Ein Beispiel zur Äquivalenz von Singularitäten in der Charakteristik p

(Semiquasihomogeneous Singularities with Saito-invariant 1)

Ein (mittlerweile klassischer) Satz von Grauert über die Äquivalenz von Singularitäten ergibt u.a., daß für eine normale Flächensingularität mit einer glatten Kurve als exzeptionellem Ort der Isomorphietyp der Kurve zusammen mit dem Normalenbündel den analytischen Typ der Singularität bestimmt.

Neuere Untersuchungen von L. Badescu, bzw. F. Russo widmen sich Äquivalenzaussagen vom Grauert'schen Typ für den Fall eines Grundkörpers der Charakteristik $p > 0$.

Hier soll am Beispiel einfach elliptischer Singularitäten eine Grenze für die Gültigkeit solcher Vergleichssätze aufgezeigt werden; insbesondere wird für $p = 2$ eine nichttriviale Familie mit festem exzeptionellem Ort (in der Dimension 3) angegeben.

1. Saito's Invariante für sqh - Hyperflächensingularitäten
2. Normalformen für sqh-Singularitäten mit $s \leq 1$
3. Deformationsbeziehungen

The weights ADE

$$
\begin{aligned}
A_n &= \left(\frac{1}{n+1}, \frac{1}{2}, \dots, \frac{1}{2}\right), \quad n \geq 1 \\
D_n &= \left(\frac{1}{n-1}, \frac{n-2}{2(n-1)}, \frac{1}{2}, \dots, \frac{1}{2}\right), \quad n \geq 4 \\
E_6 &= \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \dots, \frac{1}{2}\right) \\
E_7 &= \left(\frac{1}{3}, \frac{2}{9}, \frac{1}{2}, \dots, \frac{1}{2}\right) \\
E_8 &= \left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, \dots, \frac{1}{2}\right).
\end{aligned}
$$

graph	type (condition)
$\circ \longrightarrow \circ \longrightarrow \dots \longrightarrow \circ \longrightarrow \circ$	$A_{2m-1},\ m \geq 1$ (S) $A_{2m},\ m \geq 1$ (NS) $E_6,\ m = 4$ (NI)
$\begin{array}{ccccccccc} &&&&&& \circ && \\ &&&&&& \uparrow && \\ \circ & \longrightarrow & \circ & \longrightarrow & \dots & \longrightarrow & \circ & \longrightarrow & \circ \\ &&&&&& \downarrow && \\ &&&&&& \circ && \end{array}$	$D_m,\ m \geq 4,\ m$ even
$\begin{array}{ccccccccc} \circ & \longrightarrow & \circ & \longrightarrow & \dots & \longrightarrow & \circ & \longrightarrow & \circ \\ &&&&&& \downarrow && \\ &&&&&& \circ && \end{array}$	$D_{m+1},\ m \geq 4,\ m$ even
$\begin{array}{ccccccc} &&&& \circ && \\ &&&& \uparrow && \\ \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ \\ && \downarrow && \downarrow && \\ && \circ && \circ && \end{array}$	$E_7,\ m = 7$
$\begin{array}{ccccccccc} &&&&&& \circ && \\ &&&&&& \uparrow && \\ \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ \\ &&&& \downarrow && \downarrow && \\ &&&& \circ && \circ && \end{array}$	$E_8,\ m = 8$

S: The exceptional locus of the last blowing up in the canonical resolution is smooth.

NS: The exceptional locus of the last blowing up in the canonical resolution is not smooth.

NI: For the quadratic suspension of dimension $d+2$, the exceptional locus of the first blowing up in the canonical resolution has nonisolated singularities.

Semiquasihomogeneous - singularities in dimension 2 having $s < 1$

I) $char k \neq 2$

$$
\begin{array}{rl}
A_n^o: & x_o^{n+1} - x_1x_2, \quad n \geq 1 \\
D_n^o: & x_o^{n-1} + x_ox_1^2 + x_2^2, \quad n \geq 4 \\
E_6^o: & x_o^3 + x_1^4 + x_2^2 \\
E_6^1: & x_o^3 + x_1^4 + x_2^2 + x_o^2x_1^2 \quad \text{(additionally for } p = 3) \\
E_7^o: & x_o^3 + x_ox_1^3 + x_2^2 \\
E_7^1: & x_o^3 + x_ox_1^3 + x_2^2 + x_o^2x_1^2 \quad \text{(additionally for } p = 3) \\
E_8^o: & x_o^3 + x_1^5 + x_2^2 \\
E_8^1: & x_o^3 + x_1^5 + x_2^2 + x_o^2x_1^3 \quad \text{(additionally for } p = 3) \\
E_8^2: & x_o^3 + x_1^5 + x_2^2 + x_o^2x_1^2 \quad \text{(additionally for } p = 3) \\
E_8^1: & x_o^3 + x_1^5 + x_2^2 + x_ox_1^4 \quad \text{(additionally for } p = 5)
\end{array}
$$

II) $char k = 2$

$$
\begin{array}{rl}
A_n^o: & x_o^{n+1} + x_1x_2 \\
D_{2n}^o: & x_o^nx_1 + x_ox_1^2 + x_2^2, \quad n \geq 2 \\
D_{2n}^r: & x_o^nx_1 + x_ox_1^2 + x_2^2 + x_o^{n-r}x_1x_2, \quad n \geq 2,\ 1 \leq r \leq n-1 \\
D_{2n+1}^o: & x_o^nx_2 + x_ox_1^2 + x_2^2, \quad n \geq 2 \\
D_{2n+1}^r: & x_o^nx_2 + x_ox_1^2 + x_2^2 + x_o^{n-r}x_1x_2, \quad n \geq 2, 1 \leq r \leq n-1 \\
E_6^o: & x_o^3 + x_1^2x_2 + x_2^2 \\
E_6^1: & x_o^3 + x_1^2x_2 + x_2^2 + x_ox_1x_2 \\
E_7^o: & x_o^3 + x_ox_1^3 + x_2^2 \\
E_7^1: & x_o^3 + x_ox_1^3 + x_2^2 + x_o^2x_1x_2 \\
E_7^2: & x_o^3 + x_ox_1^3 + x_2^2 + x_1^3x_2 \\
E_7^3: & x_o^3 + x_ox_1^3 + x_2^2 + x_ox_1x_2 \\
E_8^o: & x_o^3 + x_1^5 + x_2^2 \\
E_8^1: & x_o^3 + x_1^5 + x_2^2 + x_ox_1^3x_2 \\
E_8^2: & x_o^3 + x_1^5 + x_2^2 + x_ox_1^2x_2 \\
E_8^3: & x_o^3 + x_1^5 + x_2^2 + x_1^3x_2 \\
E_8^4: & x_o^3 + x_1^5 + x_2^2 + x_ox_1x_2
\end{array}
$$

Quasihomogeneous singularities having $s = 1$
in characteristic 2

Theorem: Let $f \in k[X]$ be a polynomial defining an isolated singularity such that f is quasihomogeneous for some weight w with $s = 1$.

Then w is (up to permutation) one of the weights

$$\tilde{E}_6 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \ldots, \frac{1}{2}),\ \tilde{E}_7 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \ldots, \frac{1}{2}),\ \tilde{E}_8 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, \ldots, \frac{1}{2})$$

and f is stable-equivalent with one of the following polynomials ($t \in k$ denotes a parameter):

Case A: $char(k) \neq 2$

$\tilde{E}_6$: $f = X_1(X_1 - X_0)(X_1 - tX_0) - X_0X_2^2, \quad t \neq 0, 1$

$\tilde{E}_7$: $f = X_0X_1(X_1 - X_0)(X_1 - tX_0), \quad t \neq 0, 1$

$\tilde{E}_8$: $f = X_0(X_0 - X_1^2)(X_0 - tX_1^2), \quad t \neq 0, 1$

Case B: $char(k) = 2$

1. n odd

$\tilde{E}_6(0)$: $X_0^3 + X_1^2X_2 + X_1X_2^2 + X_3^2$

$\tilde{E}_6(t)$: $X_0^3 + tX_2^3 + X_1^2X_2 + X_0X_1X_2 + X_3^2, \quad t \neq 0$

$\tilde{E}_7(t)$: $X_0X_1(X_1 + X_0)(X_1 + tX_0), \quad t \neq 0,\ 1$

$\tilde{E}_8(t)$: $X_0(X_0 + X_1^2)(X_0 + tX_1^2), \quad t \neq 0,\ 1$

2. n even

$\tilde{E}_6(0)$: $X_0^3 + X_1^2X_2 + X_1X_2^2$

$\tilde{E}_6(t)$: $X_0^3 + tX_2^3 + X_1^2X_2 + X_0X_1X_2, \quad t \neq 0$

$\tilde{E}_{7,1}(t)$: $X_0^2 + X_0X_1^2 + X_1X_2^2(tX_1 + X_2)$

$\tilde{E}_{7,2}(t)$: $X_0^2 + X_0X_1X_2 + X_1X_2(tX_1 + X_2)^2, \quad t \neq 0$

$\tilde{E}_8(t)$: $X_0^2 + X_0X_1X_2 + X_1(X_1 + X_2^2)(X_1 + tX_2^2), \quad t \neq 0$

Tjurina-algebra of $\tilde{E}_8$ in dimension 1

The equation is
$$X_0\left(-tX_0X_1^2+tX_1^4+X_0^2-X_0X_1^2\right)$$
characteristic is
$$2$$
Groebner base of the Tjurina - ideal is:
$$\left\{tX_1^4+X_0^2,X_1^6\right\};$$
there are
$$16$$
monomials not congruent 0, namely
$$\left\{X_0X_1^5,X_1^5,X_0X_1^4,X_0^3X_1,X_1^4,X_0X_1^3,X_1^3,X_0X_1^2,X_0^2X_1,X_0^3,X_0^2,X_1^2,X_0X_1,X_1,X_0,1\right\}.$$
having
$$12$$
relations, namely
$$\{tT_4+T_1,tT_9+T_2,tT_{10}+T_3,tT_{11}+T_5,T_6,T_7,T_8,T_{12},T_{13},T_{14},T_{15},T_{16}\}.$$
There are
$$2$$
superdiagonal monomials, namely
$$\left\{X_0^3X_1,X_0X_1^5\right\}$$
having the relation
$$\{tT_1+T_2\}$$
There are
$$2$$
diagonal monomials, namely
$$\left\{X_0^3,X_0X_1^4\right\}$$
having the relation
$$\{tT_1+T_2\}.$$

Tjurina-Algebra of $\tilde{E}_6(\frac{1}{j})$ in dimension 2

equation is

$$\left(jX_0^3 + jX_0X_1X_2 + jX_1^2X_2 + X_2^3\right)/j$$

Charakteristik is

$$2$$

Groebnerbase of the Tjurina ideals:

$$\left\{X_0^2 + X_1X_2, jX_0X_1 + jX_1^2 + X_2^2, X_0X_2, jX_1^3 + X_2^3, X_2\left(jX_1^2 + X_2^2\right), X_1X_2^2, X_2^4\right\}$$

There are

$$14$$

Monomials inkongruent 0:

$$\left\{X_2^3, X_1^2X_2, X_1^3, X_0X_1^2, X_0^2X_1, X_0^2, X_1^2, X_0X_1, X_2^2, X_1X_2, X_2, X_1, X_0, 1\right\}$$

having

$$8$$

relations:

$$\{jT_1 + T_2 + T_3 + T_4 + T_5, T_{10} + T_6, jT_9 + T_7, jT_9 + T_8, T_{11}, T_{12}, T_{13}, T_{14}\}$$

There are no superdiagonal monomials.
There are

$$5$$

diagonale monomials:

$$\left\{X_0^2X_1, X_0X_1^2, X_1^3, X_1^2X_2, X_2^3\right\}$$

with the relation

$$\{jT_5 + T_1 + T_2 + T_3 + T_4\}$$

Equations of the 1-semiquasihomogeneous series

semiquasihomogeneous singularities with $s = 1$ in characteristic 2			
type	Tjurina-number	maximal set of linearly independent superdiagonal monomials	total number
case 1: dimension $\equiv 1$ *mod* 2			
$\tilde{E}_6(0)$	16	$X_0X_1X_3,\ X_1X_2X_3,\ X_0X_2X_3,\ X_0X_1X_2X_3$	4
$\tilde{E}_6(t)$	16	$X_1^2X_3,\ X_2^2X_3,\ X_1X_2X_3,\ X_2^3X_3$	4
$\tilde{E}_7$	9	$\emptyset$	0
$\tilde{E}_8$	12	$X_0X_1^5$	1
case 2: dimension $\equiv 0$ *mod* 2			
$\tilde{E}_6(0)$	8	$\emptyset$	0
$\tilde{E}_6(t)$	8	$\emptyset$	0
$\tilde{E}_{7,1}(t)$	10	$\emptyset$	0
$\tilde{E}_{7,2}(t)$	10	$\emptyset$	0
$\tilde{E}_8(t)$	10	$\emptyset$	0

Thus e.g. for n odd, the 1-sqh singularities with first term $\tilde{E}_8$ (as in the theorem of section 1) are given by adding a constant multiple the monomial $X_0X_1^5$.
If the coefficient is not zero, an easy coordinate transformation leads to the only non quasihomogeneous 1-sqh singularity of that weight; it is given by the equation $X_0(X_0 + X_1^2)(X_0 + tX_1^2) + X_0X_1^5 = 0$ ($t \notin \{0, 1\}$) with Tjurina number 11.