Algebraic structure in renormalization of combinatorially non-local field theories

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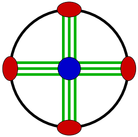
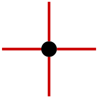
Perturbative field theory

Fields $\phi : \mathbb{R}^D \to \mathbb{R}$ with covariance/propagator $P(\boldsymbol{x}, \boldsymbol{x}') = \int \mathrm{d}\mu[\phi]\phi(\boldsymbol{x})\phi(\boldsymbol{x}')$:

$$G^{(n)}(\boldsymbol{x}_1, ..., \boldsymbol{x}_n) = \int \mathrm{d}\mu[\phi] \mathrm{e}^{\mathrm{i}S_{\text{IA}}[\phi]} \prod_{i=1}^{n} \phi(\boldsymbol{x}_i)$$

$$S_{\text{IA}}[\phi] = \int_{\mathbb{R}^D} \mathrm{d}\boldsymbol{x}\, \lambda_k \phi(\boldsymbol{x})^k = \lambda_k \int_{\mathbb{R}^D} \prod_{i=1}^{k} \mathrm{d}\boldsymbol{q}_i\, \delta\left(\sum_{i=1}^{k} \boldsymbol{q}_i\right) \prod_{i=1}^{k} \tilde{\phi}(\boldsymbol{q}_i)$$

Point-like interactions, e.g. quartic $k = 4$:  $\cong$ 

Perturbative exp. $\mathrm{e}^{\mathrm{i}S_{\text{IA}}[\phi]} = \sum_l \frac{(\mathrm{i}S_{\text{IA}})^l}{l!} \Rightarrow$ formal power series over graphs γ:

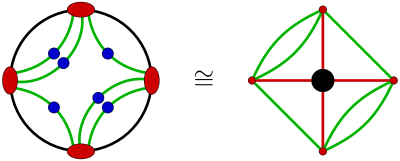
$$G^{(n)}(\boldsymbol{p}_1, ..., \boldsymbol{p}_n) = \sum_{\substack{\gamma \in \mathbf{G}_1, \\ N_\gamma^e = n}} \frac{1}{|\mathrm{Aut}\, \gamma|} \prod_{e \in \mathcal{E}_\gamma} \int \mathrm{d}\boldsymbol{q}_e \tilde{P}(\boldsymbol{q}_e) \prod_{v \in \mathcal{V}_\gamma} \mathrm{i}\lambda_v \delta\left(\sum_{e @ v} \boldsymbol{q}_e\right)$$

Combinatorially non-local field theory (cNLFT)

Combinatorially non-local interactions for fields $\phi : (\mathbb{R}^d)^r \to \mathbb{R}$:

$$S_{\text{IA}}[\phi] = \lambda_\gamma \int \prod_{i=1}^{k} \mathrm{d}\boldsymbol{q}_i \prod_{(ia,jb)} \delta(q_i^a - q_j^b) \prod_{i=1}^{k} \tilde{\phi}(\boldsymbol{q}_i)$$

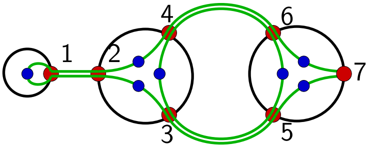
pairwise convolution of individual entries $q^a \in \mathbb{R}^d$, $a = 1, ..., r$



Combinatorics of interaction: vertex graph $\gamma =$  ^{c}, not just $k = V_\gamma$

Perturbation theory: 2-graphs

Perturbative series over “ribbon graphs”, “stranded diagramms” ...
here general concept: *2-graphs* $\Gamma \in \mathbf{G}_2$



$$G^\gamma(\boldsymbol{p}_1, ..., \boldsymbol{p}_{V_\gamma}) = \sum_{\substack{\Gamma \in \mathbf{G}_2, \\ \partial\Gamma = \gamma}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \prod_{v \in \mathcal{V}_\Gamma} \mathrm{i}\lambda_{\gamma_v} \prod_{f \in \mathcal{F}_\Gamma^{\mathrm{int}}} \int_{\mathbb{R}^d} \mathrm{d}q_f \prod_{\{i,j\} \in \mathcal{E}_\Gamma} \tilde{P}(\boldsymbol{q}_i)$$

Feynman rules:

1. coupling $\mathrm{i}\lambda_{\gamma_v}$ for each vertex $v \in \mathcal{V}_\Gamma$ with vertex graph γ_v
2. propagator $\tilde{P}(\boldsymbol{q}_i)$ for each internal edge $e = \{i, j\} \in \mathcal{E}_\Gamma$,
3. Lebesgue integral $\int_{\mathbb{R}^d} \mathrm{d}q_f$ for internal face $f \in \mathcal{F}_\Gamma^{\mathrm{int}}$ ($q_f = q_i^a$ identified)

Renormalization

Integrals might not converge $\rightarrow$ renormalization needed

- various prescriptions how to remove infinite part of the integral
- always necessary: forest formula to subtract subdivergences
- universally described by the Connes-Kreimer Hopf algebra
- principle of locality needed for this

Hopf algebra of Feynman graphs generalizes to 2-graphs in cNLFT

Locality is captured by vertex graphs

Applications of cNLFT:

- Probability & combinatorics: random matrices $\rightarrow$ random tensors
- Quantum field theory: non-trivial solvable models, e.g. Grosse-Wulkenhaar model (topology the missing ingredient in QFT?)
- Quantum gravity: Generating function of lattice gauge theories ("group field theory")

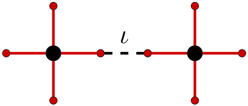
Outline

1. Non-local field theory
   - Combinatorial non-locality
2. 2-graphs
   - From 1-graphs to 2-graphs
   - Contraction and boundary
   - Algebra
3. Renormalization in cNLFT
   - Renormalizable field theories
   - The BPHZ momentum scheme
   - Outlook: combinatorial DSE

Half-edge graphs + strands

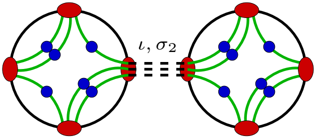
A *1-graph* is a tuple $g = (\mathcal{V}, \mathcal{H}, \nu, \iota)$ with

- a set of *vertices* $\mathcal{V}$
- a set of *half-edges* $\mathcal{H}$
- an adjacency map $\nu : \mathcal{H} \to \mathcal{V}$
- an involution $\iota : \mathcal{H} \to \mathcal{H}$ pairing *edges* (fixed points are external edges)



A *2-graph* is $G = (\mathcal{V}, \mathcal{H}, \nu, \iota; \mathcal{S}, \mu, \sigma_1, \sigma_2)$ with

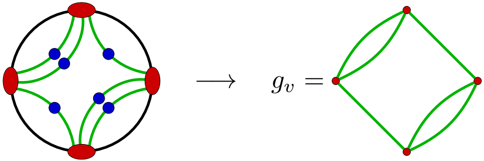
- a set of strand sections $\mathcal{S}$
- an adjacency map $\mu : \mathcal{S} \to \mathcal{H}$
- a fixed-point free involution $\sigma_1 : \mathcal{S} \to \mathcal{S}$ with $\forall s \in \mathcal{S}$: $\nu \circ \mu \circ \sigma_1(s) = \nu \circ \mu(s)$
- an involution $\sigma_2 : \mathcal{S} \to \mathcal{S}$ pairing strands at edges: $\forall s \in \mathcal{S}$: $\iota \circ \mu(s) = \mu \circ \sigma_2(s)$ and s is fixed point of σ_2 iff $\mu(s)$ is fixed point of ι.



Involutions $\iota, \sigma_1, \sigma_2$ are equivalent to edge sets $\mathcal{E} \subset \mathbf{2}^{\mathcal{H}}$ and $\mathcal{S}^v, \mathcal{S}^e \in \mathbf{2}^{\mathcal{S}}$

Vertex-graph representation

Vertex graph $g_v = (\mathcal{V}_v, \mathcal{H}_v, \nu_v, \iota_v) := \left(\nu^{-1}(v), (\nu \circ \mu)^{-1}(v), \mu|_{\mathcal{H}_v}, \sigma_1|_{\mathcal{H}_v}\right)$



Represent 2-graphs via vertex graphs

$$\pi_{\text{vg}} : (\mathcal{V}, \mathcal{H}, \nu, \iota; \mathcal{S}, \mu, \sigma_1, \sigma_2) \mapsto \Big(\bigsqcup_{v \in \mathcal{V}} g_v, \iota, \sigma_2 \Big)$$

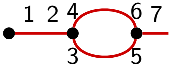
Not bijective! In general $g_v = \sqcup_i g_v^{(i)}$, vertex belonging information lost...

$$\beta_{\text{vg}} : (\mathcal{V}, \mathcal{H}, \nu, \iota; \mathcal{S}, \mu, \sigma_1, \sigma_2) \mapsto \left(\{g_v\}_{v \in \mathcal{V}}, \iota, \sigma_2\right) \text{ is bijection}$$

Example: combinatorial maps

A combinatorial map is a triple $(\mathcal{H}, \sigma, \iota)$ with permutation $\sigma : \mathcal{H} \to \mathcal{H}$ whose cycles define (oriented) vertices

Example:

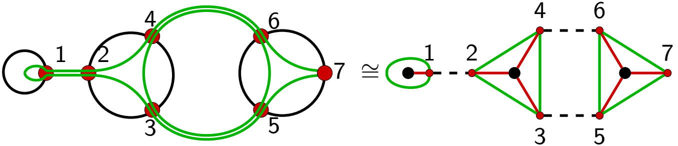
$(\mathcal{H}, \sigma, \iota) = (\{1,2,3,4,5,6,7\}, (1)(234)(576), (12)(35)(46)(7)) =$ 

Maps are 2-graphs with 2 strands per edge

Strands encoded in oriented vertices, e.i. σ-cycles $v = (h_1, h_2, ..., h_n)$ define

$$g_v = (\{h_1, h_2, ..., h_n\}, \{s_{1n}, s_{12}, s_{21}, s_{23}, ...s_{n\,n-1}, s_{n1}\}, \nu_v, \iota_v)$$

and ι_v pairs s_{ij} with s_{ji} iff $\sigma(h_i) = h_j$.

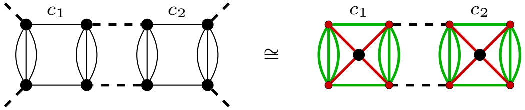


Example: edge-coloured graphs

Feynman diagrams of rank-r tensor theories: regular edge-coloured graphs

$(r+1)$-coloured graphs are 2-graphs with r strands per edge

- colour $c = 0$ edges $\rightarrow$ 2-graph edges
- colour $c \neq 0$ subgraph components $\rightarrow$ vertex graphs
- stranding of edges σ_2 fixed by colour preservation



Bijective only for connected vertex graphs (usual case in tensor theories)

Topology of edge-coloured graphs

$(r+1)$-coloured graphs $\Longleftrightarrow$ r-dimensional pseudo manifolds [Gurau '11]
(abstract simplicial complexes)

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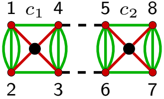
Subgraphs

For a 2-graph G, a *subgraph* H is a 2-graph differing from G in $\mathcal{E}_H \subset \mathcal{E}_G$ and $\mathcal{S}^e_H \subset \mathcal{S}^e_G$. Then one writes $H \subset G$.

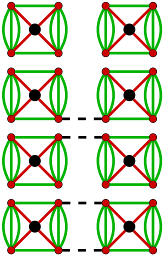
Example

2^{E_G} subgraphs per 2-graph G, for example for

$G =$ 

:

$H_0 =$, $H_1 =$, $H_2 =$, $H_3 =$

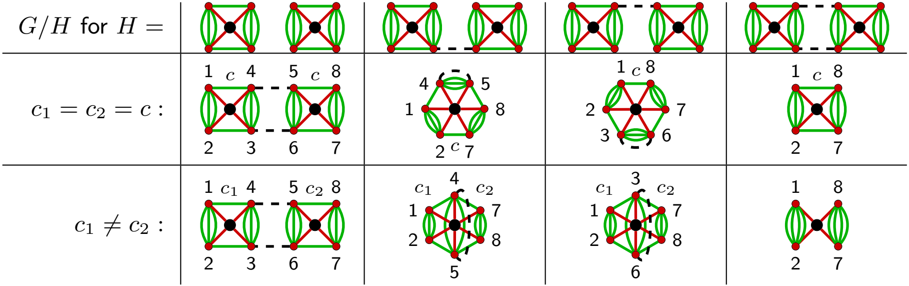


Contraction

Contraction of $H \subset G$: shrinking all stranded edges of H:

- $\mathcal{V}_{G/H} = \mathcal{K}_H$ the set of connected components of H
- $\mathcal{H}_{G/H} = \mathcal{H}_H^{\text{ext}}$, $\mathcal{S}_{G/H} = \mathcal{S}_H^{\text{ext}}$, only external half-edges of H remain
- $\mathcal{S}^v_{G/H} = \{\{s_1, s_{2n}\} | (s_1 ... s_{2n}) \in \mathcal{F}_H^{\text{ext}}\}$, external faces are shrunken to the strands at the new contracted vertices.
- $\mathcal{E}_{G/H} = \mathcal{E}_G \setminus \mathcal{E}_H$, $\mathcal{S}^e_{G/H} = \mathcal{S}^e_G \setminus \mathcal{S}^e_H$ (deleting stranded edges of H)

Example:



Labelled vs. Unlabelled

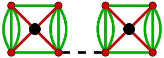
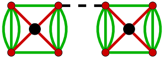
Unlabelled 2-graphs

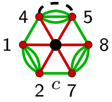
Isomorphism $j : G_1 \to G_2$ is a triple of bijections $j = (j_\mathcal{V}, j_\mathcal{H}, j_\mathcal{S})$ s.t.:

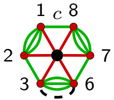
- $\nu_{G_2} = j_\mathcal{V} \circ \nu_{G_1} \circ {j_\mathcal{H}}^{-1}$ and $\mu_{G_2} = j_\mathcal{H} \circ \mu_{G_1} \circ {j_\mathcal{S}}^{-1}$
- $\iota_{G_2} = j_\mathcal{H} \circ \iota_{G_1} \circ {j_\mathcal{H}}^{-1}$
- $\sigma_{1G_2} = j_\mathcal{S} \circ \sigma_{1G_1} \circ {j_\mathcal{S}}^{-1}$ and $\sigma_{2G_2} = j_\mathcal{S} \circ \sigma_{2G_1} \circ {j_\mathcal{S}}^{-1}$

Then equivalence $G_1 \cong G_2$, *unlabelled 2-graph*, $\Gamma = [G_1]_\cong = [G_2]_\cong$.
Compatible with contractions.

Example:

$H_1 =$  $\cong H_2 =$ 

$\Rightarrow [G/H_1]_\cong = \Big[$ 

$\Big]_\cong = [G/H_2]_\cong = \Big[$ 

$\Big]_\cong$

Boundary and external structure

Residue and skeleton

2-graph has two characteristic 2-graphs without edges $\mathbf{R}^* \subset \mathbf{G}_2$:

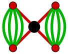
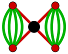
- $\mathrm{res} : \mathbf{G}_2 \to \mathbf{R}^*, \Gamma \mapsto \Gamma/\Gamma$, the "external structure"
- $\mathrm{skl} : \mathbf{G}_2 \to \mathbf{R}^*, \Gamma \mapsto \Theta_0$, the subgraph without edges

Boundary and vertex graphs

Can be used to define the boundary 1-graph of a 2-graph:

- $\partial : \mathbf{G}_2 \to \mathbf{G}_1, \quad \Gamma \mapsto \partial\Gamma := \pi_{\mathrm{vg}}(\mathrm{res}(\Gamma))$
- $\varsigma : \mathbf{G}_2 \to \mathbf{G}_1\,, \quad \Gamma \mapsto \varsigma\Gamma \; := \pi_{\mathrm{vg}}(\mathrm{skl}(\Gamma)) = \bigsqcup_{v \in \mathcal{V}_\Gamma} \gamma_v$

For r-coloured 2-graphs: indeed $(r-1)$-dimensional boundary ps. manifolds

External structure must be sensitive to con. comp. (e.g.  $\sqcup$ ):

- $\widetilde{\partial} : \mathbf{G}_2 \to \mathcal{P}(\mathbf{G}_1),\, \Gamma = \bigsqcup_i \Gamma_i \mapsto \widetilde{\partial}\Gamma := \{\partial\Gamma_i\}_i \quad = \beta_{\mathrm{vg}}(\mathrm{res}(\Gamma))$
- $\widetilde{\varsigma} : \mathbf{G}_2 \to \mathcal{P}(\mathbf{G}_1)\,, \qquad \Gamma \mapsto \widetilde{\varsigma}\Gamma := \{\gamma_v\}_{v \in \mathcal{V}_\Gamma} = \beta_{\mathrm{vg}}(\mathrm{skl}(\Gamma))$

Outline

Algebra

Let $\mathcal{G} := \langle \mathbf{G}_2 \rangle$ be the $\mathbb{Q}$-algebra generated by all 2-graphs $\Gamma \in \mathbf{G}_2$ with

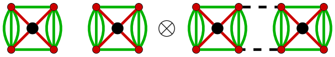
$$m : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \quad , \quad \Gamma_1 \otimes \Gamma_2 \mapsto \Gamma_1 \sqcup \Gamma_2$$

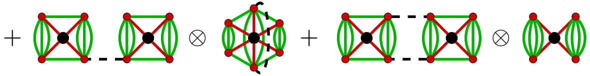
Unital commutative algebra with $u : \mathbb{Q} \to \mathcal{G}, q \mapsto q\mathbb{1}$ ($\mathbb{1}$ empty 2-graph)

Coalgebra

$$\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}, \quad \Gamma \mapsto \sum_{\Theta \subset \Gamma} \Theta \otimes \Gamma / \Theta$$

Associative counital coalgebra with counit $\epsilon = \chi_{\mathbf{R}^*} : \mathcal{G} \to \mathbb{Q}$
In fact, also bialgebra (all proofs completely parallel to 1-graphs)

Example: $\Delta\Gamma =$ 



Subalgebras

Contraction closure

Let $\mathbf{P}, \mathbf{K} \subset \mathbf{G}_2$.

- $\mathbf{P}$-*contraction closure* ${}^{\mathbf{P}}\overline{\mathbf{K}} := \{\Gamma = \Gamma'/\Theta | \Theta \subset \Gamma' \in \mathbf{K}, \Theta \in \mathbf{P}\}$
- *contraction closure* $\overline{\mathbf{K}} := {}^{\mathbf{G}_2}\overline{\mathbf{K}}$

2-graph subbialgebra

- 2-graphs of restricted vertex types $\mathbf{V}$: $\mathbf{G}_2(\mathbf{V}) := \{\Gamma \in \mathbf{G}_2 \,|\, \widetilde{\varsigma}\Gamma \in \mathcal{P}(\mathbf{V})\}$
- Prop: $\langle\overline{\mathbf{G}_2(\mathbf{V})}\rangle$ is a subbialgebra of $\mathcal{G}$.
- for field theory with interactions $\mathbf{V} \in \mathbf{G}_1$: "theory space" $\langle\overline{\mathbf{G}_2(\mathbf{V})}\rangle$

Example: Matrix/Tensor field theory

- 2-graphs characterized by fixed # of strands at edges = tensor rank r
- for rank-r interactions $\mathbf{V}_r$: $\overline{\mathbf{G}_2(\mathbf{V}_r)} = \mathbf{G}_2(\mathbf{V}_r)$ contraction closed
- r-coloured diagrams generate subbialgebra $\langle\mathbf{G}_2(\mathbf{V}_r)\rangle$

Hopf algebra of 2-graphs

interest: group structure on algebra homomorphisms $\phi, \psi : \mathcal{G} \to \mathcal{A}$ wrt

convolution product: $\phi * \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{G}}$

Hopf algebra of 2-graphs

- The bialgebra of 2-graphs $\mathcal{G}$ is a Hopf algebra, i.e. there is a *coinverse* S:

$$S * \mathrm{id} = \mathrm{id} * S = u \circ \epsilon \,.$$

- The set $\Phi_{\mathcal{A}}^{\mathcal{G}}$ of algebra homomorpisms from $\mathcal{G}$ to a unital commutative algebra $\mathcal{A}$ is a group with inverse $S^{\phi} = \phi \circ S$ for every $\phi \in \Phi_{\mathcal{A}}^{\mathcal{G}}$,

$$S^{\phi} * \phi = \phi * S^{\phi} = u_{\mathcal{A}} \circ \epsilon_{\mathcal{G}} .$$

- The subbialgebra $\langle \overline{\mathbf{G}_2(\mathbf{V})} \rangle$ for specific vertex graphs $\mathbf{V} \subset \mathbf{G}_1$ is a Hopf subalgebra of $\mathcal{G}$.

Outline

Renormalizability

cNLFT $T = (\mathbf{E}, \mathbf{V}, \omega, d)$ given by dimension $d \in \mathbb{N}$, $\mathbf{E}, \mathbf{V} \subset \mathbf{G}_1$, weights

$$\omega : \mathbf{E} \cup \mathbf{V} \to \mathbb{Z}$$

Feynman diagrams $\mathbf{G}_2^T := \mathbf{G}_2(\mathbf{V})$ generate a Hopf algebra $\mathcal{G}_T := \langle \overline{\mathbf{G}_2^T} \rangle$

Hopf algebra of divergent Feynman 2-graphs

- *Superficial degree of divergence* $\omega^{\mathrm{sd}}(\Gamma) = \sum_{v \in \mathcal{V}_\Gamma} \omega(\gamma_v) - \sum_{e \in \mathcal{E}_\Gamma} \omega(\gamma_e) + d \cdot F_\Gamma$
- T is *renormalizable* iff $\boxed{\omega^{\mathrm{sd}}(\Gamma) = \omega(\partial\Gamma)}$ for all Γ with $\omega^{\mathrm{sd}}(\Gamma) > 0$

$$\mathbf{P}_T^{\mathrm{s.d.}} := \left\{ \Gamma = \bigsqcup\nolimits_{i \in I} \Gamma_i \in \mathbf{G}_2^T \ \mathrm{1PI} \ | \forall i \in I : \Gamma_i \notin \mathbf{R} \Rightarrow \omega^{\mathrm{sd}}(\Gamma_i) \geq 0 \right\}$$

- $\mathcal{H}_T^{\mathrm{f2g}} = \langle \mathbf{P}_T^{\mathrm{s.d.}} \rangle$ is the Hopf algebra of divergent 2-graphs of T
- Hopf subalgebra of $\mathcal{G}_T$ since contraction closed due to renormalizablity.

Example: Matrix field theory

ϕ^n_D *matrix field theory* [Wulkenhaar'19][Hock'20]:

- dimension $d = D/2$ (D spacetime dim. of related non-commutative QFT)
- interactions $\mathbf{V}$ are polygons up to n vertices, $\omega|_{\mathbf{V}} = 0$
- Feynman diagrams Γ are 2-graphs bijective to combinatorial maps

$$\begin{aligned}\omega^{\mathrm{sd}}(\Gamma) &= d \cdot F_\Gamma - E_\Gamma \\ &= -d(V_\Gamma - 1) + \frac{d-1}{2}\left(\sum_{k=1}^{n} k V_\Gamma^{(k)} - V_{\partial\Gamma}\right) - d\,(2g_\Gamma + K_{\partial\Gamma} - 1)\end{aligned}$$

Grosse-Wulkenhaar model $\equiv \phi^4_D$ matrix field theory

$$2\omega^{\mathrm{sd}}(\Gamma) = D - \frac{D-2}{2} V_{\partial\Gamma} + (D-4)V_\Gamma - D(2g_\Gamma + K_{\partial\Gamma} - 1)\,.$$

- just renormalizable in $D = 4$
- only planar maps ($g_\Gamma = 0$) with single boundary ($K_{\partial\Gamma} = 1$) divergent

Tensorial field theory

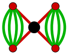
$\phi^n_{d,r}$ *tensorial field theory* [BenGeloun'14]:

- similar to $d_r = d(r-1)$ dimensional local field theory
- interactions $\mathbf{V}$ are r-coloured graphs, $\omega(\gamma_v) = d_r - \frac{d_r - 2\zeta}{2} V_{\gamma_v}$
- Feynman diagrams Γ are 2-graphs bijective to $(r+1)$-coloured graphs

Divergence degree (for general propagator $\omega(\gamma_e) = 2\zeta$):

$$\omega^{\mathrm{sd}}(\Gamma) = d_r - \frac{d_r - 2\zeta}{2} V_{\partial\Gamma} - d\left(\frac{2\omega^{\mathrm{G}}_{\Gamma} - 2\omega^{\mathrm{G}}_{\partial\Gamma}}{(r-1)!} + K_{\partial\Gamma} - 1\right).$$

Gurau degree $\omega^{\mathrm{G}} = \sum_J g_J$ generalizes genus g (J generalize Heegaard surfaces)

- theories renormalizable for interactions up to $n = \lfloor \frac{2d_r}{d_r - 2\zeta} \rfloor$
- just-renormalizable $\phi^4_{d,r}$ theories: $d_r = 4\zeta$ (e.g. $\zeta = \frac{1}{2}$: $\phi^4_{2,2}$, $\phi^4_{1,3}$)
- coproduct preserves ω^{G} [Raasakka/Tanasa'13] $\Rightarrow$ renormalizability for $\omega^{\mathrm{G}}_{\Gamma} > 0$
- $K_{\partial\Gamma} > 1$ possible: e.g. $\phi^6_{1,4}$ theory [BenGeloun/Rivasseau'13] needs  $\in \mathbf{V}$

Outline

Momemtum scheme in cNLFT

algebra homomorphism $A : \mathcal{G} \to \mathcal{A}$ to the algebra $\mathcal{A}$ of integrals with rational integrands

$$A_\Gamma = A(\Gamma) : \{p_f\}_{f \in \widetilde{\mathcal{F}}_\Gamma^{\text{ext}},} \mapsto A_\Gamma(\{p_f\}) := \prod_{v \in \mathcal{V}_\Gamma} \lambda_{\gamma_v} \prod_{f \in \mathcal{F}_\Gamma^{\text{int}}} \int_{\mathbb{R}^d} \mathrm{d}q_f \prod_{\{i,j\} \in \mathcal{E}_\Gamma} \tilde{P}(\boldsymbol{q}_i)$$

Superficial degree of divergence: $\omega^{\text{sd}}(\Gamma) = d \cdot F_\Gamma - 2\zeta E_\Gamma$

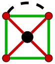
Momentum subtraction operator

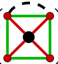
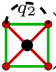
$$R[A_\Gamma](\{p_f\}) := \left(T^\omega_{\{p_f\}} A_\Gamma\right)(\{p_f\}) = \sum_{|\vec{k}| \leq \omega^{\text{sd}}(\Gamma)} \frac{1}{\vec{k}!} \frac{\partial^{|\vec{k}|} A_\Gamma^\Lambda}{\prod_f \partial p_f^{k_f}}(0) \prod_{f \in \widetilde{\mathcal{F}}_\Gamma^{\text{ext}}} p_f^{k_f}$$

Renormalized amplitude for primitive divergent 2-graphs (no subdivergences):

$$A_{\text{R}}(\Gamma) := (A - R \circ A)(\Gamma)$$

Example: Tadpole diagrams in tensorial theories

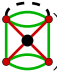
$\phi^4_{d=2,r=2}$ theory with $\tilde{P}(\boldsymbol{p}) = \frac{1}{|p_1|+|p_2|+1}$ (i.e. $\omega(\gamma_e) = 1$):  $\cong$ 

A_{R} () $(p_1) \equiv A_{\mathrm{R}}$ (p_1 )

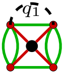
$= \lambda$  $(1 - T^1_{p_1}) \int_{\mathbb{R}^2} \mathrm{d}q_2 \, \frac{1}{|p_1| + |q_2| + 1}$

$= 2\pi\lambda$  $\Big((|p_1| + 1) \log(|p_1| + 1) - |p_1| \Big)$

$\phi^4_{d=1,r=3}$ theory with $\tilde{P}(\boldsymbol{p}) = \frac{1}{|p_1|+|p_2|+|p_3|+1}$: two tadpoles

A_{R} (p_1 ) $= \lambda$  $_c \; (1 - T^1_{p_1}) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathrm{d}q_2 \mathrm{d}q_3}{|p_1| + |q_2| + |q_3| + 1}$

$= 4\lambda$  $_c \; \Big((|p_1| + 1) \log(|p_1| + 1) - |p_1| \Big)$

A_{R} (p_2, p_3 )

$= \lambda$  $_c \; (1 - T^0_{p_2,p_3}) \int_{\mathbb{R}} \frac{\mathrm{d}q_1}{|q_1| + |p_2| + |p_3| + 1}$

$= -2\lambda$  $_c \; \log(|p_2| + |p_3| + 1)$

Subdivergences

In a renormalizable local field theory T:

- BPHZ: $\forall\Gamma$ with $\omega^{\mathrm{sd}}(\Gamma) \geq 0$ there is a counter term s.t. $A_{\mathrm{R}}(\Gamma)$ converges
- Zimmermann: forest formula for counter term of nested subdivergences
- Kreimer: counter term $S^{\mathrm{A}}_{\mathrm{R}} : \mathcal{H}^{\mathrm{fg}} \to \mathcal{A}$ from antipode S in Hopf alg. $\mathcal{H}^{\mathrm{fg}}$:

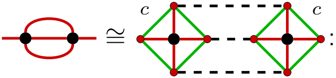
$$A_{\mathrm{R}} = S^{\mathrm{A}}_{\mathrm{R}} * A$$

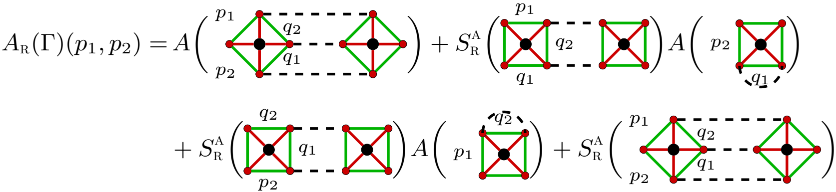
$$S^{\mathrm{A}}_{\mathrm{R}}(\Gamma) = -R\left[(S^{\mathrm{A}}_{\mathrm{R}} * A \circ P)(\Gamma)\right] = -\sum_{\substack{\Theta \in \mathcal{H}^{\mathrm{fg}} \\ \Theta \subsetneq \Gamma}} R\Big[S^{\mathrm{A}}_{\mathrm{R}}(\Theta) A(\Gamma/\Theta)\Big]$$

Renormalization in cNLFT

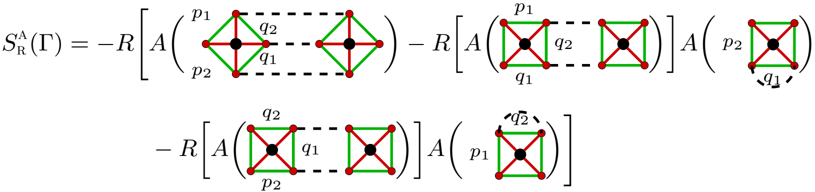
- counter term $S^{\mathrm{A}}_{\mathrm{R}}$ in the same way on the Hopf algebra of 2-graphs
- if cNLFT T is renormalizable, $A_{\mathrm{R}} = S^{\mathrm{A}}_{\mathrm{R}} * A$ on $\mathcal{H}^{\mathrm{f2g}}_{T}$ gives ren. amplitudes
- BPHZ momentum scheme: $S^{\mathrm{A}}_{\mathrm{R}}$ is algebra homomorphism since R is a Rota-Baxter operator $\big(R[AB] + R[A]R[B] = R[R[A]B + A\,R[B]]\big)$ as in local QFT

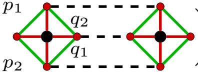
Example: sunrise diagram in $\phi^4_{2,2}$ theory

Sunrise 2-graph $\Gamma =$ 



Last counter term calculated recursively:



$A_{\mathrm{R}}\Big($  $\Big)$

$= \lambda^2$ 

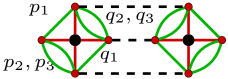
$$(1 - T^1_{p_1,p_2}) \int_{\mathbb{R}^2} \mathrm{d}q_1 \int_{\mathbb{R}^2} \mathrm{d}q_2 \Bigg(\frac{1}{|p_1|+|q_2|+1} \frac{1}{|q_1|+|q_2|+1} \frac{1}{|q_1|+|p_2|+1} + \frac{1}{|q_1|+|p_2|+1} \left(-T^0_{p_1,q_1}\right) \frac{1}{|p_1|+|q_2|+1} \frac{1}{|q_1|+|q_2|+1} + \frac{1}{|p_1|+|q_1|+1} \left(-T^0_{q_2,p_2}\right) \frac{1}{|q_1|+|q_2|+1} \frac{1}{|q_2|+|p_2|+1} \Bigg)$$

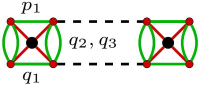
$= \lambda^2$ 

$$\frac{4\pi^2}{|p_1|+|p_2|+1} \Bigg[|p_1||p_2|\zeta_2 + (|p_1|+|p_2|+1) \sum_{i=1,2} \Big((|p_i|+1)\log(|p_i|+1) - |p_i| \Big) - \prod_{i=1,2} (|p_i|+1)\log(|p_i|+1) + \sum_{i=1,2} |p_i|(|p_i|+1)\mathrm{Li}_2(-|p_i|) \Bigg]$$

- in agreement with [Hock2020]
- multiple polylogarithms as in local QFT, but $\zeta_2 = \pi^2/6$ is peculiar

Example: sunrise diagram in $\phi^4_{1,3}$ theory

Sunrise diagram  has

- only logarithmic divergence $\omega^{\mathrm{sd}}(\Gamma) = 3 - 3 = 0$
- only one proper divergent 1PI subgraph 

- $\Rightarrow$ no overlapping divergence $\Rightarrow$ factorizing A_{R}

$A_{\mathrm{R}}(p_1, p_2, p_3) = \lambda^2$  $c\ 1$

$$\left(1 - T^0_{p_1,p_2,p_3}\right) \int_{\mathbb{R}} \mathrm{d}q_1 \frac{1}{|q_1| + |p_2| + |p_3| + 1}$$

$$\times \left(1 - T^0_{p_1,q_1}\right) \int_{\mathbb{R}} \mathrm{d}q_2 \int_{\mathbb{R}} \mathrm{d}q_3 \frac{1}{|q_1| + |q_2| + |q_3| + 1} \frac{1}{|p_1| + |q_2| + |q_3| + 1}$$

- more restricted set of LO diagrams ("melonic") in tensorial theories
- conjecture: all amplitudes are multiple polylogarithms

Outline

Combinatorial Dyson-Schwinger equations

Comb. Green's fct. expanded in # faces F (=# loops for planar maps in MFT):

$$X^\gamma = r_\gamma \pm \sum_{\substack{\Gamma \in \mathcal{H}_T^{\mathrm{f2g}} \\ \partial\Gamma = \gamma}} \alpha^{F_\Gamma} \frac{\Gamma}{|\mathrm{Aut}\,\Gamma|} = r_\gamma \pm \sum_{j=1}^{\infty} \alpha^j c_j^\gamma$$

Comb. Dyson-Schwinger eq. hold with usual comb. factors in B_+^Γ [Kreimer '08]:

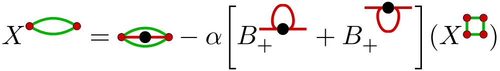
$$X^\gamma = r_\gamma \pm \sum_{k \geq 1} \alpha^k \Bigg[\sum_{\substack{\Gamma \text{ prim.} \\ F_\Gamma = k \\ \partial\Gamma = \gamma}} B_+^\Gamma \Bigg] (X^\gamma Q_\gamma)$$

by construction Hochschild 1-cocycles

Example: cDSE in quartic matrix field theory

simplifications:

- $|\mathrm{Aut}\,\Gamma| = 1$ for any map with boundary
- only two primitives for the 2-point function (due to planarity):



cDSE hold!! (no mixing with planar irregular sector necessary [Tanasa/Kreimer '13])

Challenges:

- still infinitely many primitives for X 
- factor # maximal forests in B_+ non-trivial: practical use of cDSE?
- possible to identify the reason for solvability here???
- necessary to modify the cDSE framework?

Outlook

- Result: algebraic structure of renormalization generalizes to cNLFT
- Random geometry/quantum gravity occurs at criticality
  $\rightarrow$ understand non-perturbative cNLFT!
- identify algebraic structure underlying solvability of matrix field theory
- apply to tensors of rank $r > 2$

Thanks for your attention!