{ii}(v,u)}).$$

We again use the Binet-Cauchy theorem and obtain

$$\det(L^{\mathrm{ii}(u,v)}) = \sum_{\substack{E \subseteq \mathcal{E}_G \\ |E| = V^{\mathrm{int}} - 1}} \det(I_E^{(v)}) \det(I_E^{(u)}),$$

where the superscripts (u), (v) mean that we delete the corresponding column in I_E. From (2.11) and (2.12) we see that removing one column (say v) from I_E is equivalent to interpreting v as an external vertex. From (2.13) we obtain

$$\det(I_E^{(v)}) = \begin{cases} \pm \prod_{e\in F} \alpha_e^{1/2} & \text{if } F \in \mathcal{F}_G^{v,z_1,\ldots,z_{V^{\text{ext}}}}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the product $\det(I_E^{(v)})\det(I_E^{(u)})$ has only contributions from forests in the intersection $\mathcal{F}_G^{u,z_1,\ldots,z_{V^{\text{ext}}}} \cap \mathcal{F}_G^{v,z_1,\ldots,z_{V^{\text{ext}}}} = \mathcal{F}_G^{uv,z_1,\ldots,z_{V^{\text{ext}}}}$. We obtain the inverse of L^{ii} up to signs

$$\left(L^{\text{ii}-1}\right)_{u,v} = \pm\frac{1}{\tilde{\Psi}}\tilde{\Psi}_G^{uv,z_1,\ldots,z_{V^{\text{ext}}}}.$$

Since L^{ii} is positive definite (for $\alpha_e > 0$), symmetric with non-positive off-diagonal entries it is a Stieltjes matrix. In general, the inverse of a Stieltjes matrix has only non-negative entries (see e.g. Corollary 3.24 in [14]). So, in the above formula we have plus signs.

Now we proceed to calculate $\mathcal{I}_i$ in (2.8). From (2.9) we obtain

$$(\tilde{\Psi} z^t L^{\text{ei}} L^{\text{ii}-1} L^{\text{ie}} z)(\alpha) = \sum_{k,\ell=1}^{V^{\text{ext}}} z_k^i z_\ell^i \sum_{\substack{e=\{z_k,u\}\\ f=\{z_\ell,v\}}} \tilde{\Psi}_G^{uv,z_1,\ldots,z_{V^{\text{ext}}}}(\alpha)\alpha_e\alpha_f. \tag{2.14}$$

Here the $z_\bullet^i$ are ith coordinates of the d-dimensional vector $z_\bullet$. We want to interpret the second sum in terms of subgraphs of G. We have to distinguish three cases

(1) $k \neq \ell$: Adding the two edges e, f to the spanning forest connects the three trees $T_{z_k} \ni z_k$, $T_{z_\ell} \ni z_\ell$, and $T_{uv} \ni u, v$. This gives a tree $T_{z_k z_\ell}$ that connects z_k and z_ℓ. Conversely in each tree $T_{z_k z_\ell}$ we have a unique path connecting z_k and z_ℓ. The edges e, f are unique in this path such that $z_k \in e$ and $z_\ell \in f$. Summing over u and v we obtain

$$\sum_{\substack{e=\{z_k,u\}\\ f=\{z_\ell,v\}}} \tilde{\Psi}_G^{uv,z_1,\ldots,z_{V^{\text{ext}}}}(\alpha)\alpha_e\alpha_f = \tilde{\Psi}_G^{z_k z_\ell,(z_m)_{m\neq k,\ell}}(\alpha). \tag{2.15}$$

(2) $k = \ell$ and $e \neq f$: Adding the two edges e, f connects T_{uv} and T_{z_k} to a graph C_{z_k} with one cycle which contains z_k. We obtain a spanning subgraph $C_{z_k} \cup \bigcup_{m\neq k} T_{z_m}$ with trees T_{z_m}. Let $\mathcal{CF}_k$ denote the set of all such spanning subgraphs. A graph $g \in \mathcal{CF}_k$ uniquely defines the pair of edges e, f adjacent to z_k in the cycle of g. Upon interchanging e and f we obtain every graph in $\mathcal{CF}_k$ twice. The sum over u and v gives

$$\sum_{\substack{e=\{z_k,u\}\neq\\ f=\{z_k,v\}}} \tilde{\Psi}_G^{uv,z_1,\ldots,z_{V^{\text{ext}}}}(\alpha)\alpha_e\alpha_f = 2\sum_{g\in\mathcal{CF}_k}\prod_{e\in g}\alpha_e. \tag{2.16}$$

(3) $k = \ell$ and $e = f$: In this case $u = v$ and e connects T_u and T_{z_k} to a tree T that contains u and z_k. In T there exists a unique path that connects u with z_k. The edge in this path that is adjacent to z_k is counted twice. Summing over u gives

$$\sum_{e=\{z_k,u\}} \tilde{\Psi}_G^{u,z_1,\ldots,z_{V^{\text{ext}}}}(\alpha)\alpha_e^2 = \sum_{F\in\mathcal{F}_G^{z_1,\ldots,z_{V^{\text{ext}}}}} \Big(\prod_{e\in F}\alpha_e\Big) \sum_{\substack{f\in F\\ f \text{ adjacent to } z_k}} \alpha_f. \tag{2.17}$$

If T_{z_k} in $\mathcal{F}_G^{z_1,\ldots,z_{V^{\mathrm{ext}}}}$ is the isolated vertex z_k then the sum over f on the right hand side is empty and vanishes (by definition).

Because G has no edges between external vertices L^{ee} is diagonal (see (2.9)). From (2.14) we have to subtract (see (2.8))

$$(\tilde{\Psi} z^t L^{\mathrm{ee}} z)(\alpha) = \sum_{k=1}^{V^{\mathrm{ext}}} (z_k^i)^2 \sum_{F \in \mathcal{F}_G^{z_1,\ldots,z_{V^{\mathrm{ext}}}}} \Big(\prod_{e \in F} \alpha_e\Big) \sum_{f \text{ adjacent to } z_k} \alpha_f.$$

Again we have to distinguish three cases:

(1) $f \notin F$, $f \cup F$ is a forest. Then f connects two trees in F. Because the path between z_k and z_ℓ in $T_{z_k z_\ell}$ is unique there exists a unique edge f in $T_{z_k z_\ell}$ with $z_k \in f$ such that $T_{z_k z_\ell} \backslash f$ does not connect z_k and z_ℓ. Therefore

$$\sum_{F \in \mathcal{F}_G^{z_1,\ldots,z_{V^{\mathrm{ext}}}}} \Big(\prod_{e \in F} \alpha_e\Big) \sum_{\substack{f \text{ adjacent to } z_k \\ f \notin F,\, f \cup F \text{ is a forest}}} \alpha_f = \sum_{\ell=1}^{V^{\mathrm{ext}}} \tilde{\Psi}_G^{z_k z_\ell, (z_m)_{m \neq k,\ell}}(\alpha).$$

(2) $f \notin F$, $f \cup F$ contains a cycle. Because z_k is adjacent to f the cycle contains z_k and $f \cup F \in \mathcal{CF}_k$. In a cycle two edges are adjacent to z_k. Therefore we obtain every $g \in \mathcal{CF}_k$ twice. This part of the sum over F gives the right hand side of (2.16).

(3) $f \in F$. This is the right hand side of (2.17).

In (2.8) the contributions from cases (2) and (3) cancel. From case (1) we obtain

$$\mathcal{I}_i = \tilde{\Psi}^{-1/2} \exp\Big(- \tilde{\Psi}^{-1} \sum_{k,\ell=1}^{V^{\mathrm{ext}}} ((z_k^i)^2 - z_k^i z_\ell^i) \tilde{\Psi}_G^{z_k z_\ell, (z_m)_{m \neq k,\ell}} \Big).$$

The terms with $k = \ell$ cancel. We split the sum into $k < \ell$ and $k > \ell$ and interchange k with ℓ in the second case. Summing over i gives the polynomial $\tilde{\Phi}_G$ in (2.1),

$$\mathcal{I} = \tilde{\Psi}^{-d/2} \exp(-\tilde{\Phi}_G / \tilde{\Psi}).$$

The polynomial $\tilde{\Psi}$ has degree V^{int} whereas $\tilde{\Phi}_G$ has degree $V^{\mathrm{int}} + 1$ in α. Infrared convergence for $g = G$ ensures that we have at least one edge (say edge 1) with positive weight. We now assume $n_1 = 0$ and return to the case $n_1 > 0$ later. For all edges $e \neq 1$ we substitute α_e by $\alpha_e \alpha_1$ in (2.7) and obtain for $(-1)^{\sum_e n_e} \prod_e \Gamma(n_e + \lambda\nu_e) f_G^{(\lambda)}(z)$ the expression

$$\int_0^\infty \cdots \int_0^\infty \alpha_1^{M_G - 1} \Big(\prod_{e \neq 1} \alpha_e^{n_e + \lambda\nu_e - 1} \partial_{\alpha_e}^{n_e}\Big) \tilde{\Psi}^{-d/2} \exp\Big(- \alpha_1 \frac{\tilde{\Phi}_G}{\tilde{\Psi}}\Big) \prod_e \mathrm{d}\alpha_e,$$

where $\tilde{\Phi}_G = \tilde{\Phi}_G(1, \alpha_2, \ldots, z)$ and $\tilde{\Psi} = \tilde{\Psi}(1, \alpha_2, \ldots)$ are evaluated at $\alpha_1 = 1$. Using (2.6) for $n = 0$ to evaluate the integral over α_1 we obtain

$$f_G^{(\lambda)}(z) = \frac{(-1)^{\sum_e n_e} \Gamma(M_G)}{\prod_e \Gamma(n_e + \lambda\nu_e)} \int_0^\infty \cdots \int_0^\infty \Big(\prod_{e \neq 1} \alpha_e^{n_e + \lambda\nu_e - 1} \partial_{\alpha_e}^{n_e}\Big) \frac{\prod_{e \neq 1} \mathrm{d}\alpha_e}{\tilde{\Phi}_G^{M_G} \tilde{\Psi}^{\lambda + 1 - M_G}}.$$

The integrand has degree $1 - \lambda\nu_1 - E_G$ in α (where $E_G = |\mathcal{E}_G|$ is the number of edges of G). It hence lifts to the projective integral (2.4).

To prove the case $n_1 > 0$ by induction we use the affine chart $\alpha_2 = 1$ (where the orientation of Δ is opposite to the canonical order) and integrate by parts in α_1.

Finally, we prove that (2.4) remains valid if G has edges between external vertices. Let G have an edge e that connects the external vertices z_1 and z_2. Because $e \notin F$ for all $F \in \mathcal{F}_G^{z_1,\ldots,z_{V^{\mathrm{ext}}}}$ we have

$$\tilde{\Psi}_G^{z_1,\ldots,z_{V^{\mathrm{ext}}}} = \tilde{\Psi}_{G\backslash e}^{z_1,\ldots,z_{V^{\mathrm{ext}}}}.$$

Likewise, for $\{k,\ell\} \neq \{1,2\}$,

$$\tilde{\Psi}_G^{z_k z_\ell,(z_m)_{m\neq k,\ell}} = \tilde{\Psi}_{G\backslash e}^{z_k z_\ell,(z_m)_{m\neq k,\ell}},$$

whereas the forests $F \in \mathcal{F}_G^{z_1 z_2,(z_m)_{m\neq 1,2}}$ split into two sets depending on whether or not e is in F. This yields

$$\tilde{\Psi}_G^{z_1 z_2,(z_m)_{m\neq 1,2}} = \tilde{\Psi}_{G\backslash e}^{z_1 z_2,(z_m)_{m\neq 1,2}} + \alpha_e \tilde{\Psi}_{G\backslash e}^{z_1,\ldots,z_{V^{\mathrm{ext}}}}.$$

For $\tilde{\Phi}_G$ we obtain the formula

$$\tilde{\Phi}_G = \tilde{\Phi}_{G\backslash e} + \alpha_e \tilde{\Psi}_{G\backslash e}^{z_1,\ldots,z_{V^{\mathrm{ext}}}} ||z_1 - z_2||^2.$$

We use the affine chart $\alpha_f = 1$ for an $f \neq e$ to prove (2.4) for G. With the elementary integral formula

$$\int_0^\infty \alpha^{n+\lambda\nu-1}(-\partial_\alpha)^n (A+\alpha B)^{-M_G}\mathrm{d}\alpha = \frac{\Gamma(n+\lambda\nu)\Gamma(M_G-\lambda\nu)}{\Gamma(M_G)A^{M_G-\lambda\nu}B^{\lambda\nu}}$$

we can evaluate the integral over α_e and arrive with $M_{G\backslash e} = M_G - \lambda\nu_e$ at $Q_e^{-\lambda\nu_e}$ times the dual parametric representation for $G\backslash e$. Hence, the parametric representation is valid for G. By induction over the number of edges between external vertices the result follows. $\square$

Theorem 1.4 follows as a corollary from Theorem 2.1.

Proof of Theorem 1.4. We set $n_e = 0$ for all edges e of G. We use the affine chart $\alpha_1 = 1$ in (2.4) and invert all α_e, $e > 1$. By (2.3) this gives the integrand in (1.13) for $\alpha_1 = 1$. It has degree $\lambda(\nu_1 - 1) - E_G$, where E_G is the number of edges in G. The projective version of this integral is (1.13). $\square$

3. PROOF OF THEOREM 1.1

In this section we prove the real analyticity of graphical functions on $\mathbb{C}\backslash\{0,1\}$. We first stay in the general setup of the previous section and write for the squared distance of the V^{ext} external vertices

$$s_{i,j} = ||z_i - z_j||^2.$$

Assume G is a graph such that the graphical function $f_G^{(\lambda)}$ exists. Because by (2.1) the polynomial $\tilde{\Phi}_G$ naturally depends on the $s_{i,j}$ we may use the dual parametric representation (2.4) to consider $f_G^{(\lambda)} = f_G^{(\lambda)}(s)$ as a function of the $s_{i,j}$. We want to study the analytic continuation of $f_G^{(\lambda)}(s)$. It is singular on its Landau surface, which in general contains the divisors $s_{i,j} = 0$ but also additional components.

However, we meet no divergences in the region $\mathrm{Re}\, s_{i,j} > 0$:

Theorem 3.1. *Let G be a graph with V^{ext} external vertices such that the graphical function $f_G^{(\lambda)}$ exists. Then $f_G^{(\lambda)}$ admits a single-valued analytic continuation onto the domain where $\mathrm{Re}\, s_{i,j} > 0$ for all $i,j \in \{1,\ldots,V^{\mathrm{ext}}\}$.*

In the special case of three external vertices, this implies the real analyticity of $f_G^{(\lambda)}(z)$ on $\mathbb{C}\backslash\{0,1\}$:

Proof of Theorem 1.1. Let $z \in \mathbb{C}\backslash\{0,1\}$. With the three external labels 0, 1, z we have $s_{0,1} = 1 > 0$, $s_{0,z} = z\overline{z} > 0$, and $s_{1,z} = (z-1)(\overline{z}-1) > 0$ (see (1.7)). By Theorem 3.1 we obtain that $f_G^{(\lambda)}(z)$ is a composition of analytic functions and hence analytic. This proves (G3).

The identity (G1) is immediate from (2.4). To prove (G2) it is sufficient to see that in the neighborhood of any closed path γ in $\mathbb{C}\backslash\{0,1\}$ the graphical function $f_G^{(\lambda)}$ is real analytic. Hence, along γ, the analytic continuation of $f_G^{(\lambda)}$ equals the evaluation of $f_G^{(\lambda)}$. The evaluation of $f_G^{(\lambda)}$ is single-valued. □

For the proof of Theorem 3.1 we cite the following theorem from [12], Theorem 2.12.

Theorem 3.2. *Let $\Theta \subset \mathbb{R}^m$ and $\Omega \subset \mathbb{C}^n$ denote domains in the respective spaces of dimensions $m, n \in \mathbb{N}$. Furthermore, let*

$$f = f(t,z) = f(t_1,\ldots,t_m,z_1,\ldots,z_n) : \Theta \times \Omega \longrightarrow \mathbb{C} \in C^0(\Theta \times \Omega, \mathbb{C})$$

represent a continuous function with the following properties:

(a) *For each fixed vector $t \in \Theta$ the function*

$$\Phi(z) = f(t,z), \quad z \in \Omega$$

is holomorphic.

(b) *We have a continuous integrable function $F(t) : \Theta \longrightarrow [0,+\infty) \in C^0(\Theta,\mathbb{R})$ satisfying*

$$\int_\Theta F(t)\mathrm{d}t < +\infty,$$

which represents a uniform mayorant to our function $f = f(t,z)$ - that means

$$|f(t,z)| \le F(t) \quad \textit{for all} \quad (t,z) \in \Theta \times \Omega.$$

Then the function

$$\varphi(z) := \int_\Theta f(t,z)\mathrm{d}t, \quad z \in \Omega$$

is holomorphic in Ω.

For the proof of Theorem 3.1 we need the following generalizations of degree and of low degree to non-polynomial functions:

Definition 3.3. *Let g be a graph with edge set $\mathcal{E}_g$ and let $F : \mathbb{R}^{|\mathcal{E}_g|} \longrightarrow \mathbb{C}$ be a function of the edge variables α_e, $e \in \mathcal{E}_g$. The (low) degree ($\underline{\deg}_g(F)$) $\deg_g(F)$ of F is defined by*

$$\underline{\deg}_g(F) = c \Leftrightarrow \lim_{t\to 0} t^{-c}F(t\alpha) \in \mathbb{C}^\times, \quad \deg_g(F) = c \Leftrightarrow \lim_{t\to\infty} t^{-c}F(t\alpha) \in \mathbb{C}^\times. \tag{3.1}$$

Proposition 3.4. *Let g be a subgraph of a graph G with external vertices. Let $\tilde{\Psi}_G^p(\alpha)$ be a dual spanning forest polynomial (2.2) for some partition p of external vertices. Then*

$$\underline{\deg}_g(\tilde{\Psi}_G^p) \ge V_g^{\mathrm{int}}, \quad \deg_g(\tilde{\Psi}_G^p) \le V_g - 1, \tag{3.2}$$

where V_g^{int} and V_g are defined in (1.5) and (1.6), respectively.

Proof. Let $F \in \mathcal{F}_G^p$ be a spanning forest of G. For every tree T in F we choose an external vertex $v_T \in T$ as a root. We orient the edges in T such that they point towards the root v_T. Because F is spanning, every internal vertex u in g has at least one outgoing edge in F. Conversely every edge in F has unique vertex u as source. Therefore

$$\underline{\deg}_g(\tilde{\Psi}_G^p) = \min_{|\mathcal{E}_{g\cap F}|, F\in\mathcal{F}_G^p} \geq V_g^{\mathrm{int}}.$$

By graph homology for any non-empty forest F with V_F vertices and $h_0(F)$ trees we have $|\mathcal{E}_F| = V_F - h_0(F) \leq V_F - 1$. Therefore

$$\deg_g(\tilde{\Psi}_G^p) = \max_{|\mathcal{E}_{g\cap F}|, F\in\mathcal{F}_G^p} \leq V_{g\cap F} - 1 = V_g - 1.$$

□

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. We first derive Theorem 3.1 from (2.4) in the case that all $n_e = 0$. As affine chart of Δ we choose the standard coordinate simplex $\{\sum_e \alpha_e = 1, \alpha_e \geq 0\}$. Because the integration domain is compact the integral converges if the singularities of the integrand are integrable. We consider the integrand as a function on $s = s_{i,j}$ which assume values in the complex domain ($\varepsilon > 0$)

$$\Omega^\varepsilon = \big\{s\colon \text{ Re } \ s_{i,j} \geq \varepsilon \quad \text{for all } 1 \leq i < j \leq V^{\mathrm{ext}}\big\} \subset \mathbb{C}^{V^{\mathrm{ext}}(V^{\mathrm{ext}}-1)/2}.$$

The integrand can have singularities if $\alpha_e = 0$, $\tilde{\Phi}_G(\alpha, s) = 0$, or $\tilde{\Psi}(\alpha) = 0$. In the polynomials $\tilde{\Phi}_G$ and $\tilde{\Psi}$ every monomial in α has a coefficient with strictly positive real part. Hence, these polynomials can only vanish if every monomial vanishes. The zeros of these polynomials are non-trivial coordinate subspaces $\{\alpha : \alpha_e = 0$ for all $e \in E_0 \subset \mathcal{E}_G\}$ (see [1] for a more detailed discussion). Similarly, the low degree $\underline{\deg}_{E_0}$ of the integrand does not depend on the choice of $s \in \Omega^\varepsilon$. Hence the graphical function $f_G^{(\lambda)}(s)$ exists for all $s \in \Omega^\varepsilon$ and in particular for the constant vector $s^\varepsilon_{i,j} = \varepsilon$. Because $\alpha \in \mathbb{R}_+^E$ and

$$|\tilde{\Phi}_G(\alpha, s)| \geq \mathrm{Re}\, \tilde{\Phi}_G(\alpha, s) \geq \tilde{\Phi}_G(\alpha, s^\epsilon)$$

for any $s \in \Omega^\varepsilon$ the integrand $F(\alpha, s) \leq F(\alpha, s^\epsilon)$. Therefore the integrable function $F(\alpha, s^\epsilon)$ uniformly majorizes the integrand and Theorem 3.2 implies the analyticity of $f_G^{(\lambda)}$ in Ω^ε for any $\varepsilon > 0$.

Now we consider the case $n_e > 0$. We choose $s \in \Omega^\varepsilon$ and perform the derivatives in (2.4), yielding the integrand

$$F = \left[\prod_e \alpha_e^{n_e + \lambda\nu_e - 1}\right] \frac{\sum_m \alpha^m q_m(s)}{\tilde{\Phi}_G(\alpha, s)^{M_G + \sum_e n_e} \tilde{\Psi}(\alpha)^{d/2 - M_G + \sum_e n_e}}, \tag{3.3}$$

where we expanded the numerator polynomial into its monomials $\alpha^m = \prod_e \alpha_e^{m_e}$ of Schwinger parameters and their coefficients $q_m \in \mathbb{Q}[s]$. The integrand F is homogeneous in α of degree $-|\mathcal{E}_G|$. Because ∂_{α_e} reduces the degree by one,

$$\sum_e m_e - \Big(\deg_G(\tilde{\Phi}_G) + \deg_G(\tilde{\Psi}_G)\Big) \sum_e n_e = -\sum_e n_e.$$

The polynomials $\tilde{\Phi}_G$ and $\tilde{\Psi}_G$ have degrees $V^{\mathrm{int}} + 1$ and V^{int} in α. Hence $\sum_e m_e = 2V^{\mathrm{int}} \sum_e n_e$. With this identity we see that $F(\alpha) = \sum_m q_m(s) F_m(\alpha)$ is a linear combination of integrands F_m which are the integrands of the dual parametric representation $f_G^{(\lambda')}$ in $(2\lambda'+2) = d + 4\sum_e n_e$ dimensions with weights $\lambda'\nu'_e = \lambda\nu_e + n_e + m_e > 0$. With the first part of the proof it suffices to show that $f_G^{(\lambda')}$ is a convergent graphical function. The infrared (1.5) and ultraviolet (1.6) conditions generalize to an arbitrary number of external vertices. Because differentiation ∂_{α_e} for $e \in \mathcal{E}_g$ can lower the low degree by at most one we obtain

$$\sum_{e\in g} m_e - (\underline{\deg}_g(\tilde\Phi_G) + \underline{\deg}_g(\tilde\Psi_G)) \sum_{e\in G} n_e \geq -\sum_{e\in g} n_e.$$

From the convergence of $f_G^{(\lambda)}$ and from Proposition 3.4 we obtain

$$\sum_{e\in g} \lambda'\nu'_e = \sum_{e\in g}(\lambda\nu_e + n_e + m_e) > \lambda V_g^{\mathrm{int}} + 2V_g^{\mathrm{int}} \sum_{e\in G} n_e = \lambda' V_g^{\mathrm{int}},$$

proving infrared convergence. Likewise differentiation ∂_{α_e} for $e \in \mathcal{E}_g$ lowers the degree by at least one, yielding

$$\sum_{e\in g} m_e - (\deg_g(\tilde\Phi_G) + \deg_g(\tilde\Psi_G)) \sum_{e\in G} n_e \leq -\sum_{e\in g} n_e.$$

Now,

$$\sum_{e\in g} \lambda'\nu'_e = \sum_{e\in g}(\lambda\nu_e + n_e + m_e) < (\lambda + 2\sum_{e\in G} n_e)(V_g - 1) = \lambda'(V_g - 1)$$

proves ultraviolet convergence. This completes the proof of Theorem 3.1. □

Remark 3.5. *We may consider a graphical function $f_G^{(\lambda)}(z)$ as a function of two complex variables z and $\overline{z}$ and analytically continue away from the locus where $\overline{z}$ is the complex conjugate of z. In this case Theorem 3.1 states that $f_G^{(\lambda)}$ is analytic in z and $\overline{z}$ if $\mathrm{Re}\, z\overline{z} > 0$ and $\mathrm{Re}\,(z-1)(\overline{z}-1) > 0$.*

However, after analytic continuation additional singularities will appear, notably on $z = \overline{z}$ which corresponds to the vanishing of the Källén function

$$(z-\overline{z})^2 = s_{0,z}^2 + s_{1,z}^2 + s_{0,1}^2 - 2s_{0,z}s_{1,z} - 2s_{0,z}s_{0,1} - 2s_{1,z}s_{0,1}.$$

4. PROOF OF THEOREM 1.5

Planar duality is specific to three external labels for which we use 0, 1, z.

Definition 4.1. *Let G be a graph with three external labels 0, 1, z. Let G_v be the graph that we obtain from G by adding an extra vertex v which is connected to the external vertices of G by edges $\{0,v\}$, $\{1,v\}$, $\{z,v\}$, respectively. We say that G is (externally) planar if and only if G_v is planar.*

Let G_v be planar and $G_v^\star$ a planar dual of G_v. The edges $e^\star$ of $G_v^\star$ are in one to one correspondence with the edges e of G_v. A planar dual of G is given by $G_v^\star$ minus the triangle $\{0,v\}^\star$, $\{1,v\}^\star$, $\{z,v\}^\star$ with external labels 0, 1, z corresponding to the faces $1zv$, $0zv$, $01v$, respectively. The edge weights of $G_v^\star$ are related to the edge weights of G by (1.18): $\nu_e + \nu_{e^\star} = 1 + \lambda^{-1}$.

We can draw an externally planar graph G with the external labels 0, 1, z in the outer face. A dual $G^\star$ then has also the labels in the outer face, 'opposite' to the labels of G, see Figure 2.

Another alternative way to construct a dual of G is to add three edges $e_{01} = \{0,1\}$, $e_{0z} = \{0,z\}$, $e_{1z} = \{1,z\}$ to obtain G_e. A dual $G_e^\star$ of G_e is given by the dual of G_v upon replacing the triangle $\{0,v\}^\star$, $\{1,v\}^\star$, $\{z,v\}^\star$ by a star $e_{01}^\star$, $e_{0z}^\star$, $e_{1z}^\star$.

From $G_e^\star$ we obtain a dual of G by removing the star and labeling the endpoints of the star by z, 1, 0, respectively. Clearly any construction leads to the same dual which proves the following lemma:

Lemma 4.2. *Let G be externally planar with dual $G^\star$. Then $G^\star$ is externally planar and G is a dual of $G^\star$.*

Proof of Theorem 1.5. Because the edge weights are positive we can use $n_e = 0$ in (2.4). From $M_G = \lambda + 1$ we obtain (see (1.12) and (1.18))

$$M_{G^\star} = \sum_e (\lambda + 1 - \lambda\nu_e) - (\lambda+1)V_{G^\star}^{\text{int}} = (\lambda+1)(E_G - V_{G^\star}^{\text{int}} - V_G^{\text{int}} - 1),$$

where E_G is the number of edges of G. Now,

$$V_{G^\star}^{\text{int}} = V_{G_v^\star} - 4 = h_1(G_v) - 3 = h_1(G),$$

where $h_1(X)$ is the number of independent cycles in the graph X. Because $V_G^{\text{int}} = V_G - 3$ we obtain from Euler's identity for connected graphs $V_G - E_G + h_1(G) = 1$ that $M_{G^\star} = \lambda + 1 = M_G$. Comparing (2.4) for the graph G with (1.13) for the graph $G^\star$ leads to (1.19) provided

$$\tilde{\Phi}_G = \Phi_{G^\star},$$

where we assume $\alpha_e = \alpha_{e^\star}$ for all edges e. By (1.11) and (2.1) we have to show the equality $\tilde{\Psi}_G^{ij,k} = \Psi_{G^\star}^{ij,k}$ of spanning forest polynomials for all $\{i,j,k\} = \{0,1,z\}$. This is equivalent to a one to one correspondence of 2-forests:

$$F \in \mathcal{F}_G^{ij,k} \longleftrightarrow F^\star := \{e^\star : e \notin F\} \in \mathcal{F}_{G^\star}^{ij,k}.$$

Whitney's planarity criterion ([15], Theorem 29) states that a graph is planar if and only if it has an algebraic dual. As Tutte points out in [13], Theorem 2.64, this is equivalent to the statement that every spanning tree of a planar graph corresponds to the complement of a spanning tree in its dual graph. Using this argument we can construct the desired correspondence as follows: Let $F \in \mathcal{F}_G^{ij,k}$. Adding the two edges $\{i,v\}$ and $\{k,v\}$ gives a spanning tree T_i in G_v. Similarly adding the two edges $\{j,v\}$ and $\{k,v\}$ gives a spanning tree T_j in G_v. The complements $T_i^\star$, $T_j^\star$ of these trees are a spanning tree in $G_v^\star$. We have $\{j,v\}^\star \in T_i^\star$ and $\{i,v\}^\star \in T_j^\star$. Except for these two edges the trees $T_i^\star$ and $T_j^\star$ are identical. Hence $T_i^\star \backslash \{j,v\}^\star = T_j^\star \backslash \{i,v\}^\star =: F^\star$. Clearly, $F^\star$ is a two forest in $G^\star$. The edge $\{j,v\}^\star$ connects the external vertices i and k in $G^\star$. Because $F^\star = T_i^\star \backslash \{j,v\}^\star$ the 2-forest $F^\star$ does not connect the external vertices i and $k \in G^\star$. Likewise (interchanging i and j) $F^\star$ does not connect the external vertices j and k. Therefore $F \in \mathcal{F}_{G^\star}^{ij,k}$. By symmetry with respect to taking duals the map $F \longrightarrow F^\star$ is one to one. □

Remark 4.3. *One can also use Theorem 1.19 for externally planar graphs G in the case that $M_G \neq \lambda + 1$: One may add in G (or in $G^\star$) an edge $\{0,1\}$ of weight $(\lambda + 1 - M_G)/\lambda$ (see Figure 2 and Example 1.6). This gives a new graph G' with the same graphical function as G and $M_{G'} = \lambda + 1$. Dualizing leads to a graph with a single edge of weight M_G/λ (the dual of the edge $\{0,1\}$) that connects the external vertex z in $G'^\star$ with the vertex z in $G^\star$ (which becomes internal in $G'^\star$). If $M_G = \lambda$ the new edge in $G'^\star$ has weight 1 and the graphical function of $G'^\star$ can be obtained from the graphical function of $G^\star$ by solving a differential equation (see* [11]*, Section 3.5).*

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