Character groups of Hopf algebras from renormalisation as Lie groups

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Nord University Levanger

The broad picture

Recently much interest in special Hopf algebras generated by combinatorial objects (e.g. graphs, shuffles, trees etc.)

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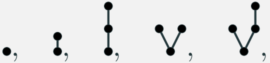
Common theme in these examples

Hopf algebra encodes combinatorics and “dual objects”, i.e. **character groups**, carry additional relevant information

Butcher-Connes-Kreimer Hopf algebra

Build a Hopf algebra of rooted trees:

$\mathcal{T} := \Big\{$  $\ldots \Big\}$

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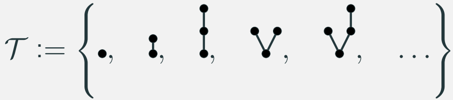

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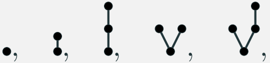
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Hopf algebra has a dual notion to the product arising from disassembling trees into subtrees.

Subtrees of a tree

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(subtree nodes colored red)

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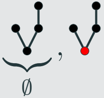
$\tau =$  subtrees of $\tau = \Big\{$  $\Big\}$

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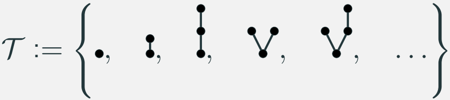

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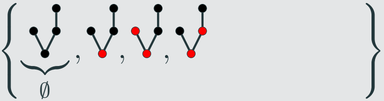
$\tau =$  subtrees of $\tau =$ 

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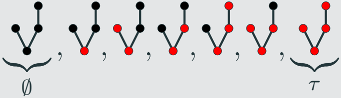
$\tau =$  subtrees of $\tau = \{$ 

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Dualise to pass to Lie theory (Milnor-Moore theorem!)

The dual picture: Character groups

Hopf algebra characters

$\mathcal{H}$ Hopf algebra, B a commutative algebra.

A **character** is an unital algebra morphism $\phi\colon \mathcal{H} \to B$.

An **infinitesimal character** is a linear map $\psi\colon \mathcal{H} \to B$ which satisfies $\psi(xy) = \epsilon(x)\psi(y) + \psi(x)\epsilon(y)$ ($\epsilon =$counit).

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Infinitesimal characters form a Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ with bracket

$$[\eta, \psi] := \eta \star \psi - \psi \star \eta.$$

Why are Hopf algebra characters interesting?

(regularized/renormalized) Feynman rules are characters of a Hopf algebra generated by the 1PI diagrams

Perturbative renormalisation of QFT (cf. Connes/Marcolli 2007)

Characters of the Hopf algebra $\mathcal{H}_{FG}$ of Feynman graphs are called "diffeographisms", the diffeographism group acts on the coupling constants via formal diffeomorphisms.

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[1] $G(\mathcal{H}, \mathbb{R})$ as "Lie group" implicitely used in Hairer, Wanner, Lubich *Geometric Numerical Integration* 2006.

Infinite-dimensional structures

(beyond Banach spaces)

Calculus beyond Banach spaces

Bastiani calculus

Let E, F be **locally convex spaces** $f\colon E \supseteq U \to F$ is C^1 if
open

$$df\colon U \times E \to F, \quad df(x,v) := \lim_{h\to 0} h^{-1}(f(x+hv) - f(x))$$

exists and is continuous.

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Infinite-dimensional Lie group

A group G is a (infinite-dimensional) Lie group if it carries a manifold structure (modelled on locally convex spaces) making the group operations smooth (in the sense of Bastiani calculus).

Structure theory for character groups

Theorem (Bogfjellmo, Dahmen, S.)

Let $\mathcal{H}$ be a graded Hopf algebra $\mathcal{H} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}_n$ with $\dim \mathcal{H}_0 < \infty$ and B be a commutative Banach algebra, then $G(\mathcal{H}, B)$ is a Lie group.

$\mathcal{H} = \bigoplus H_n$ $\quad$ $\dim H_0 < \infty$

If $\mathcal{H}$ BCK-Hopf algebra

$\mathcal{H}_0 = \mathbb{K}\emptyset$ $\quad$ if $\dim H_0 = 1$

we say H is connected

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Lie theoretic properties of $G(\mathcal{H}, B)$

- $(\mathfrak{g}(\mathcal{H}, B), [-,-])$ is the Lie algebra of $G(\mathcal{H}, B)$
- $\exp\colon \mathfrak{g}(\mathcal{H}, B) \to G(\mathcal{H}, B), \psi \mapsto \sum_{n=0}^{\infty} \frac{\psi^{\star n}}{n!}$ is the Lie group exponential
- $G(\mathcal{H}, B)$ is a Baker-Campbell-Hausdorff Lie group
- If B is finite dimensional, $G(\mathcal{H}, B)$ is the projective limit of finite dimensional groups

$g = h * h$

Subgroups associated to Hopf ideals

A subset J of a Hopf algebra $\mathcal{H}$ is a Hopf ideal if J is a two-sided algebra ideal

$$\varepsilon(J) = 0, \quad \Delta(J) \subseteq J \otimes \mathcal{H} + \mathcal{H} \otimes J, \text{ and } S(J) \subseteq J$$

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Let $\mathcal{H}$ be a graded and connected Hopf algebra and $J \subseteq \mathcal{H}$ a Hopf ideal, B a commutative locally convex algebra. Then

$$G(\mathcal{H}/J, B) \cong \mathrm{Ann}(J, B) \cap G(\mathcal{H}, B) \subseteq G(\mathcal{H}, B)$$

is a closed Lie subgroup.

Neeb: Towards a Lie theory for lc. cvx groups

Why is this interesting?

Several relations in renormalisation of QFT yield Hopf ideals, e.g. (van Suijlekom):

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So we get

- Lie group structures of the character groups $\mathcal{H}/J$,
- can realise these groups as Lie subgroups of character group of $\mathcal{H}$.

Note that in general the Hopf algebra $\mathcal{H}/J$ need not be graded (if J is not homogeneous)!

The infinite dimensional picture

Infinite-dimensional Lie-theory admits pathologies not present in the finite dimensions, e.g.

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- the Lie theorems connecting Lie algebra and Lie group are in general wrong. E.g. there are infinite dimensional Lie algebras which are not the Lie algebra of a Lie group

To recover Lie theoretic tools one studies the class of "regular" Lie groups.

Regularity for Lie groups

Differential equations of "Lie-type" can be solved on the group and depend smoothly on parameters

For a Lie group G with Lie algebra $\mathbf{L}(G)$, we call the (ordinary) differential equation

$$\dot{\gamma}(t) = \gamma(t).\eta(t) := T\lambda_{\gamma(t)}(\eta(t))$$

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Beyond Banach spaces it is not clear that these differential equations admit solutions.

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G is called **regular** (in the sense of Milnor) if for each smooth curve $\eta \colon [0,1] \to \mathbf{L}(G)$ the initial value problem

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$$\mathrm{evol} \colon C^\infty([0,1], \mathbf{L}(G)) \to G, \quad \eta \mapsto \mathrm{Evol}(\eta)(1)$$

is smooth.

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Let B be a commutative Banach algebra and $\mathcal{H} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}_n$ a graded Hopf algebra with $\dim \mathcal{H}_0 < \infty$. Then $G(\mathcal{H}, B)$ is regular in the sense of Milnor.

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Why ist this interesting?

Renormalization of QFT

Regularity of character groups is the mathematical backbone to study the renormalization group flow.

Numerical analysis (Murua/Sanz-Serna)

Lie type equations on the Butcher group and related groups are used in numerical analysis (word series).

Why care about regularity?

Time ordered exponentials in CK-renormalisation

Consider the *time ordered exponentials*

$$\mathbf{1} + \sum_{n=1}^{\infty} \int_{a \leq s_1 \leq \cdots s_n \leq b} \alpha(s_1) \cdots \alpha(s_n) \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

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However: Time ordered exponentials are solutions to Lie type equations on $G(\mathcal{H}_{FG}, \mathbb{C})$

Thank you for your attention!

More information in Bogfjellmo, S.: The geometry of characters of Hopf algebras, arxiv:1704.01099