Markus Reiß

Course *Stochastic Processes*
Winter 2009/10
Humboldt-Universität zu Berlin

Exercises: sheet 1

1. Prove: Let X be Poisson(s) and Y be Poisson(t) distributed. If X and Y are independent, then $X+Y$ is Poisson($t+s$) distributed ($t, s > 0$). This means that the property of a convolution semigroup of measures $(P(t))_{t>0}$ holds: $P(s) * P(t) = P(t+s)$, $s, t > 0$. Which measure $P(0)$ is the neutral element of such a convolution semigroup?

2. Let $(N_t,\, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$ and let $(Y_k)_{k \geqslant 1}$ be a sequence of i.i.d. random variables, independent of N. Then $X_t := \sum_{k=1}^{N_t} Y_k$ is called *compound Poisson process* ($X_t := 0$ if $N_t = 0$).

   (a) Show that (X_t) has independent and stationary increments. Infer that the laws $P(t) = \mathbb{P}^{X_t}$ define a convolution semigroup (as in (1)).

   (b) Determine expectation and variance of X_t in the case $Y_k \in L^2$.

3. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities μ and λ. Show that the arrival of flying beasts form a Poisson process of intensity $\lambda+\mu$ (*Superposition*). The probability that an arriving fly is a blow-fly is p. Does the arrival of blow-flies also form a Poisson process? (*Thinning*)

4. The number of busses that arrive until time t at a bus stop follows a Poisson process with intensity $\lambda > 0$ (in our model). Adam and Berta arrive together at time $t_0 > 0$ at the bus stop and discuss how long they have to wait in the mean for the next bus.
   Adam: Since the waiting times are Exp(λ)-distributed and the exponential distribution is memoryless, the mean is λ^{-1}.
   Berta: The time between the arrival of two busses is Exp(λ)-distributed and has mean λ^{-1}. Since on average the same time elapses before our arrival and after our arrival, we obtain the mean waiting time $\frac{1}{2}\lambda^{-1}$ (at least assuming that at least one bus had arrived before time t_0).
   What is the correct answer to this *waiting time paradoxon*?

Submit before the lecture on Tuesday, 27 October 2009

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Exercises: sheet 2

1. Let $(P(t))_{t\geqslant 0}$ be the transition matrices of a continuous-time, time-homogeneous Markov chain with finite state space. Assume that the transition probabilities $p_{ij}(t)$ are differentiable for $t \geqslant 0$. Prove:
   (a) The derivative satisfies $p'_{ij}(0) \geqslant 0$ for $i \neq j$, $p'_{ii}(0) \leqslant 0$ and $\sum_j p'_{ij}(0) = 0$.
   (b) With the matrix (*generator*) $A = (p'_{ij}(0))_{i,j}$ we obtain the *forward* and *backward equation*:
   $$P'(t) = P(t)A, \quad P'(t) = AP(t), \quad t \geqslant 0.$$
   (c) The generator A defines uniquely $P(t)$: $P(t) = e^{At} := \sum_{k\geqslant 0} A^k t^k / k!$.
   (d*) Find conditions to extend these results to general countable state space.

2. Let $(X_n,\, n \geqslant 0)$ be a discrete-time, time-homogeneous Markov chain and let $(N_t,\, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$, independent of X. Show that $Y_t := X_{N_t}$, $t \geqslant 0$, is a continuous-time, time-homogeneous Markov chain. Determine its transition probabilities and its generator.
   Remark: Under regularity conditions this gives all continuous-time, time-homogeneous Markov chains.

3. Let $C([0,\infty))$ be equipped with the topology of uniform convergence on compacts using the metric $d(f,g) := \sum_{k\geqslant 1} 2^{-k} (\sup_{t\in[0,k]} |f(t) - g(t)| \wedge 1)$. Prove:
   (a) $(C([0,\infty)), d)$ is Polish.
   (b) The Borel σ-algebra is the smallest σ-algebra such that all coordinate projections $\pi_t : C([0,\infty)) \to \mathbb{R}$, $t \geqslant 0$, are measurable.
   (c) For any continuous stochastic process $(X_t,\, t \geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ the mapping $\bar{X} : \Omega \to C([0,\infty))$ with $\bar{X}(\omega)_t := X_t(\omega)$ is Borel-measurable.
   (d) The law of $\bar{X}$ is uniquely determined by the finite-dimensional distributions of X.

4. Prove the regularity lemma: Let P be a probability measure on the Borel σ-algebra $\mathfrak{B}$ of any metric (or topological) space. Then
   $$\mathcal{D} := \Big\{ B \in \mathfrak{B} \,\Big|\, P(B) = \sup_{K\subseteq B \text{ compact}} P(K) = \inf_{O\supseteq B \text{ open}} P(O) \Big\}$$
   is closed under set differences and countable unions ($\mathcal{D}$ is a *σ-ring*). If P is tight, then $\mathcal{D}$ is a σ-algebra.

Submit before the lecture on Tuesday, 3 November 2009

Markus Reiß

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Exercises: sheet 3

1. A discrete-time *Markov process* with state space $(S, \mathscr{S})$ is specified by an initial distribution μ^0 on $(S, \mathscr{S})$ and a *transition kernel* $P : S \times \mathscr{S} \to [0, 1]$ (i.e. $B \mapsto P(x, B)$ is a probability measure for all $x \in S$ and $x \mapsto P(x, B)$ is measurable for all $B \in \mathscr{S}$). Show:

   (a) If we put iteratively $P^n(x, B) := \int_S P^{n-1}(y, B)\, P(x, dy)$ for $n \geqslant 2$ and $P^1 := P$, then each P^n is again a transition kernel.

   (b) Put for all $n \geqslant 1$, $A \in \mathscr{S}^{\otimes n}$

   $$Q_n(A) := \int_{S^n} \mathbf{1}_A(x_0, x_1, \dots, x_{n-1}) \mu^0(dx_0) P(x_0, dx_1) \cdots P(x_{n-2}, dx_{n-1}).$$

   Then $(Q_n)_{n\geqslant 1}$ defines a projective family on $S^{\mathbb{N}}$.

   (c) Let $(S, \mathscr{S})$ be Polish. Then for each initial distribution μ_0 and each transition kernel P there exists a stochastic process $(X_n,\ n \geqslant 0)$ satisfying $\mathbb{P}^{X_0} = \mu_0$ and $\mathbb{P}^{(X_0,\dots,X_{n-1})} = Q_n$, $n \geqslant 1$ (the Markov process).

2. A Gaussian process $(X_t,\ t \in T)$ is a process whose finite-dimensional distributions are (generalized) Gaussian, i.e. $(X_{t_1}, \dots, X_{t_n}) \sim N(\mu_{t_1,\dots,t_n}, \Sigma_{t_1,\dots,t_n})$ with $\Sigma_{t_1,\dots,t_n} \in \mathbb{R}^{n\times n}$ positive semi-definite.

   (a) Why are the finite-dimensional distributions of X uniquely determined by the expectation function $t \mapsto \mathbb{E}[X_t]$ and the covariance function $(s, t) \mapsto \operatorname{Cov}(X_s, X_t)$?

   (b) Prove that for an arbitrary function $\mu : T \to \mathbb{R}$ and any symmetric, positive (semi-)definite function $C : T^2 \to \mathbb{R}$, i.e. $C(t, s) = C(s, t)$ and

   $$\forall n \geqslant 1;\ t_1, \dots, t_n \in T;\ \lambda_1, \dots, \lambda_n \in \mathbb{R} :\ \sum_{i,j=1}^{n} C(t_i, t_j)\lambda_i\lambda_j \geqslant 0,$$

   there is a Gaussian process with expectation function μ and covariance function C.

3. Let (X,Y) be a two-dimensional random vector with Lebesgue density $f^{X,Y}$.

   (a) For $x \in \mathbb{R}$ with $f^X(x) > 0$ ($f^X(x) = \int f^{X,Y}(x,\eta)\,d\eta$) consider the *conditional density* $f^{Y\,|\,X=x}(y) := f^{X,Y}(x,y)/f^X(x)$. Which condition on $f^{X,Y}$ ensures for any Borel set B

   $$\lim_{h\downarrow 0} \mathbb{P}(Y \in B \,|\, X \in [x, x+h]) = \int_B f^{Y\,|\,X=x}(y)\,dy \quad ?$$

   (b) Show that for $Y \in L^2$ (without any condition on $f^{X,Y}$) the function

   $$\varphi_Y(x) := \begin{cases} \int y f^{Y\,|\,X=x}(y)\,dy, & f^X(x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

   minimizes the L^2-distance $\mathbb{E}[(Y - \varphi(X))^2]$ over all measurable functions φ. We write $\mathbb{E}[Y \,|\, X = x] := \varphi_Y(x)$ and $\mathbb{E}[Y \,|\, X] := \varphi_Y(X)$.

   (c) Prove that φ_Y is $\mathbb{P}^X$-a.s. uniquely (among all $\varphi : \mathbb{R} \to \mathbb{R}$ measurable) characterized by solving

   $$\forall A \in \mathfrak{B}_{\mathbb{R}} : \mathbb{E}[\varphi(X)\mathbf{1}_A(X)] = \mathbb{E}[Y\mathbf{1}_A(X)].$$

4. In the situation of exercise 3 prove the following properties:

   (a) $\mathbb{E}[\mathbb{E}[Y \,|\, X]] = \mathbb{E}[Y]$;

   (b) if X and Y are independent, then $\mathbb{E}[Y \,|\, X] = \mathbb{E}[Y]$ holds a.s.;

   (c) if $Y \geqslant 0$ a.s., then $\mathbb{E}[Y \,|\, X] \geqslant 0$ a.s.;

   (d) for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ we have $\mathbb{E}[\alpha Y + \beta \,|\, X] = \alpha\,\mathbb{E}[Y \,|\, X] + \beta$ a.s.;

   (e) if $\varphi : \mathbb{R} \to \mathbb{R}$ is such that $(x,y) \mapsto (x, y\varphi(x))$ is a diffeomorphism and $Y\varphi(X) \in L^2$, then $\mathbb{E}[Y\varphi(X) \,|\, X] = \mathbb{E}[Y \,|\, X]\varphi(X)$ a.s.

Submit before the lecture on Tuesday, 10 November 2009

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 4

1. Let $\Omega = \bigcup_{n\in\mathbb{N}} B_n$ be a measurable, countable partition for given $(\Omega, \mathcal{F}, \mathbb{P})$ and put $\mathcal{B} := \sigma(B_n,\, n \in \mathbb{N})$. Show:

   (a) Every $\mathcal{B}$-measurable random variable X can be written as $X = \sum_n \alpha_n \mathbf{1}_{B_n}$ with suitable $\alpha_n \in \mathbb{R}$. For $Y \in L^1$ we have $\mathbb{E}[Y \,|\, \mathcal{B}] = \sum_{n:\,\mathbb{P}(B_n)>0} \Big(\frac{1}{\mathbb{P}(B_n)} \int_{B_n} Y \, d\,\mathbb{P}\Big) \mathbf{1}_{B_n}$.

   (b) Specify $\Omega = [0,1)$ with Borel σ-algebra and $\mathbb{P} = U([0,1))$, the uniform distribution. For $Y(\omega) := \omega$, $\omega \in [0,1)$, determine $\mathbb{E}[Y \,|\, \sigma([(k-1)/n, k/n),\, k = 1, \dots, n)]$. For $n = 1, 3, 5, 10$ plot the conditional expectations and Y itself as functions on Ω.

2. Let (X, Y) be a two-dimensional $N(\mu, \Sigma)$-random vector.

   (a) For which $\alpha \in \mathbb{R}$ are X and $Y - \alpha X$ uncorrelated?

   (b) Conclude that X and $Y - (\alpha X + \beta)$ are independent for these values α and for arbitrary $\beta \in \mathbb{R}$ such that $\mathbb{E}[Y|X] = \alpha X + \beta$ with suitable $\beta \in \mathbb{R}$.

3. For $Y \in L^2$ define the *conditional variance* of Y given X by

$$\operatorname{Var}(Y|X) := \mathbb{E}[(Y - \mathbb{E}[Y \,|\, X])^2 \,|\, X].$$

   (a) Why is $\operatorname{Var}(Y|X)$ well defined?

   (b) Show $\operatorname{Var}(Y) = \operatorname{Var}(\mathbb{E}[Y \,|\, X]) + \mathbb{E}[\operatorname{Var}(Y|X)]$.

   (c) Use (b) to prove for independent random variables $(Z_k)_{k\geqslant 1}$ and N in L^2 with (Z_k) identically distributed and N $\mathbb{N}$-valued:

$$\operatorname{Var}\Big(\sum_{k=1}^{N} Z_k\Big) = \mathbb{E}[Z_1]^2 \operatorname{Var}(N) + \mathbb{E}[N] \operatorname{Var}(Z_1).$$

4. For a convex function $\varphi : \mathbb{R} \to \mathbb{R}$ (i.e. $\varphi(\alpha x + (1-\alpha)y) \leqslant \alpha\varphi(x) + (1-\alpha)\varphi(y))$ for $x, y \in \mathbb{R}$, $\alpha \in (0,1)$) show:

   (a) $D(x,y) := \frac{\varphi(y)-\varphi(x)}{y-x}$, $x \neq y$, is non-decreasing in x and y, which implies that φ is differentiable from the right and from the left and that φ is continuous.

   (b) Using the right-derivative φ'_+, we have:

$$\begin{aligned} \forall\, x, y \in \mathbb{R} : \qquad & \varphi(y) \geqslant \varphi(x) + \varphi'_+(x)(y-x), \\ \forall\, y \in \mathbb{R} : \qquad & \varphi(y) = \sup_{x \in \mathbb{Q}} \big(\varphi(x) + \varphi'_+(x)(y-x)\big). \end{aligned}$$

   (c) Assume $Y, \varphi(Y) \in L^1$. Then $\mathbb{E}[\varphi(Y) \,|\, \mathcal{G}] \geqslant \varphi(x) + \varphi'_+(x)(\mathbb{E}[Y \,|\, \mathcal{G}] - x)$ holds for all $x \in \mathbb{R}$. Infer Jensen's inequality: $\mathbb{E}[\varphi(Y) \,|\, \mathcal{G}] \geqslant \varphi(\mathbb{E}[Y \,|\, \mathcal{G}])$.

Submit before the lecture on Tuesday, 17 November 2009

Markus Reiß

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Exercises: sheet 5

1. *Doubling strategy*: In each round a fair coin is tossed, for *heads* the player receives his double stake, for *tails* he loses his stake. His initial capital is $K_0 = 0$. At game $n \geqslant 1$ his strategy is as follows: if *heads* has appeared before, his stake is zero (he stops playing); otherwise his stake is 2^{n-1} Euro.

   (a) Argue why his capital K_n after game n can be modeled with independent (X_i) such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ via

   $$K_n = \begin{cases} -(2^n - 1), & X_1 = \cdots = X_n = -1, \\ 1, & \text{otherwise.} \end{cases}$$

   (b) Represent K_n as martingale transform.

   (c) Prove $\lim_{n\to\infty} K_n = 1$ a.s. although $\mathbb{E}[K_n] = 0$ for all $n \geqslant 0$ holds.

2. Let T be an $\mathbb{N}_0$-valued random variable and $S_n := \mathbf{1}_{\{n \geqslant T\}}$, $n \geqslant 0$. Show:

   (a) The natural filtration satisfies $\mathcal{F}_n^S = \sigma(\{T = k\},\, k = 0, \ldots, n)$.

   (b) (S_n) is a submartingale with respect to $(\mathcal{F}_n^S)$ and

   $$\mathbb{E}[S_{n+1} \,|\, \mathcal{F}_n^S] = \mathbf{1}_{\{S_n = 1\}} + \mathbb{P}(T = n+1 \,|\, T \geqslant n+1)\mathbf{1}_{\{S_n = 0\}} \ \mathbb{P}\text{-a.s.}$$

   (c) Determine the Doob decomposition of (S_n). Sketch for geometrically distributed T ($\mathbb{P}(T = k) = (1-p)p^k$) the sample paths of (S_n), its compensator and their difference.

3. Prove the *Höffding inequality*: Let (M_n) be a martingale with $M_0 = 0$ and $|M_n(\omega) - M_{n-1}(\omega)| \leqslant K_n$, $\omega \in \Omega$, $n \geqslant 1$. Then:

   $$\mathbb{P}(|M_n| \geqslant x) \leqslant 2 \exp\Big(- \frac{x^2}{2 \sum_{i=1}^n K_i^2} \Big), \quad x > 0.$$

   Proceed stepwise:

   (a) From $\mathbb{E}[Z] = 0$ and $|Z| \leqslant 1$ we deduce $e^{\eta Z} \leqslant \cosh(\eta) + Z \sinh(\eta)$ and $\mathbb{E}[e^{\eta Z}] \leqslant \cosh(\eta) \leqslant e^{\eta^2/2}$ for all $\eta \in \mathbb{R}$.

   (b) This implies $\mathbb{E}[\exp(\eta M_n) \,|\, \mathcal{F}_{n-1}] \leqslant \exp(\eta M_{n-1} + \eta^2 K_n^2/2)$.

   (c) By induction we obtain $\mathbb{E}[\exp(\eta M_n)] \leqslant \exp(\eta^2 \sum_{i=1}^n K_i^2/2)$.

   (d) Use the (generalized) Markov inequality and optimize over η to conclude.

4. Your winnings per unit stake on game game n are ε_n, where (ε_n) are independent random variables with $\mathbb{P}(\varepsilon_n = 1) = p$, $\mathbb{P}(\varepsilon_n = -1) = 1 - p$ for $p > 1/2$. Your stake X_n on game n must lie between zero and C_{n-1}, your capital at time $n - 1$. For some $N \in \mathbb{N}$ and $C_0 > 0$ your objective is to maximize the expected interest rate $\mathbb{E}[\log(C_N/C_0)]$.
   Show that for any predictable strategy X the process $\log(C_n) - n\alpha$ is a supermartingale with respect to $\mathscr{F}_n := \sigma(\varepsilon_1, \ldots, \varepsilon_n)$ where

$$\alpha := p \log p + (1-p)\log(1-p) + \log 2 \ (\textit{entropy}).$$

   Hence, $\mathbb{E}[\log(C_N/C_0)] \leqslant N\alpha$ always holds. Find an optimal strategy such that $\log(C_n) - n\alpha$ is even a martingale.
   Remark: This is the martingale approach to optimal control.

Submit before the lecture on Tuesday, 24 November 2009

Markus Reiß

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Exercises: sheet 6

1. Let $(\mathcal{F}_n^X)_{n\geqslant 0}$ be the natural filtration of a process $(X_n)_{n\geqslant 0}$ and consider a finite stopping time τ with respect to $(\mathcal{F}_n^X)$.

   (a) Prove $\mathcal{F}_\tau = \sigma(\tau,\, X_{\tau\wedge n},\, n \geqslant 0)$.
   Hint: for '$\subseteq$' write $A \in \mathcal{F}_\tau$ as $A = \bigcup_n A \cap \{\tau = n\}$.

   (b*) Do we even have $\mathcal{F}_\tau = \sigma(X_{\tau\wedge n},\, n \geqslant 0)$?

2. Let $(S_n)_{n\geqslant 0}$ be a simple random walk with $\mathbb{P}(S_n - S_{n-1} = 1) = p$, $\mathbb{P}(S_n - S_{n-1} = -1) = q = 1 - p$, $p \in (0,1)$. Prove:

   (a) Put $M(\lambda) = pe^{\lambda} + qe^{-\lambda}$, $\lambda \in \mathbb{R}$. Then the process

   $$Y_n^\lambda := \exp\Big(\lambda S_n - n\log(M(\lambda))\Big), \quad n \geqslant 0,$$

   is a martingale (w.r.t. its natural filtration).

   (b) For $a, b \in \mathbb{Z}$ with $a < 0 < b$ and the stopping time(!) $\tau := \inf\{n \geqslant 0 \,|\, S_n \in \{a, b\}\}$ we have

   $$e^{a\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau = a\}}] + e^{b\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau = b\}}] = 1 \text{ if } M(\lambda) \geqslant 1.$$

   (c) This implies for all $s \in (0,1]$ (put $s = M(\lambda)^{-1}$)

   $$\mathbb{E}[s^\tau \mathbf{1}_{\{S_\tau = a\}}] = \frac{\lambda_+(s)^b - \lambda_-(s)^b}{\lambda_+(s)^b\lambda_-(s)^a - \lambda_+(s)^a\lambda_-(s)^b},$$
   $$\mathbb{E}[s^\tau \mathbf{1}_{\{S_\tau = b\}}] = \frac{\lambda_-(s)^a - \lambda_+(s)^a}{\lambda_+(s)^b\lambda_-(s)^a - \lambda_+(s)^a\lambda_-(s)^b}$$

   with $\lambda_\pm(s) = (1 \pm \sqrt{1 - 4pqs^2})/(2ps)$.

   (d) Now let $a \downarrow -\infty$ and infer that the generating function of the first passage time $\tau_b := \inf\{n \geqslant 0 \,|\, S_n = b\}$ is given by

   $$\varphi_{\tau_b}(s) := \mathbb{E}[s^{\tau_b}\mathbf{1}_{\{\tau_b < \infty\}}] = \Big(\frac{1 - \sqrt{1 - 4pqs^2}}{2qs}\Big)^b, \quad s \in (0,1].$$

   In particular, we have $\mathbb{P}(\tau_b < \infty) = \varphi_{\tau_b}(1) = \min(1, p/q)^b$.

Submit before the lecture on Tuesday, 1 December 2009

Markus Reiß

Course *Stochastic Processes*
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Humboldt-Universität zu Berlin

Exercises: sheet 7

1. Let $(X_n)_{n\geqslant 0}$ be an $(\mathscr{F}_n)$-adapted family of random variables in L^1. Show that $(X_n)_{n\geqslant 0}$ is a martingale if and only if for all bounded $(\mathscr{F}_n)$-stopping times τ the identity $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ holds.

2. Prove that a family $(X_i)_{i\in I}$ of random variables is uniformly integrable if and only if $\sup_{i\in I} \|X_i\|_{L^1} < \infty$ holds as well as
$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 :\ \mathbb{P}(A) < \delta \Rightarrow \sup_{i\in I} \mathbb{E}[|X_i|\mathbf{1}_A] < \varepsilon.$$

3. Give a martingale proof of Kolmogorov's 0-1 law:
   (a) Let $(\mathscr{F}_n)$ be a filtration and $\mathscr{F}_\infty = \sigma(\mathscr{F}_n, n \geqslant 0)$. Then for $A \in \mathscr{F}_\infty$ we have $\lim_{n\to\infty} \mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_n] = \mathbf{1}_A$ a.s.
   (b) For a sequence $(X_k)_{k\geqslant 1}$ of independent random variables consider the natural filtration $(\mathscr{F}_n)$ and the terminal σ-algebra $\mathscr{T} := \bigcap_{n\geqslant 1} \sigma(X_k,\, k \geqslant n)$. Then for $A \in \mathscr{T}$ we deduce $\mathbb{P}(A) = \mathbb{E}[1_A \,|\, \mathscr{F}_n] \to \mathbf{1}_A$ a.s. such that $P(A) \in \{0, 1\}$ holds.

4. A monkey types at random the 26 capital letters of the Latin alphabet. Let τ be the first time by which the monkey has completed the sequence ABRACADABRA. Prove that τ is almost surely finite and satisfies
$$\mathbb{E}[\tau] = 26^{11} + 26^4 + 26.$$
   How much time does it take on average if one letter is typed every second?
   Hint: You may look at a fair game with gamblers G_n arriving before times $n = 1, 2, \ldots$ G_n bets 1 Euro on 'A' for letter n; if he wins, he puts 26 Euro on 'B' for letter $n+1$, otherwise he stops. If he wins again, he puts 26^2 Euro on 'R', otherwise he stops etc.

Submit before the lecture on Tuesday, 8 December 2009

Markus Reiß

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Exercises: sheet 8

1. Suppose $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_2$ are probability measures on $(\Omega, \mathscr{F})$. Show:
   (a) If $\mathbb{P}_2 \ll \mathbb{P}_1 \ll \mathbb{P}_0$ holds, then $\frac{d\mathbb{P}_2}{d\mathbb{P}_0} = \frac{d\mathbb{P}_2}{d\mathbb{P}_1}\frac{d\mathbb{P}_1}{d\mathbb{P}_0}$ holds $\mathbb{P}_0$-a.s.
   (b) $\mathbb{P}_0$ and $\mathbb{P}_1$ are *equivalent* (i.e. $\mathbb{P}_1 \ll \mathbb{P}_0$ and $\mathbb{P}_0 \ll \mathbb{P}_1$) if and only if $\mathbb{P}_1 \ll \mathbb{P}_0$ and $\frac{d\mathbb{P}_1}{d\mathbb{P}_0} > 0$ holds $\mathbb{P}_0$-a.s. In that case we have $\frac{d\mathbb{P}_0}{d\mathbb{P}_1} = \left(\frac{d\mathbb{P}_1}{d\mathbb{P}_0}\right)^{-1}$ $\mathbb{P}_0$-a.s. and $\mathbb{P}_1$-a.s.

2. Prove in detail for $\mathbb{Q} \ll \mathbb{P}$, $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$ and $Y \in L^1(\mathbb{Q})$ the identity $\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ]$. Give an example where $Y \in L^1(\mathbb{Q})$, but not $Y \in L^1(\mathbb{P})$ holds.

3. Let $(Z_n)_{n\geqslant 0}$ be a non-negative martingale on $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_n))$ with $\mathbb{E}_{\mathbb{P}}[Z_0] = 1$. Prove:
   (a) $\mathbb{Q}_n(A) := \mathbb{E}[Z_n \mathbf{1}_A]$, $A \in \mathscr{F}_n$, defines a probability measure with $\mathbb{Q}_n \ll \mathbb{P}|_{\mathscr{F}_n}$ for all $n \geqslant 0$. For $m > n$ we have the consistency $\mathbb{Q}_n = \mathbb{Q}_m|_{\mathscr{F}_n}$.
   (b) Conversely, if $(\mathbb{Q}_n)$ is a sequence of probability measures satisfying this consistency property and $\mathbb{Q}_n \ll \mathbb{P}|_{\mathscr{F}_n}$ for all $n \geqslant 0$, then $Z_n := \frac{d\mathbb{Q}_n}{d\mathbb{P}|_{\mathscr{F}_n}}$, $n \geqslant 0$, forms a $\mathbb{P}$-martingale.
   (c) The following change-of-measure rule is valid for $n \leqslant m$ and $Y \in L^1(\Omega, \mathscr{F}_m, \mathbb{Q}_m)$:
   $$\mathbb{E}_{\mathbb{Q}_m}[Y \mid \mathscr{F}_n] = \frac{\mathbb{E}_{\mathbb{P}}[YZ_m \mid \mathscr{F}_n]}{Z_n} \qquad \mathbb{P}\text{-a.s. and } \mathbb{Q}_m\text{-a.s.}$$
   Here, the right hand side is set to zero on $\{Z_n = 0\}$.

4. Let $Z_n(x) = (3/2)^n \sum_{k\in\{0,2\}^n} \mathbf{1}_{I(k,n)}(x)$, $x \in [0,1]$, with intervals $I(k,n) := [\sum_{i=1}^n k_i 3^{-i}, \sum_{i=1}^n k_i 3^{-i} + 3^{-n}]$. Show:
   (a) $(Z_n)_{n\geqslant 0}$ with $Z_0 = 1$ forms a martingale on $([0,1], \mathfrak{B}_{[0,1]}, \lambda, (\mathscr{F}_n))$ with Lebesgue measure λ on $[0,1]$ and $\mathscr{F}_n := \sigma(I(k,n),\ k \in \{0,1,2\}^n)$.
   (b) (Z_n) converges λ-a.s., but not in $L^1([0,1], \mathfrak{B}_{[0,1]}, \lambda)$ (Sketch!).
   (c) Interpret Z_n as the density of a probability measure $\mathbb{P}_n$ with respect to λ. Then $(\mathbb{P}_n)$ converges weakly to some probability measure $\mathbb{P}_\infty$ ($\mathbb{P}_\infty$ is called *Cantor measure*). There is a Borel set $C \subseteq [0,1]$ with $\mathbb{P}_\infty(C) = 1$, $\lambda(C) = 0$.
   Hint: Show that the distribution functions converge to a limit distribution function, which is λ-a.e. constant.

Submit before the lecture on Tuesday, 15 December 2009

Markus Reiß

Course *Stochastic Processes*
Winter 2009/10
Humboldt-Universität zu Berlin

Exercises: sheet 9

1. Let $(X_t,\, t \in T)$ be a Gaussian process for $T = \mathbb{R}^+$ or $T = \mathbb{N}_0$. Show that X is stationary if and only if X is *weakly stationary*, i.e. its mean function is constant and its covariance function c satisfies $c(t,s) = c(t-s,0)$ for all $t \geqslant s$.

   Find an example of a non-Gaussian process, which is weakly stationary, but not stationary in the strict sense.

2. Let $X_0 \sim N(\mu, \sigma_0^2)$ and $\varepsilon_t \sim N(0, \sigma^2)$, $t \geqslant 1$, be independent random variables. Then for $a \in \mathbb{R}$ the *autoregressive process* X is defined recursively by
$$X_t = aX_{t-1} + \varepsilon_t, \quad t \geqslant 1.$$
   (a) Why is $(X_t,\, t \in \mathbb{N}_0)$ a Gaussian process?
   (b) Determine the mean and the covariance function of X.
   (c) For which parameter values $a, \mu, \sigma_0^2, \sigma^2$ is X stationary?
   (d*) Simulate some trajectories of $(X_t,\, 0 \leqslant t \leqslant 100)$ for $\mu = 0$, $\sigma_0^2 = \sigma^2 = 1$ and $a \in \{0; -0.5; 1; -2\}$.

3. Prove the following result for the invariant σ-algebra $\mathscr{I}_T$ with respect to some measure-preserving transformation T on $(\Omega, \mathscr{F}, \mathbb{P})$:

   (a) A (real-valued) random variable Y is $\mathscr{I}_T$-measurable if and only if it is $\mathbb{P}$-a.s. invariant, i.e. $\mathbb{P}(Y \circ T = Y) = 1$. In particular, T is ergodic if and only if each $\mathbb{P}$-a.s. invariant and bounded random variable is $\mathbb{P}$-a.s. constant.
   (b) For each invariant event $A \in \mathscr{I}_T$ there exists a strictly invariant event B (i.e. with $T^{-1}(B) = B$ exactly) such that $\mathbb{P}(A \Delta B) = 0$.

4. Let $(X_n,\, n \geqslant 0)$ be a real-valued ergodic process, canonically constructed on $(\mathbb{R}^{\mathbb{N}_0}, \mathfrak{B}_{\mathbb{R}}^{\otimes \mathbb{N}_0}, \mathbb{P})$ (i.e. the corresponding left-shift T is measure-preserving and ergodic). Prove:

   (a) For any $m \geqslant 1$ the $\mathbb{R}^2$-valued process $((X_n, X_{n+m}),\, n \geqslant 0)$ is also ergodic.
   (b) Suppose $X_n \in L^2$ for all $n \geqslant 0$. Then the following estimators for the mean $\mathbb{E}[X_0]$ and the covariance $\operatorname{Cov}(X_0, X_m)$ are strongly consistent (i.e. converge a.s. to the true value for $n \to \infty$):
$$\hat{\mu}_n := \frac{1}{n}\sum_{k=0}^{n-1} X_k, \quad \hat{C}_n(m) := \frac{1}{n}\sum_{k=0}^{n-m-1} X_k X_{k+m} - \hat{\mu}_n^2.$$

Submit before the lecture on Tuesday, 12 January 2010

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Exercises: sheet 10

1. Prove von Neumann's ergodic theorem: For measure-preserving T and $X \in L^p$, $p \geqslant 1$, we have that $A_n := \frac{1}{n}\sum_{i=0}^{n-1} X \circ T^i$ converges to $\mathbb{E}[X \mid \mathscr{I}_T]$ in L^p.
   Hint: Show that $|A_n|^p$ is uniformly integrable.

2. Show that a measure-preserving map T on $(\Omega, \mathscr{F}, \mathbb{P})$ is ergodic if and only if for all $A, B \in \mathscr{F}$
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(A \cap T^{-k}B) = \mathbb{P}(A)\,\mathbb{P}(B).$$
   Hint: For one direction apply an ergodic theorem to $\mathbf{1}_B$.

3. *Gelfand's Problem:* Does the decimal representation of 2^n ever start with the initial digit 7? Study this as follows:
   (a) Determine the relative frequencies of the initial digits of $(2^n)_{1\leqslant n\leqslant 30}$.
   (b) Let $A \sim U([0,1])$. Prove that the initial digit k in $(10^A 2^n)_{1\leqslant n\leqslant m}$ converges as $m \to \infty$ a.s. to $\log_{10}(k+1) - \log_{10}(k)$ (consider $X_n = A + n\log_{10}(2)$ mod 1!).
   (c) Prove that the convergence in (b) even holds everywhere. In particular, the relative frequency of the initial digit 7 in the powers of 2 converges to $\log_{10}(8/7) \approx 0{,}058$.
   Hint: Show for trigonometric polynomials $p(a) = \sum_{|m|\leqslant M} c_m e^{2\pi i m a}$ that $\frac{1}{n}\sum_{k=0}^{n-1} p(a+k\eta) \to \int_0^1 p(x)dx$ holds for all $\eta \in \mathbb{R} \setminus \mathbb{Q}$, $a \in [0,1]$ (calculate explicitly for monomials!) and approximate.

4. Consider the Ehrenfest model, i.e. a Markov chain on $S = \{0, 1, \dots, N\}$ with transition probabilities $p_{i,i+1} = (N-i)/N$, $p_{i,i-1} = i/N$.
   (a) Show that $\mu(\{i\}) = \binom{N}{i} 2^{-N}$, $i \in S$, is an invariant initial distribution.
   (b) Is the Markov chain starting in μ ergodic?
   (*c) Simulate the Ehrenfest model with initial value $i_0 \in \{N/2; N\}$, $N = 100$ for $T \in \{100; 100{,}000\}$ time steps. Plot the relative frequencies of visits to each state in S and compare with μ. Explain what you see!

Submit before the lecture on Tuesday, 19 January 2010

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Exercises: sheet 11

1. Decide whether for $n \to \infty$ the probability $\mathbb{P}_n$ with the following Lebesgue densities f_n on $\mathbb{R}$ converge in total variation distance, weakly or not at all:

$$f_n(x) = ne^{-nx}\mathbf{1}_{[0,\infty)}(x), \quad f_n(x) = \tfrac{n+1}{n}x^{1/n}\mathbf{1}_{[0,1]}(x), \quad f_n(x) = \tfrac{1}{n}\mathbf{1}_{[0,n]}(x).$$

2. Consider real-valued random variables $(X_n)_{n\geqslant 1}$, $(Y_n)_{n\geqslant 1}$, X. Prove:

   (a) If $\mathbb{P}(X=a)=1$ holds for some $a\in\mathbb{R}$, then $X_n \xrightarrow{d} X \iff X_n \xrightarrow{\mathbb{P}} X$.

   (b) If $Y_n \xrightarrow{d} a$, $a\in\mathbb{R}$, and $X_n \xrightarrow{d} X$, then $(X_n,Y_n)\xrightarrow{d}(X,a)$, in particular $X_nY_n \xrightarrow{d} aX$.

3. We say that a family of real-valued random variables $(X_i)_{i\in I}$ is *stochastically bounded*, notation $X_i = O_{\mathbb{P}}(1)$, if

$$\lim_{R\to\infty}\sup_{i\in I}\mathbb{P}(|X_i|>R)=0.$$

   (a) Show $X_i = O_{\mathbb{P}}(1)$ if and only if the laws $(\mathbb{P}^{X_i})_{i\in I}$ are uniformly tight.

   (b) Prove that any L^p-bounded family of random variables is stochastically bounded, hence has uniformly tight laws.

   (c) If $X_n \xrightarrow{\mathbb{P}} 0$ holds, then we write $X_n = o_{\mathbb{P}}(1)$. Check the symbolic rules $O_{\mathbb{P}}(1)+o_{\mathbb{P}}(1)=O_{\mathbb{P}}(1)$ and $O_{\mathbb{P}}(1)o_{\mathbb{P}}(1)=o_{\mathbb{P}}(1)$.

4. Prove: Every relatively (weakly) compact family $(\mathbb{P}_i)_{i\in I}$ of probability measures on a Polish space $(S,\mathfrak{B}_S)$ is uniformly tight. Proceed as follows:

   (a) For $k\geqslant 1$ consider open balls $(A_{k,m})_{m\geqslant 1}$ of radius $1/k$ that cover S. If $\limsup_{M\to\infty}\inf_i \mathbb{P}_i(\bigcup_{m=1}^M A_{k,m}) < 1$ were true, then by assumption and by the Portmanteau Theorem we would have $\limsup_{M\to\infty}\mathbb{Q}(\bigcup_{m=1}^M A_{k,m}) < 1$ for some limiting probability measure $\mathbb{Q}$, which is contradictory.

   (b) Conclude that for any $\varepsilon>0$, $k\geqslant 1$ there are indices $M_{k,\varepsilon}\geqslant 1$ such that $\inf_i \mathbb{P}_i(K) > 1-\varepsilon$ holds with $K := \bigcap_{k\geqslant 1}\bigcup_{m=1}^{M_{k,\varepsilon}} A_{k,m}$. Moreover, K is relatively compact in S, which suffices.

Submit before the lecture on Tuesday, 26 January 2010

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Exercises: sheet 12

1. The *Brownian bridge* $(X_t, t \in [0,1])$ is a centered and continuous Gaussian process with $\mathrm{Cov}(X_s, X_t) = s(1-t)$ for $0 \leqslant s \leqslant t \leqslant 1$. Show that it has the same law on $C([0,1])$ as $(B_t - tB_1,\, t \in [0,1])$, B a Brownian motion.
   Optional: Simulate 100 trajectories of a Brownian bridge. Use conditional densities to show that X is the process obtained from $(B_t,\, t \in [0,1])$ conditioned on $\{B_1 = 0\}$.

2. Prove: If the random vector $X_n \in \mathbb{R}^{d_1}$ is independent of the random vector $Y_n \in \mathbb{R}^{d_2}$ for all $n \geqslant 1$ and $X_n \xrightarrow{d} N(\mu_1, \Sigma_1)$, $Y_n \xrightarrow{d} N(\mu_2, \Sigma_2)$ hold, then $(X_n, Y_n) \xrightarrow{d} N(\mu_1, \Sigma) \otimes N(\mu_2, \Sigma_2) = N((\mu_1, \mu_2), \mathrm{diag}(\Sigma_1, \Sigma_2))$ follows.
   Hint: Check that $(X_n, Y_n)_{n \geqslant 1}$ has tight laws and identify the limiting laws on cartesian products.
   Optional: Show a more general result for independent laws on Polish spaces.

3. Let $(S, \mathscr{S})$ be a measurable space, T an uncountable set.
   (a) Show that for each $B \in \mathscr{S}^{\otimes T}$ there is a countable set $I \subseteq T$ such that
   $$\forall x \in S^T,\, y \in B : \; (x(t) = y(t) \text{ for all } t \in I) \Rightarrow x \in B.$$
   Hint: Check first that sets B with this property form a σ-algebra.
   (b) Conclude for a metric space S with at least two elements that the set $C := \{f : [0,1] \to S \mid f \text{ continuous}\}$ does not belong to $\mathscr{S}^{\otimes [0,1]}$.

4. Consider the simple symmetric random walk $(S_n,\, n \geqslant 0)$ and the stopping time $\tau_a := \inf\{n \geqslant 0 \mid S_n = a\}$ for $a \in \mathbb{N}$.
   (a) Prove the *reflection principle* (sketch!): $\mathbb{P}(S_n > a) = \mathbb{P}(S_n > a,\, \tau_a < n) = \mathbb{P}(S_n < a,\, \tau_a < n)$.
   (b) Conclude for $M_n = \max\{S_0, \ldots, S_n\}$:
   $$\mathbb{P}(M_n \geqslant a) = \mathbb{P}(\tau_a \leqslant n) = \mathbb{P}(S_n = a) + 2\,\mathbb{P}(S_n > a).$$

Submit before the lecture on Tuesday, 2 February 2010