Approximation of Nonsmooth Optimization Problems and Elliptic Variational Inequalities with Applications to Elasto-Plasticity

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to manifold applications.*

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Contents

Introduction

The subject of this thesis is the efficient solution of elliptic variational inequality problems and optimization problems with convex constraints defined over some (infinite-dimensional) real Banach space X. Given some operator $A\colon X \to X^*$, the abstract variational inequality problem is defined as follows:

$$\text{Find } u \in X: \quad \langle A(u), v-u\rangle + j(v) - j(u) \geq \langle l, v-u\rangle, \qquad \forall\, v \in X, \tag{0.0.1}$$

where $j\colon X \to \mathbb{R} \cup \{+\infty\}$ is a nonsmooth convex function and $l \in X^*$ denotes a bounded linear functional on X. A large class of variational inequalities is determined as the optimality condition to optimization problems of the type

$$\inf \quad F(v) - \langle l, v\rangle + j(v) \quad \text{over } v \in X, \tag{0.0.2}$$

where $F : X \to \mathbb{R}$ is a Gâteaux differentiable but not necessarily convex function with $F' = A$. In particular, the case of a convex constraint set $K \subset X$ is formally contained in the above formulation by setting $j(v) = +\infty$ for all $v \notin K$. The abstract problems (0.0.1) and (0.0.2) arise in a myriad of applications involving physics, engineering, finance, life sciences and many more. In this thesis, the focus is mainly on applications in quasi-static associative elasto-plasticity, where the variable v describes the material behavior subject to a given external loading l. Material properties and model parameters determine the operator A and the functional j.

In general, these problems cannot be solved directly apart from special cases. On one hand, the fact that the above problems are posed in some infinite-dimensional Banach space already underlines the necessity of an approximation in terms of a suitable finite-dimensional problem, for example, using finite element methods. On the other hand, it is often preferable to replace the original problem by some perturbed version even if a suitable discretization is established. This may be motivated by the fact that the discrete problem is not solvable without unreasonable effort or that the associated solver displays undesirable properties like mesh-dependence or lack of robustness. Thus, the analysis of (0.0.1) (or (0.0.2)) is intimately linked to the investigation of the behavior of the problem with respect to certain classes of perturbations including different discretization and regularization approaches. In this context, the main paradigm of this work is that the understanding of the underlying infinite-dimensional problem structure is crucial for both, the analysis of the consistency of perturbation methods, as well as the properties of corresponding solution algorithms. Here, we generally employ the term consistency whenever solutions of a perturbed version converge (in a sense to specify) to a solution of the original problem.

Following a preliminary discussion of basic tools from Functional Analysis and Optimization in Part I, Part II begins with a unified consistency analysis of standard perturbation methods comprising, among others, Galerkin approximation, Moreau-Yosida-regularization, singular perturbation or a combination of the latter. This will be achieved by introducing the general class of *quasi-monotone perturbations*, both, on the level of the variational inequality and the constrained optimization problem. Using the concepts of Γ-convergence [37] and Mosco-convergence [95] in this general framework, it turns out that certain density properties of intersections of the convex constraint set K with dense subspaces Y of the Banach space X naturally arise as a necessary and sufficient consistency criterion. The corresponding Γ- and Mosco limits, which can be understood as appropriate limit problems to a sequence of perturbations, are determined by the closure of these intersections.

In the literature on variational inequality problems, density properties classically appear as intermediate steps towards the convergence results of finite element methods under minimal regularity [89, 53, 61]. In the context of plasticity problems, density results represent an important ingredient for relaxation approaches [7, 38, 15]. However, the consideration of density properties in such a general framework appears to be novel.

In Chapter 5, we mainly focus on the case where $X = X(\Omega)$ is a usual (vector-valued) Lebesgue or Sobolev space over a bounded domain $\Omega \subset \mathbb{R}^N$, $Y = Y(\Omega)$ is the subspace of continuous or smooth vector fields and K prescribes a pointwise bound on the norm of the function value via an obstacle function α defined on Ω, i.e.,

$$K = K(X(\Omega)) = \{w \in X(\Omega) : |w(x)| \leq \alpha(x) \text{ a.e. in } \Omega\}. \tag{0.0.3}$$

We further introduce the problem formulation and give some new density results in Lebesgue and Sobolev spaces for continuous obstacles relying on the theory of mollification. These statements extend recent results from [69] which seems to be the only available reference which is specifically dedicated to abstract density properties of sets of the type (0.0.3). With the help of Mosco-convergence of convex sets, these results carry over to lower semicontinuous obstacles. A different approach is proposed for upper bounds α which are supersolutions of an elliptic partial differential equation (PDE). In the latter case, the smoothing is achieved by solving a sequence of elliptic PDEs. It is further shown that results of this type cannot be expected in general if the obstacle is just a Sobolev function, and for this purpose a concrete counterexample is given. In Chapter 6, we make use of the preceding density results for the continuous setting to prove the Mosco-convergence of various types of finite element-discretizations of K in the spirit of [53]. As a consequence, the discretized problems are consistent with the corresponding infinite-dimensional limit problem; see, for instance, [95, 53] or Theorem 3.1 and Theorem 6.3. In the last section, we return to the infinite-dimensional setting by studying density properties in the context of dualization in total variation based image restoration. The majority of the results in Part II have been obtained within a joint project with M. Hintermüller and C.N. Rautenberg, and as of May 2016 the results are not yet published.

In Part III, we consider an elasto-plastic contact problem under the small strain assumption. The problem is characterized by the presence of a rigid obstacle which restricts the deformation of the material under a loading process. In contrast to elasticity, plastic material behaviour is irreversible and the set of admissible stresses is constraint by definition. While the literature on discrete plasticity is extensive (see Chapter 7 for many references), infinite-dimensional algorithms are rather scarce and essentially only contain linearly convergent subspace correction methods [24]. In Chapter 9, an infinite-dimensional semismooth Newton method is proposed for the one-step time-discretized contact problem of quasi-static elasto-plasticity with combined kinematic-isotropic hardening. Neglecting friction, the combination of both effects, contact and plasticity, leads to the problem of a variational inequality of the mixed kind, i.e., the functional j in (0.0.1) is nonsmooth on its effective domain and the latter is a proper convex subset of X.

We note that the semismooth Newton method based on the notion of Newton- or slant differentiability [31, 75] has received considerable attention throughout the last decade as it has proved to be a remarkably efficient method, notably for the solution of various problems in PDE-constrained optimization [75, 66, 67] and of variational inequalities [41, 68, 87], to mention only a few. The fact that the original elasto-plastic contact problem does not allow for a Newton differentiable reformulation motivates the consideration of a special Fenchel dual problem (Chapter 8) of the so-called primal problem of quasi-static plasticity [61]. This dual problem turns out to be amenable to a regularization scheme where the associated solution path is induced by a coupled

Moreau-Yosida/Tikhonov regularization. The approach is presented in Chapter 9. Its consistency hinges on the density of the intersection of certain convex sets, and we rely on the results from Part II to prove that the sequence of solutions to the regularized problems converges strongly to the optimal displacement-stress-strain triple of the original elasto-plastic contact problem in the space-continuous setting. It is also argued that the mappings associated with the resulting optimality conditions are Newton- or slantly differentiable in the continuous setting. As a consequence, each regularized subsystem can be solved at a local superlinear rate of convergence and mesh-independently upon discretization. The latter property marks the crucial difference to the purely discrete approaches from the literature. For efficiency purposes, an inexact path-following approach is proposed and the discretized version is derived with the help of a conforming finite element approach. The chapter is closed with a numerical validation of the theoretical results. The main results of Part III can also be found in the author's joint publication with M. Hintermüller [72] as well as the short summary [71].

Part IV is devoted to the time-discretized problem of quasi-static evolution in perfect plasticity. In contrast to hardening plasticity, many difficulties arise from the fact that optimal displacements, which in general are not uniquely determined, have to be sought in the non-reflexive Banach space of functions with bounded deformation [82]. The time-dependent Prandtl-Reuss Model as well as Johnson's weak formulation [78] and the notion of quasi-static evolution in perfect plasticity [38] are introduced in Chapter 10. In Chapter 11, we first consider the corresponding time-incremental primal formulation from [38], which is a convex but nonsmooth constrained minimization problem. An equivalent inf-sup problem in a standard reflexive Lebesgue space is derived based on a reduced formulation of the primal problem. It is shown that the standard incremental stress problem from [78] can be determined as a Fenchel dual problem to this reduced formulation. In this way, the classical duality results from [7, 119] for *Hencky plasticity*, which is a simplified plasticity model of limited practical relevance, are extended to Prandtl-Reuss plasticity, and neccesary and sufficient optimality conditions can be derived. As an alternative to the approximation by plasticity problems with vanishing hardening [15], a primal-dual stabilization scheme based on a modified version of the visco-plastic regularization is proven to be consistent with the initial problem. As a consequence, not only stresses but also displacement and strains are shown to converge to a solution of the original problem in a suitable topology without imposing a higher regularity of the displacements or the plastic strains. The resulting scheme gives rise to a well-defined Fenchel dual problem, which is a modification of the usual stress problem in perfect plasticity. Moreover, the dual problem has a simple structure and turns out to be well-suited for numerical purposes. For the corresponding subproblems, we propose a path-following semismooth Newton method in infinite-dimensions, which is based on the modified stress problem. For details, see Chapter 12. Part IV is based on a preprint with M. Hintermüller which is not yet submitted.

Part I

Preliminaries: Functional Analysis, Optimization and Variational Inequalities

1 Functional Analysis

In this chapter, we briefly introduce the general notation as well as the basic function spaces for this text.

1.1 General Notation

All vector spaces in this thesis are defined over the field of real numbers $\mathbb{R}$. Let X be an arbitrary Banach space with norm $\| \,.\, \|_X$. The open ball around an element x with radius r in the norm topology is indicated by $B_r(x) := \{\tilde{x} \in X : \|x - \tilde{x}\|_X < r\}$. The topological dual space of X is denoted by X^*, and we write $\langle x^*, x\rangle_{(X^*,X)}$ for the duality pairing of an element $x^* \in X^*$ with $x \in X$. If X is a Hilbert space with inner product $(\,.\, , \,.\,)_X$, the Riesz Theorem states that for any element $x^* \in X^*$ there exists a unique element $x_0 \in X$ such that $\langle x^*, x\rangle_{(X^*,X)} = (x_0, x)_X$ for all $x \in X$, and we may identify $X \simeq X^*$ isometrically. For an arbitrary subset $A \subset X$ we denote by

$$\overline{A}^X, \ \partial_X A, \ \mathrm{int}_X\, A$$

the closure, the boundary and the interior of A in X with respect to the norm topology, respectively. The complement of A is denoted by A^c. Whenever the context leaves no ambiguity, we simply write $\| \,.\, \|, (\,.\, , \,.\,), \langle \,.\, , \,.\, \rangle$ and $\overline{A}, \partial A, \mathrm{int}\, A$ without indicating the specific space. The functions

$$i_A : X \to \mathbb{R} \cup \{+\infty\}, \quad i_A(x) = 0, x \in A, \ i_A(x) = +\infty, x \notin A,$$

and

$$\chi_A : X \to \mathbb{R} \cup \{+\infty\}, \quad \chi_A(x) = 1, x \in A, \ \chi_A(x) = 0, x \notin A,$$

are referred to as indicator and characteristic function of A with respect to X, respectively. We also need the space $\mathcal{L}(X, W)$ of bounded linear operators between two Banach spaces X and W. For $\Lambda \in \mathcal{L}(X, W)$, we designate by $\ker \Lambda := \{x \in X : \Lambda x = 0\}$ and $\mathrm{ran}\, \Lambda := \{\Lambda x : x \in X\}$ the kernel and the range of Λ, respectively, and we denote by $\Lambda^* \in \mathcal{L}(W^*, X^*)$ its adjoint operator,

$$\langle \Lambda^* w^*, x\rangle_{(X^*,X)} = \langle w^*, \Lambda x\rangle_{(W^*,W)}, \quad w^* \in W^*, x \in X.$$

We further write

$$W \hookrightarrow X$$

if the Banach space W embeds into the Banach space X, i.e., if there exists an injective mapping $\iota \in \mathcal{L}(W, X)$. For example, if $W \hookrightarrow X$ and the embedding ι has dense range, then the adjoint operator

$$\iota^* : X^* \to W^*, \quad \langle \iota^* x^*, w\rangle = \langle x^*, \iota w\rangle,$$

is also an embedding. In the special case where X is a Hilbert space with $W \subset X$, $\iota(x) = x, \ x \in W$, one obtains the Gelfand triple

$$W \overset{\iota}{\hookrightarrow} X \simeq X^* \overset{\iota^*}{\hookrightarrow} W^*,$$

and ι^* is simply the restriction operator,

$$\iota^* : X \ni x \mapsto (x, \,.\,)|_W \in W^*.$$

The identity mapping in X is written as id_X. The strong convergence of a sequence $(x_k) \subset X$ to $x \in X$ is denoted by $x_k \to x$. On many occasions, we also endow X with its weak topology, and we write $x_k \rightharpoonup x$ whenever the sequence (x_k) converges to x in the weak topology. If $X = W^*$ is the dual space of a normed space W, we may alternatively equip X with the weak*-topology and sequential convergence in this space is denoted by $x_k \overset{*}{\rightharpoonup} x$. By slight abuse of notation, we write $(x_k) \subset X$ to indicate that $\{x_k\} \subset X$. On some occasions we also use Urysohn's principle, which states that a sequence (x_k) converges to x if and only if each subsequence of (x_k) has a further subsequence converging to x. In order to avoid ambiguities, n-tuples $[x_1, \ldots, x_n]$ are denoted in brackets. For a function $F : X \to \mathbb{R} \cup \{+\infty\}$ on the Banach space X, we also introduce the lower semicontinuous (l.s.c.) envelopes in the norm and weak topology;

$$\operatorname{sc}^- F(x) = \sup\{G(x) : G \leq F, G \text{ is l.s.c.}\}, \tag{1.1.1}$$
$$\operatorname{sc}^-_{\mathrm{w}} F(x) = \sup\{G(x) : G \leq F, G \text{ is weakly l.s.c.}\}. \tag{1.1.2}$$

In this text, $|\,.\,|$ stands for an arbitrary norm on $\mathbb{R}^N$ whereas $|\,.\,|_p$ designates the standard p-norm on $\mathbb{R}^N$ for $1 \leq p \leq \infty$. The dual norm $|\,.\,|_*$ of $|\,.\,|$ is given by

$$|x^*|_* = \sup_{x \in \mathbb{R}^N, |x|=1} x^* \cdot x.$$

We further make use of the space $\mathbb{M}^{N\times N}$ which consists of all symmetric matrices of dimension $N \times N$, as well as its subspace

$$\mathbb{M}_0^{N\times N} := \{A \in \mathbb{M}^{N\times N} : \ \mathrm{tr}(A) := \sum_{i=1}^{N} A_{ii} = 0\}.$$

The orthogonal projection onto the orthogonal complement of $\mathbb{M}_0^{N\times N}$ in $\mathbb{M}^{N\times N}$ is referred to as the deviator of A, and it holds that

$$\operatorname{dev} A = A - \tfrac{\mathrm{tr}(A)}{N} I_N, \tag{1.1.3}$$

where I_N is the identity matrix of dimension $N \times N$. On $\mathbb{M}^{N\times N}$, we usually consider the Frobenius norm $|A|_F = \sqrt{\sum_{ij} A_{ij}^2}$.

1.2 Some Frequently Used Function Spaces

1.2.1 Spaces of smooth functions

We start by introducing several standard function spaces defined on a nonempty open subset $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$, which are frequently used in the present text. Let W be a Banach space. We denote by

$$C^k(\Omega; W) = \{f : \Omega \to W : \ f \text{ is } k\text{-times continously differentiable on } \Omega\}$$

the space of k-times continuously Fréchet differentiable functions on Ω with values in W, and we write $C(\Omega; W) = C^0(\Omega; W)$ for the space of continuous functions on Ω as well as

$$C^\infty(\Omega; W) = \bigcap_{k=0}^{\infty} C^k(\Omega; W).$$

For $\Omega \subset \mathbb{R}^N$ nonempty, open and bounded, the space of k-times continuously differentiable functions up to the boundary is given by

$$\begin{aligned} C^k(\overline{\Omega}; W) = \{ f : \Omega \to W : \ f \in C^k(\Omega; W), \\ \partial_s f \text{ has a continuous extension to } \overline{\Omega} \ \forall s \in \mathbb{N}, 0 \le |s|_1 \le k \}. \end{aligned}$$

Here, $\partial_s f$ designates the classical partial derivative of a function $f : \Omega \to W$ with respect to the multi-index $s \in \mathbb{N}_0^N$. Equipped with the norm

$$\|f\|_{C^k(\overline{\Omega};W)} = \sum_{|s|_1 \le k} \|\partial_s f\|_{C(\overline{\Omega};W)},$$

where $\|\partial_s f\|_{C(\overline{\Omega};W)} := \max_{x \in \overline{\Omega}} \|\partial_s f(x)\|_W$, the vector space $C^k(\overline{\Omega}; W)$ becomes a Banach space. A function $f : \Omega \to W$ is called κ-Hölder-continuous $(0 < \kappa < 1)$, if

$$\sup_{x,y \in \Omega} \tfrac{\|f(x) - f(y)\|_W}{|x-y|^\kappa} < +\infty, \tag{1.2.1}$$

and Lipschitz-continuous, if (1.2.1) holds for $\kappa = 1$. Furthermore, we designate by $C^{k,\kappa}(\overline{\Omega}; W)$ the space of functions in $C^k(\overline{\Omega}; W)$ such that the partial derivatives of order equal to k are κ-Hölder-continuous.

Moreover, the subspace of functions in $C^k(\Omega; W)$ with compact support in Ω is denoted by $C_c^k(\Omega; W)$ and the vector space $C_c^\infty(\Omega; W)$ is defined by intersection as above. The space of restrictions to Ω of smooth functions on $\mathbb{R}^N$ is designated by

$$\mathcal{D}(\overline{\Omega}; W) := \{ f|_\Omega : f \in C_c^\infty(\mathbb{R}^N; W) \}.$$

For an arbitrary nonempty subset $\Omega \subset \mathbb{R}^N$, one further defines the space $C_0(\Omega; W)$ of continuous functions vanishing at infinity, which means that for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that

$$\sup_{x \in \Omega \setminus K_\varepsilon} \|f(x)\|_W \le \varepsilon.$$

This vector space becomes a Banach space when equipped with the norm

$$\|f\|_{C_0(\Omega;W)} = \sup_{x \in \Omega} \|f(x)\|_W.$$

Note that $C_0(\Omega; W)$ is the completion of $C_c(\Omega; W)$ with respect to $\| \, . \, \|_{C_0(\Omega;W)}$. Finally, we define the space $AC(I; W)$ of W-valued absolutely continuous functions on the interval $I \subset \mathbb{R}^N$.

We further stipulate the notational convention that the indication of W in the definition of the above spaces is omitted whenever $W = \mathbb{R}$.

1.2.2 Smoothness of domains

A nonempty, open and connected subset of $\mathbb{R}^N$ is called domain. Many of the theoretical results associated with function spaces on domains demand a certain regularity of the boundary $\partial\Omega$ of Ω. The following definition gives rise to a categorization of domains in terms of the smoothness of their boundary.

Definition 1.1 ($C^{k,\kappa}$-domain)**.** We say that a bounded domain Ω has a $C^{k,\kappa}$-boundary, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, if there exists $J \in \mathbb{N}$ such that for each $j \in \{1,\dots J\}$ there exists an orthonormal basis $\{e_1^j,\dots,e_N^j\}$ of $\mathbb{R}^N$ together with a reference point $y^j \in \mathbb{R}^{N-1}$, constants $r^j > 0$ and $h^j > 0$ as well as a function $g^j \in C^{k,\kappa}(\overline{B_{r^j}(y^j)})$ such that the sets

$$\Omega_j := \{x = \sum_{i=1}^{N} x_i^j e_i^j \in \mathbb{R}^N : |\tilde{x}^j - y^j| < r^j, \quad |x_N^j - g^j(\tilde{x}^j)| < h^j \in \mathbb{R}^N\},$$

with $\tilde{x}^j = [x_1^j,\dots,x_{N-1}^j]$, fulfill the following set of conditions:

(i) $\partial\Omega \subset \bigcup_{j=1}^{J} \Omega_j$,

(ii) $\Omega_j \cap \partial\Omega = \{x \in \mathbb{R}^N : |\tilde{x}^j - y^j| < r^j, \quad x_N^j = g^j(\tilde{x}^j)\}$,

(iii) $\Omega_j \cap \Omega = \{x \in \mathbb{R}^N : |\tilde{x}^j - y^j| < r^j, \quad g^j(\tilde{x}^j) < x_N^j\}$,

(iv) $\Omega_j \cap \Omega^c = \{x \in \mathbb{R}^N : |\tilde{x}^j - y^j| < r^j, \quad g^j(\tilde{x}^j) > x_N^j\}$.

Conditions (i)-(iv) ensure, that $\partial\Omega$ is locally the graph of a $C^{k,\kappa}$-function and that the domain is locally situated on one side of its boundary. In particular, this excludes domains with slits. Domains with a $C^{0,1}$-smooth boundary are referred to as Lipschitz domains, and in this case the domain cannot have any cusp-like features.

1.2.3 Spaces derived from distribution theory

Lebesgue and Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be nonempty and open, and let W be a Banach space. Following [4, 127], we define $L^p(\Omega; W), 1 \leq p \leq \infty$, as the space (of equivalence classes) of Bochner-Lebesgue measurable W-valued functions for which the norm

$$\|u\|_{L^p(\Omega;W)} = \left(\int_\Omega \|u(x)\|_W^p \, dx\right)^{1/p}, \quad 1 \leq p < \infty,$$
$$\|u\|_{L^\infty(\Omega;W)} = \operatorname*{ess\,sup}_{x\in\Omega} \|u(x)\|_W,$$

is finite. Usually, we just need the definition for $W = \mathbb{R}$ or $W = \mathbb{R}^N$, which then reduces to the conventional meaning of the Lebesgue space and we write $L^p(\Omega) = L^p(\Omega;\mathbb{R})$. In some cases it is also necessary to consider the following more general function space: A function $u : \Omega \to W$ is called weakly measurable if the mappings

$$\Omega \ni \ x \mapsto \langle w^*, u(x)\rangle_{(W^*,W)} \ \in \mathbb{R},$$

are Lebesgue measurable in the usual sense for any $w^* \in W^*$. The space $L^p_w(\Omega; W)$, $1 \le p \le \infty$, is then defined as above, with the only exception that the functions in $L^p_w(\Omega; W)$ are only required to be weakly measurable.

We further denote by $W^{k,p}(\Omega)$ the usual Sobolev space of real-valued functions which possess distributional partial derivatives ∂_s in the Lebesgue space $L^p(\Omega)$ up to order k. The corresponding local space $W^{k,p}_{\text{loc}}(\Omega)$ is given by

$$W^{k,p}_{\text{loc}}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable} : u \in W^{k,p}(\Omega_0) \text{ for all } \Omega_0 \Subset \Omega \text{ open}\},$$

where $\Omega_0 \Subset \Omega$ means that Ω_0 is compactly contained in Ω, i.e., $\overline{\Omega}_0$ is compact and $\overline{\Omega}_0 \subset \Omega$. The subspace $W^{k,p}_0(\Omega)$ of $W^{k,p}(\Omega)$ is given by the closure of $C^\infty_c(\Omega)$ with respect to the norm in $W^{k,p}(\Omega)$, which is defined by

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|s|_1 \le k} \|\partial_s u\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

For $p = 2$, the Hilbert spaces $W^{k,2}(\Omega)$ and $W^{k,2}_0(\Omega)$ are denoted by $H^k(\Omega)$ and $H^k_0(\Omega)$, respectively. The vector-valued versions $W^{k,p}(\Omega; \mathbb{R}^d)$, $W^{k,p}_0(\Omega; \mathbb{R}^d)$ are defined componentwise. If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then there exists a trace mapping

$$\tau \in \mathcal{L}(W^{1,p}(\Omega), L^p(\partial\Omega))$$

such that $\tau(u) = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$. As is customary, in the definition of the space $L^p(\partial\Omega)$ it is tacitly assumed that $\partial\Omega$ is equipped with the $(N-1)$-dimensional Hausdorff measure denoted by $\mathcal{H}^{N-1}$.

Time-dependent Sobolev spaces on a given time-interval $(0, T)$, $T > 0$, with values in a Banach space X are defined as follows. Let $u \in L^1(0, T; X)$. A function $v \in L^1(0, T; X)$ is called weak derivative of u, if and only if,

$$\int_{(0,T)} u\varphi' \, dx = -\int_{(0,T)} v\varphi \, dx \quad \forall\, \varphi \in C^\infty_c(0, T);$$

in this case, we write $\dot{u} := v$. Note that the definition of weak derivative only requires local integrability of u and v. In accordance with the spatial case, we set

$$\begin{aligned} W^{1,p}(0, T; X) &:= \{u \in L^p(0, T; X) : u \text{ has a weak derivative } \dot{u} \in L^p(0, T; X)\}, \\ \|u\|_{W^{1,p}(0,T;X)} &:= (\|u\|^p_{L^p(0,T;X)} + \|\dot{u}\|^p_{L^p(0,T;X)})^{1/p}, \end{aligned}$$

and $H^1(0, T; X) = W^{1,2}(0, T; X)$. By virtue of the embedding

$$H^1(0, T; X) \hookrightarrow C([0, T]; X),$$

initial (and final) values of a function $u \in H^1(0, T; X)$ are well-defined. These properties and many further details on Sobolev spaces can be found, for instance, in [1, 48].

Spaces related to the divergence operator

Many variational problems in mechanics or fluid dynamics require spaces that are related to the regularity of the distributional divergence. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We define the spaces

$$\begin{aligned} H(\operatorname{div};\Omega) &:= \{v \in L^2(\Omega;\mathbb{R}^N) : \operatorname{div} v \in L^2(\Omega)\}, \\ Q &:= L^2(\Omega;\mathbb{M}^{N\times N}), \\ H(\operatorname{Div};\Omega) &:= \{\sigma \in Q : \operatorname{Div}\sigma \in L^2(\Omega;\mathbb{R}^N)\} = H(\operatorname{div};\Omega)^2 \cap Q, \end{aligned}$$

where we distinguish between the scalar-valued divergence operator div and its vector-valued counterpart Div;

$$\operatorname{div} v := \sum_{i=1}^{N} \partial_i v_i, \quad [\operatorname{Div}\sigma]_k := \sum_{i=1}^{N} \partial_i \sigma_{ik},\ k = 1,\dots,N.$$

These spaces become Hilbert spaces when endowed with the scalar products

$$\begin{aligned} (v,\tilde{v})_{H(\operatorname{div};\Omega)} &= (v,\tilde{v})_{L^2(\Omega)^N} + (\operatorname{div} v, \operatorname{div}\tilde{v})_{L^2(\Omega)}, \\ (\sigma,\tilde{\sigma})_{H(\operatorname{Div};\Omega)} &= (\sigma,\tilde{\sigma})_Q + (\operatorname{Div}\sigma, \operatorname{Div}\tilde{\sigma})_{L^2(\Omega)^N}. \end{aligned}$$

To establish boundary values and Green's formulae for these spaces, we rely on various trace spaces which are introduced in Section 1.2.4. If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, there exists a normal trace operator

$$\tau_\nu : H(\operatorname{div};\Omega) \to H^{-1/2}(\partial\Omega) := H^{1/2}(\partial\Omega)^*,$$

which extends $\tau_\nu|_{C^1(\overline{\Omega};\mathbb{R}^N)}$ by continuity such that the Green's formula

$$\langle \tau_\nu v, u\rangle_{(H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega))} = (\operatorname{div} v, u)_{L^2(\Omega)} + (v,\nabla u)_{L^2(\Omega)^N} \tag{1.2.2}$$

holds for any $v \in H(\operatorname{div};\Omega)$ and $u \in H^1(\Omega)$. In an analogous fashion, the normal trace operator on the space $H(\operatorname{Div};\Omega)$ is defined by extension of $\tau_\nu|_{C^1(\overline{\Omega};\mathbb{M}^{N\times N})}$. This yields an operator

$$\tau_\nu : H(\operatorname{Div};\Omega) \to H^{-1/2}(\partial\Omega)^N.$$

At this point, we remark that the notation τ_ν is used for normal trace operators on different spaces; the domain should always be clear from the context. The vectorial version of (1.2.2) is given by

$$\langle \tau_\nu \sigma, u\rangle_{(H^{-1/2}(\partial\Omega)^N,H^{1/2}(\partial\Omega)^N)} = (\operatorname{Div}\sigma, u)_{L^2(\Omega)^N} + (\sigma,\varepsilon(u))_Q \tag{1.2.3}$$

for any $\sigma \in H(\operatorname{Div};\Omega)$ and $u \in H^1(\Omega)^N$, where

$$\varepsilon(u) := \tfrac{1}{2}(\nabla u + \nabla u^\top)$$

denotes the symmetrized gradient of u.

Now assume that $\partial\Omega$ is decomposed according to

$$\partial\Omega = \Gamma_0 \cup \Sigma \cup I, \quad \Gamma_0 \cap \Sigma = \emptyset,$$

where Γ_0 and Σ are nonempty, relatively open and disjoint, and $I = \partial\Sigma = \partial\Gamma_0$ denotes a common Lipschitz interface in the sense of Definition 1.1. The restriction of τ_ν to Σ has to be understood in the sense of $H_{00}^{-1/2}(\Sigma) = H_{00}^{1/2}(\Sigma)^*$; for the definition of the space $H_{00}^{1/2}(\Sigma)$ we refer to (1.2.16). The trace operators

$$\tau_\nu^\Sigma : H(\mathrm{div};\Omega) \to H_{00}^{-1/2}(\Sigma)$$

and

$$\tau_\nu^\Sigma : H(\mathrm{Div};\Omega) \to H_{00}^{-1/2}(\Sigma)^N$$

are well-defined by

$$\langle \tau_\nu^\Sigma v, u\rangle_{(H_{00}^{-1/2}(\Sigma),H_{00}^{1/2}(\Sigma))} = (\mathrm{div}\, v, u)_{L^2(\Omega)} + (v, \nabla u)_{L^2(\Omega)^N}, \tag{1.2.4}$$

for any $v \in H(\mathrm{div};\Omega), u \in H^1_{0,\Gamma_0}(\Omega)$, and

$$\langle \tau_\nu^\Sigma \sigma, u\rangle_{(H_{00}^{-1/2}(\Sigma)^N,H_{00}^{1/2}(\Sigma)^N)} = (\mathrm{Div}\, \sigma, u)_{L^2(\Omega)^N} + (\sigma, \varepsilon(u))_Q, \tag{1.2.5}$$

for any $\sigma \in H(\mathrm{Div};\Omega), u \in H^1_{0,\Gamma_0}(\Omega)^N$, respectively. Here, the space of H^1-functions with partially vanishing trace is given by

$$H^1_{0,\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : \tau(u) = 0 \text{ a.e. on } \Gamma_0\}, \tag{1.2.6}$$

where 'a.e.' stands for almost everywhere. For the definition of various (trace) spaces for mixed boundary value problems, we again refer to Section 1.2.4.

Spaces related to Borel measures

For a Borel measurable subset $\Omega \subset \mathbb{R}^N$, $M(\Omega;\mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued Borel measures. Note that in the literature, the notion of Borel measure is not unified. Here, we refer to a Borel measure as an $\mathbb{R}^d$-valued measure on $\mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ is the Borel σ-Algebra of Ω. In other words, a mapping $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^d$ is a Borel measure, if and only if,

$$\mu(\emptyset) = 0, \quad \mu(\cup_{k\in\mathbb{N}} B_k) = \sum_{k\in\mathbb{N}} \mu(B_k),$$

for all pairwise disjoint sets $(B_k)_{k\in\mathbb{N}}$ with $B_k \in \mathcal{B}(\Omega)$.

Given a Borel measure $\mu \in M(\Omega;\mathbb{R}^d)$ and a p-norm $|\,.\,|_p$ on $\mathbb{R}^d$, we denote by $|\mu|_p$ the total variation, which is a (nonnegative) measure ($\mu \in M_+(\Omega)$) defined by

$$|\mu|_p(B) := \sup\{\sum_{k\in\mathbb{N}} |\mu(B_k)|_p : (B_k) \subset \mathcal{B}(\Omega) \text{ pairwise disjoint}, B = \cup_{k\in\mathbb{N}} B_k\} \tag{1.2.7}$$

for all $B \in \mathcal{B}(\Omega)$. Equipped with the norm $\|\mu\|_{M(\Omega;\mathbb{R}^d)} := |\mu|_p(\Omega)$, $M(\Omega;\mathbb{R}^d)$ becomes a Banach space. Moreover, the Riesz-Alexandrov Theorem allows to identify $M(\Omega;\mathbb{R}^d)$ and $[C_0(\Omega;\mathbb{R}^d)]^*$ via the isomorphism

$$M(\Omega;\mathbb{R}^d) \ni \mu \mapsto \left[\varphi \mapsto \int_\Omega \varphi \cdot d\mu\right] \in C_0(\Omega;\mathbb{R}^d)^*. \tag{1.2.8}$$

In fact, (1.2.8) defines an isometric isomorphism if $C_0(\Omega;\mathbb{R}^d)$ is endowed with the norm

$$\|\varphi\|_{C_0(\Omega;\mathbb{R}^d)} = \sup_{x\in\Omega} |\varphi(x)|_{p'}, \quad p' = \tfrac{p}{p-1};$$

see for instance [6, Prop. 1.47]. Moreover, we often consider $L^1(\Omega;\mathbb{R}^d)$ as a subspace of $M(\Omega;\mathbb{R}^d)$ by understanding any $v \in L^1(\Omega;R^d)$ as a density with respect to the Lebesgue measure λ on $\mathbb{R}^N$:

$$L^1(\Omega;\mathbb{R}^d) \ni v \mapsto \left[\varphi \mapsto \int_\Omega \varphi \cdot v \, dx\right] \in C_0(\Omega;\mathbb{R}^d)^*.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The space of functions with bounded variation on Ω is denoted by

$$BV(\Omega) = \{u \in L^1(\Omega) : \partial_i u \in M(\Omega) \;\; \forall\, i = 1, \ldots, N\},$$

and we set

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|_{M(\Omega;\mathbb{R}^N)}.$$

Based on the Riesz-Alexandrov isomorphism, we may consider Du as an $\mathbb{R}^N$-valued distribution which is also continuous on $C_c^\infty(\Omega;\mathbb{R}^N)$ equipped with the supremum norm. Likewise, the space of functions with bounded deformation is defined by

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^N) : \varepsilon(u) \in M(\Omega;\mathbb{M}^{N\times N})\},$$

where the standard norm on $BD(\Omega)$ is defined by

$$\|u\|_{BD(\Omega)} = \|u\|_{L^1(\Omega;\mathbb{R}^N)} + \|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})}.$$

Further observe that $BV(\Omega)^N \subset BD(\Omega)$, and we recall that

$$BD(\Omega) \hookrightarrow L^p(\Omega;\mathbb{R}^N), \quad p \leq \tfrac{N}{N-1}, \tag{1.2.9}$$

the embedding being compact for any $p < N/(N-1)$. If $\partial\Omega \in C^1$, functions in $BD(\Omega)$ admit an integrable trace on the boundary, i.e., $u \in L^1(\partial\Omega)$. In this case, the following Green's formula for functions $u \in BD(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$ is available:

$$\int_\Omega \varphi \, d\varepsilon_{ij}(u) = -\tfrac{1}{2}\int_\Omega u_i\partial_j\varphi + u_j\partial_i\varphi \, dx + \int_{\partial\Omega} [\gamma(u) \odot \nu]_{ij} \, \varphi \, d\mathcal{H}^{N-1}, \tag{1.2.10}$$

where $a \odot b = \frac{1}{2}(ab^\top + ba^\top)$ denotes the symmetrized outer product of two vectors a and b. Moreover, it can be shown that the functional

$$u \mapsto \|u\|_{L^1(\Gamma_0;\mathbb{R}^N)} + \|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})}$$

defines an equivalent norm on $BD(\Omega)$ whenever $\Gamma_0 \subset \partial\Omega$ is nonempty and open. The space $BD(\Omega)$ can be characterized as the dual space of a separable normed space giving rise to a weak*-topology. For a sequence $(u_n) \subset BD(\Omega)$, it is known that (u_n) converges weakly* to u in $BD(\Omega)$, if and only if,

$$u_n \to u \text{ in } L^1(\Omega), \quad \varepsilon(u_n) \overset{*}{\rightharpoonup} \varepsilon(u) \text{ in } M(\Omega;\mathbb{M}^{N\times N}). \tag{1.2.11}$$

For these results and further details on the space $BD(\Omega)$, we refer to [119, 116].

The space $BV(0,T;X)$ denotes the space of X-valued functions with bounded variation on $(0,T)$, i.e., $u \in BV(0,T;X)$ if and only if the total variation

$$\sup\{\sum_{j=1}^{\tilde{J}} \|u(t_j) - u(t_{j-1})\|_X : \tilde{J} \in \mathbb{N}, 0 = t_0 \leq t_1 \leq \ldots \leq t_{\tilde{J}} = t\}$$

is finite.

1.2.4 Sobolev spaces on manifolds

Trace Spaces

In the present work, we will often make use of the function spaces associated to boundary traces of Sobolev functions. In this section, we generally assume that Ω is a bounded Lipschitz domain. We start by defining the common Hilbert space of trace functions,

$$H^{1/2}(\partial\Omega) = \tau(H^1(\Omega)),$$

where $\tau : H^1(\Omega) \to L^2(\partial\Omega)$ is the usual trace operator for Sobolev functions. This space is endowed with the norm induced by the fractional 1/2-seminorm on $\partial\Omega$,

$$|g|_{1/2,\partial\Omega} = \left(\int_{\partial\Omega}\int_{\partial\Omega} \tfrac{|g(x)-g(y)|^2}{|x-y|^N}\,dx\,dy\right)^{1/2}, \tag{1.2.12}$$

i.e.,

$$\|g\|_{H^{1/2}(\partial\Omega)} = \left(\|g\|^2_{L^2(\partial\Omega)} + |g|^2_{1/2,\partial\Omega}\right)^{1/2}.$$

An equivalent norm on $H^{1/2}(\partial\Omega)$ can be obtained by using the quotient space norm of $H^1(\Omega)/H^1_0(\Omega)$, i.e.,

$$\|g\|_{H^{1/2}(\partial\Omega)} = \inf_{\substack{\tilde g\in H^1(\Omega)\\ \tilde g = g \text{ on } \partial\Omega}} \|\tilde g\|_{H^1(\Omega)},$$

because τ is also continuous as a mapping $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$.

For a relatively open subset $\Gamma \subset \partial\Omega$, the situation is more delicate. Following [57], one defines analogously to (1.2.12) the space

$$H^{1/2}(\Gamma) = \{g \in L^2(\Gamma) : |g|_{1/2,\Gamma} < +\infty\}$$

with

$$\|g\|_{H^{1/2}(\Gamma)} = \left(\|g\|^2_{L^2(\Gamma)} + |g|^2_{1/2,\Gamma}\right)^{1/2}, \quad |g|^2_{1/2,\Gamma} = \int_\Gamma\int_\Gamma \tfrac{|g(x)-g(y)|^2}{|x-y|^N}\,dx\,dy. \tag{1.2.13}$$

Alternatively, many authors, e.g. [33, 39, 93], define $H^{1/2}(\Gamma)$ to be the space of restrictions of functions in $H^{1/2}(\partial\Omega)$. In analogy to the case of an open subset of $\mathbb{R}^N$, the two definitions are equivalent if $\partial\Gamma$ is Lipschitz, cf. [99, Theorem 1.3.1], and in this case the quotient norm

$$\|g\|_{H^{1/2}(\Gamma)} = \inf_{\substack{\tilde g\in H^{1/2}(\partial\Omega)\\ \tilde g = g \text{ on } \Gamma}} \|\tilde g\|_{H^{1/2}(\partial\Omega)},$$

represents an equivalent norm on $H^{1/2}(\Gamma)$. The trace space $H^{1/2}(\Gamma)$ turns out to be well-behaved with regard to superposition and multiplication with Lipschitz functions.

Lemma 1.2 (superposition). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $\Gamma \subset \partial\Omega$ relatively open. Then the superposition operator*

$$\theta : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \quad g \mapsto \theta(g).$$

is well-defined and continuous for any Lipschitz function $\theta \in C^{0,1}(\mathbb{R})$.

Proof. The assertion follows from the definition of the 1/2-seminorm in (1.2.13). □

Lemma 1.3 (multiplication)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $\Gamma \subset \partial\Omega$ nonempty and open. Then the following assertions hold true.*

(i) It holds that $g \cdot \varphi \in H^{1/2}(\Gamma)$ for all $g \in H^{1/2}(\Gamma)$ and $\varphi \in C^{0,1}(\Gamma)$.

(ii) If $u \in H^1(\Omega)$ and $\varphi \in C^{0,1}(\Omega)$, then $\tau(u\varphi) = \tau(u)\tau(\varphi)$.

Proof. (i) The assertion follows from the equivalent definition of $H^{1/2}(\Gamma)$ via the pullback operation onto the local charts and basic results on multiplication with smooth functions in Sobolev spaces; see for example [57, 1].

(ii) Standard results on the multiplication of Sobolev functions prove that $u\varphi \in H^1(\Omega)$. Since the equation is satisfied for all $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, the assertion holds by a density argument. □

Corollary 1.4 (normal trace)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with a nonempty, relatively open $C^{1,1}$-boundary portion $\Gamma \subset \partial\Omega$. Then $u_\nu := u \cdot \nu \in H^{1/2}(\Gamma)$ for all $u \in H^1(\Omega)^N$.*

Proof. Since Γ is $C^{1,1}$-smooth, the unit outer normal ν to Ω is Lipschitz on Γ and the assertion follows from Lemma 1.3(i). □

Under the conditions of Corollary 1.4, the vector-valued space $H^{1/2}(\Gamma)^N$ can be decomposed into a tangential and a normal component,

$$H^{1/2}(\Gamma)^N = H^{1/2}_T(\Gamma)^N \oplus H^{1/2}_\nu(\Gamma)^N, \tag{1.2.14}$$

where

$$\begin{aligned} H^{1/2}_T(\Gamma)^N &:= \{g \in H^{1/2}(\Gamma)^N : g\nu = 0 \text{ on } \Gamma\}, \\ H^{1/2}_\nu(\Gamma)^N &:= \{g \in H^{1/2}(\Gamma)^N : \ \exists \hat{g} \in H^{1/2}(\Gamma) \text{ with } g = \hat{g}\nu \text{ on } \Gamma\}. \end{aligned}$$

These spaces need to be suitably adapted for mixed Dirichlet-Neumann problems. Suppose the boundary $\partial\Omega$ has the decomposition

$$\partial\Omega = \Gamma_0 \cup \Sigma \cup I, \quad \Gamma_0 \cap \Sigma = \emptyset, \tag{1.2.15}$$

with a nonempty relatively open Dirichlet boundary part Γ_0, a nonempty relatively open complementary boundary portion Σ and a common Lipschitz interface $I = \partial\Sigma = \partial\Gamma_0$, which has zero surface measure as a consequence. If the Dirichlet boundary condition posed on Γ_0 is homogeneous, one defines the space

$$H^1_{0,\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : \tau(u) = 0 \text{ on } \Gamma_0\},$$

equipped with the $H^1(\Omega)$-norm. The image of this space under the trace operator is a strict subspace of $H^{1/2}(\Sigma)$ denoted by

$$H^{1/2}_{00}(\Sigma) := \{g \in L^2(\Sigma) : \ \exists\, u \in H^1_{0,\Gamma_0}(\Omega) \text{ with } \tau(u) = g \text{ on } \Sigma\}. \tag{1.2.16}$$

Again, this space becomes a Banach space when endowed with the quotient space norm

$$\|g\|_{H^{1/2}_{00}(\Sigma)} = \inf_{\substack{\tilde{g} \in H^1_{0,\Gamma_0}(\Omega) \\ \tilde{g} = g \text{ on } \Sigma}} \|\tilde{g}\|_{H^1(\Omega)},$$

such that the associated trace operator restricted to the complement of the Dirichlet boundary

$$\tau^\Sigma : H^1_{0,\Gamma_0}(\Omega) \to H^{1/2}_{00}(\Sigma), \quad \tau^\Sigma(u) = \tau(u)|_\Sigma,$$

is surjective and continuous by definition. This implies that $H^{1/2}_{00}(\Sigma)$ corresponds to the space of the same name from [90] which is defined for C^∞-submanifolds Σ with C^∞-boundary $\partial\Sigma$ independently of an associated domain Ω. In case these smoothness properties are not given, the definition of this space and its relation to the space from (1.2.16) remains rather obscure. For related issues, we refer to [62] and [98, Chap. 5].

In case the trace operator is further restricted to a nonempty relatively open set Γ with $\overline{\Gamma} \subset \Sigma$ and Lipschitz boundary $\partial\Gamma$, a standard cut-off function argument shows that any function $g \in H^{1/2}(\Gamma)$ can be extended to a function $g \in H^{1/2}_{00}(\Sigma)$ such that the trace mapping restricted to Γ,

$$\tau^\Gamma : H^1_{0,\Gamma_0}(\Omega) \to H^{1/2}(\Gamma), \quad \tau^\Gamma(u) = \tau(u)|_\Gamma, \tag{1.2.17}$$

is surjective. With the help of the decomposition of $H^{1/2}(\Gamma)^N$ into normal and tangential components, we can derive the following statement.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with a boundary decomposition according to (1.2.15) and a nonempty open $C^{1,1}$-boundary portion Γ with $\overline{\Gamma} \subset \Sigma$ and Lipschitz boundary $\partial\Gamma$. Then the normal trace mapping restricted to Γ,*

$$\tau^\Gamma_\nu : H^1_{0,\Gamma_0}(\Omega)^N \to H^{1/2}(\Gamma), \quad \tau^\Gamma_\nu(u) = (\tau(u) \cdot \nu)|_\Gamma,$$

where τ^Γ is applied componentwise to u, is surjective.

Proof. Note that the decomposition (1.2.14) defines an isomorphism between $H^{1/2}(\Gamma)^N$ and $H^{1/2}_T(\Gamma)^N \times H^{1/2}(\Gamma)$. Applied componentwise, the surjectivity of τ^Γ (1.2.17) yields the assertion. □

In contact problems, a constraint of the type $u \cdot \nu \leq \psi$ is imposed on a given nonempty subset Γ_c of Σ. If Γ_c does not have a positive surface measure, the inequality understood in the pointwise almost everywhere sense represents a void condition. As a remedy, one considers the following more general ordering in the space $H^{1/2}_{00}(\Sigma)$, which is similar to the well known ordering in $H^1(\Omega)$ appearing in the theory of capacity, cf. [80, 103] for an introduction. In the following, we simply write $u_\nu = u \cdot \nu$ for the normal trace on a subset of the boundary.

Definition 1.6 (Cone of nonnegative functions in $H^{1/2}_{00}$)**.** We say that a function $g \in H^{1/2}_{00}(\Sigma)$ is nonnegative ($g \geq 0$) on Γ_c in the sense of $H^{1/2}_{00}(\Sigma)$, if there exists a sequence (φ_n) of functions $\varphi_n \in C^{0,1}(\Sigma) \cap H^{1/2}_{00}(\Sigma), n \in \mathbb{N}$, with

$$\varphi_n \to g \text{ in } H^{1/2}_{00}(\Sigma), \quad \varphi_n(x) \geq 0 \ \ \forall\, x \in \Gamma_c.$$

For a given upper bound $\psi \in H^{1/2}_{00}(\Sigma)$ and under the assumption that

$$\Sigma \text{ is } C^{1,1}\text{-smooth}, \ \Gamma_c \subset \Sigma \text{ arbitrary}, \tag{1.2.18}$$

the definition of the cone is then used to define the constraint set

$$K_1 = \{u \in H^1_{0,\Gamma_0}(\Omega)^N : u_\nu \leq \psi \ \text{ on } \Gamma_c \text{ in } H^{1/2}_{00}(\Sigma)\}, \tag{1.2.19}$$

where the inequality is defined in the obvious way using Definition 1.6. To justify this definition, we state the following corollary of Lemma 1.3(ii).

Corollary 1.7. *Let Ω be a bounded Lipschitz domain with the boundary decomposition* (1.2.15) *with $C^{1,1}$-smooth boundary portion Σ. Then the normal trace mapping restricted to Σ,*

$$\tau_\nu^\Sigma : H^1_{0,\Gamma_0}(\Omega)^N \to H^{1/2}_{00}(\Sigma), \quad \tau_\nu^\Sigma(u) = (\tau(u)\cdot\nu)|_\Sigma,$$

is well-defined and surjective.

Proof. Let $u \in H^1_{0,\Gamma_0}(\Omega)^N$. By Kirszbraun's Theorem [81] we may extend the field ν of unit outer normals to a Lipschitz function $\tilde{\nu} \in C^{0,1}(\overline{\Omega})$. Then it holds that $u \cdot \tilde{\nu} \in H^1(\Omega)$ and Lemma 1.3(ii) implies that $\tau(u\tilde{\nu})$ vanishes on Γ_0 and equals u_ν on Σ, such that $u_\nu \in H^{1/2}_{00}(\Sigma)$, which proves that τ_ν^Σ is well-defined. The surjectivity can be argued analogously to the proof of Corollary 1.5. □

Alternatively, if

$$\Gamma_c \subset \partial\Omega \text{ is open }, \ \partial\Gamma_c \text{ Lipschitz}, \ \overline{\Gamma}_c \subset \Sigma, \tag{1.2.20}$$

it is customary to impose the constraint in the usual pointwise almost everywhere sense,

$$K_1 = \{u \in H^1_{0,\Gamma_0}(\Omega)^N : u_\nu \le \psi \ \text{ a.e. on } \Gamma_c\}. \tag{1.2.21}$$

In this case, only Γ_c is required to be $C^{1,1}$-smooth. We refer to the classical monographs on elastic contact problems [98, Chap. 5-6] and [76] for further details.

Sobolev spaces on smooth manifolds

The trace spaces from the preceding section may be considered in the much more general context of Sobolev spaces on manifolds. Let $\Omega \subset \mathbb{R}^N$ be a $C^{k,\kappa}$-domain. Using the local parametrizations given by Definition 1.1, the boundary $\partial\Omega$ defines an $(N-1)$-dimensional $C^{k,\kappa}$-submanifold of $\mathbb{R}^N$ in a canonical way, see [126, Theorem 2.15]. When defining a Sobolev space on an open subset $\Gamma \subset \partial\Omega$, three major complications arise which do not occur in the Euclidean case:

(i) The smoothness of the manifold limits the order of the distribution.

(ii) The definition of the space has to be independent of the atlas (i.e., the local parametrizations).

(iii) The definition of the (distributional) gradient has to be independent of the atlas.

Based on the smoothness of the localized mapping, the space $C^l(\Gamma)$ is only well-defined (i.e., independent of the local chart) for $l \le k$, as, by definition, coordinate changes are only in $C^{k,\kappa}$. Accordingly, the definition of the Sobolev space $W^{l,q}(\Gamma)$ on Γ has to be based on lower order distributions. Alternatively, some authors define the spaces $W^{l,q}(\Gamma)$ via the corresponding regularity of the localized mapping using the local chart. Again, the regularity of the Sobolev space is limited by the smoothness of Γ, cf. [57]. However, the literature on the calculus in these spaces is rather fragmentary, see [14, p.353 ff.], [57, p.20]. In particular, the definition of the (distributional) gradient, Poincaré type inequalities and embedding properties are not available.

By contrast, Sobolev spaces on smooth manifolds are easier to handle. Therefore assume that Γ is an $(N-1)$-dimensional C^∞-submanifold of $\mathbb{R}^N$. More precisely, since Ω is assumed to be some $C^{k,\kappa}$-domain, $\partial\Omega$ (possibly after an appropriate orthogonal coordinate transformation) is given locally by the graph of functions $g^j \in C^{k,\kappa}, i = 1,\dots,m$, on a bounded open subset of $\mathbb{R}^{N-1}$ (Definition 1.1); we assume that those g^j whose graph have nonempty intersection with Γ are not only in $C^{k,\kappa}$ but in C^∞. Consequently, for each $x \in \Gamma$, the tangent space $T_x\Gamma$ at x then represents an $(N-1)$-dimensional subspace of $\mathbb{R}^N$ and we may define the *Riemannian metric*

$$\mathfrak{g}_x : T_x\Gamma \times T_x\Gamma \to \mathbb{R}, \quad \mathfrak{g}_x(v,w) := (v,w)_{\mathbb{R}^N},$$

using the Euclidean inner product. The resulting construction $(\Gamma, \mathfrak{g})$ is the standard example for a *Riemannian manifold*. In connection with the associated *Riemannian measure* (which corresponds to the surface measure in this case), this approach leads to the definition of Sobolev spaces on Riemannian manifolds to which the distribution theory from the Euclidean case carries over, cf. [56]. For the alternative approach via the completion of smooth functions with respect to the $W^{k,p}$-norm see [63]. Following the former reference, we define the space

$$\overset{\rightarrow}{L^p}(\Gamma), \quad 1 \leq p < +\infty,$$

as the set of equivalence classes of measurable vector fields

$$u : \Gamma \to T\Gamma, \quad u(x) \in T_x\Gamma \ \ \forall\, x \in \Gamma,$$

for which the norm

$$\|u\|_{\overset{\rightarrow}{L^p}(\Gamma)} := \left(\int_\Gamma |u|^p \, d\mathcal{H}^{n-1} \right)^{1/p} \tag{1.2.22}$$

is finite. Analogously to the Euclidean case, the space $W^{1,p}(\Gamma)$ is then defined as the subspace of $L^p(\Gamma)$ which consists of all elements with p-integrable distributational gradient:

$$W^{1,p}(\Gamma) := \{u \in L^p(\Gamma) : \nabla u \in \overset{\rightarrow}{L^p}(\Gamma)\}.$$

2 Optimization in Banach Spaces

Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Then

$$\operatorname{dom} F := \{x \in X : F(x) < +\infty\}$$

stands for the effective domain of F, and F is called proper if $\operatorname{dom} F \neq \emptyset$.

2.1 Abstract Existence Results

In this section, we briefly introduce the standard paradigm of the existence theory for optimization problems in Banach spaces which is known as the *direct method in the calculus of variations*. To begin with, let X be an arbitrary nonempty set and, in order to include constrained problems, the functional $F : X \to \mathbb{R} \cup \{+\infty\}$ is assumed to be an extended real-valued function. Consider the general optimization problem (without topology)

$$\inf \quad F(x) \quad \text{ over } x \in X. \tag{2.1.1}$$

A nonstandard way to formulate the existence result is the following:

Theorem 2.1. *If there exists a topology $\mathfrak{t}$ on X such that*

(i) there exists $c > \inf_{x\in X} F(x)$ such that the lower level set

$$\{x \in X : F(x) \leq c\}$$

is sequentially compact and

(ii) F is sequentially l.s.c.,

both with respect to $\mathfrak{t}$, then (2.1.1) has a solution.

In this way, the richness of the topology $\mathfrak{t}$ is supposed to represent an appropriate trade-off between the two competing goals (i) and (ii).

Proof. Let (x_n) be an infimizing sequence, i.e., $F(x_k) \to \inf_{x\in X} F(x)$. Hence, it holds that $F(x_k) \leq c$ for almost every $k \in \mathbb{N}$. Using the sequential compactness of the lower level set, we find a converging subsequence (x_{k_l}) such that $x_{k_l} \xrightarrow{\mathfrak{t}} \bar{x}$, i.e., (x_{k_l}) converges to $\bar{x}$ with respect to $\mathfrak{t}$. Using the sequential lower semicontinuity of F, one obtains that

$$\inf_{x\in X} F(x) = \liminf_{l\to\infty} F(x_{k_l}) \geq F(\bar{x}) > -\infty,$$

which implies that $\bar{x}$ is indeed a minimizer of F on X. $\square$

Remark 2.2. For the existence of a minimizer to (2.1.1), it is sufficient that the objective function is l.s.c. along *one* infimizing sequence.

In this work, we mainly focus on optimization problems in Banach spaces $(X, \|\,.\,\|_X)$ where the objective function F is coercive, i.e.,

$$F(x) \to +\infty \text{ if } \|x\|_X \to \infty.$$

In this case, the lower level sets are necessarily bounded, and we may evoke classical results on the sequential compactness of bounded sets in the weak topology (Eberlein-Šmulian Theorem) or in the weak*-topology (Banach-Alaoglu-Bourbaki-Theorem); see for example [127].

Corollary 2.3. *Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be coercive. If one of the two following conditions is fulfilled, then* (2.1.1) *has a solution.*

(i) $(X, \|\,.\,\|_X)$ *is reflexive and F is sequentially weakly l.s.c..*

(ii) $(X, \|\,.\,\|_X)$ *is the topological dual space of a separable normed space and F is sequentially weakly* l.s.c..*

In these cases, the existence result simply follows from Theorem 2.1 by choosing $\mathfrak{t}$ to be the weak or the weak* topology, respectively.

2.2 Convex Optimization and Fenchel Duality

In this section, we collect some standard results from convex optimization. Let X be a Banach space and suppose that $F : X \to \mathbb{R} \cup \{+\infty\}$ is proper, l.s.c. and convex. The usual convex subdifferential $\partial F(x)$ at $x \in X$ is defined by

$$\partial F(x) := \{x^* \in X^* : F(x) + \langle x^*, \tilde{x} - x\rangle \leq F(\tilde{x}) \ \ \forall \tilde{x} \in X\},$$

such that any solution $\bar{x}$ to the optimization problem (2.1.1) is characterized by

$$0 \in \partial F(\bar{x}).$$

Under these assumptions, the (sequential) weak lower semicontinuity of F is given by Mazur's Theorem such that the existence of $\bar{x}$ is guaranteed by Corollary 2.3(i) provided, in addition, that F is coercive and X is reflexive. In this text, we make consistent use of the following well-known property [46]: It holds that

$$x^* \in \partial F(x) \iff x \in \partial F^*(x^*) \iff \langle x^*, x\rangle = F^*(x^*) + F(x),$$

where $F^* : X^* \to \mathbb{R} \cup \{+\infty\}$ denotes the *Fenchel conjugate function* to F;

$$F^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - F(x)\}.$$

As an important example for the computation of conjugate functions, we mention the following result.

Lemma 2.4. *Let $1 \leq p \leq \infty$ and $d \in \mathbb{N}$. Let $\beta : \Omega \to \mathbb{R}$ be measurable with $\beta(x) \geq 0$ a.e. and $|\,.\,| : \mathbb{R}^d \to \mathbb{R}$ an arbitrary norm on $\mathbb{R}^d$ with dual norm $|\,.\,|_*$. Let $j : L^p(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$j(u) := \int_\Omega \beta(x)|u(x)|\, dx.$$

Then the convex conjugate of j is given by $j^* : L^{p'}(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ with

$$j^*(u^*) = i_K(u^*) \text{ with } K = \{u^* \in L^{p'}(\Omega)^d : |u^*|_* \leq \beta \text{ a.e. in } \Omega\}.$$

Proof. Since the function $\hat{j}(x,u) := \beta(x)\,|u|$ is a Carathéodory integrand, we obtain by [46, IX, Prop. 2.1],

$$j^*(u^*) = \int_\Omega \hat{j}^*(x, u^*(x))\,dx,$$

where $\hat{j}^*$ denotes the Fenchel conjugate of $\hat{j}$ with respect to u. It is further easy to show that $\hat{j}^*(x,u^*) = i_{\hat{K}_x}(u^*)$ with $\hat{K}_x = \{v^* \in \mathbb{R}^d : |v^*|_* \leq \beta(x)\}$. □

Let W be another Banach space and $\Lambda \in \mathcal{L}(X,W)$. Let $G : W \to \mathbb{R} \cup \{+\infty\}$ be proper, l.s.c. and convex. Consider the abstract optimization problem

$$\begin{cases} \inf & F(x) + G(\Lambda x) \\ \text{over} & x \in X. \end{cases} \tag{2.2.1}$$

In order to derive optimality conditions and duality results for (2.2.1), the following constraint qualification is required:

$$0 \in \operatorname{int}(\operatorname{dom} G - \Lambda \operatorname{dom} F). \tag{2.2.2}$$

In the literature, one often encounters the following constraint qualification as a particular case.

$$\exists\, x_0 \in X : \quad F(x_0) < +\infty,\ G(\Lambda x_0) < +\infty,\ G \text{ is continuous at } \Lambda x_0, \tag{2.2.3}$$

cf. [46]. In fact, (2.2.3) is a stronger condition then the constraint qualification (2.2.2). Provided (2.2.2) is satisfied, the following chain rule for the subdifferential of the objective in (2.2.1) is valid:

$$\partial(F + G \circ \Lambda)(x) = \partial F(x) + \Lambda^* \partial G(\Lambda x);$$

see [11, IV, Theorem 5]. In this case, the necessary and sufficient optimality condition for $\bar{x}$ to be a solution to (2.2.1) is characterized by the existence of an element

$$\bar{w}^* \in \partial G(\Lambda \bar{x})$$

such that

$$0 \in \partial F(\bar{x}) + \Lambda^* \bar{w}^*.$$

In some situations, it is further favorable to consider the *Fenchel dual* problem to (2.2.1). This problem is defined by

$$\begin{cases} \inf & F^*(-\Lambda^* w^*) + G^*(w^*) \\ \text{over} & w^* \in W^*. \end{cases} \tag{2.2.4}$$

The analogous constraint qualification for the dual problem reads

$$0 \in \operatorname{int}(\Lambda^* \operatorname{dom} G^* + \operatorname{dom} F^*), \tag{2.2.5}$$

and the following result relates the two problems, see [11, p. 221].

Theorem 2.5. *Under the above hypotheses on the spaces X, W as well as F and G, the following assertions hold true.*

(i) Assume that (2.2.2) is satisfied. Then there exists a solution $\bar{w}^*$ to the dual problem (2.2.4) and there is no duality gap, i.e., it holds that

$$\inf (2.2.1) = -\inf (2.2.4), \quad \textit{the infima being finite.} \tag{2.2.6}$$

(ii) Let X be reflexive and assume that (2.2.5) is satisfied. Then there exists a solution $\bar{x}$ to the primal problem (2.2.1) and (2.2.6) is fulfilled.

In case one can show that there exists no duality gap, the primal-dual solution pairs are characterized by the following statement.

Lemma 2.6. *Under the standing assumptions on F, G and X, W, the following assertions are equivalent.*

(i) $\bar{x}$ is a solution of (2.2.1) and $\bar{w}^*$ is a solution of (2.2.4) and there is no duality gap, i.e., (2.2.6) holds true.

(ii) $[\bar{x}, \bar{w}^*]$ is a solution to the following system of inclusions:

$$0 \in \partial F(\bar{x}) + \Lambda^* \bar{w}^*, \quad \Lambda \bar{x} \in \partial G^*(\bar{w}^*). \tag{2.2.7}$$

Proof. Suppose (i) is given. By (2.2.6), one obtains

$$\begin{aligned} 0 &= F(\bar{x}) + F^*(-\Lambda^* \bar{w}^*) + G(\Lambda \bar{x}) + G^*(\bar{w}^*) \\ &= (F(\bar{x}) + F^*(-\Lambda^* \bar{w}^*) - \langle -\Lambda^* \bar{w}^*, \bar{x} \rangle) + (-\langle \bar{w}^*, \Lambda \bar{x} \rangle + G(\Lambda \bar{x}) + G^*(\bar{w}^*)). \end{aligned} \tag{2.2.8}$$

By definition of the Fenchel conjugate function, each expression in parentheses is nonnegative, which implies (ii).

The other implication can be shown as follows: First note that the biconjugate of the convex marginal function

$$h : W \to \mathbb{R} \cup \{+\infty\}, \quad h(w) := \inf_{x \in X} (F(x) + G(\Lambda x + w))$$

is given by

$$h^{**}(w) = -\inf_{w^* \in W^*} F^*(-\Lambda^* w^*) + G^*(w^*) - \langle w^*, w \rangle,$$

cf. [11, Chap. IV], such that the extremal values of the primal and the negative dual problem are given by the value of h and its biconjugate h^{**} at 0. From the inequality $h^{**}(0) \leq h(0)$, one obtains

$$-(F^*(-\Lambda^* w^*) + G^*(w^*)) \leq F(x) + G(\Lambda x), \quad \forall\, x \in X, \; \forall\, w^* \in W^*. \tag{2.2.9}$$

Suppose $[\bar{x}, \bar{w}^*]$ is a solution to the system of inclusions (2.2.7). Then it follows from (2.2.8) that

$$-(F^*(-\Lambda^* \bar{w}^*) + G^*(\bar{w}^*)) = F(\bar{x}) + G(\Lambda \bar{x}),$$

and together with (2.2.9) one obtains statement (i). □

2.3 The Semismooth Newton Method

2.3.1 Basic properties and calculus

The notion of Newton differentiability that is of interest in this work can be found in [75] and reads as follows.

Definition 2.7 (Newton differentiability)**.** Let X, W be Banach spaces and $U \subset X$ an open set. A mapping $\Psi : U \to W$ is called Newton differentiable in U if there exists a family of mappings $G_\Psi : U \to \mathcal{L}(X, W)$ which satisfy

$$\lim_{h\to 0} \frac{\|\Psi(x+h) - \Psi(x) - G_\Psi(x+h)h\|_W}{\|h\|_X} = 0,$$

for all $x \in U$.

Note that the Newton derivative G_Ψ is not necessarily uniquely determined, e.g., the mapping

$$\Psi : \mathbb{R} \to \mathbb{R}, \quad \Psi(x) := |x| - b,\ b \in \mathbb{R},$$

has the Newton derivative

$$G_\Psi(x) = \begin{cases} -1, & x < 0, \\ \delta, & x = 0, \\ 1, & x > 0, \end{cases}$$

where $\delta \in \mathbb{R}$ is arbitrary.

Now suppose that we want to solve the equation

$$\Psi(x) = 0 \tag{2.3.1}$$

for some (nonlinear) operator $\Psi : X \to W$. For instance, the operator Ψ may represent the stationarity condition of the optimization problem (2.1.1). Assuming that the nonlinear equation (2.3.1) admits a solution $\bar{x} \in X$ and that Ψ is Newton differentiable in an open neighborhood $U(\bar{x})$ of $\bar{x}$ with nonsingular Newton derivative $G_\Psi(x)$ for all $x \in U$, we consider the generalized Newton iteration

$$x^{(k+1)} = x^{(k)} - G_\Psi(x^{(k)})^{-1}\Psi(x^{(k)}), \quad k \in \mathbb{N}, \tag{2.3.2}$$

for some starting point $x^{(0)} \in U(\bar{x})$. The following local convergence result is well known, cf. [31, 75]:

Theorem 2.8. *If $\{\|G_\Psi(x)^{-1}\|, x \in U(\bar{x})\}$ is bounded, then there exists a radius $r > 0$ such that the sequence $(x^{(k)})$ generated by the generalized Newton iteration* (2.3.2) *is well-defined and converges superlinearly to $\bar{x}$ provided $x^{(0)} \in B_r(\bar{x}) := \{\tilde{x} \in X : \|\tilde{x} - \bar{x}\|_X < r\}$.*

Various applications of the semismooth Newton method in infinite dimensions, notably for the solution of PDE-constrained optimization [75, 66, 67] and variational inequality problems [41, 68, 87], have been investigated in the recent past.

We further recall two important calculus rules related to the Newton differentiability of several nonsmooth functions. Let $\Omega \subset \mathbb{R}^N$ be a given domain. For $1 \le q \le p \le \infty$, consider the Nemytskii operator $[\,.\,]^+$ defined by

$$\begin{aligned} [\,.\,]^+ : L^p(\Omega) &\to L^q(\Omega), \\ v &\mapsto (x \mapsto \max(0, v(x))). \end{aligned}$$

The following remarkable result, which can be found in [75], represents a striking contrast to the semismoothness of its finite-dimensional analogue in that the operator $[\,.\,]^+$ is only Newton differentiable for special combinations of domain and image space.

Lemma 2.9 (Newton differentiability of the pointwise maximum)**.** *The pointwise maximum function* $\Psi(\,.\,) := [\,.\,]^+$,

$$\Psi : L^p(\Omega) \to L^q(\Omega),$$

is Newton differentiable for $1 \le q < p \le +\infty$. *A corresponding Newton derivative is given by*

$$G_\Psi(u)h := \begin{cases} 0, & \text{on } \mathcal{I}(u), \\ h, & \text{on } \mathcal{A}(u), \end{cases}$$

where $\mathcal{A}(u) := \{x \in \Omega : u(x) > 0\}$ *and* $\mathcal{I}(u) := \Omega \setminus \mathcal{A}(u)$.

The analogous result is valid for Lebesgue spaces on $\mathcal{H}^{N-1}$-measurable subsets of the boundary of a Lipschitz domain.

In optimization problems with pointwise constraints on the norm, it is further customary to treat (the indicator function of) a constraint of the form

$$|u(x)|_2 \le \beta(x), \quad u \in L^2(\Omega)^d, \beta \in L^2(\Omega), \beta \ge 0 \text{ a.e. in } \Omega,$$

by a Moreau-Yosida regularization in $L^2(\Omega)$; for an overview of the most important properties of the Moreau-Yosida regularization, we refer to [10, Prop. 17.2.1]. The resulting regularized functional involves the Fréchet differentiable mapping

$$u \mapsto \tfrac{1}{2}\|[|u|_2 - \beta]^+\|^2_{L^2(\Omega)},$$

whose derivative is given by

$$\mathfrak{m}(u) := [|u|_2 - \beta]^+ \mathfrak{q}(u),$$

where $\mathfrak{q}(\,.\,) : L^2(\Omega)^d \to L^\infty(\Omega)^d$ is defined by

$$\mathfrak{q}(v) := \begin{cases} \frac{v}{|v|_2} & \text{if } |v|_2 > 0, \\ 0 & \text{else.} \end{cases} \tag{2.3.3}$$

The following result on the Newton differentiability of the mapping $\mathfrak{m}$ is available; see [68].

Lemma 2.10 (Newton differentiability of a generalized maximum function)**.** *Let* $\beta \in L^\infty(\Omega)$ *with* $\beta(x) \ge c > 0$ *a.e. in* Ω. *Then the mapping*

$$\mathfrak{m} : L^p(\Omega)^d \to L^s(\Omega)^d$$

is Newton differentiable for $3 \le 3s \le p < +\infty$. *A corresponding Newton derivative is given by*

$$G_{\mathfrak{m}}(u) := \chi_{\mathcal{A}(u)} \cdot \mathfrak{M}(u)$$

where

$$\begin{cases} \rho(u) & := [|u|_2 - \beta]^+ \frac{1}{|u|_2}, \\ \mathfrak{M}(u)(\,.\,) & := \rho(u)(\,.\,) + (1 - \rho(u)) \frac{u u^\top(\,.\,)}{|u|_2^2}, \\ \mathcal{A}(u) & := \{x \in \Omega : |u(x)|_2 > \beta(x)\}. \end{cases} \tag{2.3.4}$$

Again, the contrast to the discrete situation is blatant.

2.3.2 Mesh independence

Let $\Psi : X \to W$ be a nonlinear operator. Consider again the problem of finding an $x \in X$ that solves the equation

$$\Psi(x) = 0. \tag{2.3.5}$$

Assume that the finite-dimensional counterpart of (2.3.5) is to find an $x_h \in X_h$ such that

$$\Psi_h(x_h) = 0, \tag{2.3.6}$$

where $\Psi_h : X_h \to W_h$ is an approximation of Ψ, and the finite-dimensional spaces X_h, W_h represent appropriate discretized versions of the spaces X and W, respectively. The parameter h is associated with a mesh size $h > 0$ which characterizes the discrete spaces. Further suppose that we are given an algorithm for the solution of (2.3.5) that generates a sequence of iterates $(x^{(k)})$ converging to a solution $\bar{x}$ of equation (2.3.6), as well as a family of discrete versions of this algorithm that generate sequences $(x_h^{(k)})$ converging to a solution $\bar{x}_h$ of (2.3.6). A desirable feature of this algorithm is mesh-independence. From a computational point of view, mesh-independent convergence is often characterized by iteration numbers of the underlying problem solver that are uniformly bounded, or, in the ideal case, essentially constant with respect to a decrease in the mesh size. More precisely, the concept is related to the local property that the convergence quotients

$$\frac{\|x_h^{k+1} - \bar{x}_h\|_{X_h}}{\|x_h^k - \bar{x}_h\|_{X_h}}$$

are, in a certain sense, stable with respect to mesh refinement. For instance, in the context of semismooth Newton (SSN) methods, mesh-independence typically refers to the property that for any given linear convergence rate $\theta \in (0,1)$, there exists a radius $\rho > 0$ and a mesh width h_0 such that

$$\begin{aligned}\|x^{k+1} - \bar{x}\|_X &\leq \theta \|x^k - \bar{x}\|_X,\\ \|x_h^{k+1} - \bar{x}_h\|_{X_h} &\leq \theta \|x_h^k - \bar{x}_h\|_{X_h},\end{aligned}$$

provided that $h \leq h_0$ and $\max(\|x^0 - \bar{x}\|_X, \|x_h^0 - \bar{x}_h\|_{X_h}) \leq \rho$. For many solvers of variational inequality or constrained optimization problems, mesh-independence cannot be proven rigorously. However, semismooth Newton methods do admit mesh-independence results; cf. [65],[64]. In this context, mesh-independent convergence requires the generalized differentiability (Definition 2.7) of the nonlinear mapping associated with the root finding problem in infinite dimensions (2.3.5); see [74, Theorem 3].

For variational problems, the property of Newton differentiability is closely related to a sufficiently high regularity of the solution (or the Lagrange multipliers) in order to find a reformulation that fulfills the norm gap requirement; cf. e.g. Lemma 2.9 and Lemma 2.10. Such an increased regularity is often not available. This may result in a considerable computational overhead when computing on very fine meshes; see for instance [16, Table 5.3] for the case of state-constrained optimal control problems. In this regard, it is indispensable to analyze the infinite-dimensional algorithm rather than just the corresponding discrete counterpart.

2.4 Γ- and Mosco-convergence

The general concept of Γ-convergence, which was introduced by De Giorgi (cf. [40]) in the 1970s, is a very useful tool to analyze the stability of an optimization problem

$$\inf \quad F(x), \quad \text{over } x \in X,$$

with respect to certain perturbed problems

$$\inf \quad F_n(x), \quad \text{over } x \in X,$$

in a very general framework. Here, we denote by X an arbitrary topological space and

$$F : X \to \mathbb{R} \cup \{+\infty\}, \quad F_n : X \to \mathbb{R} \cup \{+\infty\},$$

are extended real-valued functions on X. For any $x \in X$, we denote by $\mathcal{N}(x)$ the set of all open neighborhoods of x. Taking the perturbed problems as a starting point, Γ-convergence can be considered as a way to define a limit function F which is suitable from an optimization point of view. In this section, we give a basic account of the theory of Γ-convergence and its relation to pointwise convergence. For a detailed discussion we refer to the monographs [37, 21].

Definition 2.11 (Γ-limits)**.** The Γ-lower and the Γ-upper limit of (F_n) at $x \in X$ are defined by

$$(\Gamma\text{-}\liminf_{n\to+\infty} F_n)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y\in U} F_n(y) \tag{2.4.1}$$

$$(\Gamma\text{-}\limsup_{n\to+\infty} F_n)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{n\to\infty} \inf_{y\in U} F_n(y), \tag{2.4.2}$$

respectively. (F_n) is said to Γ-converge to $F : X \to \mathbb{R} \cup \{+\infty\}$, if and only if,

$$F(x) = (\Gamma\text{-}\liminf_{n\to+\infty} F_n)(x) = (\Gamma\text{-}\limsup_{n\to+\infty} F_n)(x),$$

for all $x \in X$. In this case, we write $F(x) = (\Gamma\text{-}\lim_{n\to+\infty} F_n)(x)$.

If X is a Banach space, it is further convenient to write $\Gamma_w\text{-}\limsup_{n\to+\infty} F_n$ and $\Gamma_w\text{-}\liminf_{n\to+\infty} F_n$ for the Γ-upper and Γ-lower limit of (F_n), respectively, in the weak topology of X. We also write

$$\Gamma_w\text{-}\lim_{n\to+\infty} F_n = \Gamma_w\text{-}\limsup_{n\to+\infty} F_n = \Gamma_w\text{-}\liminf_{n\to+\infty} F_n$$

for the weak Γ-limit of (F_n) provided the latter equality is satisfied.

Furthermore, it is sufficient to use a local neighborhood base instead of the set $\mathcal{N}(x)$ in the definition of the Γ-limits (2.4.1) and (2.4.2). For example, for the Γ-limits in the norm topology of the Banach space X, one may use the open balls with center x, and for the limits in the weak topology one may resign to sets of the form

$$U = U(x) = \{y \in X : |\langle x_i^*, y - x\rangle| < r,\ i \in I\},$$

with a finite index set I, $x_i^* \in X^*$ for all $i \in I$ and $r > 0$; see, e.g., [10, Prop. 2.4.5]. In particular, the fact that the strong topology is finer than the weak topology implies that

$$\Gamma_w\text{-}\liminf_{n\to+\infty} F_n \le \Gamma\text{-}\liminf_{n\to+\infty} F_n, \quad \Gamma_w\text{-}\limsup_{n\to+\infty} F_n \le \Gamma\text{-}\limsup_{n\to+\infty} F_n; \tag{2.4.3}$$

see also [37, Prop. 6.3].

An alternative sequential definition of Γ-convergence is also customary.

Definition 2.12 (Sequential Γ-convergence)**.** The sequence (F_n) sequentially Γ-con-verges to F, if and only if,

(i) $F(x) \leq \liminf_{n\to\infty} F_n(x_n) \quad \forall\, x_n \to x$,

(ii) $F(x) = \lim_{n\to\infty} F_n(y_n) \quad$ for some $y_n \to x$

In this case, we write $F = \Gamma^s\text{-}\lim_{n\to+\infty} F_n$.

Note that there is at most one sequential Γ-limit of the sequence (F_n). Following [37, Chap. 8], we summarize some sufficient conditions which ensure that $\Gamma\text{-}\lim_{n\to+\infty} F_n$ and $\Gamma^s\text{-}\lim_{n\to+\infty} F_n$ coincide.

Proposition 2.13. *The following relations between Γ-convergence and sequential Γ-con-vergence hold true:*

(i) Let the topology of X be first countable. Then (F_n) Γ-converges to F if and only if (F_n) sequentially Γ-converges to F.

(ii) Let X be a Banach space equipped with its weak topology whose normed dual X^* is separable. If there exists a coercive function $H : X \to \mathbb{R} \cup \{+\infty\}$ with $F_n \geq H$ for all $n \in \mathbb{N}$, then (F_n) Γ-converges to F if and only if (F_n) sequentially Γ-converges to F.

(iii) Let X be a reflexive Banach space equipped with its weak topology. Assume there exists a weakly l.s.c. and coercive function $H : X \to \mathbb{R} \cup \{+\infty\}$ such that $F_n \geq H$ for all $n \in \mathbb{N}$. If the functional F fulfills conditions (i) and (ii) from Definition 2.12, i.e., $F = \Gamma^s\text{-}\lim_{n\to+\infty} F_n$, then (F_n) Γ-converges to F.

A point x_ε is called *ε-minimizer* of F if

$$F(x_\varepsilon) \leq \inf_{x\in X} F(x) + \varepsilon, \quad \text{if } \inf_{x\in X} F(x) > -\infty.$$

Otherwise, if $\inf_{x\in X} F(x) = -\infty$, any $x_\varepsilon \in X$ with $F(x_\varepsilon) \leq -\frac{1}{\varepsilon}$ qualifies as an ε-minimizer. For a sequence (x_n) in a general topological space X, each point $x \in X$ is called cluster point of (x_n) if any $U \in \mathcal{N}(x)$ contains infinitely many sequence members of (x_n). The following theorem shows that the Γ-limit $\Gamma\text{-}\lim_{n\to+\infty} F_n$ is defined such that it inherits its minimizers from appropriate limit points of generalized minimizers of the functions F_n. The proofs can be found in the literature; see for instance [10, Theorem 12.1.1], [37, Corollary 7.2].

Theorem 2.14 (Convergence of minimizers)**.** *Let (x_n) be a sequence of ε_n-minimizers of F_n where $\varepsilon_n \to 0$, $\varepsilon_n > 0$ for all $n \in \mathbb{N}$. The following assertions hold true:*

(i) Assume (F_n) Γ-converges to F. Then each cluster point of (x_n) in X is a minimizer of F, and it holds that

$$\limsup_{n\to\infty} F_n(x_n) = \inf_{x\in X} F(x).$$

(ii) Assume (F_n) sequentially Γ-converges to F. Then the limit of each converging subsequence (x_{n_k}) in X is a minimizer of F, and it holds that

$$\lim_{k\to\infty} F_{n_k}(x_{n_k}) = \inf_{x\in X} F(x).$$

It is further important to know the relation between Γ-convergence and pointwise convergence. At this point we recall that sc^{-} denotes the lower semicontinuous envelope; cf. (1.1.1).

Proposition 2.15 (Relation to pointwise convergence)**.** *The following assertions hold true:*

(i) If (F_n) is an increasing sequence, then $\Gamma\text{-}\lim_{n\to+\infty} F_n = \lim_{n\to\infty} \mathrm{sc}^{-} F_n$.

(ii) If (F_n) is a decreasing sequence converging pointwise to F, then $\Gamma\text{-}\lim_{n\to+\infty} F_n = \mathrm{sc}^{-} F$.

Another important concept is the notion of Mosco-convergence which is widely used in the perturbation analysis of convex functions and variational inequalities including regularization, penalization and discretization methods. The following definition was first stated in [95].

Definition 2.16 (Mosco-convergence)**.** Let X be a Banach space. The sequence (F_n) Mosco-converges to F if

(i) $F(x) \leq \liminf_{n\to\infty} F_n(x_n) \quad \forall\, (x_n) \subset X$ with $x_n \rightharpoonup x$,

(ii) $F(x) = \lim_{n\to\infty} F_n(y_n)$ for some $(y_n) \subset X$ with $y_n \to x$.

In other words, (F_n) Mosco-convergences to F if and only if F is the sequential Γ-limit of (F_n) in both, the weak and strong topology of X. The sequence (y_n) from Definition 2.16(ii) is called recovery sequence.

Proposition 2.17. *Let X be a Banach space. Suppose (F_n) Γ-converges to F in both, the strong and weak topology. Then (F_n) Mosco-converges to F.*

Proof. First notice that Proposition 2.13(i) implies that F coincides with the sequential Γ-limit in the norm topology and thus (ii) of Definition 2.16 is valid. Secondly, let $x \in X$ and $x_n \rightharpoonup x$. For given $U \in \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the set of all weakly open neighborhoods of x, there exists $n_0 \in \mathbb{N}$ with $x_n \in U$ for all $n \geq n_0$. Consequently, one obtains

$$\liminf_{n\to\infty} \inf_{y\in U} F_n(y) \leq \liminf_{n\to\infty} F_n(x_n)$$

for all $U \in \mathcal{N}(x)$ such that

$$\Gamma_w\text{-}\liminf_{n\to+\infty} F_n(x) \leq \liminf_{n\to\infty} F_n(x_n).$$

Together with $F = \Gamma_w\text{-}\lim_{n\to+\infty} F_n = \Gamma_w\text{-}\liminf_{n\to+\infty} F_n$, one obtains

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n),$$

which is precisely part (i) of Definition 2.16. □

As an example, we consider the following abstract class of perturbations of the indicator function $i_K : X \to \mathbb{R} \cup \{+\infty\}$ to a nonempty, closed and convex subset $K \subset X$ of a Banach space X. This general class turns out to be very useful when several types of perturbation methods, such as discretization, regularization or penalization are considered or possibly combined.

Definition 2.18 (Quasi-monotone perturbation)**.** A sequence of mappings

$$R_n : X \to \mathbb{R} \cup \{+\infty\}$$

is called a *quasi-monotone perturbation of the indicator function i_K with respect to a dense subspace Y of X* if there exist functions $\underline{R_n} : X \to \mathbb{R} \cup \{+\infty\}$, $\overline{R_n} : X \to \mathbb{R} \cup \{+\infty\}$ such that

$$0 \leq \underline{R}_n \leq R_n \leq \overline{R}_n \quad \forall\, n \in \mathbb{N},$$

having the additional properties

$$\begin{cases} \underline{R_n} \leq \underline{R_{n+1}} \;\; \forall\, n \in \mathbb{N}, \quad \lim_{n\to+\infty} \underline{R_n}(x) = i_K(x) \;\; \forall\, x \in X, \\ \underline{R_n} \text{ is sequentially weakly l.s.c. } \; \forall\, n \in \mathbb{N}, \end{cases} \tag{2.4.4}$$

and

$$\overline{R}_n \geq \overline{R}_{n+1}, \;\; \forall\, n \in \mathbb{N}, \quad \lim_{n\to+\infty} \overline{R}_n(x) = i_{K\cap Y}(x) \;\; \forall\, x \in X. \tag{2.4.5}$$

Note that no additional assumptions are assumed for R_n.

Proposition 2.19. *Let X be a Banach space and $K \subset X$ a nonempty, convex and closed subset. Suppose that (R_n) is a quasi-monotone perturbation of the indicator function i_K with respect to some dense subspace $Y \subset X$. Additionally, assume that the lower bound $\underline{R_n}$ from Definition 2.18 is weakly l.s.c.. Then if*

$$\overline{K \cap Y}^X = K,$$

i.e., $K \cap Y$ is dense in K with respect to the norm of X, then (R_n) Mosco-converges to i_K.

Proof. The proof is identical to the proof of Proposition 4.6 because the additional weak lower semicontinuity assumption on $\underline{R}_n$ implies $\mathrm{sc}^-_{\mathrm{w}}\, \underline{R}_n = \underline{R}_n$, such that (4.1.5) holds true with $F := 0$. □

3 Variational Inequalities

3.1 Some Generalities

Consider the following optimization problem on a Hilbert space X:

$$\inf \quad F(v) - \langle l, v\rangle + j(v) \qquad \text{over } v \in X. \tag{3.1.1}$$

Here, $F : X \to \mathbb{R}$ is assumed to be Gâteaux-differentiable, $j : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued proper and convex functional and $l \in X^*$. Note that this general setting absorbs nonsmooth- and convex-constrained problems at the same time. The necessary optimality condition (the existence of a solution not being established yet) for a solution u to (3.1.1) is given by

$$\langle A(u), v - u\rangle + j(v) - j(u) \geq \langle l, v - u\rangle, \quad \forall\, v \in X, \tag{3.1.2}$$

where $A := F'$, and (3.1.2) is even a sufficient optimality criterion whenever F additionally is convex. This type of problem falls into the class of variational inequality (VI) problems which generalizes (3.1.2) to operators A which are not necessarily of potential type.

For the study of the existence theory for variational inequality problems, the following standard class of variational inequalities suffices for our purposes: Let $a : X \times X \to \mathbb{R}$ be a continuous and elliptic bilinear form, i.e., there exists $c, \kappa > 0$ such that

$$|a(u, v)| \leq c\|u\|_X \|v\|_X, \quad a(u, u) \geq \kappa \|u\|_X^2,$$

for all $u, v \in X$. Further suppose that j is l.s.c., proper and convex. Then the Lions-Stampacchia-Theorem states that the variational inequality problem of finding

$$u \in X : \quad a(u, v - u) + j(v) - j(u) \geq \langle l, v - u\rangle, \quad \forall\, v \in X, \tag{3.1.3}$$

has a unique solution. For a general convex function j, the variational inequality in (3.1.3) is usually referred to as *of second kind* whereas the case $j = i_K$ for some convex set $K \subset X$ gives rise to a variational inequality *of the first kind*. Existence results for the problem class (3.1.2) may be obtained for nonlinear operators A which fulfill much weaker continuity and monotonicity assumptions, and we refer to [128] for an extensive study of this matter.

3.2 Approximation of Variational Inequalities

The following result can be considered as a generalization of known approximation results for variational inequalities, cf. [53, 89, 61, 54]. Furthermore, the method of proof is analogous to the latter references but the formulation avoids the lower semicontinuity of the perturbed function. This allows to also include singular perturbations of the elliptic operator governing the inequality which is not contained in the original work on Mosco-convergence, cf. [95, Theorem B]. Here, all possible perturbations are understood as perturbations of the nonsmooth function j. Moreover, we call a sequence of functionals $j_n : X \to \mathbb{R} \cup \{+\infty\}$ uniformly proper if there exists $\hat{l} \in X^*$ and $c \in \mathbb{R}$ such that

$$j_n(v) \geq \langle \hat{l}, v\rangle + c, \quad \forall\, v \in X. \tag{3.2.1}$$

Theorem 3.1 (Perturbation of VIs of the second kind)**.** *Let X be a Hilbert space, $a : X \times X \to \mathbb{R}$ a continuous and elliptic bilinear form, $j : X \to \mathbb{R} \cup \{+\infty\}$ convex, l.s.c, proper and $l \in X^*$. Consider the elliptic VI of the second kind,*

$$\text{find } u \in X : \quad a(u, v-u) + j(v) - j(u) \geq \langle l, v-u \rangle, \quad \forall\, v \in X, \tag{VI}$$

and the perturbed versions,

$$\text{find } u_n \in X : \quad a(u_n, v-u_n) + j_n(v) - j_n(u_n) \geq \langle l, v-u_n \rangle, \quad \forall\, v \in X, \tag{VI$_n$}$$

for a sequence of uniformly proper functions $j_n : V \to \mathbb{R} \cup \{+\infty\}$. Assume that

(i) (j_n) Mosco-converges to j (Definition 2.16), and that

(ii) each problem (VI$_n$) admits a solution u_n.

Then it holds that

$$u_n \to u \text{ in } X \text{ and } j_n(u_n) \to j(u).$$

Proof. (i) Boundedness of (u_n):

Let $v \in X$. By assumption (i), there exists a sequence $(v_n) \subset X$ with $v_n \to v$ and $j_n(v_n) \to j(v)$. From (VI$_n$) it follows that

$$a(u_n, u_n) + j_n(u_n) \leq a(u_n, v_n) + j_n(v_n) - \langle l, v_n - u_n \rangle. \tag{3.2.2}$$

Since $(j_n(v_n))$ and (v_n) are uniformly bounded and (j_n) is uniformly proper (3.2.1), one obtains from (3.2.2), using the properties of a,

$$\|u_n\|^2 \leq c_1 \|u_n\| + c_2,$$

for some constants $c_1, c_2 \in \mathbb{R}$. This implies the boundedness of (u_n).

(ii) Weak limit of (u_n):

Since (u_n) is bounded, there exists a weakly convergent subsequence, which by abuse of notation is also denoted by (u_n), with $u_n \rightharpoonup \tilde{u}$ in X. By taking the lim inf in (3.2.2) and making use of Definition 2.16(i), one obtains

$$\begin{aligned} a(\tilde{u}, \tilde{u}) + j(\tilde{u}) &\leq \liminf_{n \to \infty} (a(u_n, v_n) + j_n(v_n) - \langle l, v_n - u_n \rangle) \\ &= a(\tilde{u}, v) + j(v) - \langle l, v - \tilde{u} \rangle, \end{aligned}$$

i.e., $\tilde{u} = u$ is the solution of (VI), and by Urysohn's principle the entire sequence (u_n) converges weakly to u.

(iii) Strong convergence of (u_n):

The assumptions on a as well as (3.2.2) yield

$$\begin{aligned} \kappa \|u_n - u\|^2 + j_n(u_n) &\leq a(u_n, u_n) + j_n(u_n) - a(u_n, u) - a(u, u_n) + a(u, u) \\ &\leq a(u_n, v_n) + j_n(v_n) - l(v_n - u_n) - a(u_n, u) - a(u, u_n) + a(u, u), \end{aligned}$$

which implies

$$\begin{aligned}
j(u) &\leq \liminf_{n\to\infty} j_n(u_n) \leq \limsup_{n\to\infty} j_n(u_n) \\
&\leq \limsup_{n\to\infty} (\kappa \|u_n - u\|^2 + j_n(u_n)) \\
&\leq \limsup_{n\to\infty} (a(u_n, v_n) + j_n(v_n) - l(v_n - u_n) - a(u_n, u) - a(u, u_n) + a(u, u)) \\
&= a(u, v - u) + j(v) - \langle l, v - u \rangle,
\end{aligned}$$

on account of $u_n \rightharpoonup u$, $v_n \to v$ and $j_n(v_n) \to j(v)$ as $n \to \infty$. Since v was arbitrary, it is possible to set $v = u$ in the last estimate which proves $j_n(u_n) \to j(u)$ and thus also $\|u_n - u\| \to 0$ as $n \to \infty$. $\square$

We remark that the assumption (ii) compensates for the fact that j_n is neither assumed to be l.s.c. nor convex. It is further standard to generalize the above result to nonlinear strongly monotone operators on reflexive Banach spaces [128].

Part II

Density of Convex Intersections and Applications

4 Motivation

The analysis of optimization problems and variational inequalities over a convex subset K of a Banach space X as well as the design of suitable solution algorithms often involve the general concepts of dualization and perturbation methods where the latter may comprise regularization, penalization or discretization approaches, or possibly a combination of the latter. While keeping the abstract framework in this chapter, it is shown that the stability properties of the respective problem class with regard to a large class of perturbations rely on the closure property

$$\overline{K(Y)}^X = K, \tag{4.0.1}$$

where Y is some dense subspace of X with respect to the norm topology of X and $K(Y)$ is given by

$$K(Y) := \{x \in Y : x \in K\} = K \cap Y.$$

In general, for a Banach space X, an arbitrary dense subspace $Y \subset X$ as well as a convex and closed subset $K \subset X$ the inclusion

$$K \cap Y \subset K \cap X \tag{4.0.2}$$

is not necessarily dense even for a linear subspace K: In fact, consider the following example.

Example 4.1. Let $\Omega = (-1, 1), X = L^2(\Omega), Y = H^1(\Omega)$ and $K = \{c\, f_{\mathrm{HS}} : c \in \mathbb{R}\}$, where f_{HS} denotes the Heaviside-function,

$$f_{\mathrm{HS}}(\omega) := 0 \ \text{ for } \omega < 0, \quad f_{\mathrm{HS}}(\omega) := 1 \ \text{ for } \omega \geq 0.$$

It follows that $K \cap Y = \{0\}$ and the density property (4.0.1) is violated.

4.1 Optimization Problems with Convex Constraints

In many variational problems one seeks the solution in a given convex, closed and nonempty subset K of a Banach space $(X, \|.\|)$. To illustrate the problem, let us consider the following abstract class of optimization problems:

$$\begin{cases} \inf & F(x), \quad \text{over } x \in X, \\ \text{s.t.} & x \in K. \end{cases} \tag{4.1.1}$$

We assume that $F : X \to \mathbb{R}$ is continuous, coercive and sequentially weakly lower semicontinuous but not necessarily convex. Thus, if X is reflexive, problem (4.1.1) admits a solution. The problem class (4.1.1) is ubiquitous, encompassing numerous fields, such as the variational form of partial differential equations, variational inequality problems, optimal control of partial differential equations with constraints on the state and/or control and many other.

The starting point of our analysis is the conjecture that the stability of (4.1.1) with respect to a large class of perturbations depends on the density condition (4.0.1). In order to substantiate this conjecture, we investigate the consistency of various perturbations with the help of the theory of Γ-convergence, which is briefly introduced in Section 2.4. To begin with, we consider two important examples which are particular instances of the class of quasi-monotone perturbations, cf. Definition 2.18. Thereupon, the more general case will be discussed. Note also that by convexity of K there is no difference between the closure with respect to the norm in (4.0.1) and the weak and the sequential weak closure taken in X. We will therefore write $\overline{K(Y)}$ instead of $\overline{K(Y)}^X$ when no confusion may occur.

Example 4.2 (Tikhonov-Regularization)**.** Let $(Y, \|\,.\,\|_Y)$ be a Banach space which is densely and continuously embedded into X. For a sequence of positive non-decreasing parameters (γ_n) with $\gamma_n \to +\infty$ and fixed $\alpha > 0$, consider a Tikhonov regularization of (4.1.1) defined by

$$\inf \quad F(x) + R_n(x), \qquad \text{over } x \in X, \tag{4.1.2}$$

where $R_n(x) := i_K(x) + \frac{1}{2\gamma_n}\|x\|_Y^\alpha$, and it is understood that $R_n(x) = +\infty$ if $x \notin Y$. Setting $R_n(x) = i_K(x) + \frac{1}{2\gamma_n}\|x\|_Y^\alpha$, it is easily seen that (R_n) is a quasi-monotone perturbation according to Definition 2.18: In fact, set $\underline{R_n} := i_K$ for all $n \in \mathbb{N}$ and $\overline{R_n} := R_n$. Obviously, (2.4.4) and (2.4.5) are satisfied.

The density property (4.0.1) naturally arises when considering the Γ-limit of the objective function in (4.1.2).

Proposition 4.3. *Let X be a Banach space which is reflexive or has a separable dual space X^*. Let $(Y, \|\,.\,\|_Y)$ be a subspace of X which is densely and continuously embedded in X. Assume $K \subset X$ to be nonempty, closed and convex and $F : X \to \mathbb{R}$ continuous, coercive and sequentially weakly l.s.c.. Then it holds that*

$$\Gamma\text{-}\lim_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) = \Gamma_w\text{-}\lim_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) = F + i_{\overline{K\cap Y}},$$

i.e, the objective function in Example 4.2 Γ-converges to $F + i_{\overline{K\cap Y}}$ in both, the weak and strong topology of X.

Proof. Using the definition of the Γ-limits as well as its relation to pointwise convergence, cf. Proposition 2.15, one obtains

$$\begin{aligned}\Gamma\text{-}&\limsup_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) \\ &= \Gamma\text{-}\lim_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) = \operatorname{sc}^-(F + i_{K\cap Y}) = F + i_{\overline{K\cap Y}},\end{aligned}$$

where the last equality follows from the additivity of the lower semicontinuous envelope (F is continuous) and the fact that $\operatorname{sc}^- i_{K\cap Y} = i_{\overline{K\cap Y}}$. From the definition of the Γ-limits, it follows that the strong Γ-lower limit is bounded below by the weak Γ-lower limit, cf. (2.4.3), such that

$$\begin{aligned}\Gamma\text{-}\liminf_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) &\geq \Gamma_w\text{-}\liminf_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) \\ &\geq \Gamma_w\text{-}\liminf_{n\to+\infty}(F + i_{K\cap Y}) = \operatorname{sc}_w^-(F + i_{K\cap Y}) \geq F + i_{\overline{K\cap Y}}.\end{aligned}$$

To justify the last estimate, note that the coercivity and the sequential weak lower semicontinuity of F implies that the level sets $\{u \in X : F(u) \leq t\}$, $t \in \mathbb{R}$, are bounded and sequentially weakly closed. Under the stated conditions on X, the level sets are also closed, since in these cases, the sequential weak closure of bounded subsets coincides with the weak closure, see [37, Prop. 8.7, Prop. 8.14]. This implies that F is weakly lower semicontinuous. With

$$F + i_{\overline{K\cap Y}} \leq \Gamma\text{-}\liminf_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) \leq \Gamma\text{-}\limsup_{n\to+\infty}(F + i_K + \tfrac{1}{2\gamma_n}\|\,.\,\|_Y^\alpha) \leq F + i_{\overline{K\cap Y}},$$

the assertion is proven. □

Example 4.4 (Galerkin approximation)**.** Let X be a separable Banach space. Let (X_n) be a Galerkin approximation scheme of nested finite-dimensional subspaces X_n, i.e., $X_n \subset X$ and $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$ with the Galerkin approximation property

$$\overline{\bigcup_{n\in\mathbb{N}} X_n}^X = X.$$

Consider the discrete version of problem (4.1.1) given by

$$\inf \quad F(x) + R_n(x), \qquad \text{over } x \in X, \tag{4.1.3}$$

where $R_n := i_{K\cap X_n}$. Again, (R_n) fits into the framework of quasi-monotone perturbations: In fact, setting $\underline{R_n} := i_K$, (2.4.4) is clearly fulfilled. Let $Y = \bigcup_{n\in\mathbb{N}} X_n$, then (2.4.5) is fulfilled with $\overline{R_n} := R_n$.

In some cases, for example PDE problems with curved boundaries, it is hardly possible to ensure that the subspaces (X_n) are nested. In this situation, density results are still useful; see section 4.2.

Proposition 4.5. *Let X be a separable Banach space which is reflexive or has a separable dual space. Let (X_n) be a nested Galerkin approximation scheme and $K \subset X$ nonempty, closed and convex. Further assume that $F : X \to \mathbb{R}$ is continuous, coercive and sequentially weakly l.s.c.. Then it holds that*

$$\Gamma\text{-}\lim_{n\to+\infty}(F + i_{K\cap X_n}) = \Gamma_w\text{-}\lim_{n\to+\infty}(F + i_{K\cap X_n}) = F + i_{\overline{K\cap Y}},$$

i.e, the objective function in Example 4.4 Γ-converges to $F + i_{\overline{K\cap Y}}$ in both, the weak and strong topology of X.

Proof. The proof is analogous to the one of Proposition 4.3. □

At this point, the remarkable feature of Examples 4.2 and 4.4 should be emphasized: Although the pointwise limit of the perturbed problems is given in both examples by

$$\inf \quad F(x) + i_K(x), \quad \text{over } x \in Y,$$

which is the same problem as (4.1.1) only posed on a dense subset of X, we in general do not retrieve the same infimum. This aspect stands in certain analogy with the Lavrentiev phenomenon in the calculus of variations, cf. [35].

We now turn to the general case. To subsume as many different perturbation methods as possible we consider the sequence of perturbed problems

$$\inf \quad F(u) + R_n(u), \quad \text{over } u \in X, \tag{4.1.4}$$

defined by a given quasi-monotone perturbation

$$R_n : X \to \mathbb{R} \cup \{+\infty\}$$

of the indicator function $i_K : X \to \mathbb{R} \cup \{+\infty\}$ with respect to a dense subspace $Y \subset X$. The following result extends the previous propositions to the abstract class of quasi-monotone perturbations.

Proposition 4.6. *Let X be a Banach space which is reflexive or which has a separable dual space X^*. If the density property (4.0.1) holds true, then $F + i_K$ is the Γ-limit of $(F + R_n)$ in both, the weak and strong topology, and, in particular, $(F + R_n)$ Mosco-converges to $F + i_K$.*

Proof. Using (2.4.3) and the relations between Γ- and pointwise convergence (Proposition 2.15), one obtains with (2.4.5) and the continuity of F,

$$\begin{aligned}\Gamma_w\text{-}\limsup_{n\to+\infty}(F+R_n) &\le \Gamma\text{-}\limsup_{n\to+\infty}(F+R_n)\\ &\le \Gamma\text{-}\limsup_{n\to+\infty}(F+\overline{R_n}) = \operatorname{sc}^-(F+i_{K\cap Y}) = F + i_{\overline{K\cap Y}},\end{aligned}$$

since $(\overline{R_n})$ is monotonically decreasing and pointwise converging to $i_{K\cap Y}$. Similarly, (2.4.4) implies

$$\begin{aligned}\Gamma_w\text{-}\liminf_{n\to+\infty}(F+R_n) &\ge \Gamma_w\text{-}\liminf_{n\to+\infty}(F+\underline{R_n})\\ &= \lim_{n\to+\infty}\operatorname{sc}^-_{\mathrm{w}}(F+\underline{R_n}) = \lim_{n\to+\infty}(F+\underline{R_n}) = F+i_K,\end{aligned} \tag{4.1.5}$$

where the second last equality follows from the fact that $F+\underline{R_n}$ is weakly l.s.c.. This property can be argued as in the proof of Proposition 4.3. Eventually, it holds that

$$F+i_K \le \Gamma_w\text{-}\liminf_{n\to+\infty}(F+R_n) \le \Gamma_w\text{-}\limsup_{n\to+\infty}(F+R_n) \le \Gamma\text{-}\limsup_{n\to+\infty}(F+R_n) \le F+i_{\overline{K\cap Y}},$$

such that $\Gamma\text{-}\lim_{n\to+\infty}(F+R_n) = \Gamma_w\text{-}\lim_{n\to+\infty}(F+R_n) = F+i_K$ if (4.0.1) holds true. The second assertion follows directly from Proposition 2.17. □

Under the density assumption (4.0.1), one may infer that each (weak) cluster point of a sequence of (generalized) minimizers (x_n) of the perturbed problem (4.1.4) is a minimizer of (4.1.1). For details see Section 2.4.

Algorithmically, it is often more favorable to replace the constrained problem (4.1.1) by a sequence of unconstrained problems. For this purpose, we combine the approximation methods of the examples above with a penalty approach using the Moreau-Yosida-regularization from convex analysis. The resulting perturbations are shown to belong to the general class of quasi-monotone perturbations and thus pertain to the abstract sequence of problems (4.1.4).

Example 4.7 (Conformal discretization and Moreau-Yosida regularization)**.** Let X be a separable Hilbert space and (X_n) a nested Galerkin scheme as in Example 4.4. With each $n \in \mathbb{N}$, we also associate an arbitrary sequence (γ_n) of positive non-decreasing parameters converging to $+\infty$. The combination of regularization and discretization leads to the definition

$$R_n(x) := \tfrac{\gamma_n}{2}\inf_{y\in K}\|x-y\|^2 + i_{X_n}(x), \tag{4.1.6}$$

where the spaces X_n are defined as in the previous example. Setting $\underline{R}_n(x) := \frac{\gamma_n}{2}\inf_{y\in K}\|x-y\|^2$, (2.4.4) is fulfilled owing to standard properties of the Moreau-Yosida regularization; see [10, Prop. 17.2.1]. Defining $\overline{R_n} := i_{K\cap X_n}$, (2.4.5) is fulfilled for $Y := \bigcup_{n\in\mathbb{N}} X_n$ and the framework of (4.1.4) applies.

Consequently, the perturbation approach of the preceding Example 4.7 is stable with respect to (4.1.1) provided the density result (4.0.1) is satisfied. In particular, this is true for *any* combination of n and γ_n in (4.1.6). Let us give a different perspective on this result: One may show the existence of a suitable coupling of n and γ_n in order to retrieve $F+i_K$ as the sequential Γ-limit of $(F+R_n)$ in the strong topology without invoking the density property, see [94, Proposition 2.4.6]. However, the proof is non-constructive and thus this coupling is not useful for algorithmic purposes. On the other hand, if the density property (4.0.1) is not fulfilled there also exists a strictly increasing sequence (γ_n) with $\gamma_n \to +\infty$ such that the $F + i_K$ is not the Γ-limit in the strong topology. This is a consequence of the following result.

Lemma 4.8. *Let the assumptions of Example 4.7 be satisfied. Further suppose that $\overline{K \cap Y} \subsetneq K$. Then for all $x \in K \setminus \overline{K \cap Y}$ there exists a strictly increasing sequence (γ_n) with $\gamma_n \to \infty$ such that there exists no strong recovery sequence at x, i.e.,*

$$F(y_n) + R_n(y_n) \nrightarrow F(x)$$

for all $(y_n) \subset X$ with $y_n \to x$, where (R_n) is given by (4.1.6).

Proof. Let $x \in K \setminus \overline{K \cap Y}$ and $\rho > 0$ such that $\overline{B_\rho(x)} \cap \overline{K \cap Y} = \emptyset$ where $B_\rho(x) := \{\tilde{x} \in X : \|x - \tilde{x}\| < \rho\}$.

(a) We first prove the following result:

$$\forall\, n \in \mathbb{N} \;\; \exists\, \gamma_n > 0: \quad \left[\left(y \in Y \wedge \operatorname{dist}(y, K \cap \overline{B_\rho(x)})^2 < \tfrac{1}{\gamma_n}\right) \Longrightarrow y \notin X_n\right]. \tag{4.1.7}$$

Assume the opposite, i.e.,

$$\exists\, n_0 \in \mathbb{N}: \quad \left[\forall\, n \in \mathbb{N}\, \exists\, x_n \in X_{n_0}, v_n \in K \cap \overline{B_\rho(x)}: \quad \|x_n - v_n\|^2 \le \tfrac{1}{n}\right].$$

Since $v_n \in \overline{B_\rho(x)} \cap K$ for all $n \in \mathbb{N}$ and $\overline{B_\rho(x)} \cap K$ is convex, bounded and closed, there exists a subsequence (v_{n_k}) of (v_n) with $v_{n_k} \rightharpoonup v$ and $v \in \overline{B_\rho(x)} \cap K$. As $x_n - v_n \to 0$, one also obtains $x_{n_k} \rightharpoonup v$ and thus $v \in X_{n_0}$. Hence, $v \in X_{n_0} \cap K \cap \overline{B_\rho(x)} = \emptyset$, which is a contradiction.

(b) Non-existence of a strong recovery sequence:

Choose (γ_n) according to (4.1.7) and assume that there exists a recovery sequence (y_n) to x which means that $y_n \to x$ and $F(y_n) + \frac{\gamma_n}{2} \operatorname{dist}(y_n, K)^2 + i_{X_n}(y_n) \to F(x)$. The continuity of F implies that $y_n \in X_n$ and $\frac{\gamma_n}{2} \operatorname{dist}(y_n, K)^2 \to 0$. Consequently, using $y_n \to x$ and $x \in K$, there exists $n_1 \in \mathbb{N}$ such that

$$\operatorname{dist}(y_n, K)^2 = \operatorname{dist}(y_n, K \cap B_\rho(x))^2 \le \tfrac{1}{\gamma_n}$$

for all $n \ge n_1$. With the help of part (a), we conclude that $y_n \notin X_n$ for all $n \ge n_1$ which is a contradiction.

$\square$

Example 4.9 (Combined Moreau-Yosida-Tikhonov-Regularization)**.** Let X be a Hilbert space and $(Y, \|\,.\,\|_Y)$ a Banach space which is densely and continuously embedded into X. For two sequences of positive non-decreasing parameters $(\gamma_n), (\gamma'_n)$ with $\gamma_n, \gamma'_n \to +\infty$ and fixed $\alpha > 0$, consider the simultaneous Moreau-Yosida and Tikhonov regularization,

$$R_n(x) := \tfrac{\gamma_n}{2} \inf_{y \in K} \|x - y\|^2 + \tfrac{1}{2\gamma'_n} \|x\|_Y^\alpha, \tag{4.1.8}$$

with $\alpha > 0$ fixed, where it is understood that $R_n(x) = +\infty$ if $x \notin Y$. Setting $\underline{R_n}(x) := \frac{\gamma_n}{2} \inf_{y \in K} \|x - y\|^2$ and $\overline{R_n}(x) := i_K(x) + \frac{1}{2\gamma_n}\|x\|_Y^\alpha$, (2.4.4) and (2.4.5) are verified as in the previous example.

Again, it is possible to show that the density property is a necessary condition for the existence of a strong recovery sequence. This is the purpose of the following result which is similar to Lemma 4.8.

Lemma 4.10. *Let the assumptions of Example 4.9 be satisfied. Further suppose that $\overline{K \cap Y} \subsetneq K$. Let the corresponding sequence (γ'_n) be fixed. Then for all $x \in K \setminus \overline{K \cap Y}$ there exists a strictly increasing sequence*

(γ_n) with $\gamma_n \to \infty$ such that there exists no strong recovery sequence at x, i.e.,

$$F(y_n) + R_n(y_n) \nrightarrow F(x)$$

for all $(y_n) \subset X$ with $y_n \to x$, where (R_n) is given by (4.1.8).

Proof. Let $x \in K \setminus \overline{K \cap Y}$ and $\rho > 0$ such that $\overline{B_\rho(x)} \cap \overline{K \cap Y} = \emptyset$.

(a) We first prove the following result:

$$\forall\, n \in \mathbb{N} \ \ \exists\, \gamma_n > 0 : \quad \left[y \in Y \ \wedge \ \operatorname{dist}(y, K \cap \overline{B_\rho(x)})^2 < \tfrac{1}{\gamma_n} \implies \|y\|_Y^2 > \gamma_n' \right]. \tag{4.1.9}$$

Assume the opposite, i.e., there exists $n_0 \in \mathbb{N}$ with

$$\forall\, n \in \mathbb{N}\ \exists\, y_n \in Y, v_n \in K \cap \overline{B_\rho(x)} : \quad \left[\|y_n - v_n\|^2 \le \tfrac{1}{n} \wedge \|y_n\|_Y^2 \le \gamma_{n_0} \right].$$

As in the proof of Lemma 4.8 it follows that there exists a subsequence (v_{n_k}) of (v_n) with $v_{n_k} \rightharpoonup v$ and $v \in \overline{B_\rho(x)} \cap K$. As $y_n - v_n \to 0$, one also obtains $y_{n_k} \rightharpoonup v$. From $\|y_n\|_Y^2 \le \gamma_{n_0}$ for all $n \in \mathbb{N}$, one deduces that $v \in Y$ and thus $v \in Y \cap K \cap \overline{B_\rho(x)} = \emptyset$, which is a contradiction.

(b) Non-existence of a strong recovery sequence:

Choose (γ_n) according to (4.1.9) and assume that there exists a recovery sequence (y_n) to x which means that $y_n \to x$ and

$$F(y_n) + \tfrac{\gamma_n}{2} \operatorname{dist}(y_n, K)^2 + \tfrac{1}{2\gamma_n'} \|y_n\|_Y^2 \to F(x).$$

The continuity of F implies that $\frac{1}{2\gamma_n'}\|y_n\|_Y^2 \to 0$ and $\frac{\gamma_n}{2} \operatorname{dist}(y_n, K)^2 \to 0$. Consequently, using $y_n \to x$, there exists $n_1 \in \mathbb{N}$ such that

$$\operatorname{dist}(y_n, K)^2 = \operatorname{dist}(y_n, K \cap B_\rho(x))^2 \le \tfrac{1}{\gamma_n}$$

for all $n \ge n_1$. With the help of part (a), we conclude that $\|y_n\|^2 \ge \gamma_n'$ for all $n \ge n_1$ which contradicts $\frac{1}{2\gamma_n'}\|y_n\|_Y^2 \to 0$.

□

As a consequence of the above statements, the density property $\overline{K \cap Y} = K$ is also a necessary condition for the consistency of the perturbation approaches (4.1.6) and (4.1.8) with respect to the limit problem (4.1.1) in the norm topology. It should also be emphasized that these examples only represent an assorted variety of perturbations which fit into the problem class (4.1.4).

4.2 Elliptic Variational Inequalities

Closure properties of convex intersections of the type (4.0.1) are also of fundamental importance for the analysis of perturbations of variational inequalities. Let X be a Hilbert space and $K \subset X$ a nonempty, closed and convex subset. In this section we consider the general variational inequality problem of the first kind:

$$\text{Find } u \in X : \quad \langle A(u), v - u \rangle \ge \langle l, v - u \rangle \quad \forall\, v \in K, \tag{4.2.1}$$

for some, in general, nonlinear operator $A : X \to X^*$ and $l \in X^*$. This problem can be equivalently reformulated using the indicator function i_K to K.

$$\text{Find } u \in X: \quad \langle A(u), v-u\rangle + i_K(v) - i_K(u) \geq \langle l, v-u\rangle \quad \forall\, v \in X. \tag{4.2.2}$$

The operator A is assumed to be Lipschitz continuous, i.e., there exists $L > 0$ with

$$\|A(v) - A(u)\| \leq L\|v-u\| \quad \forall\, u, v \in X,$$

and strongly monotone, i.e., there exists $\kappa > 0$ with

$$\langle A(v) - A(u), v-u\rangle \geq \kappa\|v-u\|^2 \quad \forall\, u, v \in X.$$

Under the standing assumptions, the Lions-Stampacchia-Theorem ensures that problem (4.2.1) has a unique solution $\bar{u}$. In the following, we investigate three main classes of perturbations of (4.2.1) and their relation to the density properties of K.

4.2.1 Quasi-monotone perturbation

Suppose (R_n) is a quasi-monotone perturbation of i_K (Definition 2.18). Consider the perturbed variational inequality problem,

$$\text{find } u_n \in X: \quad \langle A(u_n), v-u_n\rangle + R_n(v) - R_n(u_n) \geq \langle l, v-u_n\rangle \quad \forall\, v \in X, \tag{4.2.3}$$

where (R_n) is a quasi-monotone perturbation of i_K with respect to a dense subspace Y of X according to Definition 2.18. The stability of the approximation scheme (4.2.3) hinges on the density property (4.0.1). If the latter condition is fulfilled and, additionally, the lower bound $\underline{R}_n$ is weakly l.s.c., then Proposition 2.19 implies that (R_n) Mosco-converges to i_K. Thus one may invoke Theorem 3.1 to conclude the consistency of the perturbation scheme.

4.2.2 Galerkin approximation

In general, finite-dimensional approximations of K are neither conformal nor nested as was the case in Example 4.4 and Example 4.7, where K was 'discretized' by $K \cap X_n$ which is numerically realizable only in special cases. Instead, it is often more favorable to consider non-nested approximations $K_n \subset X_n$ with $K_n \not\subseteq K$ in general, such that the finite-dimensional variational inequality problems,

$$\text{find } u_n \in X: \quad \langle A(u_n), v-u_n\rangle + i_{K_n}(v) - i_{K_n}(u_n) \geq \langle l, v-u_n\rangle \quad \forall\, v \in X, \tag{4.2.4}$$

do not fit into the framework of quasi-monotone perturbations. According to Theorem 3.1, the Mosco-convergence of (K_n) to K, or equivalently, the weak and strong sequential Γ-convergence of i_{K_n} to i_K, suffices to ensure that the approximation (4.2.4) is stable with respect to the limit problem (4.2.2). This property is maintained in a very general context, that is, the monotone operator A and the right hand side f may also be perturbed. Under mild monotonicity assumptions on A and its possible perturbations one may even derive strong convergence for the discrete solutions (u_n), see [95] for details. However, Mosco-convergence requires the existence of a recovery sequence for any element $u \in K$. To construct this sequence in the context of finite element methods, one typically uses an interpolation procedure which is only defined on the (supposedly) dense subset $K \cap Y$ of K where typically $Y = C^\infty(\overline{\Omega})$ or $Y = C(\overline{\Omega})$, cf. [53], which again leads to problem (4.0.1). At this point, we refer to Section 6.1 for details and a variety of examples.

4.2.3 Singular perturbation

In the context of variational inequalities, the closure property (4.0.1) also plays a role in the theory of singular perturbations. Let $A_1 : Y \to Y^*$ be a Lipschitz continuous and strongly monotone operator on the Hilbert space $(Y, \|\,.\,\|_Y)$ which is supposed to embed densely and continuously into X. For a sequence of regularization parameters (γ_n) with $\gamma_n \to +\infty$ consider the perturbed problems,

$$\text{find } u_n \in K \cap Y : \quad \langle (A + \tfrac{1}{\gamma_n} A_1)(u_n), v - u_n \rangle \geq \langle l, v - u_n \rangle \quad \forall\, v \in K \cap Y. \tag{4.2.5}$$

Provided $K \cap Y$ is closed in Y, observe that problem (4.2.5) admits a unique solution $u_n \in K \cap Y$. In this case, the appropriate limit problem is given by

$$\text{Find } u \in \overline{K \cap Y}^X : \quad \langle A(u), v - u \rangle \geq \langle l, v - u \rangle \quad \forall\, v \in \overline{K \cap Y}^X, \tag{4.2.6}$$

which corresponds to the initial variational inequality problem if the density property (4.0.1) holds true. In this case the sequence (u_n) converges strongly in X to the solution of (4.2.6). Here, the assumptions on A_1 may be alleviated. For details, [103, Section 4.9] may be consulted. In a similar fashion, the closure of $K \cap Y$ also plays an important role in the analysis and the design of algorithms for hyperbolic variational inequalities of first order in that it determines the limit of vanishing viscosity approaches; see, e.g., [104, p.160 ff.].

5 Density Results for Pointwise Constraint Sets in Sobolev Spaces

From the discussion of the preceding chapter it follows that density properties of the given convex constraint set K represent the basis for the consistency of various perturbation methods to solve associated variational inequality or optimization problems over K. In this chapter, we are primarily interested in density properties of pointwise constraint sets in Sobolev spaces which typically arise in many variational problems involving PDEs and comprise a myriad of applications such as elasto-plasticity, image restoration or Bingham flow problems, to mention only a few. They also naturally appear as a physical or budgetary restriction in optimal control problems.

5.1 Cone Constraints

Before elaborating on sets involving pointwise constraints on the norm, we deal with the simpler setting where the pointwise constraint represents a cone constraint. The subsequent results are implicitly used on many occasions in this text, especially in the context of contact constraints in elasto-plasticity. The most basic situation occurs when the metric projection is known to preserve the regularity. This setting applies to some unilateral constraints.

Lemma 5.1. *Let X be a Hilbert space and $Y \subset X$ a dense subset of X. Let $K \subset X$ be nonempty, convex and closed. If the projection mapping $\pi_K : X \to K$ is Y-invariant, i.e.,*

$$\pi_K(Y) \subset Y, \tag{5.1.1}$$

then $\overline{K \cap Y}^X = K$, i.e., $K \cap Y$ is dense in K with respect to the norm in X.

Proof. By density, there exists for $x \in K$ a sequence $(x_n) \subset Y$ with $x_n \to x$. Now, $\pi_K(x_n) \in Y$ for all n by assumption, such that

$$\|\pi_K(x_n) - x\|_X = \|\pi_K(x_n) - \pi_K(x)\|_X \leq \|x_n - x\|_X \to 0,$$

as $n \to \infty$.

□

Example 5.2. Consider the space $X = L^2(\Omega)$ for a Lipschitz domain $\Omega \subset \mathbb{R}^N$ and the dense subspace $Y = H^1(\Omega)$. Define $K = L^2_-(\Omega) := \{u \in L^2(\Omega) : u \leq 0 \text{ a.e. in } \Omega\}$ as the cone of nonpositive functions in $L^2(\Omega)$. Hence, the projection onto K is given by $\pi_K(u)(x) = \min(0, u(x))$, $x \in \Omega$, where $\pi_K(u) \in Y$ for all $u \in Y$. Thus it holds that

$$\overline{\{u \in H^1(\Omega) : u \leq 0 \text{ a.e. in } \Omega\}}^{L^2(\Omega)} = L^2_-(\Omega).$$

Example 5.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Consider the space $X = L^2(\Gamma)$ on a nonempty open subset $\Gamma \subset \partial\Omega$. Define $K = L^2_-(\Gamma) := \{z \in L^2(\Gamma) : z \leq 0 \text{ a.e. on } \Gamma\}$ as the cone of nonpositive functions in $L^2(\Gamma)$. The projection onto K is given by $\pi_K(z)(x) = \min(0, z(x))$, $x \in \Gamma$.

As a consequence of Lemma 1.2, it holds that $\pi_K(u) \in Y = H^{1/2}(\Gamma)$ for any $u \in Y$ which implies

$$\overline{\{z \in H^{1/2}(\Gamma) : z \leq 0 \text{ a.e. on } \Gamma\}}^{L^2(\Omega)} = L^2_-(\Omega). \tag{5.1.2}$$

In general, the Y-invariance (5.1.1) of the projection operator is not given. Therefore, density properties require alternative proof strategies. Consider, for instance, the following result which provides a stronger statement than Example 5.2. For Sobolev spaces on manifolds, we refer to Section 1.2.4.

Lemma 5.4. *Let Γ be a k-dimensional C^∞-submanifold of $\mathbb{R}^N$ ($k \leq N$) and consider $(\Gamma, \mathfrak{g})$, $\mathfrak{g} := \langle\, . \,, \,.\, \rangle_{\mathbb{R}^N}$, as a Riemannian manifold. Then the density property,*

$$\overline{L^2_-(\Gamma) \cap C^\infty_c(\Gamma)}^{L^2(\Gamma)} = L^2_-(\Gamma) \tag{5.1.3}$$

holds for $L^2_-(\Gamma) := \{u \in L^2(\Gamma) : u \leq 0 \text{ a.e. on } \Gamma\}$.

Proof. Let $u \in L^2_-(\Gamma)$. Since $C^\infty_c(\Gamma)$ is dense in $L^2(\Gamma)$ [56] there exists a sequence $(\varphi_k) \subset C^\infty_c(\Gamma)$, such that $\varphi_k \to u$ in $L^2(\Gamma)$. We further denote by

$$\psi_k \in C^{0,1}(\mathbb{R}) \cap C^\infty(\mathbb{R}), k \in \mathbb{N},$$

non-positive functions with uniformly bounded Lipschitz modules L_k, i.e. $\sup_k L_k < +\infty$, which satisfy

$$\psi_k(t) \to \min(0, t) \quad \forall\, t \in \mathbb{R}.$$

Such a sequence (ψ_k) can be easily constructed [56, Example 5.3]. Using the triangle inequality we infer

$$\|u - \psi_k(\varphi_k)\|_{L^2(\Gamma)} \leq \underbrace{\|\min(0, u) - \psi_k(u)\|_{L^2(\Gamma)}}_{\to 0} + \underbrace{\|\psi_k(u) - \psi_k(\varphi_k)\|_{L^2(\Gamma)}}_{\leq L_k \|u - \varphi_k\|_{L^2(\Gamma)}}$$

where the convergence of the left summand follows from the Dominated Convergence Theorem. This completes the proof. $\square$

If $(X, (\,.\,, \,.\,))$ is a Hilbert space, the above results may be employed to draw useful conclusions on the closure of $K \cap Y$ in the topology of the dual space. For the statement of the subsequent result, we denote by

$$K^\circ := \{x \in X : (x, v) \leq 0 \ \ \forall\, v \in K\},$$

the polar cone to a subset $K \subset X$ in X upon the identification $X \simeq X^*$ provided by the Riesz isomorphism.

Lemma 5.5. *Let $(Y, \|\,.\,\|_Y)$ be a reflexive Banach space which embeds densely and continuously into the Hilbert space $(X, (\,.\,, \,.\,))$ with embedding operator $\iota : Y \hookrightarrow X$ and adjoint $\iota^* : X \to Y^*$. Let $K \subset X$ be a nonempty convex cone. Provided the density property (4.0.1) is satisfied, it holds that*

$$\overline{\iota^*(K \cap Y)}^{Y^*} = (K^\circ \cap Y)^*,$$

where

$$(K^\circ \cap Y)^* := \{y^* \in Y^* : \langle y^*, y \rangle_{(Y^*, Y)} \leq 0 \ \ \forall\, y \in K^\circ \cap Y\},$$

i.e., $(K^\circ \cap Y)^*$ denotes the polar cone of $K^\circ \cap Y$ with respect to the pairing (Y^*, Y).

Proof. By the Bipolar Theorem [102], it holds that $\overline{\iota^*(K \cap Y)}^{Y^*} = (\iota^*(K \cap Y))^{**}$, where the bipolar cone $(\iota^*(K \cap Y))^{**}$ is defined by

$$(\iota^*(K \cap Y))^{**} = \{y^* \in Y^* : \langle y^*, y \rangle_{(Y^*,Y)} \leq 0 \ \ \forall\, y \in (\iota^*(K \cap Y))^*\}.$$

Here, we employ the identification $Y^{**} \simeq Y$ for the elements of $(\iota^*(K \cap Y))^* \subset Y^{**}$. For that reason it suffices to show that

$$\iota^*(K \cap Y)^* = K^\circ \cap Y,$$

and indeed, the definition of the polar cone implies that any element $y \in \iota^*(K \cap Y)^*$ is characterized by

$$\langle y, \iota^* \tilde{y} \rangle_{(Y,Y^*)} = (y, \tilde{y}) \leq 0 \quad \forall\, \tilde{y} \in K \cap Y.$$

With the density assumption one retrieves $(y, x) \leq 0$ for all $x \in K$ and thus $y \in K^\circ \cap Y$, which ends the proof. ☐

Example 5.6. By virtue of the density property (5.1.2), we obtain

$$\overline{\iota^*(\{z \in H^{1/2}(\Gamma) : z \leq 0 \text{ a.e. on } \Gamma\})}^{H^{-1/2}(\Gamma)} = H^{1/2}_+(\Gamma)^*,$$

where

$$H^{1/2}_+(\Gamma)^* = \{z^* \in H^{-1/2}(\Gamma) : \langle z^*, z \rangle \leq 0 \ \ \forall\, z \in H^{1/2}(\Gamma), z \geq 0 \text{ a.e. on } \Gamma\}$$

denotes the polar cone to the nonnegative functions in $H^{1/2}(\Gamma)$ and ι is the canonical embedding of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$.

5.2 Continuous Obstacles

From now on, we mainly focus on the case where $X = X(\Omega) \subset L^1(\Omega)^d$ is a usual (possibly vector-valued) Lebesgue or Sobolev space over a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and $Y = Y(\Omega)$ is a dense subspace of continuous or smooth vector fields. The constraint set K_Λ prescribes a pointwise bound on the norm of the function value, the gradient or the divergence, i.e.,

$$K_\Lambda(X(\Omega)) := \{w \in X(\Omega) : |\Lambda w(x)| \leq \alpha(x) \text{ a.e. in } \Omega\}, \tag{5.2.1}$$

with $\Lambda \in \{\mathrm{id}, \nabla, \mathrm{div}\}$. Here, $|\,.\,|$ designates an arbitrary norm on $\mathbb{R}^d$, $\mathbb{R}^{N\times d}$ or $\mathbb{R}$, respectively, and $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ is assumed to be a nonnegative Lebesgue measurable function. In the case $\Lambda = \mathrm{id}$, we simply write $K = K_{\mathrm{id}}$. The space of functions that are restrictions to Ω of smooth functions with compact support on $\mathbb{R}^N$ is denoted by $\mathcal{D}(\overline{\Omega})$, that is,

$$\mathcal{D}(\overline{\Omega}) = \{\varphi|_\Omega : \varphi \in C^\infty_c(\mathbb{R}^N)\}.$$

For real-valued uniformly continuous obstacles with

$$\operatorname*{ess\,inf}_{x\in\Omega} \alpha(x) > 0, \tag{5.2.2}$$

the following important result can be found in the recent paper [69].

Theorem 5.7. *Let $1 \leq p < +\infty$ and $\alpha \in C(\overline{\Omega})$ with (5.2.2). Then the following density result for*

$X(\Omega) \in \{L^p(\Omega;\mathbb{R}^d), W_0^{1,p}(\Omega;\mathbb{R}^d), H_0(\mathrm{div};\Omega)\}$, holds true:

$$\overline{K(C_c^\infty(\Omega)^d)}^{X(\Omega)} = K(X(\Omega)), \tag{5.2.3}$$

where $d = N$ if $X(\Omega) = H_0(\mathrm{div};\Omega)$. Moreover, it holds that

$$\overline{K_\Lambda(C_c^\infty(\Omega)^d)}^{X(\Omega)} = K_\Lambda(X(\Omega)),$$

for $\Lambda = \nabla, X(\Omega) = W_0^{1,p}(\Omega)^d$ and $\Lambda = \mathrm{div}, X(\Omega) = H_0(\mathrm{div})$, where $d = N$ in the latter case.

To analyze the case without homogeneous Dirichlet boundary conditions, a small modification of the approximating sequence constructed in [69] is sufficient in order to arrive at the following statement.

Theorem 5.8. *Let $1 \le p < +\infty$. Let $\alpha \in C(\overline{\Omega})$ fulfill (5.2.2). Then it holds that*

$$\overline{K(\mathcal{D}(\overline{\Omega})^d)}^{W^{1,p}(\Omega)^d} = K(W^{1,p}(\Omega)^d), \tag{5.2.4}$$

i.e., $K(\mathcal{D}(\overline{\Omega})^d)$ is dense in $K(W^{1,p}(\Omega)^d)$ with respect to the norm topology in $W^{1,p}(\Omega)^d$.

Proof. Let $w \in K(W^{1,p}(\Omega)^d)$. Since Ω is a bounded Lipschitz domain we may extend w to a function in $W^{1,p}(\mathbb{R}^N)^d$ using for each component the extension-by-reflection operator. The resulting operator $E : W^{1,p}(\Omega)^d \to W^{1,p}(\mathbb{R}^N)^d$ has the properties $Ew|_\Omega = w$ and $E \in \mathcal{L}(W^{1,p}(\Omega)^d, W^{1,p}(\mathbb{R}^N)^d)$. The extension is constructed as follows [4]: We first choose a partition of unity $\eta_j, j = 0, \dots J$,

$$\eta_j \in C_c^\infty(\Omega_j), \quad \eta_j \ge 0, \quad \sum_{j=0}^J \eta_j(x) = 1 \ \ \forall\, x \in \overline{\Omega},$$

subordinate to the covering

$$\overline{\Omega} \subset \bigcup_{j=0}^N \Omega_j, \tag{5.2.5}$$

where Ω_j, $j = 1, \dots, J$, are the sets given by the Lipschitz regularity of $\partial\Omega$ (Definition 1.1) supplemented by an open set Ω_0 with $\Omega_0 \Subset \Omega$ such that (5.2.5) holds true. The extension problem is localized writing

$$w = \sum_{j=0}^J \eta_j w. \tag{5.2.6}$$

Since η_0 has compact support in Ω, we have $\eta_0 w \in W_0^{1,p}(\Omega)^d$ and we define

$$E_0 : W_0^{1,p}(\Omega)^d \to W^{1,p}(\mathbb{R}^N)^d$$

as the componentwise extension-by-zero operator. With the notation from Definition 1.1, we further note that since $B_{r^j}(y^j)$ has a smooth boundary, the local parametrizations g^j can be extended to functions $\tilde{g}^j$ in $C^{k,\kappa}(\mathbb{R}^{n-1})$ [51, Lemma 6.36]. For $j \in \{1, \dots, J\}$, one observes that $\eta_j w \in W^{1,p}(\tilde{\Omega}_j)^d$, where $\tilde{\Omega}_j$ denotes the interior of the epigraph of $\tilde{g}^j$, i.e.,

$$\tilde{\Omega}_j = \{x \in \mathbb{R}^N : \tilde{g}^j(\tilde{x}^j) < x_N^j\}.$$

Recall that $\tilde{x}^j = [x_1^j, \ldots, x_{N-1}^j]$ where $x = [x_1^j, \ldots, x_N^j]$ are the coordinates of x in the j-th local coordinate system according to Definition 1.1. The functions $\eta_j w$ are then extended to $\mathbb{R}^N$ by reflection on the graph of $\tilde{g}_j$:

$$E_j w(x) := \begin{cases} w(x), & x \in \tilde{\Omega}_j, \\ w(\tilde{x}^j, 2\tilde{g}^j(\tilde{x}) - x_N^j), & \tilde{g}^j(\tilde{x}^j) > x_N^j, \end{cases}$$

such that $E_j w|_{\tilde{\Omega}_j} = w$. Moreover, it can be shown that $E_j \in \mathcal{L}(W^{1,p}(\tilde{\Omega}_j)^d, W^{1,p}(\mathbb{R}^N)^d)$, cf. [4]. Using (5.2.6), we set

$$Ew = \sum_{j=0}^{J} E_j(\eta_j w), \tag{5.2.7}$$

which represents the desired extension operator $E \in \mathcal{L}(W^{1,p}(\Omega)^d, W^{1,p}(\mathbb{R}^N)^d)$. From the definition of the operators E_j, it can be deduced that (5.2.7) also defines an extension operator $E_{C(\overline{\Omega})} : C(\overline{\Omega}) \to C(\mathbb{R}^N)$ which satisfies $(E_{C(\overline{\Omega})}\alpha)|_\Omega = \alpha$, and, since $E_j(\eta_j \alpha_j) \in C_c(\mathbb{R}^N)$, also $E_{C(\overline{\Omega})}\alpha \in C_c(\mathbb{R}^N)$. In addition,

$$|E_0(\eta_0 w)(x)| \leq E_0(\eta_0 \alpha)(x), \quad |E_j(\eta_j w)(x)| \leq E_j(\eta_j \alpha)(x), \; j = 1, \ldots J,$$

for a.e. $x \in \mathbb{R}^N$. Hence, we have

$$|Ew(x)| \leq \sum_{j=0}^{J} |E_j(\eta_j w)(x)| \leq \sum_{j=0}^{J} E_j(\eta_j \alpha)(x) = E_{C(\overline{\Omega})}\alpha(x), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{5.2.8}$$

For a sequence (ρ_n) of smooth mollifiers $\rho_n(x) = n^N \rho(nx)$ where

$$\rho \in \mathcal{D}(\mathbb{R}^N), \rho \geq 0, \rho(x) = 0 \text{ for } |x| \geq 1, \int_{\mathbb{R}^N} \rho(x)\, dx = 1, \tag{5.2.9}$$

we define the approximating sequence $S_n(w, \Omega)$ to w by

$$S_n(w, \Omega)(x) := (\rho_n * Ew)(x) = \int_{\mathbb{R}^N} Ew(y)\, \rho_n(x - y)\mathrm{d}y, \quad x \in \mathbb{R}^N. \tag{5.2.10}$$

It is well known that $S_n(w, \Omega)|_\Omega \to w$ in $W^{1,p}(\Omega)^d$ as $n \to \infty$ and, since Ew has compact support in $\mathbb{R}^N$, it holds that $S_n(w, \Omega) \in C_c^\infty(\mathbb{R}^N)^d$ and in particular $S_n(w, \Omega)|_\Omega \in \mathcal{D}(\overline{\Omega})^d$. In order to achieve feasibility, we use the scaling sequence

$$\beta_n := \left(1 + \frac{\sup_{x \in \mathbb{R}^N} |\alpha_n(x) - E_{C(\overline{\Omega})}\alpha(x)|}{\min_{x \in \overline{\Omega}} \alpha(x)}\right)^{-1} \in (0, 1],$$

where $\alpha_n(x) := ((E_{C(\overline{\Omega})}\alpha) * \rho_n)(x), \; x \in \mathbb{R}^N$. Since $E_{C(\overline{\Omega})}\alpha \in C_c(\mathbb{R}^N)$, (α_n) converges to $E_{C(\overline{\Omega})}\alpha$ uniformly in $\mathbb{R}^N$ and thus $\beta_n \to 1$ as $n \to \infty$. In addition, (5.2.8) together with (5.2.10) yields $|S_n(w, \Omega)| \leq \alpha_n(x)$ for $x \in \mathbb{R}^N$ and thus

$$\beta_n^{-1}\alpha(x) = \alpha(x) + \frac{\sup_{x \in \mathbb{R}^N} |\alpha_n(x) - E_{C(\overline{\Omega})}\alpha(x)|}{\min_{x \in \overline{\Omega}} \alpha(x)} \alpha(x) \geq \alpha_n(x) \geq |S_n(w, \Omega)|, \tag{5.2.11}$$

for all $x \in \Omega$. Consequently, $\beta_n S_n(w, \Omega) \in K(\mathcal{D}(\overline{\Omega})^d)$ and $\beta_n S_n(w, \Omega) \to w$ in $W^{1,p}(\Omega)^d$ which accomplishes the proof. □

Remark 5.9. In [69], an additional reparametrization appears in the definition of the approximating sequence (5.2.10) in order to safeguard the homogeneous Dirichlet boundary condition. This is not necessary in the context of Theorem 5.8.

5.3 Discontinuous Obstacles

5.3.1 A counterexample for obstacles in Sobolev spaces

Note that Theorem 5.8 requires continuous obstacles. In some applications, such as in the regularization and discretization of elasto-plasticity or image restoration problems, it may be useful to consider obstacles which are not continuous. Under such circumstances, the results of this section show that density properties of the type (5.2.3) or (5.2.4) cannot be expected if the obstacle is just a Sobolev function.

To begin with, the following Lemma generalizes the construction in [48, Example 4, p.247].

Lemma 5.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\{x_k : k \in \mathbb{N}\} \subset \Omega$ a countable dense subset, i.e.,*

$$\overline{\{x_k : k \in \mathbb{N}\}} = \overline{\Omega}.$$

Further let $1 \leq p < +\infty$. Then it holds that for any $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and any absolutely convergent series $\sum_{k\in\mathbb{N}} a_k, a_k \in \mathbb{R}$, the function g defined by

$$g(x) = \sum_{k=1}^{\infty} a_k u(x - x_k), \tag{5.3.1}$$

is well-defined and belongs to $W^{1,p}(\Omega)$.

Proof. We write $\Omega - \{x_k\} := \{x - x_k : x \in \Omega\}$. Since Ω is bounded, there is an open set $\Omega_0 \Subset \mathbb{R}^N$ which fulfills $\bigcup_{k\in\mathbb{N}}(\Omega - \{x_k\}) \subset \Omega_0$. To prove the convergence of the series $\sum_{k=1}^{\infty} a_k u(\,.\, - x_k)$ in $W^{1,p}(\Omega)$, we first note that $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ implies that $\|u(\,.\, - x_k)\|_{W^{1,p}(\Omega)}$ is bounded uniformly in k:

$$\|u(\,.\, - x_k)\|^p_{L^p(\Omega)} = \int_\Omega |u(x - x_k)|^p \, dx = \int_{\Omega - \{x_k\}} |u(x)|^p \, dx \leq \int_{\Omega_0} |u(x)|^p \, dx.$$

The same argument is valid for $\partial_i(u(\,.\, - x_k)) = \partial_i u(\,.\, - x_k)$. Therefore one obtains

$$\sum_{k=1}^{n} \|a_k u(\,.\, - x_k)\|_{W^{1,p}(\Omega)} \leq \|u\|_{W^{1,p}(\Omega_0)} \sum_{k=1}^{n} |a_k|,$$

which means that the series $\sum_{k=1}^{\infty} a_k u(\,.\, - x_k)$ is absolutely convergent and, since $W^{1,p}(\Omega)$ is a Banach space, also convergent in $W^{1,p}(\Omega)$. $\square$

Remark 5.11. Since any absolutely convergent series in $L^p(\Omega)$ is also converging pointwise a.e. to its $L^p(\Omega)-$limit function [29, Theorem 2.9], the function g defined in (5.3.1) is also the pointwise (a.e.) limit of the series $\sum_{k=1}^{\infty} a_k u(x - x_k)$.

We are now ready to construct the counterexample. Without loss of generality, assume $0 \in \Omega \subset \mathbb{R}^N$ with $N \geq 2$ and denote by

$$B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x - y|_2 \leq \varepsilon\},$$

the open ball with radius $\varepsilon > 0$ and center $x \in \mathbb{R}^N$ with respect to the Euclidean norm $|\,.\,|_2$ in $\mathbb{R}^N$. Let $\{x_k : k \in \mathbb{N}\}$ be a countable dense subset, i.e.,

$$\overline{\{x_k : k \in \mathbb{N}\}} = \overline{\Omega},$$

and $r > 0$ such that $B_r(0) \subset \Omega$. Consider the function

$$\varphi(x) := \tilde{\varphi}(x) \cdot \ln(\ln(c|x|_2^{-1})), \quad c \geq er \ \text{ fixed}, \tag{5.3.2}$$

where $\tilde{\varphi} \in C_c^\infty(B_r(0))$ is a smooth cut-off function with $\tilde{\varphi}(x) \geq 0$ for all $x \in B_r(0)$ and $\tilde{\varphi} \equiv 1$ on $B_{r/2}(0)$. We note that φ is nonnegative with a singularity at the origin and it belongs to $W^{1,N}(\mathbb{R}^N)$, cf. [1, Example 4.43]. Further set

$$g(x) := \sum_{k=1}^{\infty} k^{-2} \varphi(x - x_k), \tag{5.3.3}$$

and note that $g \in W^{1,N}(\Omega)$ with g being unbounded at each x_k, see Lemma 5.10. Further take a function $\phi \in C^1(\mathbb{R})$ with $0 \leq \phi(t) < 1$, $\phi(t) \to 1$ for $t \to +\infty$ and $|\phi'(t)| \leq c$ for all $t \in \mathbb{R}$. Then the obstacle

$$\alpha := 2 - \phi \circ g \tag{5.3.4}$$

belongs to $W^{1,N}(\Omega)$ (see, e.g., [80, Lemma A.3]). Notice also that α is bounded away from zero and it is basically equal to 1 on the dense set $\{x_k : k \in \mathbb{N}\}$. Consequently, any continuous function w with $w \leq \alpha$ a.e. in Ω fulfills $w \leq 1$ on Ω:

Assume the latter is not the case. Then, there exist $k_0 \in \mathbb{N}$ as well as $\mu > 0, \delta > 0$ such that

$$w(x) \geq 1 + \mu \quad \forall\, x \in B_\delta(x_{k_0}). \tag{5.3.5}$$

Let $R > 0$ be such that $\phi(t) \geq 1 - \frac{\mu}{2}$ for all $t \geq R$. By continuity, there also exists $\delta' > 0$ such that $\varphi(x - x_{k_0}) \geq Rk_0^2$ a.e. in $B_{\delta'}(x_{k_0})$ such that

$$g(x) \geq k_0^{-2}\varphi(x - x_{k_0}) \geq R, \quad \text{ a.e. } x \in B_{\delta'}(x_{k_0}),$$

which implies

$$w(x) \leq \alpha(x) = 2 - \phi(g(x)) \leq 1 + \tfrac{\mu}{2} \quad \text{ a.e. } x \in B_{\delta'}(x_{k_0}),$$

contradicting (5.3.5). Hence, any sequence of continuous functions approximating α from below is bounded above by 1. However, as $\alpha(x) > 1$ for a.e. $x \in \Omega$ by definition, and convergence in the norm topology of $L^p(\Omega)$ implies convergence pointwise a.e. (along a certain subsequence), we obtain

$$\alpha \in K(L^p(\Omega)) \setminus \overline{K(C(\Omega) \cap L^p(\Omega))}^{L^p(\Omega)}, \tag{5.3.6}$$

for any $1 \leq p \leq +\infty$ and

$$\alpha \in K(W^{1,p}(\Omega)) \setminus \overline{K(C(\Omega) \cap W^{1,p}(\Omega))}^{W^{1,p}(\Omega)}, \tag{5.3.7}$$

for all $p \leq N$ where α is defined by (5.3.4). At this point, we recall that

$$K(X(\Omega)) = \{w \in X(\Omega) : |w(x)| \leq \alpha(x) \text{ a.e. on } \Omega\}, \quad X(\Omega) \subset L^1(\Omega),$$

in accordance with the notation from (5.2.1).

Remark 5.12 (Complements on the counterexample)**.** An interesting point in the preceding counterexample is the structure of the set of singularities $\mathcal{S}$ where $g(x)$ is not well-defined as a real number by the infinite sum (5.3.3) if φ from (5.3.2) is understood as the function (and not its equivalence class), which is well-defined and continuous on $\Omega \setminus \{0\}$. Extending φ to Ω by setting $\varphi(0) = +\infty$, we obtain $g(x_k) = +\infty$ for all $k \in \mathbb{N}$ and, understanding $g : \Omega \to \mathbb{R} \cup \{+\infty\}$ as an extended real-valued function, we arrive at the following definition:

$$\mathcal{S} := \{x \in \Omega : \; g(x) = +\infty \text{ with } g(x) \text{ defined by (5.3.3) where } \varphi(0) = +\infty\}.$$

By definition, the set $\{x_k : k \in \mathbb{N}\}$ is contained in $\mathcal{S}$. Besides, it is certain that $\mathcal{S}$, and thus the points where the infinite series does not converge, must have measure zero. If not, then the limit function g which is also the pointwise a.e. limit, see Remark 5.11, would be equal to $+\infty$ on a set of positive measure, which contradicts the fact that $g \in H^1(\Omega)$. On the other hand, $\mathcal{S}$ is in a certain sense much "bigger" than $\{x_k : k \in \mathbb{N}\}$. A first indication of this fact is the observation that the set $\{x_k : k \in \mathbb{N}\}$ is strictly contained in $\mathcal{S}$. Otherwise, we could consider the concrete representative of α from (5.3.4) given by

$$\alpha(x) = \begin{cases} 1, & \text{on } \{x_k : k \in \mathbb{N}\} \\ 2 - \phi(g(x)), & \text{on } \Omega \setminus \{x_k : k \in \mathbb{N}\}. \end{cases}$$

This well-defined function $\alpha : \Omega \to \mathbb{R}$ is continuous on $\{x_k : k \in \mathbb{N}\}$ and discontinuous on $\Omega \setminus \{x_k : k \in \mathbb{N}\}$ by the density property of $\{x_k : k \in \mathbb{N}\}$ in Ω and the fact that $\alpha(x) > 1$ for all $x \notin \{x_k : k \in \mathbb{N}\}$. However, this is a contradiction, as the following standard topological argument shows: It is well known that the set of discontinuities of a function defined on a metric space is an F_σ-set [9, Theorem 8.2.6], i.e., it can be expressed as the countable union of closed subsets. Applying this result to the above representative $\alpha : \Omega \to \mathbb{R}$, one obtains a sequence of closed sets $F_i \subset \mathbb{R}^N$ such that

$$\Omega \setminus \{x_k : k \in \mathbb{N}\} = \bigcup_{i \in \mathbb{N}} (F_i \cap \Omega) = \bigcup_{i \in \mathbb{N}} \left(F_i \cap \bigcup_{j \in \mathbb{N}} G_j \right) = \bigcup_{i,j \in \mathbb{N}} (F_i \cap G_j), \tag{5.3.8}$$

where the second equality in (5.3.8) makes use of suitable sets $G_j \subset \mathbb{R}^N$ which are closed in $\mathbb{R}^N$ and fulfill $\Omega = \bigcup_{j \in \mathbb{N}} G_j$. Also note that $F_i \cap G_j$ is closed and nowhere dense in $\mathbb{R}^N$. Finally, we obtain the countable decomposition

$$\Omega = \bigcup_{k \in \mathbb{N}} \{x_k\} \cup \bigcup_{i,j \in \mathbb{N}} (F_i \cap G_j)$$

of Ω into nowhere dense subsets which is inconsistent with the Baire category theorem. Consequently, it holds that $\{x_k : k \in \mathbb{N}\} \subsetneq \mathcal{S}$.

The set $\mathcal{S}$ even turns out to be nonmeager. At this point we remark that in the literature related to the Baire category theorem, a nonmeager set is often called of second category, i.e, it cannot be expressed as the countable union of nowhere dense subsets of $\mathbb{R}^N$. To prove this property, we define the nested sets

$$\mathcal{S}_n := \bigcup_{k \in \mathbb{N}} B_{r_{nk}}(x_k) \cap \Omega, \quad r_{nk} := ce^{-e^{nk^2}},$$

which consist of the union of open balls with diminishing radius around the points x_k. It can be verified that

$$\bigcap_{n\in\mathbb{N}} \mathcal{S}_n \subset \mathcal{S}.$$

To show this, let $x \in \bigcap_{n\in\mathbb{N}} \mathcal{S}_n$ and $n \in \mathbb{N}$ arbitrary. By definition, there exists an index k_0 with $x \in B_{r_{nk_0}}(x_{k_0}) \cap \Omega$. Hence,

$$g(x) = \sum_{k=1}^{\infty} k^{-2}\varphi(x - x_k) \geq k_0^{-2} \ln\ln(c|x - x_{k_0}|_2^{-1}) \geq n.$$

Letting $n \to \infty$ yields $g(x) = +\infty$ and thus $x \in \mathcal{S}$. On the other hand, we observe that any complement $\mathcal{S}_n^c$ is closed and a similar argument as in (5.3.8) shows that

$$\left(\bigcap_{n\in\mathbb{N}} \mathcal{S}_n\right)^c \cap \Omega = \bigcup_{n\in\mathbb{N}} (\mathcal{S}_n^c \cap \Omega) = \bigcup_{j,n\in\mathbb{N}} (\mathcal{S}_n^c \cap G_j).$$

Since all sets $\mathcal{S}_n$ contain the dense set $\{x_k : k \in \mathbb{N}\}$, $\mathcal{S}_n^c \cap G_j$ also has empty interior. Therefore the complement of $\bigcap_{n\in\mathbb{N}} \mathcal{S}_n$ in Ω is meager, or, in other words, of first category. The Baire category theorem implies that $\bigcap_{n\in\mathbb{N}} \mathcal{S}_n$, and thus $\mathcal{S}$, must be nonmeager. To summarize, the set $\mathcal{S}$ of singularities of g has the following properties:

- for $x \in \Omega$, $\sum_{k=1}^{\infty} k^{-2}\varphi(x - x_k)$ diverges, if and only if, $x \in \mathcal{S}$,
- $\lambda(\mathcal{S}) = 0$,
- $\mathcal{S}$ is nonmeager (of second category),
- $\{x_k : k \in \mathbb{N}\} \subsetneq \bigcap_{n\in\mathbb{N}} \mathcal{S}_n \subset \mathcal{S}$.

Remark 5.13 (Erratum)**.** The preceding counterexample shows that the density property

$$\overline{K(C_c^\infty(\Omega)^d)}^{L^2(\Omega)^d} = K(L^2(\Omega)^d)$$

which is stated in [72, Lemma B.3] does not hold for arbitrary obstacles $\alpha \in L^2(\Omega)$ which are bounded away from zero unless further conditions on α are imposed, cf. below. Using the notation of the latter reference, the proof of that Lemma fails in the case where the subsets

$$K_j^\delta := \{x \in K_j : \operatorname{dist}(x, \Omega \setminus K_j) < \delta\},$$

do not fulfill

$$\lambda(K_j^\delta) \to 0 \quad \text{as } \delta \to 0. \tag{5.3.9}$$

Here, $K_j = K_j(\delta) \subset \Omega$ denotes a compact set such that the restriction to K_j of the approximated function $w \in K(L^2(\Omega)^d)$ is continuous and $\lambda(\Omega \setminus K_j) < \delta$. The existence of K_j is ensured by Lusin's theorem. Note that (5.3.9) fails, for example, if $\mathring{K}_j = \emptyset$. In this situation, the continuous cut-off function

$$w_{\delta,j}(x) := \tfrac{\min(\delta, \operatorname{dist}(x, \Omega\setminus K_j))}{\delta} w_j,$$

where w_j are the components of w, simply vanishes everywhere on Ω and the suggested approximating sequence defined in the proof of [72, Lemma B.3] does not fulfill the desired approximation property. Observe that this is the case in the above counterexample, i.e., for $d = 1$ and $w = \alpha$, as the respective K_j from Lusin's theorem cannot contain interior points since no representative of α is continuous on an open subset of Ω. However, the results of the subsequent sections in this chapter indicate that the density property remains valid for a large class of discontinuous obstacles.

We summarize the preceding results on general discontinuous obstacles in the following statement.

Theorem 5.14. *The following density results hold true:*

(i) Let $N \geq 2$ and $1 \leq p \leq +\infty$. Then there exists an obstacle $\alpha \in W^{1,N}(\Omega) \cap L^\infty(\Omega)$ satisfying (5.2.2) such that,

$$\overline{K(C(\Omega) \cap L^p(\Omega))}^{L^p(\Omega)} \subsetneq K(L^p(\Omega)),$$

the inclusion being strict.

(ii) Let $N \geq 2$ and $1 \leq p \leq N$. Then there exists an obstacle $\alpha \in W^{1,N}(\Omega) \cap L^\infty(\Omega)$ satisfying (5.2.2) such that,

$$\overline{K(C(\Omega) \cap W^{1,p}(\Omega))}^{W^{1,p}(\Omega)} \subsetneq K(W^{1,p}(\Omega)),$$

the inclusion being strict.

(iii) Let $N < p < +\infty$ or $p = N = 1$. For any measurable obstacle function $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ which satisfies (5.2.2), it holds that

$$\overline{K(\mathcal{D}(\overline{\Omega})^d)}^{W^{1,p}(\Omega)^d} = K(W^{1,p}(\Omega)^d).$$

Proof. We only prove assertion (iii) since (i) and (ii) follow immediately from (5.3.6) and (5.3.7). As a consequence of the Sobolev Imbedding Theorem, any $w \in K(W^{1,p}(\Omega)^d)$ is contained in $C(\overline{\Omega})^d$. Let $w \in K(W^{1,p}(\Omega)^d)$. Setting

$$\hat{\alpha}(x) = \max(|w(x)|, \operatorname*{ess\,inf}_{x \in \Omega} \alpha(x)),$$

it follows that $|w(x)| \leq \hat{\alpha}(x)$ a.e. in Ω. Since $\hat{\alpha} \in C(\overline{\Omega})$ and (5.2.2) holds with $\hat{\alpha}$ instead of α, we may invoke Theorem 5.8 to infer that there exists a sequence $(w_n) \subset \mathcal{D}(\overline{\Omega})^d$ with $w_n \to w$ in $W^{1,p}(\Omega)^d$ and $|w_n(x)| \leq \hat{\alpha}(x) \leq \alpha(x)$ such that $w_n \in K(\mathcal{D}(\overline{\Omega})^d)$ which accomplishes the proof. $\square$

Adapting the approximation sequence in a suitable way, one may also infer the corresponding statements for Sobolev spaces incorporating homogeneous Dirichlet boundary conditions.

Corollary 5.15. *The following density results hold true:*

(i) Let $N \geq 2$ and $1 \leq p \leq N$. Then there exists an obstacle $\alpha \in W^{1,N}(\Omega) \cap L^\infty(\Omega)$ satisfying (5.2.2) such that,

$$\overline{K(C(\Omega) \cap W_0^{1,p}(\Omega))}^{W_0^{1,p}(\Omega)} \subsetneq K(W_0^{1,p}(\Omega)),$$

the inclusion being strict.

(ii) Let $N < p < +\infty$ or $p = N = 1$. For any measurable obstacle function $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ which satisfies (5.2.2), it holds that

$$\overline{K(C_c^\infty(\Omega)^d)}^{W_0^{1,p}(\Omega)^d} = K(W_0^{1,p}(\Omega)^d).$$

Proof. (i) For the upper bound α from (5.3.4), the set $K(W_0^{1,p}(\Omega))$ contains $\alpha \cdot \hat{\varphi}$ for a suitable cut-off function $\hat{\varphi} \in C_c^\infty(\Omega), 0 \le \hat{\varphi} \le 1$ with $\hat{\varphi} \equiv 1$ except on a small neighborhood of $\partial\Omega$. The assertion now follows directly from the discussion preceding Remark 5.12.

(ii) Taking account of Theorem 5.7, statement (ii) can be proven analogously to Theorem 5.14(iii). At this point we want to give an alternative proof of (ii) which serves to clarify the proof of Theorem 5.7 and its dependence on the regularity of Ω. Let $w \in K(W_0^{1,p}(\Omega)^d)$. As a consequence of the Sobolev Imbedding theorem, w is contained in $C_0(\Omega)^d$.

Step 1: On star-shaped bounded Lipschitz domains, we consider the approximating sequence from [69, Lemma 2] which preserves zero boundary conditions: After a suitable translation, we may assume that Ω is star-shaped with respect to the origin. For $\varepsilon > 0$ let $\lambda \in (0,1)$ such that $w_\lambda := \lambda w$ fulfills $\|w_\lambda - w\|_{W^{1,p}(\Omega)^d} < \varepsilon/2$. Define $\delta_\lambda := (1-\lambda)\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$ and observe that

$$|w_\lambda(x)| \le \alpha(x) - \delta_\lambda \quad \text{for a.e. } x \in \Omega. \tag{5.3.10}$$

Induced by a sequence of monotonically increasing parameters (θ_n) with $\theta_n \uparrow 1$, we consider the following approximating sequence to w_λ,

$$S_n(w_\lambda, \Omega) := \rho_n * \tilde{w}_\lambda^{\theta_n}. \tag{5.3.11}$$

Here, ρ_n is defined as in (5.2.9) and $w_\lambda^{\theta_n}(x) := \tilde{w}_\lambda(\frac{x}{\theta_n})$ where $\tilde{w}_\lambda$ denotes the extension by zero of w_λ to $\mathbb{R}^N$. This implies $S_n(w_\lambda, \Omega) \in C_c^\infty(\Omega)^d$ for sufficiently large n. Furthermore, [69, Lemma 2] entails that

$$S_n(w_\lambda, \Omega) \to w_\lambda \text{ in } W_0^{1,p}(\Omega)^d, \quad S_n(w_\lambda, \Omega) \to w_\lambda \text{ in } C_0(\Omega)^d, \tag{5.3.12}$$

as $n \to \infty$. Moreover, let $n = n(\lambda)$ be sufficiently large to yield

$$\|S_n(w_\lambda, \Omega) - w_\lambda\|_{W^{1,p}(\Omega)^d} < \varepsilon/2, \quad \|S_n(w_\lambda, \Omega) - w_\lambda\|_{C(\overline{\Omega})^d} < \delta_\lambda.$$

Consequently, we obtain $S_n(w_\lambda, \Omega) \in K(C_c^\infty(\Omega)^d)$ from (5.3.10) and

$$\|S_n(w_\lambda, \Omega) - w\|_{W^{1,p}(\Omega)^d} < \varepsilon$$

by the triangle inequality.

Step 2: For a general bounded Lipschitz domain Ω, we cover $\partial\Omega$ by open sets $O_j, j = 1, \dots J$, such that $\Omega \cap O_j$ is an open set with Lipschitz boundary which is star-shaped with respect to one of its points, see [120]. Supplementing O_j by an open set $O_0 \Subset \Omega$ such that

$$\overline{\Omega} \subset \bigcup_{j=0}^{J} O_j, \tag{5.3.13}$$

we consider a partition of unity $(\eta_j), \eta_j \in C_c^\infty(O_j)$, subordinate to (5.3.13) and set

$$w_{\lambda,n} := \rho_n * (\widetilde{\eta_0 w_\lambda}) + \sum_{j=1}^{J} S_n(\eta_j w_\lambda, \Omega \cap O_j),$$

where S_n is well-defined by (5.3.11) since $\eta_j w_\lambda \in W_0^{1,p}(\Omega \cap O_j)^d$ and $\Omega \cap O_j$ is star-shaped. Using (5.3.12), this implies

$$w_{\lambda,n} \to w_\lambda \text{ in } W_0^{1,p}(\Omega)^d, \quad w_{\lambda,n} \to w_\lambda \text{ in } C_0(\Omega)^d,$$

as $n \to +\infty$. Employing the estimate (5.3.10), we obtain $w_{\lambda,n} \in K(C_c^\infty(\Omega)^d)$ for sufficiently large n and an $\varepsilon/2$-argument ends the proof. □

5.3.2 Lower semicontinuous obstacles and L^p-spaces

Up to now, only the uniform continuity of the obstacle guarantees the validity of density properties of the type (5.2.3) or (5.2.4). The following idea allows to enlarge the space of obstacles which are compatible with the density property: If a set can be approximated in an appropriate way by a sequence of sets for which the density result holds, then the density property is transferred to the limit set. Employing this strategy directly leads to the following class of obstacles fulfilling a kind of generalized lower semicontinuity requirement.

Definition 5.16. The set of functions $\mathbb{LC}(\Omega)$ comprises all measurable functions $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ for which there exists a sequence of functions $\alpha_n : \Omega \to \mathbb{R}$ with

$$\alpha_n \in C(\overline{\Omega}), \quad \inf_{x \in \Omega} \alpha_n(x) > 0, \quad \alpha_n \leq \alpha, \tag{5.3.14}$$

for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \alpha_n(x) \to \alpha(x)$ for a.e. $x \in \Omega$.

Theorem 5.17. *Let $1 \leq p < +\infty$. If $\alpha \in \mathbb{LC}(\Omega)$, then it holds that*

$$\overline{K(C_c^\infty(\Omega)^d)}^{L^p(\Omega)^d} = K(L^p(\Omega)^d)$$

Proof. Let $w \in K(L^p(\Omega)^d)$ for $\alpha \in \mathbb{LC}(\Omega)$. For a sequence (α_n) given by Definition 5.16 consider the functions

$$w_n(x) := \min\{|w(x)|, \alpha_n(x)\} \tfrac{w(x)}{|w(x)|},$$

where it is understood that $w_n(x) := 0$ if $w(x)$=0. It follows from Lebesgue's theorem on dominated convergence that $w_n \to w$ in $L^p(\Omega)^d$. Further observe that $w_n \in K_n(L^p(\Omega)^d)$ where

$$K_n(X(\Omega)) = \{w \in X(\Omega) : |w(x)| \leq \alpha_n(x) \text{ a.e. on } \Omega\}.$$

By the properties of α_n, w_n can be approximated by smooth functions $\tilde{w}_n \in K_n(C_c^\infty(\Omega)^d)$ with respect to the norm in $L^p(\Omega)^d$ according to (5.2.3). Since $\alpha_n \leq \alpha$ for all $n \in \mathbb{N}$, we also have $\tilde{w}_n \in K(C_c^\infty(\Omega)^d)$ and an $\varepsilon/2$-argument completes the proof. □

Under these assumptions, the statement of [72, Lemma B.3] holds, too. We proceed by considering some important special cases.

Corollary 5.18. *Let $1 \leq p < +\infty$. Let $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and fulfill* (5.2.2). *Then it holds that*

$$\overline{K(C_c^\infty(\Omega)^d)}^{L^p(\Omega)^d} = K(L^p(\Omega)^d).$$

Proof. Let $w \in K(L^p(\Omega)^d)$. Without loss of generality, we may assume that α is proper and that

$$\inf_{x \in \Omega} \alpha(x) > 0.$$

Denote by $\tilde{\alpha}$ the extension by zero of α, i.e.,

$$\tilde{\alpha} = \alpha, \text{ on } \Omega, \quad \tilde{\alpha} = 0 \text{ on } \mathbb{R}^N \setminus \Omega,$$

which is l.s.c. on $\mathbb{R}^N$. The Lipschitz regularization of $\tilde{\alpha}$,

$$\alpha_n(x) = \inf_{y \in \mathbb{R}^N} \{\tilde{\alpha}(y) + n\|x - y\|\},$$

is known to yield functions $\alpha_n \in C^{0,1}(\mathbb{R}^N)$ with $\inf_{x\in\Omega} \alpha_n(x) \geq \inf_{x\in\Omega} \alpha(x) > 0$ and $\alpha_n(x) \uparrow \alpha(x)$ for all $x \in \mathbb{R}^N$. Hence $\alpha \in \mathbb{LC}(\Omega)$ and Theorem 5.17 applies. □

For instance, an important class of obstacles are the piecewise continuous functions: Suppose there exists a partition of the bounded Lipschitz domain Ω into open subsets $\Omega_l \subset \Omega$ with Lipschitz boundary such that

$$\overline{\Omega} = \cup_{l=1}^{L} \overline{\Omega}_l, \ \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \quad \alpha|_{\Omega_l} \in C(\overline{\Omega}_l). \tag{5.3.15}$$

The preceding theorem shows that for obstacles of this class the density result in the norm topology of the L^p-spaces holds true, provided the obstacle is bounded away from zero. Additionally, constraints that are only imposed on a regular subset of Ω may also be handled.

Corollary 5.19. *Let $1 \leq p < +\infty$. Let $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ fulfill (5.2.2) and one of the following additional assumptions.*

(i) α is piecewise continuous in the sense of (5.3.15),

(ii) α is lower semicontinuous on a Lipschitz domain $\Omega_0 \subset \Omega$ and $\alpha = +\infty$ on $\Omega \setminus \overline{\Omega_0}$.

Then the density property

$$\overline{K(C_c^\infty(\Omega)^d)}^{L^p(\Omega)^d} = K(L^p(\Omega)^d)$$

is satisfied.

Proof. It suffices to observe that in both cases, α may be modified on a subset of measure zero to be l.s.c. on Ω. □

5.3.3 Lower semicontinuous obstacles and Sobolev spaces

Conditions on the obstacle α so that the density results for Sobolev spaces hold can be relaxed from assuming that $\alpha \in C(\overline{\Omega})$ to lower regularity requirements with the aid of Mosco-convergence of closed and convex sets. The following definition goes back to [95].

Definition 5.20 (Mosco-convergence)**.** Let X be a reflexive Banach space and (K_n) a sequence of closed convex subsets with $K_n \subset X$. Then $K_n \xrightarrow{M} K$ as $n \to +\infty$, i.e., (K_n) is said to Mosco-converge to the set $K \subset X$, if and only if,

$$K \supset \{v \in X : (\ \exists\, (v_k) \subset X : v_k \in K_{n_k}\ \forall\, k \in \mathbb{N}, v_k \rightharpoonup v)\}, \tag{M1}$$

$$K \subset \{v \in X : (\ \exists\, (v_n) \subset X,\ \ \exists\, N \in \mathbb{N} : v_n \in K_n\ \forall\, n \geq N, v_n \to v)\}. \tag{M2}$$

Note that the Mosco-convergence of (K_n) to K is equivalent to the Mosco-convergence or, equivalently, the sequential weak-strong Γ-convergence, of the corresponding indicator functions, cf. Section 2.4. We further define the following unilateral constraint sets for measurable functions

$\alpha_n, \alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$:

$$\begin{aligned} K_n^-(X(\Omega)) &:= \{w \in X(\Omega) : w(x) \geq -\alpha_n \text{ a.e. in } \Omega\}, \\ K_n^+(X(\Omega)) &:= \{w \in X(\Omega) : w(x) \leq \alpha_n \text{ a.e. in } \Omega\}, \\ K_-(X(\Omega)) &:= \{w \in X(\Omega) : w(x) \geq -\alpha \text{ a.e. in } \Omega\}, \\ K_+(X(\Omega)) &:= \{w \in X(\Omega) : w(x) \leq \alpha \text{ a.e. in } \Omega\}. \end{aligned}$$

From the theory of varying obstacle problems, i.e. $X(\Omega) = W_0^{1,p}(\Omega)$, it is known that the Mosco-convergence of the unilateral sets,

$$K_n^{\pm}(W_0^{1,p}(\Omega)) \xrightarrow{M} K^{\pm}(W_0^{1,p}(\Omega)) \tag{5.3.16}$$

is characterized by the convergence of the capacities of the level sets of α_n, see [36]. In order to use the more tractable sufficient conditions for (5.3.16) from Boccardo and Murat [19, p.87], we define the following class of functions fulfilling a generalized lower semicontinuity condition in the space $W^q(\Omega)$.

Definition 5.21. For $q \geq 1$ we denote by $\mathbb{W}^q(\Omega)$ the set of functions $\alpha \in W^{1,q}(\Omega)$ for which there exists a sequence of functions (α_n) with α_n satisfying (5.2.2), $\alpha_n \leq \alpha$ a.e. in Ω and $\alpha_n \in C(\overline{\Omega}) \cap W^{1,q}(\Omega)$ for all $n \in \mathbb{N}$ such that $\alpha_n \rightharpoonup \alpha$ in $W^{1,q}(\Omega)$.

Note that the class $\mathbb{W}^q(\Omega)$ is strictly contained in $W^{1,q}(\Omega) \cap \mathbb{LC}(\Omega)$. Additionally, if the sequence (α_n) is non-decreasing, then the obstacle α is lower semicontinuous for being the pointwise limit of a non-decreasing sequence of continuous functions: In fact, $W^{1,q}(\Omega)$ embeds compactly in $L^1(\Omega)$ and hence $\alpha_{n_j}(x) \to \alpha(x)$ a.e. in Ω for $j \to \infty$ for some subsequence (α_{n_j}) (here we consider α as an extended-real valued function). However, the functions in $\mathbb{W}^q$ are not necessarily continuous: Let $N > 1$, $\Omega = B_r(0)$ and

$$\alpha(x) = \ln(\ln(c|x|^{-1})), \quad x \in \Omega,\ c \geq er. \tag{5.3.17}$$

It follows that $\alpha \in W^{1,q}(\Omega)$ for all $q \leq N$ but $\alpha \notin C(\overline{\Omega})$, cf. (5.3.2). The sequence (α_n) defined as $\alpha_n(x) = \min(\alpha(x), n)$ for $n \in \mathbb{N}$ satisfies the requirements of the definition of $\mathbb{W}^q(\Omega)$.

In order to be able to invoke the above results for the unilateral situation, we consider the case where $K(X(\Omega))$ is defined by the maximum norm, i.e.,

$$K(X(\Omega); |\,.\,|_\infty) := \{w \in X(\Omega) : |w(x)|_\infty \leq \alpha(x) \text{ a.e. in } \Omega\}. \tag{5.3.18}$$

The density result involving the class $\mathbb{W}^q(\Omega)$ for $q \geq 1$ can now be established.

Theorem 5.22. *Let $1 < p < \infty$ and $\alpha \in \mathbb{W}^q(\Omega)$ with $p < q < +\infty$. Then it holds that*

$$\overline{K(C_c^\infty(\Omega)^d; |\,.\,|_\infty)}^{W_0^{1,p}(\Omega)^d} = K(W_0^{1,p}(\Omega)^d; |\,.\,|_\infty), \tag{5.3.19}$$

where $K(X(\Omega)^d, |\,.\,|_\infty)$ is defined in (5.3.18).

Proof. Without loss of generality, consider the one-dimensional case $d = 1$. Let $w \in K(W_0^{1,p}(\Omega); |\,.\,|_\infty)$. Since $\alpha_n \rightharpoonup \alpha$ in $W^{1,q}(\Omega)$ with $q > p > 1$, one obtains the Mosco-convergence result (5.3.16) from [18, p.87]. Consequently, there exist two recovery sequences,

$$w_n^{\pm} \in K_n^{\pm}(W_0^{1,p}(\Omega)), \tag{5.3.20}$$

with $w_n^\pm \to w$ in $W_0^{1,p}(\Omega)$. Using the continuity of $\max(\,.\,,0), \min(\,.\,,0) : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, it follows that the sequence

$$w_n = \max(w_n^+, 0) + \min(w_n^-, 0),$$

converges to w in $W_0^{1,p}(\Omega)$. Moreover, it holds that $|w_n| \leq \alpha_n$. The assumptions on α_n allow to use Theorem 5.7 to infer the existence of a smooth function $\tilde{w}_n \in C_c^\infty(\Omega)$ with $|\tilde{w}_n| \leq \alpha_n \leq \alpha$ a.e. in Ω which approximates w_n arbitrarily well. Using $w_n \to w$ in $W_0^{1,p}(\Omega)^d$, the assertion follows by an $\varepsilon/2$-argument. $\square$

For piecewise continuous obstacles $\alpha : \Omega \to \mathbb{R}$ according to (5.3.15), the above result can be further refined in the following sense. If the Sobolev function which is to be approximated is continuous along the jump interfaces determining the obstacle, then it is the limit of feasible smooth functions. For instance, this is the case when the function originates from the solution of an elliptic PDE such that the singularities are only expected near the boundary of Ω, for example at a reentrant corner.

Therefore we define for $\eta > 0$ the enlarged interior boundaries of $\mathcal{I} = \cup_{k=1}^M \partial\Omega_k \setminus \partial\Omega$ as

$$\mathcal{I}_\eta := \{x \in \Omega : \operatorname{dist}(x, \mathcal{I}) \leq \eta\},$$

and we consider the space of functions $C(\mathcal{I};\Omega)$ which are uniformly continuous across $\mathcal{I}$,

$$C(\mathcal{I};\Omega) := \{f : \Omega \to \mathbb{R} : f|_{\mathcal{I}_\eta} \in C(\overline{\mathcal{I}_\eta}) \text{ for some } \eta > 0\}.$$

Theorem 5.23. *Let $1 \leq p < \infty$. Let α be piecewise continuous in the sense of* (5.3.15) *and assume that* (5.2.2) *is fulfilled. The following density result holds true:*

$$\overline{K(\mathcal{D}(\overline{\Omega})^d)}^{W^{1,p}(\Omega)^d} = \overline{K(W^{1,p}(\Omega)^d \cap C(\mathcal{I};\Omega)^d)}^{W^{1,p}(\Omega)^d}.$$

Proof. Let $w \in K(W^{1,p}(\Omega)^d)$ so that $|w| \leq \alpha$ a.e. in Ω and assume that w is uniformly continuous on $\mathcal{I}_\eta$. Consider $Ew \in W^{1,p}(\mathbb{R}^N)^d$ to be the extension of w to the entire $\mathbb{R}^N$ via the extension-by-reflection operator E defined previously in (5.2.7). Let $E\alpha : \mathbb{R}^N \to \mathbb{R}$ be the analogous extension of α. As shown in the proof of Theorem 5.8, this extension is bound-preserving: $|Ew| \leq E\alpha$ a.e. in $\mathbb{R}^N$.

Denote by $S_n(w,\Omega) := \rho_n * Ew$ and $\alpha_n = \rho_n * E\alpha$ the mollifications of Ew and $E\alpha$ from (5.2.10), respectively. Since α is continuous on $\overline{E_\eta}$ where $E_\eta := (\mathcal{I}_\eta)^c \cap \Omega$, it follows that $\alpha_n \to \alpha$ uniformly on E_η. Further define

$$\beta_n = \left(1 + \frac{\sup_{x \in E_\eta} |\alpha(x) - \alpha_n(x)|}{\operatorname{ess\,inf}_{x\in\Omega} \alpha(x)}\right)^{-1},$$

where we use that $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$. It follows that $\beta_n \uparrow 1$ as $n \to \infty$ and we have that $\beta_n \alpha_n(x) \leq \alpha(x)$ for all $x \in E_\eta$. Since $|Ew(x)| \leq E\alpha(x)$ a.e. in $\mathbb{R}^N$, this implies

$$\beta_n |S_n(w,\Omega)(x)| \leq \alpha(x), \quad \forall\, x \in E_\eta. \tag{5.3.21}$$

To enforce the feasibility on the enlarged interface set $\mathcal{I}_\eta$, we decompose $\mathcal{I}_\eta$ as $\mathcal{I}_\eta = A^+ \cap A^-$ where $A^+ = \{x \in \mathcal{I}_\eta : |w(x)| \geq s\}$ for fixed $s > 0$ with $s < \operatorname{ess\,inf}_{x\in\Omega} \alpha(x)$, and $A^- = \mathcal{I}_\eta \setminus A^+$. Define

$$\gamma_n = \left(1 + \frac{\sup_{x\in A^+} |w(x) - S_n(w,\Omega)(x)|}{s}\right)^{-1}.$$

Since $S_n(w,\Omega) \to w$ uniformly on $\mathcal{I}_\eta$, then $\gamma_n \uparrow 1$ as $n \to \infty$. Moreover, the estimate

$$\gamma_n |S_n(w,\Omega)(x)| \le |w(x)| \le \alpha(x), \quad \forall\, x \in A^+, \tag{5.3.22}$$

can be shown analogously to (5.2.11). By definition, $|w(x)| < s < \operatorname{ess\,inf}_{x\in\Omega} \alpha(x)$ for all $x \in A^-$. Using once again the uniform convergence of $S_n(w,\Omega)$ to w on $\mathcal{I}_\eta$, one observes that, for sufficiently large n,

$$|S_n(w,\Omega)| \le \operatorname*{ess\,inf}_{x\in\Omega} \alpha(x), \quad \text{for all } x \in A^-. \tag{5.3.23}$$

Finally, the sequence $w_n(x) = \gamma_n \beta_n S_n(w,\Omega)(x)$ satisfies $w_n \in \mathcal{D}(\overline{\Omega})$ for all $n \in \mathbb{N}$ and

$$w_n \to w \quad \text{in } W^{1,p}(\Omega)^d \quad \text{and} \quad |w_n(x)| \le \alpha(x), \text{ a.e. in } \Omega, \tag{5.3.24}$$

for sufficiently large n; where we have used (5.3.21), (5.3.22) and (5.3.23). This completes the proof. $\square$

5.3.4 Supersolutions of elliptic PDEs

By now, density properties for pointwise constraints in Sobolev spaces of the type

$$\overline{K(C_c^\infty(\Omega)^d)}^{W_0^{1,p}(\Omega)^d} = K(W_0^{1,p}(\Omega)^d), \text{or} \quad \overline{K(\mathcal{D}(\overline{\Omega})^d)}^{W^{1,p}(\Omega)^d} = K(W^{1,p}(\Omega)^d),$$

have been obtained on the basis of mollification and a subsequent procedure to enforce feasibility. An alternative approach is the approximation of a function via the solution of an appropriate sequence of elliptic PDEs. Using standard regularity theory, one may prove higher regularity of the approximating sequence and one is left to prove feasibility. In this section we focus on obstacles which are solutions of an elliptic PDE. Therefore consider a second order differential operator in divergence form:

$$A = \sum_{i,j=1}^{N} -\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i} + c(x) \tag{5.3.25}$$

where $a_{ij}, b_i, c \in L^\infty(\Omega)$ for $1 \le i,j \le N$, the matrix $[a_{ij}(x)]$ is symmetric a.e. and uniformly elliptic, i.e., there exists $\kappa_a > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \kappa_a |\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^N,$$

for a.e. $x \in \Omega$. It is further assumed that a_{ij}, b_i, c are such that A is strongly monotone over $H_0^1(\Omega)$, i.e., there exists $\kappa > 0$ such that

$$\langle Au, u\rangle_{(H^{-1}(\Omega), H_0^1(\Omega))} \ge \kappa \|u\|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

For example, this is the case if $b_i \equiv 0$ for $1 \le i \le N$ and $c(x) \ge 0$ a.e. in Ω. We call a function $\alpha \in H^1(\Omega)$ a weak supersolution with respect to the elliptic operator A, if $A\alpha \ge 0$ in the $H^{-1}(\Omega)$-sense, that is,

$$\langle A\alpha, v\rangle \ge 0, \quad \forall\, v \in H_0^1(\Omega), v \ge 0 \text{ a.e. in } \Omega. \tag{5.3.26}$$

The subsequent theorem covers density properties for obstacles that are weak supersolutions of an elliptic PDE of type (5.3.25).

Theorem 5.24. *Suppose that $\alpha \in H^1(\Omega)$ is a weak supersolution for some A as in (5.3.25) in the sense of (5.3.26) with $\alpha \geq 0$ on $\partial\Omega$. For $X(\Omega) \in \{L^2(\Omega)^d, H_0^1(\Omega)^d\}$ it holds that*

$$\overline{K(Y(\Omega); |\,.\,|_\infty)}^{X(\Omega)} = K(X(\Omega); |\,.\,|_\infty),$$

in the following cases.

(i) $\partial\Omega \in C^{0,1}$, $a_{ij} \in C^{0,1}(\Omega)$ *or* $a_{ij} \in C^1(\Omega)$: $\quad Y(\Omega) = (H^2_{loc}(\Omega) \cap H_0^1(\Omega))^d$,

(ii) $\partial\Omega \in C^{1,1}$ *or* Ω *convex,* $a_{ij} \in C^{0,1}(\Omega)$: $\quad Y(\Omega) = (H^2(\Omega) \cap H_0^1(\Omega))^d$,

(iii) $\partial\Omega \in C^{0,1}$, $a_{ij}, b_i, c \in C^{m+1}(\Omega)$, $m \in \mathbb{N}_0$: $\quad Y(\Omega) = (H^{m+2}_{loc}(\Omega) \cap H_0^1(\Omega))^d$,

(iv) $\partial\Omega \in C^{m+2}$, $a_{ij}, b_i, c \in C^{m+1}(\overline{\Omega})$, $m \in \mathbb{N}_0$: $\quad Y(\Omega) = (H^{m+2}(\Omega) \cap H_0^1(\Omega))^d$.

Proof. Without loss of generality, assume $d = 1$. To begin with, observe that the maximum principle implies $\alpha(x) \geq 0$ a.e. in Ω. Let $w \in K(X(\Omega))$ be arbitrary. Consider the sequence (w_n) where for $n \in \mathbb{N}$, w_n is defined as the unique solution to

$$\text{find } y \in H_0^1(\Omega): \quad \frac{1}{n}Ay + y = w \quad \text{in } H^{-1}(\Omega). \tag{5.3.27}$$

We denote by T_n the solution mapping to (5.3.27), i.e., $w_n = T_n(w)$.

Step 1: T_n-invariance of $K(H_0^1(\Omega))$. We now prove that for any $n \in \mathbb{N}$, we have that $-\alpha \leq w_n \leq \alpha$ a.e., i.e.,

$$T_n : K(L^2(\Omega)) \to K(H_0^1(\Omega)), \tag{5.3.28}$$

given that $A\alpha \geq 0$ in $H^{-1}(\Omega)$. Proceeding as in [121], we consider $(w_n - \alpha)^+$ as a test function for (5.3.27). Then,

$$\begin{aligned}\frac{\kappa}{n}\|(w_n-\alpha)^+\|^2_{H_0^1(\Omega)} + \|(w_n-\alpha)^+\|^2_{L^2(\Omega)} &\leq \langle (\frac{1}{n}A + I)(w_n - \alpha), (w_n-\alpha)^+\rangle \\ &\leq \langle w - \alpha - \frac{1}{n}A\alpha, (w_n-\alpha)^+\rangle \\ &\leq -\frac{1}{n}\langle A\alpha, (w_n-\alpha)^+\rangle \leq 0,\end{aligned}$$

where we have used that $w - \alpha \leq 0$ a.e. in Ω. Therefore, $w_n \leq \alpha$ a.e. in Ω. Analogously, we obtain that $w_n \geq -\alpha$ a.e., by considering $(-\alpha - w_n)^+$ as test function and by adding to both sides $-\langle \frac{1}{n}A\alpha + \alpha, (-\alpha - w_n)^+\rangle$. This proves (5.3.28), i.e., $w_n \in K(H_0^1(\Omega))$.

Step 2: Some convergence results for singular perturbations.

The desired convergence modes of the approximating sequences rely on standard arguments for singular perturbations, cf. [103, Theorem 9.1, Theorem 9.4] for the case of singularly perturbed variational inequalities. For the sake of coherence, we give here some details of the proofs. We first prove for $y \in L^2(\Omega)$:

$$y_n \to y \text{ in } L^2(\Omega) \quad \Longrightarrow \quad \hat{y}_n := T_n(y_n) \to y \text{ in } L^2(\Omega). \tag{5.3.29}$$

In fact, $\hat{y}_n$ solves

$$\tfrac{1}{n}A\hat{y}_n + \hat{y}_n = y_n \qquad \text{in } H^{-1}(\Omega), \tag{5.3.30}$$

for all $n \in \mathbb{N}$. Testing this equation with $\hat{y}_n$ shows that $(\hat{y}_n)$ is bounded in $L^2(\Omega)$ and $((1/\sqrt{n})\hat{y}_n)$ is bounded in $H_0^1(\Omega)$. Consequently, (5.3.30) implies that $\hat{y}_n \rightharpoonup y$ in $L^2(\Omega)$. One may further subtract $\frac{1}{n}Av + v$ from (5.3.30) to obtain

$$(\tfrac{1}{n}A + I)(\hat{y}_n - v) = y_n - (\tfrac{1}{n}A + I)v,$$

where $v \in H_0^1(\Omega)$ is arbitrary. Upon testing this equation with $(\hat{y}_n - v)$ one obtains

$$\tfrac{\kappa}{n}\|\hat{y}_n - v\|^2_{H_0^1(\Omega)} + \|\tilde{y}_n - v\|^2_{L^2(\Omega)} \leq (y_n - v, \hat{y}_n - v) - \tfrac{1}{n}\langle Av, \hat{y}_n - v\rangle,$$

which entails

$$\begin{aligned}\limsup_{n\to\infty}&\left(\tfrac{\kappa}{n}\|\hat{y}_n - v\|^2_{H_0^1(\Omega)} + \|\hat{y}_n - v\|^2_{L^2(\Omega)}\right)\\ &\leq \|y - v\|^2_{L^2(\Omega)} + \limsup_{n\to\infty} -\tfrac{1}{n}\langle Av, \hat{y}_n - v\rangle, \quad \forall\, v \in H_0^1(\Omega).\end{aligned} \tag{5.3.31}$$

Since $((1/\sqrt{n})\hat{y}_n)$ is bounded in $H_0^1(\Omega)$, and $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, estimate (5.3.31) ensues that for any $\varepsilon > 0$, there exists $v_\varepsilon \in H_0^1(\Omega)$ with

$$\limsup_{n\to\infty}\|\hat{y}_n - v_\varepsilon\|^2_{L^2(\Omega)} \leq \varepsilon^2, \quad \|y - v_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon.$$

One immediately infers that $\hat{y}_n \to y$ in $L^2(\Omega)$. Thus, (5.3.29) is true.

Secondly, for $y \in H_0^1(\Omega)$, one may also prove

$$y_n \to y \text{ in } H_0^1(\Omega) \quad \Longrightarrow \quad \hat{y}_n := T_n(y_n) \to y \text{ in } H_0^1(\Omega). \tag{5.3.32}$$

In fact, since $y_n \in H_0^1(\Omega)$ and A is strongly monotone, we observe that

$$\begin{aligned}\frac{\kappa}{n}\|\hat{y}_n - y_n\|^2_{H_0^1(\Omega)} + \|\hat{y}_n - y_n\|^2_{L^2(\Omega)} &\leq \langle\left(\frac{1}{n}A + I\right)(\hat{y}_n - y_n), \hat{y}_n - y_n\rangle\\ &= \frac{1}{n}\langle Ay_n, y_n - \hat{y}_n\rangle\\ &\leq \frac{1}{n}\|Ay_n\|_{H^{-1}(\Omega)}\|y_n - \hat{y}_n\|_{H_0^1(\Omega)},\end{aligned}$$

where we have used that $\hat{y}_n$ solves (5.3.27) with y_n as a right hand side. From this estimate, one may infer that

$$\kappa\|\hat{y}_n - y_n\|^2_{H_0^1(\Omega)} \leq c\|y_n - \hat{y}_n\|_{H_0^1(\Omega)},$$

owing to the boundedness of (y_n) in $H_0^1(\Omega)$. Hence, also $(\hat{y}_n)$ is bounded in $H_0^1(\Omega)$. Employing (5.3.29) one obtains $\hat{y}_n \rightharpoonup y$ in $H_0^1(\Omega)$ along a subsequence, and by uniqueness, it holds that $\hat{y}_n \rightharpoonup y$ for the entire sequence $(\hat{y}_n)$. Finally, from the inequalities above, we obtain that

$$\kappa\limsup_{n\to\infty}\|\hat{y}_n - y_n\|^2_{H_0^1(\Omega)} \leq \limsup_{n\to\infty}\langle Ay_n, y_n - \hat{y}_n\rangle = 0,$$

such that $\hat{y}_n = T_n(y_n) \to y$ in $H_0^1(\Omega)$ and thus (5.3.32) is proven.

Thirdly, we define for $w_n^q := T_n^q(w)$ where $T_n^q(w) := T_n(T_n^{q-1}(w))$ for $q \in \mathbb{N} \setminus \{1\}$ and $T_n^1(w) := T_n(w)$. It can be deduced from (5.3.29) and (5.3.32) by induction that, as $n \to \infty$,

$$w_n^k \to w \qquad \text{in } L^2(\Omega), \qquad \forall\, k \in \mathbb{N} \cup \{0\}, \tag{5.3.33}$$

for $w \in L^2(\Omega)$ and
$$w_n^k \to w \qquad \text{in } H_0^1(\Omega), \qquad \forall\, k \in \mathbb{N} \cup \{0\}, \tag{5.3.34}$$
for $w \in H_0^1(\Omega)$, respectively.

Step 3: Regularity and convergence of the approximating sequences.

The regularity of the $H_0^1(\Omega)$ solution $T_n(w)$ to (5.3.27) is different with respect to the statement cases: If Ω has a Lipschitz boundary $\partial\Omega$ and $a_{ij} \in C^{0,1}(\Omega)$ or $a_{ij} \in C^1(\Omega)$ for $1 \leq i,j \leq N$, the solution $T_n(w)$ belongs to $H_0^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ (see [96] for the first case and [48] for the second one). The solution $T_n(w)$ belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ if $\partial\Omega$ is $C^{1,1}$-smooth [96] or when Ω is convex [57].

In case $w \in K(L^2(\Omega))$, (5.3.29) with $y_n \equiv w$ ensures that $w_n \to w$ in $L^2(\Omega)$. In conjunction with the regularity and the feasibility of $w_n = T_n(w)$ described above, we have then established (i) and (ii) for $X(\Omega) = L^2(\Omega)$. Secondly, note that if $w \in K(H_0^1(\Omega))$, then $w_n \to w$ in $H_0^1(\Omega)$ by (5.3.32) with $y_n \equiv w$ and as seen above, $w_n \in K(H_0^1(\Omega))$. This, together with the regularity of $w_n = T_n(w)$ established above proves in turn (i) and (ii) for $X(\Omega) = H_0^1(\Omega)$.

It is left to argue for (iii) and (iv) as follows. If $a_{ij}, b_i, c \in C^{m+1}(\Omega)$ for $1 \leq i,j \leq N$ and $\partial\Omega$ is Lipschitz, then for each $n \in \mathbb{N}$, the operator T_n has the following increasing regularity properties (see [48]),
$$w \in H^k_{\mathrm{loc}}(\Omega) \Longrightarrow T_n(w) \in H^{k+2}_{\mathrm{loc}}(\Omega) \cap H_0^1(\Omega), \quad 0 \leq k \leq m;$$
and if $a_{ij}, b_i, c \in C^{m+1}(\overline{\Omega})$ for $1 \leq i,j \leq N$ and $\partial\Omega$ is of class C^{m+2}, for each $n \in \mathbb{N}$,
$$w \in H^k(\Omega) \Longrightarrow T_n(w) \in H^{k+2}(\Omega) \cap H_0^1(\Omega), \quad 0 \leq k \leq m.$$
Finally, this proves (iii) given that $w_n^m \in H^{m+2}_{\mathrm{loc}}(\Omega) \cap H_0^1(\Omega), w_n^m \in K(H_0^1(\Omega))$, and $w_n^m \to w$ as $n \to \infty$ in $L^2(\Omega)$ or $H_0^1(\Omega)$ depending on the regularity of w, cf. (5.3.33) and (5.3.34). The analogous reasoning applies to (iv). $\square$

Let us briefly comment on the relation to the preceding density result from Theorem 5.22. First, note that we do not require the obstacle to be bounded away from zero as in the previous paragraphs. Moreover, a classical result by Trudinger [122, Cor. 5.3] for the case without lower order terms ($b_i \equiv 0, c \equiv 0$), states that any weak supersolution in the sense of (5.3.26) is upper semicontinuous. For this reason, the class of obstacles considered in Theorem 5.24 differs from the one of Theorem 5.22. On the other hand, the maximum principle implies that any weak subsolution is nonpositive on Ω (provided it vanishes on an open portion of the boundary) which is why this type of obstacles is irrelevant for the investigation of density properties.

6 Applications

6.1 Finite Elements

In this section we want to show how the density results (5.2.3) and (5.2.4) can be used to derive the Mosco-convergence of certain discretized versions K_h of $K(X(\Omega))$ associated with standard finite element spaces suitable for an approximation of $X(\Omega)$. The very general concept of Mosco-convergence is typically useful for investigating the stability of variational inequality problems that involve convex constraint sets, e.g, those of the type $K(X(\Omega))$, with regard to a suitable class of perturbations. In this context, the discretization of $K(X(\Omega))$ can be seen as a special type of perturbation. Applications are manifold and comprise, for instance, the discretization of variational problems in mechanics, such as in elasto-plasticity with hardening (Section 9.3), or in image restoration, with regard to the predual problem of total variation regularization (Section 6.2).

6.1.1 On the significance of Mosco-convergence

For the sake of convenience, we repeat at this point the notion of Mosco-convergence from Definition 5.20.

Definition 6.1 (Mosco-convergence)**.** Let X be a reflexive Banach space and (K_n) a sequence of closed convex subsets with $K_n \subset X$. Then (K_n) is said to Mosco-converge to the set $K \subset X$, if and only if,

$$K \supset \{v \in X : (\, \exists\, (v_k) \subset X : v_k \in K_{n_k} \; \forall\, k \in \mathbb{N}, v_k \rightharpoonup v)\}, \tag{M1}$$

$$K \subset \{v \in X : (\, \exists\, (v_n) \subset X, \;\; \exists\, n_0 \in \mathbb{N} : v_n \in K_n \; \forall\, n \geq n_0, v_n \to v)\} \tag{M2}$$

Here, (K_{n_k}) denotes an arbitrary subsequence of (K_n). Note that if (K_n) converges to K in the sense of Mosco then K is necessarily closed and convex, too.

Remark 6.2. In some textbooks on finite-dimensional approximations of variational inequalities, cf. e.g. [53, 61], condition (M2) is replaced by the following criterion:

> There exists a dense subset $\tilde{K} \subset K$ and an operator $r_n : \tilde{K} \to X$, (M2′)
> such that for all $v \in \tilde{K}$ it holds that $r_n v \to v$ in X and there exists $n_0 = n_0(v) \in \mathbb{N}$:
> $r_n v \in K_n$ for all $n \geq n_0$.

It is easy to show that (M2′) implies (M2). In fact, let $v \in K$ and denote by $\pi_{K_n} v$ its (not necessarily uniquely determined) projection onto K_n. By density, for $\varepsilon > 0$, there exists $v^\varepsilon \in \tilde{K}$ such that $\|v^\varepsilon - v\| \leq \varepsilon$. Thus it holds that

$$\|v - \pi_{K_n} v\| = \inf_{v^n \in K_n} \|v - v^n\| \leq \|v - r_n v^\varepsilon\| \leq \varepsilon + \|v^\varepsilon - r_n v^\varepsilon\|$$

for sufficiently large n such that $\lim_{n\to\infty} \|v - \pi_{K_n} v\| \leq \varepsilon$ where ε was arbitrary.

Condition (M2′) turns out to be convenient especially in the context of finite-dimensional approximations, where r_n is given by suitable interpolation operators which are only well-defined on a dense subset of $X(\Omega)$ giving rise to sets $\tilde{K}$ which consist of sufficiently smooth functions in

K such as $K(C(\overline{\Omega})^d)$ or $K(C_c^\infty(\Omega)^d)$. In this respect, this is precisely the point where the density results of Chapter 5 are required.

For the significance of Mosco-convergence in the context of convex constrained optimization and variational inequality problems we refer to the various examples and statements in Chapter 4. For instance, in the case of variational inequalities, it suffices to consider the situation of Theorem 3.1 with $j_n := i_{K_n}$ to argue that the Mosco-convergence of (K_n) implies the consistency of the discretization scheme. Moreover, each discretized problem is uniquely solvable by the properties of K_n. For an even sharper result, we mention the following well-known result from [103, p.99], which is a special case of the general results in [95] and allows for perturbations in the monotone operator A as well as the linear functional l.

Theorem 6.3. *Let X be a real Hilbert space and $K_n \subset X$ nonempty, closed and convex subsets. Assume $A_n : K_n \to X^*$ to be uniformly Lipschitz and strongly monotone operators that fulfill*

$$A_n v_n \to Av \quad \text{in } V^*$$

for all $(v_n) \subset X$ with $v_n \to v$ and $v_n \in K_n$ for all $n \in \mathbb{N}$. Further assume that $(l_n) \subset X^*$ with (l_n) converging to l in X^* and that (K_n) converges to K in X in the sense of Mosco; cf. (M1),(M2). *Then the sequence of unique solutions (u_n) of the problems*

$$\text{find } u_n \in K_n : \quad \langle A_n u_n, v - u_n \rangle \geq \langle l_n, v_n - u_n \rangle, \quad \forall\, v_n \in K_n,$$

converges strongly to the solution u of the limit problem

$$\text{find } u \in K : \quad \langle Au, v - u \rangle \geq \langle l, v - u \rangle, \quad \forall\, v \in K. \tag{6.1.1}$$

In the following, the perturbation is assumed to be originating from a finite-dimen-sional approximation $K_n = K_{h_n}$ of the set $K(X(\Omega))$ in the framework of classical finite element methods where the parameter n is associated with a sequence of mesh sizes (h_n) tending to zero. In this context, Mosco-convergence requires that each element of the set $K(X(\Omega))$ can be approximated by discrete feasible elements. Under this condition, Theorem 6.3 ensures that the solutions to the discrete problems converge to the solution of the original infinite-dimensional problem irrespective of the regularity of the data or the obstacle defining $K(X(\Omega))$.

In this sense, Mosco-convergence is a powerful tool whenever the discrete spaces are fixed *a priori*, i.e., irrespective of the data or the solution of the specific problem. The resulting sequence of finite-dimensional problems can be understood as an approximation of *any problem in a given problem class*. This applies, for example, to classical finite element methods.

In contrast, *adaptive* finite element methods intend to design the sets K_{h_n} in order to approximate *the solution of a specific problem*. In fact, the sets K_{h_n} (and thus the discrete problems) are successively determined during the course of the adaptive algorithm and their definition builds upon information on the preceding solution u_{n-1} and the specific data. In the case of elliptic variational inequalities, this is justified by Falk's a priori estimate [49], which shows that in order to prove convergence, it is sufficient to tailor the sets K_{h_n} in dependence on the original variational inequality for specific data, e.g., A and l in the context of problem (6.1.1). In practice, this is achieved by using a posteriori error estimators that consecutively exploit information from discrete solutions. In this way, adaptive methods aim at a reduction of the discretization error whilst enlarging the dimension of the discrete space as economically as possible. However, rigorous convergence proofs with regard to adaptive discretizations of variational inequalities are restricted to special cases and usually rely on rather strong assumptions. For instance, in the case of the obstacle problem with a piecewise affine obstacle, we mention the article [111]. Moreover, density results may still be useful in the convergence analysis of adaptive schemes which require interpolation operators, cf. [110].

6.1.2 Finite Element discretized convex sets

In this section we assume that $\Omega \subset \mathbb{R}^N$ is polyhedral. Together with Ω, a sequence of geometrically conformal affine simplicial meshes $(\mathcal{T}_h)_{h>0}$ of Ω with mesh size

$$h := \max_{T \in \mathcal{T}_h} \operatorname{diam} T$$

is assumed to be given, see [47]. In analogy to the case $N = 2$ we refer to each $\mathcal{T}_h$ as a triangulation. The N-dimensional Lebesgue measure of an element $T \in \mathcal{T}_h$ is denoted by $\lambda(T)$. We also admit the standard assumption that the sequence $(\mathcal{T}_h)$ is shape-regular, i.e.,

$$\exists\, c > 0 : \ \frac{\operatorname{diam}(T)}{\rho_T} \leq c \quad \forall\, T \in \mathcal{T}_h,\ \forall\, h, \tag{6.1.2}$$

where $\operatorname{diam}(T) = \max_{x,y \in T} |x - y|$ denotes the diameter of T and ρ_T designates the diameter of the largest ball that is contained in T. We further write x_T for the (barycentric) midpoint of an element T, and $\mathcal{M}_h = \{x_T : T \in \mathcal{T}_h\}$, $\mathcal{N}_h$ and $\mathcal{E}_h$ for the set of element midpoints, triangulation nodes and faces with respect to $\mathcal{T}_h$, respectively. By abuse of notation, we write $|\mathcal{M}_h|$ and $|\mathcal{N}_h|$ for the cardinality of the respective set. Let $\chi_T : \Omega \to \mathbb{R}$ designate the characteristic function of T with respect to Ω, that is,

$$\chi_T(x) = 0 \ \ \forall\, x \notin T, \quad \chi_T(x) = 1 \ \ \forall\, x \in T.$$

We further make use of the standard $H^1(\Omega)$-conformal finite element space of globally continuous piecewise affine functions associated to $\mathcal{T}_h$ denoted by

$$P_{1,h}(\Omega) := \{u \in C(\overline{\Omega}) : u|_T \in \mathbb{P}_1 \ \ \forall\, T \in \mathcal{T}_h\}.$$

Here, $\mathbb{P}_1$ denotes the space of polynomials of degree less than or equal one. Together with the finite-dimensional subspace $P_{1,h}(\Omega)$ and its standard nodal basis $\{\varphi_x : x \in \mathcal{N}_h\}$ we consider the global interpolation operator

$$I_h : C(\overline{\Omega}) \to P_{1,h}(\Omega), \quad I_h u := \sum_{x \in \mathcal{N}_h} u(x)\varphi_x. \tag{6.1.3}$$

We note that I_h is only defined on a dense subspace of $H^1(\Omega)$. Suitable to the discretization of variational problems in $H(\operatorname{div};\Omega)$, we also define the $H(\operatorname{div};\Omega)$-conforming space of Raviart-Thomas finite elements of lowest order,

$$RT_h(\Omega) = \{w \in L^2(\Omega)^N : w|_T \in \mathbb{RT} \ \ \forall\, T \in \mathcal{T}_h, [w \cdot \nu]|_E = 0 \ \ \forall\, E \in \mathcal{E}_h \cap \Omega\}, \tag{6.1.4}$$

where $\mathbb{RT} = \{w \in \mathbb{P}_1^N : \ \exists\, a \in \mathbb{R}^N, b \in \mathbb{R} : w(x) = a + bx\}$ and ν denotes the unit outer normal to T. To incorporate homogeneous Neumann boundary conditions, one uses the $H_0(\operatorname{div},\Omega)$-conforming subspace

$$RT_{0,h}(\Omega) := RT_h(\Omega) \cap H_0(\operatorname{div};\Omega).$$

The construction of suitable edge-based basis functions $\{\varphi_E : E \in \mathcal{E}_h\}$ can be found in the literature; cf.,e.g., [13]. As a result, the boundary condition in the definition of $RT_{0,h}(\Omega)$ can be easily accounted for. The global Raviart-Thomas interpolation operator is given by

$$I_h^{RT} : W^{1,1}(\Omega)^N \to RT_h(\Omega), \quad I_h^{RT} w := \sum_{E \in \mathcal{E}_h} \left(\int_E w \cdot \nu \, d\mathcal{H}^{N-1} \right) \varphi_E. \tag{6.1.5}$$

We emphasize that the subsequent results may be extended to higher order elements. However, higher order elements are typically useful when the solution to the variational problem (6.1.1) associated with K displays a higher regularity, which in turn relies on higher regularity of the data and the obstacle, which we do not want to assume. In the latter case, the concept of Mosco-convergence is not binding to prove the convergence of the finite element method and a priori error estimates with a rate can be derived; cf.,e.g., [23]. Further, even for simple variational inequality problems such as the classical elasto-plastic torsion problem, there is a regularity limitation for the solution regardless of the smoothness of the data; cf. [53].

Note also that the subsequently covered problems comprise situations where the discrete feasible sets K_h are not necessarily nested and non-conforming in the sense that they are in general not contained in the continuous feasible set $K(X)$. In the following, c denotes a positive constant which may take different values on different occasions.

Continuous Obstacles

We first consider uniformly continuous upper bounds α.

Lemma 6.4 (Mosco-convergence, first condition). *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain and assume that $\alpha \in C(\overline{\Omega})$ with $\alpha(x) \geq 0$ in Ω. Let (w_h) be a sequence which fulfills for all h, $w_h \in P_{1,h}(\Omega)^d$ and $|w_h(x_T)| \leq \alpha(x_T)$ for all $T \in \mathcal{T}_h$. If $w_h \rightharpoonup w$ for $h \to 0$ in $L^2(\Omega)^d$ then it holds that $|w| \leq \alpha$ a.e. in Ω.*

Proof. It suffices to show that $i_K(w) = 0$ where $K := \{w \in L^2(\Omega)^d : |w| \leq \alpha \text{ a.e.}\}$. Moreover, it holds that $i_K = j^*$ where j^* denotes the Fenchel conjugate

$$j^*(v^*) := \sup_{v \in L^2(\Omega)^d} \{(v^*, v) - j(v)\}$$

of the mapping $j : L^2(\Omega)^d \to \mathbb{R}$, $j(v) := \int_\Omega \alpha |v|_* \, dx$, see Lemma 2.4. We recall that

$$|v^*|_* = \sup_{v \in \mathbb{R}^d \setminus \{0\}} v^* \cdot v / |v|$$

denotes the dual norm of $|\,.\,|$. From the definition of j^*, we obtain that $i_K(w) = 0$ is equivalent to

$$(w, v) \leq \int_\Omega \alpha |v|_* \quad \forall \, v \in L^2(\Omega)^d. \tag{6.1.6}$$

By a density argument, it suffices to prove this result for all $v \in C_c(\Omega)^d$. Denote by

$$\alpha_h := \sum_{T \in \mathcal{T}_h} \alpha(x_T) \chi_T, \quad v_h := \sum_{T \in \mathcal{T}_h} v(x_T) \chi_T \tag{6.1.7}$$

the piecewise constant interpolants of α and v, respectively. The uniform continuity of α and v implies $\alpha_h \to \alpha$ in $L^\infty(\Omega)$ and $v_h \to v$ in $L^\infty(\Omega)^d$. By the weak convergence of w_h, the strong convergence of α_h and v_h as well as the midpoint quadrature rule, we obtain in the limit as $h \to 0$,

$$\begin{aligned}
\int_\Omega w \cdot v \, dx \leftarrow \int_\Omega w_h \cdot v_h \, dx &= \sum_{T \in \mathcal{T}_h} \int_T w_h \cdot v_h \, dx \\
&= \sum_{T \in \mathcal{T}_h} \lambda(T) \, w_h(x_T) \cdot v_h|_T \, dx \\
&\leq \sum_{T \in \mathcal{T}_h} \lambda(T) \, \alpha(x_T) \, |v_h|_T|_* \, dx \\
&= \int_\Omega \alpha_h |v_h|_* \, dx \to \int_\Omega \alpha |v|_* \, dx,
\end{aligned} \tag{6.1.8}$$

which proves (6.1.6). □

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain and assume that $\alpha \in C(\overline{\Omega})$ with $\alpha(x) \geq 0$ in Ω. Let (w_h) be a sequence which fulfills for all h, $w_h \in P_{1,h}(\Omega)^d$ and $|w_h(x)| \leq \alpha(x)$ for all $x \in \mathcal{N}_h$. If $w_h \rightharpoonup w$ for $h \to 0$ in $L^2(\Omega)^d$ then it holds that $|w| \leq \alpha$ a.e. in Ω.*

Proof. The assertion follows by a slight modification of the proof of Lemma 6.4. Instead of the piecewise constant interpolant we define α_h as the piecewise affine interpolant of α, i.e., $\alpha_h := I_h \alpha$ which fulfills $\alpha(x) = (I_h \alpha)(x)$ for all $x \in \mathcal{N}_h$ and $\alpha_h \to \alpha$ strongly in $L^\infty(\Omega)$. By (6.1.8) we obtain

$$\begin{aligned}
\int_\Omega w \cdot v \, dx \leftarrow \int_\Omega w_h \cdot v_h \, dx &= \sum_{T \in \mathcal{T}_h} \tfrac{\lambda(T)}{N+1} \sum_{x \in \mathcal{N}_h \cap T} w_h(x) \cdot v_h|_T \, dx \\
&\leq \sum_{T \in \mathcal{T}_h} \tfrac{\lambda(T)}{N+1} \sum_{x \in \mathcal{N}_h \cap T} |w_h(x)| \, |v_h|_T|_* \\
&\leq \sum_{T \in \mathcal{T}_h} \tfrac{\lambda(T)}{N+1} \sum_{x \in \mathcal{N}_h \cap T} \alpha(x) \, |v_h|_T|_* \\
&= \int_\Omega \alpha_h |v_h|_* \, dx \to \int_\Omega \alpha |v|_* \, dx.
\end{aligned}$$

□

Theorem 6.6. *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain. Assume that $\alpha \in C(\overline{\Omega})$ fulfills (5.2.2). Then the sets*

$$K_h := \{ w \in P_{1,h}(\Omega)^d : |w(x_T)| \leq \alpha(x_T) \text{ for all } T \in \mathcal{T}_h \} \tag{6.1.9}$$

Mosco-converge for $h \to 0$ to the set $K(H^1(\Omega)^d)$ in $H^1(\Omega)^d$.

Proof. Since weak convergence in $H^1(\Omega)$ implies weak convergence in $L^2(\Omega)$, the preceding Lemma 6.4 shows that (M1) is fulfilled. We now show (M2'). To prove the assertion we may use a strategy that is similar to the one in [53, Theorem 3.3] and requires (5.2.4). Note that Theorem 5.8 implies that the set

$$\tilde{K} := \{ \varphi \in C^\infty(\overline{\Omega})^d : |\varphi(x)| < \alpha(x) \text{ for all } x \in \overline{\Omega} \} \tag{6.1.10}$$

is dense in $K(H^1(\Omega)^d)$ w.r.t. the $H^1(\Omega)^d$-norm. For the global interpolation operator I_h defined in (6.1.3) we have the classical estimate,

$$\|u - I_h u\|_{L^\infty(\Omega)} \leq c h^2 \|u\|_{W^{2,\infty}(\Omega)} \quad \forall \, u \in W^{2,\infty}(\Omega). \tag{6.1.11}$$

Here, c denotes a constant independent of h on account of the shape-regularity of the triangulation (6.1.2); cf. [47, p.61]. We set

$$r_h : \tilde{K} \to P_{1,h}(\Omega)^d, \quad r_h w := [I_h w_1, \ldots, I_h w_d]$$

and it follows from [47, Corollary 1.109] that $r_h w \to w$ as $h \to 0$ in $H^1(\Omega)^d$ for all $w \in \tilde{K}$. Applying estimate (6.1.11) to the components of $w \in \tilde{K}$ and using the equivalence of norms on $\mathbb{R}^d$, one obtains that

$$\| \, |w - r_h w| \, \|_{L^\infty(\Omega)} \leq ch^2 \|w\|_{W^{2,\infty}(\Omega)^d}, \tag{6.1.12}$$

for a suitable modification of c. This implies

$$|r_h w(x)| \leq |w(x)| + ch^2 \|w\|_{W^{2,\infty}(\Omega)^d} \quad \forall\, x \in \Omega. \tag{6.1.13}$$

Since any $w \in \tilde{K}$ is uniformly bounded away from α, it follows from (6.1.13) that there exists $h_0 = h_0(w)$ such that $r_h w \in K_h \; \forall\, h \leq h_0$, which implies (M2'). □

Corollary 6.7. *Under the conditions of Theorem 6.6, the sets (K_h) defined in (6.1.9) Mosco-converge for $h \to 0$ to the set $K(L^2(\Omega)^d)$ in $L^2(\Omega)^d$.*

Proof. Again, Lemma 6.4 implies that (M1) holds true with $X = L^2(\Omega)^d$. For $\tilde{K}$ defined in (6.1.10) it holds that $\tilde{K}$ is also dense in $K(L^2(\Omega)^d)$ with respect to the $L^2(\Omega)^d$-norm, cf. (5.2.3). Thus, (M2') follows analogously to the proof of Theorem 6.6. □

Corollary 6.8. *Under the conditions of Theorem 6.6 the node-based discrete sets*

$$K_h := \{w_h \in P_{1,h}(\Omega)^d : |w_h(x)| \leq \alpha(x) \; \forall\, x \in \mathcal{N}_h\}, \tag{6.1.14}$$

Mosco-converge for $h \to 0$ to $K(H^1(\Omega)^d)$ in $H^1(\Omega)^d$.

Proof. The proof is analogous to the proof of Theorem 6.6, noticing that (6.1.13) also implies that, for any $w \in \tilde{K}$, it holds that $r_h w \in K_h \; \forall\, h \leq h_0(w)$ with (K_h) according to the node-based definition (6.1.14). □

Remark 6.9. In view of the corresponding density result for homogeneous Dirichlet boundary conditions, cf. (5.2.3), the set $P_{1,h}(\Omega)$ in the definition of the discretized sets K_h in (6.1.9) and (6.1.14) can be replaced by the space

$$P_{1,h}^{\partial\Omega} = \{u \in C(\overline{\Omega}) : u|_T \in \mathbb{P}_1 \;\; \forall\, T \in \mathcal{T}_h, \; u(x) = 0 \;\; \forall\, x \in \mathcal{N}_h \cap \partial\Omega\}.$$

The resulting discrete sets K_h incorporate the zero boundary condition and the corresponding results on Mosco-convergence for $h \to 0$ remain valid replacing $H^1(\Omega)^d$ by $H_0^1(\Omega)^d$.

For the discretization of constraint sets in $H(\mathrm{div};\Omega)$ with the help of the Raviart-Thomas finite element space (6.1.4), we state the following similar result.

Theorem 6.10. *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain. Assume $\alpha \in C(\overline{\Omega})$ fulfills (5.2.2). Then the sets*

$$K_h := \{w \in RT_{0,h}(\Omega) : |w(x_T)| \leq \alpha(x_T) \;\; \forall\, T \in \mathcal{T}_h\}$$

Mosco-converge to $K(H_0(\mathrm{div};\Omega))$ in $H(\mathrm{div};\Omega)$ and to $K(L^2(\Omega)^N)$ in $L^2(\Omega)^N$ as $h \to 0$.

Proof. Let $w_h \in K_h$ for all h. First observe that if (w_h) weakly converges to w in $H(\mathrm{div};\Omega)$ then it also weakly converges to w in $L^2(\Omega)^N$ and using Lemma 6.4 one concludes that $|w| \leq \alpha$ a.e in Ω. The continuity of the normal trace mapping

$$H(\mathrm{div};\Omega) \ni w \mapsto \langle w\nu, v\rangle_{(H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega))} \in \mathbb{R}$$

for fixed $v \in H^1(\Omega)$ implies $w\nu = 0$ in $H^{-1/2}(\partial\Omega)$ and we conclude that $w \in K(H_0(\mathrm{div};\Omega))$ whence it follows that (M1) is satisfied. Secondly, note that $K(C_c^\infty(\Omega)^N)$ is dense in $K(H_0(\mathrm{div};\Omega))$ with respect to the $H(\mathrm{div};\Omega)$-norm, cf. (5.2.3). This entails that also

$$\tilde{K} := \{w \in C_c^\infty(\Omega)^N : |w(x)| < \alpha(x) \ \ \forall\, x \in \Omega\}$$

is dense in $K(H_0(\mathrm{div};\Omega))$.

For the global Raviart-Thomas interpolation operator defined in (6.1.5), the following interpolation error estimate holds true, cf. [47, Corollary 1.115]:

$$\|u - I_h^{RT}u\|_{L^\infty(\Omega)^N} + \|\operatorname{div} u - \operatorname{div} I_h^{RT}u\|_{L^\infty(\Omega)} \leq ch\|u\|_{W^{1,\infty}(\Omega)^N} \tag{6.1.15}$$

for all $u \in W^{2,\infty}(\Omega)^N$. Setting $r_h w := I_h^{RT}w$ for any $w \in \tilde{K}$ and taking account of the fact that $I_h^{RT}w \to w$ in $H(\mathrm{div};\Omega)$ for all $w \in \tilde{K}$, we may proceed analogously to the proof of Theorem 6.6 to verify (M2′): Indeed, from the estimate (6.1.15) one deduces that $r_h w \to w$ in $H(\mathrm{div};\Omega)$ and

$$|r_h w(x)| \leq |w(x)| + ch\|u\|_{W^{1,\infty}(\Omega)^N}, \quad \forall\, x \in \Omega.$$

As a result, the definition of $\tilde{K}$ implies that there exists $h_0 = h_0(w)$ such that $r_h w \in K_h$ for all $h \leq h_0$. Consequently, (M2′) is fulfilled. □

The preceding approaches can also be applied to derive corresponding statements for constraint sets involving partial derivatives. To begin with, we consider the gradient-constraint sets

$$K_\nabla(X(\Omega)) = \{w \in X(\Omega) : |\nabla w| \leq \alpha \text{ a.e. in } \Omega\},$$

for $X(\Omega) \subset H^1(\Omega)^d$.

Theorem 6.11. *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain and assume that $\alpha \in C(\overline{\Omega})$ satisfies (5.2.2). Define*

$$K_h := \{w \in P_{1,h}^{\partial\Omega}(\Omega)^d : |\nabla w|_T| \leq \alpha(x_T) \ \ \forall\, T \in \mathcal{T}_h\}. \tag{6.1.16}$$

Then the sets (K_h) Mosco-converge to $K_\nabla(H_0^1(\Omega)^d)$ in $H_0^1(\Omega)^d$.

Proof. To prove (M1) it suffices to notice that if $w_h \rightharpoonup w$ in $H_0^1(\Omega)^d$ then $\nabla w_h \rightharpoonup \nabla w$ in $L^2(\Omega)^{N\times d}$. Similar to the proof of Lemma 6.4, one obtains for $v \in C_c(\Omega)^{N\times d}$ that

$$\int_\Omega \nabla w : v\,dx \leftarrow \int_\Omega \nabla w_h : v\,dx \leq \int_\Omega |\nabla w_h||v|_*\,dx \leq \int_\Omega \alpha_h|v|_*\,dx \to \int_\Omega \alpha|v|_*\,dx,$$

using α_h from (6.1.7). Therefore, (6.1.6) holds with ∇w in place of w and (M1) is verified.

To prove (M2′) we consider again the global interpolation operator I_h from (6.1.3). The standard estimate

$$\|\nabla u - \nabla I_h u\|_{L^\infty(\Omega)^N} \leq ch\|u\|_{W^{2,\infty}(\Omega)} \quad \forall\, u \in W^{2,\infty}(\Omega),$$

holds true, see e.g. [47, Corollary 1.109]. Note also that $K_\nabla(C_c^\infty(\Omega)^d)$ is dense in $K_\nabla(H_0^1(\Omega)^d)$ for the $H_0^1(\Omega)^d$-norm [69, Theorem 4]. Thus,

$$\tilde{K} := \{w \in C_c^\infty(\Omega)^d : |\nabla w(x)| < \alpha(x) \quad \forall\, x \in \Omega\}$$

is also dense in $K_\nabla(H^1(\Omega)^d)$. Therefore one may argue as in the proof of Theorem 6.6 to deduce (M2'). □

We now consider constraints with respect to the divergence. For $X(\Omega) \subset H(\mathrm{div};\Omega)$ let

$$K_{\mathrm{div}}(X(\Omega)) := \{w \in X(\Omega) : |\operatorname{div} w| \le \alpha \text{ a.e. in } \Omega\}. \tag{6.1.17}$$

Theorem 6.12. *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain. Assume that $\alpha \in C(\overline{\Omega})$ fulfills (5.2.2). Then the sets*

$$K_h := \{w \in RT_{0,h}(\Omega) : |\operatorname{div} w|_T| \le \alpha(x_T) \quad \forall\, T \in \mathcal{T}_h\}$$

Mosco-converge in $H_0(\mathrm{div};\Omega)$ to the set $K_{\mathrm{div}}(H_0(\mathrm{div};\Omega))$ as $h \to 0$.

Proof. Taking account of the fact that $w_h \rightharpoonup w$ in $H(\mathrm{div};\Omega)$, $w_h \in K_h$, implies $\operatorname{div} w_h \rightharpoonup \operatorname{div} w$ in $L^2(\Omega)$, (M1) follows analogously to the corresponding part of the proof of Theorem 6.11. Since $K_{\mathrm{div}}(C_c^\infty(\Omega)^N)$ is dense in $K_{\mathrm{div}}(H_0(\mathrm{div};\Omega))$ [69, Theorem 4], the set

$$\tilde{K} := \{w \in C_c^\infty(\Omega)^d : |\operatorname{div} w(x)| < \alpha(x) \;\; \forall\, x \in \Omega\}$$

is also dense in $K_{\mathrm{div}}(H_0(\mathrm{div};\Omega))$. Setting $r_h = I_h^{RT}$, the estimate (6.1.15) implies $r_h w \to w$ in $H(\mathrm{div};\Omega)$ and

$$\|\operatorname{div} w - \operatorname{div} r_h w\|_{L^\infty(\Omega)} \le ch\|w\|_{W^{2,\infty}(\Omega)^N}$$

for all w in $\tilde{K}$. In particular, one may argue as in the proof of Theorem 6.6 to verify (M2'). □

Discontinuous Obstacles

For general discontinuous upper bounds, a point-based discretization is obviously not possible. As a remedy, the construction of the discrete sets K_h typically involves some kind of averaging process. For this purpose we define the integral mean

$$⨍_T \alpha\, dx := \int_T \alpha\, dx / \lambda(T).$$

The practical computation of this integral depends on the type of discontinuity α exhibits on T. Theorem 5.14 shows that the density results (5.2.3) and (5.2.4) for continuous α, as the main ingredient to prove the consistency of the finite element approximation, may fail to hold true. However, the results from Chapter 5 indicate that the density property is still guaranteed for a large class of discontinuous obstacles. To maintain the greatest level of generality, we assume that the nonnegative measurable function $\alpha : \Omega \to \mathbb{R} \cup \{+\infty\}$ allows for the density property

$$\overline{K(C(\overline{\Omega}))}^{L^2(\Omega)^d} = K(L^2(\Omega)^d). \tag{6.1.18}$$

Here, we focus on the consistency in the L^2-topology but an extension to the other cases is possible by appropriately modifying assumption (6.1.18). We stress the fact that the assumption is fulfilled in relevant situations, cf. e.g., Corollary 5.18.

Lemma 6.13. *Let $\Omega \subset \mathbb{R}^N$ be a polyhedral domain and $\alpha \in L^2(\Omega)$ with $\alpha(x) \geq 0$ a.e. in Ω. Let (w_h) be a sequence which fulfills for all h, $w_h \in (P_{1,h}(\Omega))^d$ and $|w_h(x_T)| \leq \fint_T \alpha\, dx$ for all $T \in \mathcal{T}_h$. If $w_h \rightharpoonup w$ for $h \to 0$ in $L^2(\Omega)^d$ then it holds that $|w| \leq \alpha$ a.e. in Ω.*

Proof. The assertion follows analogously to the proof of Lemma 6.4 by a slight modification of the definition of α_h. Instead of the piecewise constant interpolant we consider the piecewise constant quasi-interpolant $\alpha_h := \sum_{T \in \mathcal{T}_h} \chi_T \fint_T \alpha\, dx$. Observe that (α_h) converges strongly to α in $L^2(\Omega)$, which is sufficient to retrace the proof of Lemma 6.4. □

Theorem 6.14. *Assume that the upper bound $\alpha \in L^2(\Omega)$ fulfills (5.2.2) and (6.1.18). Then the sets*

$$K_h := \{w \in P_{1,h}(\Omega)^d : |w(x_T)| \leq \fint_T \alpha\, dx \;\forall\, T \in \mathcal{T}_h\}$$

Mosco-converge for $h \to 0$ to the set $K(L^2(\Omega)^d)$ in $L^2(\Omega)^d$.

Proof. We only need to prove (M2′) since Lemma 6.13 implies (M1). First note that assumption (6.1.18) implies that $K(C_c^\infty(\Omega)^d)$ is also dense in $K(L^2(\Omega)^d)$: In fact, for any fixed $w \in K(C(\overline{\Omega}))$, one may use the obstacle

$$\hat{\alpha}(x) = \max(|w(x)|, \operatorname*{ess\,inf}_{x \in \Omega} \alpha(x))$$

in (5.2.3), and noting that $\hat{\alpha}(x) \leq \alpha(x)$ a.e. in Ω, the assertion readily follows. Secondly, we define the set

$$\tilde{K} := \{w \in C_c^\infty(\Omega)^d : \;\exists\, \delta = \delta(w) > 0 \text{ such that } |w(x)| \leq \alpha(x) - \delta \text{ a.e. in } \Omega\},$$

and note that $\tilde{K}$ is dense in $K(L^2(\Omega)^d)$ by (6.1.18) and (5.3.10). Furthermore, we set

$$r_h w := [I_h w_1, \ldots, I_h w_d]$$

for $w \in \tilde{K}$ and I_h as above. Integrating estimate (6.1.13) yields

$$|\fint_T r_h w\, dx| \leq \fint_T |w(x)|\, dx + \; ch^2 \|w\|_{W^{2,\infty}(\Omega)^d} \quad \forall\, T \in \mathcal{T}_h.$$

Let $w \in \tilde{K}$ be fixed. Since $r_h w$ is affine on each $T \in \mathcal{T}_h$, an application of the midpoint rule shows

$$|r_h w(x_T)| \leq \fint_T |w(x)|\, dx + ch^2 \|w\|_{W^{2,\infty}(\Omega)^d} \quad \text{for all } T \in \mathcal{T}_h,$$

which implies

$$|r_h w(x_T)| \leq \fint_T \alpha\, dx - \delta(w) + ch^2 \|w\|_{W^{2,\infty}(\Omega)^d} \quad \forall\, T \in \mathcal{T}_h. \tag{6.1.19}$$

This implies $r_h w \in K_h$ for all $w \in \tilde{K}$ and $h \leq h_0(w)$. By (6.1.11) it holds that $r_h w \to w$ in $L^2(\Omega)^d$ for $h \to 0$ which proves (M2′). □

6.2 Image Restoration

A popular mathematical model to retrieve a "good" approximation u of a true image from blurred or noisy data whilst preserving edges in the original image is defined by the problem of total variation (TV) regularization. This model has been introduced in [105]. For given data $f \in L^2(\Omega)$ on a Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a fixed blurring operator $K \in \mathcal{L}(L^2(\Omega))$, the problem reads as follows.

Problem (TV-P).

$$\inf \quad \tfrac{1}{2}\|Ku - f\|^2_{L^2(\Omega)} + \int_\Omega \alpha \, d|Du|_r \qquad \text{over } u \in BV(\Omega),$$

for fixed r with $1 \le r \le \infty$.

In the standard model, $\alpha > 0$ is a fixed parameter, such that

$$\int_\Omega \alpha \, d|Du|_r = \alpha |Du|_r(\Omega),$$

where $|Du|_r \in M_+(\Omega)$ is the total variation of the (vector-valued) finite measure $Du \in M(\Omega;\mathbb{R}^2)$ with respect to the r-norm on $\mathbb{R}^2$. The space $M_+(\Omega)$ consists of all (nonnegative) finite measures on Ω. For a detailed definition of these spaces we refer to Section 1.2. Moreover, from the vectorial version (1.2.8) of the Riesz-Alexandrov Theorem, one obtains the dual characterization of the total variation, i.e.,

$$|\mu|_r(\Omega) = \sup\{\sum_{i=1}^2 \int_\Omega \varphi_i \, d\mu_i : \varphi \in C_0(\Omega;\mathbb{R}^2),\ |\varphi(x)|_{r'} \le 1 \text{ a.e. in } \Omega\}, \tag{6.2.1}$$

for all $\mu \in M(\Omega;\mathbb{R}^2)$, where

$$1 = \tfrac{1}{r} + \tfrac{1}{r'}.$$

In order to guarantee existence and uniqueness of a solution to (TV-P) one may assume that $B := K^*K$ is invertible, otherwise an additional strictly convex and coercive term is needed. This model is well understood and efficient solvers are available, see, for instance, [30, 73].

Only recently, also heterogeneous (distributed) regularizations have gained interest. Here, the assumption that α is constant is dropped, and instead, it is assumed that $\alpha = \alpha(x)$ is a nonnegative function on Ω. From now on we consider the case of a uniformly continuous regularization parameter. For $\alpha \in C(\overline{\Omega}), \alpha \ge 0$, the integral

$$\int_\Omega \alpha \, d|Du|_r = |\alpha Du|_r(\Omega) \tag{6.2.2}$$

has to be understood as the integral of α with respect to the total variation $|Du|_r$. If, additionally, α is bounded away from zero, then there exists a unique solution to (TV-P). From an application point of view, the distributed TV-regularization is preferable in many respects [70]. However, as in the case for a constant parameter α, the variational formulation (TV-P) poses major difficulties for numerical algorithms for being nonsmooth and posed in a nonreflexive Banach space. Instead, one may consider the following problem.

Problem (TV-D).

$$\begin{cases} \min & \tfrac{1}{2}\|\operatorname{div} p + K^*f\|^2_B - \tfrac{1}{2}\|f\|^2_{L^2(\Omega)} \qquad \text{over } p \in H_0(\operatorname{div}), \\ \text{s.t.} & |p(x)|_{r'} \le \alpha(x) \text{ a.e. in } \Omega, \end{cases}$$

where $\|u\|_B^2 := (u, B^{-1}u)_{L^2(\Omega)}$ *for* $u \in L^2(\Omega)$.

With the help of the results from Section 5.2 it can be shown that problem (TV-D) is the (pre)dual problem of (TV-P); cf. [65] for the case of a constant α.

Theorem 6.15. *Let* $\alpha \in C(\overline{\Omega})$ *be a positive function, i.e.,* (5.2.2) *is assumed to hold. Then (TV-D) is a Fenchel predual problem of (TV-P) and no duality gap occurs.*

Proof. Define $F : H_0(\operatorname{div};\Omega) \to \mathbb{R} \cup \{+\infty\}$ and $G : L^2(\Omega) \to \mathbb{R}$ by

$$F(p) := i_{K(H_0(\operatorname{div};\Omega);|\,.\,|_{r'}))}(p), \quad G(u) := \tfrac{1}{2}\|u + K^* f\|_B^2 - \tfrac{1}{2}\|f\|^2_{L^2(\Omega)},$$

for

$$K(X(\Omega);|\,.\,|_{r'}) := \{p \in X(\Omega) : |p(x)|_{r'} \le \alpha(x) \text{ a.e. in } \Omega\}, \quad X(\Omega) \subset L^2(\Omega)^2.$$

Further observe that (TV-D) can be equivalently written as

$$\min \quad F(p) + G(\operatorname{div} p) \qquad \text{over } p \in H_0(\operatorname{div};\Omega). \tag{6.2.3}$$

The dual problem to (6.2.3) is given by

$$\inf \quad F^*(-\operatorname{div}^* u) + G^*(u) \qquad \text{over } u \in L^2(\Omega),$$

and the constraint qualification (2.2.3) is fulfilled such that there is no duality gap. The Fenchel conjugate G^* of G can be computed in a straightforward way and one obtains

$$G^*(u) = \tfrac{1}{2}(u, Bu)_{L^2(\Omega)} - (u, K^* f)_{L^2(\Omega)} + \tfrac{1}{2}\|f\|^2_{L^2(\Omega)} = \tfrac{1}{2}\|Ku - f\|^2_{L^2(\Omega)}, \tag{6.2.4}$$

such that it suffices to determine $F^*(-\operatorname{div}^* u)$ for $u \in L^2(\Omega)$. Two cases are distinguished.

(i) $u \in BV(\Omega)$: From the density of $K(C_0^1(\Omega;\mathbb{R}^2);|\,.\,|_{r'})$ in $K(H_0(\operatorname{div};\Omega);|\,.\,|_{r'})$ according to Theorem 5.7, one deduces that

$$\begin{aligned} F^*(-\operatorname{div}^* u) &= \sup_{p \in K(H_0(\operatorname{div};\Omega);|\,.\,|_{r'})} (u, -\operatorname{div} p) \\ &= \sup_{p \in K(C_0^1(\Omega)^2;|\,.\,|_{r'})} \langle Du, p\rangle_{(C_0(\Omega;\mathbb{R}^2)^*, C_0(\Omega;\mathbb{R}^2))} \\ &= \sup_{\substack{p \in C_0^1(\Omega)^2,\\ |p(x)|_{r'} \le 1 \text{ in } \Omega}} \langle Du, \alpha p\rangle_{(C_0(\Omega;\mathbb{R}^2)^*, C_0(\Omega;\mathbb{R}^2))}, \end{aligned}$$

where the last equality follows from (5.2.2). Using the dual characterization (6.2.1) of $|Du|_r$ as well as (6.2.2), one obtains

$$F^*(-\operatorname{div}^* u) = \sup_{\substack{p \in C_0^1(\Omega)^2,\\ |p(x)|_{r'} \le 1 \text{ in } \Omega}} \langle Du, \alpha p\rangle_{(C_0(\Omega;\mathbb{R}^2)^*, C_0(\Omega;\mathbb{R}^2))} = |\alpha Du|_r = \int_\Omega \alpha \, d|Du|_r.$$

(ii) $u \in L^2(\Omega) \setminus BV(\Omega)$: With the above density argument, one obtains with $\underline{\alpha} := \inf_{x\in\Omega} \alpha(x)$,

$$\begin{aligned} F^*(-\operatorname{div}^* u) &= \sup_{p \in K(C_0^1(\Omega)^d; |\,.\,|_{r'})} \langle Du, p\rangle_{(C_0(\Omega;\mathbb{R}^2)^*, C_0(\Omega;\mathbb{R}^2))} \\ &\geq \sup_{\substack{p\in C_0^1(\Omega)^d, \\ |p(x)|_{r'} \leq \underline{\alpha} \text{ in } \Omega}} \langle Du, p\rangle_{(C_0(\Omega;\mathbb{R}^2)^*, C_0(\Omega;\mathbb{R}^2))} = +\infty, \end{aligned}$$

as $Du \notin M(\Omega;\mathbb{R}^2)$. Together with (6.2.4), one obtains problem (TV-P) as a Fenchel dual problem to (TV-D).

□

From a numerical point view, problem (TV-D) is more favorable than the original formulation in $BV(\Omega)$ in that it features a quadratic objective functional subject to pointwise constraints on the function value. Most importantly, the unique solution to (TV-P) can be retrieved from any solution of the (pre)dual problem (TV-D) using primal-dual optimality conditions. In addition, an efficient semismooth Newton solver for the (pre)dual problem is available. For details we refer to [65, 73, 70].

As for a novel numerical procedure to approximate the predual problem posed in $H_0(\operatorname{div};\Omega)$, it may prove worthwhile to discretize (TV-D) using the well known Raviart-Thomas finite element space. To set the stage for this ansatz, assume for simplicity that Ω is polygonal. Upon establishing a sequence of geometrically conformal (see [47]) and shape-regular (see (6.1.2)) triangulations $(\mathcal{T}_h)$ of Ω induced by a sequence of mesh widths $h = (h_n)$, one obtains the following discretized problems.

Problem (TV-D$_h$).

$$\begin{cases} \min & \frac{1}{2}\|\operatorname{div} p + K^* f\|_B^2 - \frac{1}{2}\|f\|^2_{L^2(\Omega)} \qquad \textit{over } p \in RT_{0,h}(\Omega), \\ \textit{s.t.} & |p(x_T)|_{r'} \leq \alpha(x_T) \quad \forall\, T \in \mathcal{T}_h. \end{cases}$$

Observe that the pointwise constraints are realized on the midpoints of the triangulation. Note that in contrast to the node values, the midpoint values are well-defined for Raviart-Thomas functions. Using the results from Section 6.1 on the Mosco-convergence of discretized convex sets (Theorem 6.10), one concludes that the (discrete) solutions p_h converge weakly in $H_0(\operatorname{div};\Omega)$ to a solution of problem (TV-D) as $h \to 0$. Many interesting questions regarding problem (TV-D$_h$) remain to be investigated. This primarily concerns the effect of the discretization on the primal problem and its implication for the application to image restoration as well as the realization of the midpoint constraints within an efficient solver for (TV-D$_h$).

Part III

An Infinite-Dimensional Semismooth Newton Solver for Elasto-Plastic Contact Problems

7 Quasi-Static Hardening Plasticity and Contact Condition

7.1 Introduction

In this part we consider the quasi-static elasto-plasticity model with an associative flow law (sometimes called Prandtl-Reuss normality law) and von Mises hardening under the small strain assumption set forth in [61]. First investigations of the elasto-plastic problem from a mathematical point of view can be found in [44, 79], where [79] includes an existence result for the fully continuous case which extends the corresponding result in the absence of hardening from [78]. The numerical analysis of semi-discrete and fully-discrete versions can be found, for example, in [2, 60]. Appropriate discretization schemes for plasticity problems with hardening have been investigated extensively in the recent past. Here we only mention [3, 27, 26, 108] for adaptive finite element methods. Numerical solution methods comprise the multigrid approach in [123], various generalized Newton methods in finite dimensions [32, 58, 107, 123, 125], including the standard *return mapping* algorithm in [112] as well as interior point strategies, cf. e.g. [85].

A general introduction to elastic contact problems including corresponding numerical approaches can be found in the monographs [76, 98], and multigrid methods for elastic contact are analyzed, e.g., in [83] and [84, 86], where the latter references are devoted to two-body contact. For the treatment of elastic friction problems we refer to [34, 86] as well as to the efficient active set algorithm proposed in [77]. Subspace correction methods for variational inequalities of the second kind with application to frictional contact have been investigated in [12]. In [32, 59] plastic material behavior is incorporated in addition to the contact constraints. In the latter references the elasto-plastic friction problem is reformulated utilizing a nonlinear complementarity problem (NCP) function yielding a nonsmooth system which can be solved efficiently by applying a generalized Newton method in a discrete framework provided a set of damping parameters is chosen appropriately.

While some attention has been paid to infinite-dimensional methods in linear elasticity with (frictional) contact [87, 115], elasto-plastic problems are still less researched. Among the few available references we mention [24] for domain decomposition methods leading to a linear rate of convergence.

In the following sections, we introduce the model of elasto-plastic contact and the properties of the appropriate (primal) weak formulation are reviewed based on the monograph [61]. In Chapter 8, the corresponding time-incremental problem in terms of the displacement and the plastic strain is discussed. The resulting problem may be further reduced to a problem in the displacement only [58]. While the resulting optimality conditions are semismooth in the discrete setting, the approach turns out to be problematic as far as function space convergence is concerned. In fact, due to the lack of a sufficient norm gap between domain and image space of the mapping involved in the underlying nonsmooth system in the displacement variable, generalized differentiability in the sense of Section 2.3 does not hold true. The resulting lack of a well-defined infinite-dimensional generalized Newton iteration usually results in a mesh-dependent solver.

As an alternative, we consider a specific Fenchel dual problem in terms of two stress-related variables. Being equivalent to a variational inequality problem of the first kind, the dual problem turns out to be structurally simpler as the original problem, which is a variational inequality of the mixed (i.e. first and second) kind.

In Chapter 9, we introduce a suitable regularization which consists of a combination of the

Moreau-Yosida- and the Tikhonov regularization. This allows to deal with the constraints of the dual problem while achieving the necessary norm gap requirement for the application of the semismooth Newton method by an appropriate choice of the Tikhonov regularization space. With the help of the results from Part II, the regularization is shown to converge to the original problem under certain density requirements involving the intersection of the convex constraint set with the regularization space. In this regard, we may further exploit the results from Part II to verify that these density properties are indeed fulfilled in relevant cases.

In contrast to the original problem, the regularized problems from Section 9.1 can be solved by the SSN method in infinite dimensions and the convergence of the resulting solver is studied; cf. Section 9.2. It should be emphasized that the entire convergence analysis is valid in, both, the two- and three-dimensional case. In the last section, the infinite-dimensional setting is left and a simple conforming finite element discretization is proposed. We further derive the discrete version of the solver and the original problem is approximated by a path-following strategy with respect to the regularization parameters. To verify the theoretical properties of the solver, we present numerical results for three elasto-plastic contact problems in 2D which support the theoretical results of this work.

7.2 Quasi-Static Plasticity with Hardening

In elasto-plasticity one models the behavior of an elasto-plastic material subject to a given loading procedure in a time interval $[0, T]$. Adopting the point of view of continuum mechanics, the specimen is represented by a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, whose boundary smoothness will be specified in the forthcoming sections. For mathematical purposes, it will be assumed that the body adheres to a fixed part $\Gamma_0 \subset \partial\Omega$ with a positive surface measure in order to ensure that the associated bilinear forms are elliptic. On the complement $\Sigma = \partial\Omega \setminus \overline{\Gamma}_0$, a given surface load represented by the density $g = g(t, x)$ is applied. A given volume force density is denoted by $f = f(t, x)$. The quantities of interest in this respect are the displacement $u = u(t, x) \in \mathbb{R}^N$, the plastic strain $p = p(t, x) \in \mathbb{M}_0^{N\times N}$, the mechanical stress $\sigma = \sigma(t, x) \in \mathbb{M}^{N\times N}$, internal variables $\xi = \xi(t, x) \in \mathbb{R}^m$ and stress-conjugate forces $\chi = \chi(t, x)$, which together model the evolution of the given material. The variables χ and ξ are related by a given material-dependent hardening modulus $\mathbb{H}$ through $\xi = -\mathbb{H}\chi$. The fundamental difference to an elastic material is based on the fact that plastic behavior is irreversible, i.e., if the load is removed, the material will in general not return to its original state. In contrast to elasticity, the set of admissible stresses at each material point is constraint to lie in a certain closed, convex and nonempty subset of $\mathbb{K}$ which depends on the internal variables. The boundary $\partial\mathbb{K}$ is referred to as the yield surface. The admissible set

$$\mathbb{K} = \{[\sigma, \chi] \in \mathbb{M}^{N\times N} \times \mathbb{R}^m : \phi(\sigma, \chi) \leq 0\}$$

is determined by a yield function $\phi : \mathbb{M}^{N\times N} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ which is assumed to be non-differentiable, proper, l.s.c. and convex [60]. We will further consider an associative flow rule, i.e., the generalized plastic strain rate $[\dot{p}, \dot{\xi}]$ lies in the normal cone to the yield surface;

$$[\dot{p}(t, x), \dot{\xi}(t, x)] \in N_{\mathbb{K}}(\sigma(t, x), \chi(t, x)),$$

where

$$N_{\mathbb{K}}(\sigma, \chi) = \{[p, \xi] \in \mathbb{M}^{N\times N} \times \mathbb{R}^m : \\ p : (\tilde{\sigma} - \sigma) + \xi \cdot (\tilde{\chi} - \chi) \leq 0 \;\; \forall [\tilde{\sigma}, \tilde{\chi}] \in \mathbb{K}\}. \tag{7.2.1}$$

This implies that plastic straining can only occur if $[\dot{p}, \dot{\xi}]$ lies on the yield surface. It is also assumed that the yield criterion is homogeneous, i.e., $\mathbb{K}$ does not depend on the material point. Furthermore, we only consider the quasi-static case which means that inertial forces are assumed to be negligible. Consequently, in the balance of linear momentum,

$$\rho \ddot{u} = \operatorname{Div} \sigma + f,$$

the term $\rho \ddot{u}$ is omitted and one is left with the equilibrium condition

$$-\operatorname{Div} \sigma = f.$$

We will further make use of the small strain assumption in that the total strain is reasonably well approximated by the symmetric part of the displacement gradient,

$$\varepsilon(u) = \tfrac{1}{2}(\nabla u + \nabla u^\top),$$

and Hooke's law from elasticity is additively supplemented by the quantity p which accounts for the inelastic part of the strain such that

$$\varepsilon(u) = \mathbb{C}^{-1}\sigma + p.$$

In this context,

$$\mathbb{C}(x) \in \mathbb{R}^{N\times N\times N\times N}, \mathbb{C}_{ijkl} \in L^\infty(\Omega), \tag{7.2.2}$$

denotes the fourth-order elasticity tensor which is assumed to be symmetric, i.e. $\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}$ and pointwise stable, i.e., $\exists\, \kappa_1 > 0$ with

$$\mathbb{C}(x)\sigma : \sigma \geq \kappa_1 |\sigma|_F^2 \quad \forall\, \sigma \in \mathbb{M}^{N\times N} \text{ and a.e. } x \in \Omega,$$

where $A : B = \sum_{i,j=1\ldots N} a_{ij} \cdot b_{ij}$ for $A, B \in \mathbb{R}^{N\times N}$. Analogous properties are supposed to be fulfilled by the hardening modulus $\mathbb{H}(x) \in \mathbb{R}^{m\times m}$: $H_{ij}(x) \in L^\infty(\Omega)$, and $\exists\, \kappa_2 > 0$ with

$$\mathbb{H}(x)\xi \cdot \xi \geq \kappa_2 |\xi|_2^2 \quad \forall\, \xi \in \mathbb{R}^m \text{ and a.e. } x \in \Omega.$$

The above relations are complemented by appropriate initial value conditions. For a linear hardening law and a given proper, l.s.c. and convex yield function ϕ, the following set of conditions in strong form models quasi-static elasto-plastic evolution.

Problem 7.1. *Given $f = f(t,x)$ and $g = g(t,x)$ with $f(0,x) = 0$ in Ω and $g(0,x) = 0$ on Σ, find $[u, p, \sigma, \xi] = [u, p, \sigma, \xi](t,x)$ with*

$$[u, p, \sigma, \xi](0,x) = 0 \quad \text{in } \Omega,$$

such that

$$u(t,x) = 0 \quad \text{on } \Gamma_0, \tag{7.2.3}$$
$$\sigma\nu(t,x) = g(t,x) \quad \text{on } \Sigma, \tag{7.2.4}$$
$$-\operatorname{Div}\sigma(t,x) = f(t,x) \quad \text{in } \Omega, \tag{7.2.5}$$
$$\varepsilon(u)(t,x) = \mathbb{C}^{-1}(x)\sigma(t,x) + p(t,x) \quad \text{in } \Omega, \tag{7.2.6}$$
$$[\sigma(t,x), -\mathbb{H}(x)\xi(t,x)] \in \mathbb{K} \quad \text{in } \Omega, \tag{7.2.7}$$
$$[\dot{p}(t,x), \dot{\xi}(t,x)] \in N_{\mathbb{K}}(\sigma(t,x), -\mathbb{H}(x)\xi(t,x)) \quad \text{in } \Omega, \tag{7.2.8}$$

and for $t \in [0,T]$.

Here, $N_{\mathbb{K}}$ denotes the normal cone to the convex set $\mathbb{K}$ defined in (7.2.1).

7.3 Contact Condition

Often, the displacement of the body is restricted by a given rigid obstacle giving rise to an elasto-plastic contact problem. Therefore we fix a set $\Gamma_c \subset \Sigma$ which potentially contains the contact region with the obstacle. The actual contact zone is a priori unknown. To measure the gap between the initial configuration Ω and the obstacle we use a given function $\psi = \psi(x), \psi \geq 0$, defined on Γ_c, which does not vary in time; see [98]. We further assume that friction effects are negligible such that the tangential component of the normal stress vanishes on Γ_c, i.e.,

$$(\sigma\nu)_T = 0 \quad \text{on } \Gamma_c, \quad \sigma\nu = (\sigma\nu)_\nu \nu + (\sigma\nu)_T, \tag{7.3.1}$$

where $(\sigma\nu)_\nu = \sigma\nu \cdot \nu$. The contact constraint is simply incorporated by a kinematic non-penetration condition on the normal component of the displacement u:

$$u \cdot \nu =: u_\nu \leq \psi \text{ on } \Gamma_c. \tag{7.3.2}$$

Concerning the splitting of the boundary we further assume

$$\partial\Omega = \overline{\Gamma}_c \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0, \quad \Gamma_c \cap \Gamma_1 \cap \Gamma_0 = \emptyset, \tag{7.3.3}$$

where Γ_1 is the boundary portion subject to a surface force. By (7.3.3) and (7.3.1) it follows that the normal stress $\sigma\nu$ on Γ_c can only be nonzero on the actual contact zone. Moreover, as the obstacle itself is assumed to be rigid, the normal component of the stress field on Γ_c is always nonpositive which results in the following complementarity conditions of contact

$$(\sigma\nu)_\nu \leq 0, \quad u_\nu \leq \psi, \quad (\sigma\nu)_\nu(u_\nu - \psi) = 0 \quad \text{on } \Gamma_c. \tag{7.3.4}$$

Combining the frictionless obstacle problem with the plastic evolution of Section 7.2, one finally arrives at the following set of conditions.

Problem 7.2. *Given* $f = f(t,x)$ *and* $g = g(t,x)$ *with* $f(0,x) = 0$ *in* Ω *and* $g(0,x) = 0$ *on* Σ, *find* $[u,p,\sigma,\xi] = [u,p,\sigma,\xi](t,x)$, *with*

$$[u,p,\sigma,\xi](0,x) = 0 \quad \text{in } \Omega,$$

such that

$$u(t,x) = 0 \quad \text{on } \Gamma_0, \tag{7.3.5}$$
$$\sigma\nu(t,x) = g(t,x) \quad \text{on } \Gamma_1,$$
$$-\operatorname{Div}\sigma(t,x) = f(t,x) \quad \text{in } \Omega, \tag{7.3.6}$$
$$\varepsilon(u)(t,x) = \mathbb{C}^{-1}(x)\sigma(t,x) + p(t,x) \quad \text{in } \Omega, \tag{7.3.7}$$
$$[\sigma(t,x), -\mathbb{H}(x)\xi(t,x)] \in \mathbb{K} \quad \text{in } \Omega, \tag{7.3.8}$$
$$[\dot{p}(t,x), \dot{\xi}(t,x)] \in N_{\mathbb{K}}(\sigma(t,x), -\mathbb{H}(x)\xi(t,x)) \quad \text{in } \Omega, \tag{7.3.9}$$
$$(\sigma\nu)_T = 0 \quad \text{on } \Gamma_c, \tag{7.3.10}$$
$$(\sigma\nu)_\nu \le 0, \quad u_\nu \le \psi, \quad (\sigma\nu)_\nu(u_\nu - \psi) = 0 \quad \text{on } \Gamma_c. \tag{7.3.11}$$

for $t \in [0,T]$.

Note that (7.3.7)-(7.3.9) determine the plasticity behavior and (7.3.11) represents the complementarity conditions of contact; for details cf. [61, 98].

7.4 Function Space Setting and Variational Formulation

7.4.1 Elasto-plasticity

Appropriate variational formulations for Problem 7.1 together with the existence theory can for example be found [60, 78] or [61, Chapter 7,8]. Our notation is loosely based on the latter reference. First of all, we assume that Ω is a bounded domain with Lipschitz boundary $\partial\Omega$, cf. Definition 1.1, and Γ_0 is a nonempty open subset of $\partial\Omega$. To reformulate the problem of quasi-static plasticity we define the Hilbert spaces

$$V := [H^1_{0,\Gamma_0}(\Omega;\mathbb{R}^N)], \quad Q := [L^2(\Omega;\mathbb{M}^{N\times N})],$$

suitable for the displacement u and the stress σ, respectively. Note that the space V is the vector-valued version of the space of $H^1(\Omega)$-functions with trace vanishing on Γ_0, cf. (1.2.6). These spaces are endowed with the standard scalar products. In order to mirror the plastic incompressibility condition, p is required to be an element of the closed subspace Q_0 of Q defined by

$$Q_0 := \{q \in Q : \operatorname{tr}(q) = 0 \text{ a.e. in } \Omega\},$$

which inherits the scalar product of Q. To derive the so-called primal variational formulation for Problem 7.1, (7.3.7) is used to eliminate σ and the standard weak form of (7.3.5)-(7.3.7) is given by

$$\varepsilon^*\mathbb{C}(\varepsilon(u) - p) = l \quad \text{in } V^*, \tag{7.4.1}$$

where $\varepsilon \in \mathcal{L}(V,Q)$ and $l = l(t)$ is given by

$$l(\tilde{u}) := \int_\Omega f\tilde{u}\,dx + \int_\Sigma g\tilde{u}\,d\mathcal{H}^{N-1},$$

with $f = f(t) \in L^2(\Omega)$ and $g = g(t) \in L^2(\Sigma)$. Concerning the variational formulation of the flow law (7.3.9) we recall the following abstract statement which is based on standard arguments from convex analysis. For the sake of coherence, we will sketch the short proof.

Lemma 7.3. *Let $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ open and $w(x), w^*(x) \in L^2(\Omega)^d$. Let $j : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and l.s.c.. Then the following assertions are equivalent:*

(i) $w^*(x) \in \partial j(w(x))$ *for a.e.* $x \in \Omega$.

(ii) It holds that $w^* \in \partial G(w)$*, where* $G : L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ *is defined by*

$$G(w) := \int_\Omega j(w(x))\, dx.$$

Proof. Assertion (i) is equivalent to

$$j^*(w^*(x)) + j(w(x)) - w^*(x) \cdot w(x) = 0, \quad \text{for a.e. } x \text{ in } \Omega.$$

This is equivalent to

$$\int_\Omega j^*(w^*(x)) + j(w(x)) - w^*(x) \cdot w(x)\, dx = 0, \tag{7.4.2}$$

as the integrand is always nonnegative. Since j is l.s.c., it is also a Borel function and thus a normal integrand. By [46, Prop. IX.2.1], it holds that

$$G^*(w^*) = \int_\Omega j^*(w^*(x))\, dx, \quad \forall\, w^* \in L^2(\Omega)^d.$$

Hence, (7.4.2) is precisely assertion (ii). □

Applying assertion (ii) of Lemma 7.3 to the dual variant of (7.3.9) together with (7.4.1) leads to the coupled formulation

$$\varepsilon^* \mathbb{C}(\varepsilon(u) - p) = l \quad \text{in } V^*, \tag{7.4.3}$$

$$\int_\Omega i^*_{\mathbb{K}}(\tilde{p}, \tilde{\xi})\, dx \geq \int_\Omega i^*_{\mathbb{K}}(\dot{p}, \dot{\xi}) + \mathbb{C}(\varepsilon(u) - p) : (\tilde{p} - \dot{p}) - \mathbb{H}\xi \cdot (\tilde{\xi} - \dot{\xi})\, dx \qquad \forall\, [\tilde{p}, \tilde{\xi}] \in Q_0 \times L^2(\Omega)^m. \tag{7.4.4}$$

Using the notation

$$\begin{aligned} a_1([u, p, \xi], [\tilde{u}, \tilde{p}, \tilde{\xi}]) &:= (\mathbb{C}(\varepsilon(u) - p), (\varepsilon(\tilde{u}) - \tilde{p}))_Q + (\mathbb{H}\xi, \tilde{\xi})_{L^2(\Omega)^m}, \\ D(\tilde{p}, \tilde{\xi}) &:= \int_\Omega i^*_{\mathbb{K}}(\tilde{p}, \tilde{\xi})\, dx, \end{aligned} \tag{7.4.5}$$

and testing the weak form (7.4.3) of the equilibrium condition with $(\tilde{u} - \dot{u})$ one derives the equivalent formulation as a time-dependent variational inequality problem known as the primal problem of quasi-static elasto-plasticity.

Problem 7.4. *Let* $l \in H^1((0, T); V^*)$ *with* $l(0) = 0$*. Find*

$$[u, p, \xi] : [0, T] \to V \times Q_0 \times L^2(\Omega)^m,$$

with $[u, p, \xi](0) = 0$ *such that for a.e.* $t \in (0, T)$ *it holds that*

$$a_1([u, p, \xi], [\tilde{u}, \tilde{p}, \tilde{\xi}] - [\dot{u}, \dot{p}, \dot{\xi}]) + D(\tilde{\xi}, \tilde{p}) - D(\dot{p}, \dot{\xi}) \geq l(\tilde{u} - \dot{u}) \qquad \forall\, [\tilde{u}, \tilde{p}, \tilde{\xi}] \in V \times Q_0 \times L^2(\Omega)^m \tag{7.4.6}$$

It has thus been shown that any solution of Problem 7.1 which is smooth in the space variable solves the variational inequality formulation Problem 7.4. On the other hand, if the solution to Problem 7.4 is sufficiently smooth in the space variable then standard arguments using integration by parts prove that it is also a solution of Problem 7.1. The formal equivalence has thus been shown. Similarly, an alternative variational formulation in terms of $[u, \sigma, \chi]$ can be obtained if the formulation of the flow law from (7.3.9) is kept and an elimination of p through (7.3.7) is employed. The so-called dual problem can be further reduced to what is known as the stress problem in the literature on plasticity. We refer to [78], [61, Chapter 8] and Section 10.2 for details.

Based on the analysis of an abstract time-dependent variational inequality generalizing (7.4.6), the existence result for Problem 7.4 is obtained through a time-discretization process and a subsequent limiting argument for the linear interpolates of the incremental solutions as the time step goes to zero. The argument depends on the ellipticity of the bilinear form a_1 associated to (7.4.6) and requires hardening. In the following we will focus on the case of combined linearly isotropic-kinematic hardening with the von Mises yield criterion. In this case, it holds that

$$\begin{cases} \xi = [p, \eta] \in \mathbb{M}_0^{N\times N} \times \mathbb{R}, \quad \chi = [\chi_1, \chi_2] \in \mathbb{M}^{N\times N} \times \mathbb{R}, \\ \mathbb{H}[p, \eta] = \begin{bmatrix} k_1 p & 0 \\ 0 & k_2 \eta \end{bmatrix}, \quad k_1, k_2 \geq 0, \end{cases} \tag{7.4.7}$$

where $k_1, k_2 \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega(k_1 + k_2) > 0$. This is a natural condition, otherwise there would be no hardening effect and a problem of perfect plasticity arises which requires a different functional analytic framework. This is discussed in detail in Part IV. However, the resulting problem may be approximated consistently by a sequence of plasticity problems with vanishing hardening [15].

The von Mises yield function is given by

$$\phi(\sigma, \chi) = |\operatorname{dev}\sigma + \operatorname{dev}\chi_1|_F - \sigma_y + \chi_2 + i_{\mathbb{R}^-}(\chi_2), \tag{7.4.8}$$

where $\sigma_y > 0$ is a positive constant associated to the (one-dimensional) yield stress. Note that this setting also formally includes the cases of isotropic ($k_1 = 0, \xi = \eta, \chi = \chi_2$) and kinematic ($k_2 = 0, \xi = p, \chi = \chi_1$) hardening by making the appropriate dimensional adaptations in $\mathbb{H}$ and ϕ. At this point, we can state the existence result from [61, Theorem 7.3].

Theorem 7.5 (Existence and uniqueness in Bochner Space)**.** *For combined linearly isotropic-kinematic hardening there exists a unique solution*

$$[u, p, \eta] \in H^1((0, T), V \times Q_0 \times L^2(\Omega))$$

of Problem 7.4.

The analogous statement holds true for the cases of linearly kinematic and linearly isotropic hardening.

7.4.2 Elasto-plastic contact problem

In order to be consistent with the trace theory in Sobolev spaces discussed in Section 1.2.4, a couple of regularity assumptions on the different boundary portions from (7.3.3) are in order. First assume that the Lipschitz boundary $\partial\Omega$ is split according to (1.2.15), i.e.,

$$\partial\Omega = \Gamma_0 \cup \Sigma \cup I, \quad \Gamma_0 \cap \Sigma = \emptyset,$$

with a nonempty relatively open Dirichlet boundary part Γ_0, a nonempty relatively open complementary boundary portion Σ and a common Lipschitz interface $I = \partial\Sigma = \partial\Gamma_0$. The zone of potential contact is denoted by Γ_c, where $\Gamma_c \subset \Sigma$ and the gap function $\psi \in H^{1/2}_{00}(\Sigma)$ is assumed to fulfill $\psi \geq 0$ a.e. on Γ_c. In accordance with Section 1.2.4, we will assume that either (1.2.18) or (1.2.20) is fulfilled. In the latter case we define

$$K_1 := \{u \in V : u_\nu \leq \psi \text{ a.e. on } \Gamma_c\},$$

otherwise we set

$$K_1 := \{u \in V : u_\nu \leq \psi \text{ on } \Gamma_c \text{ in } H^{1/2}_{00}(\Sigma)\}.$$

The given surface force $g = g(t) \in L^2(\Gamma_1)$ is assumed to act on the remaining part $\Gamma_1 = \Sigma \setminus \overline{\Gamma}_c$. For the variational formulation of (7.3.5)-(7.3.7) together with the complementarity conditions of contact (7.3.11), the variational equality (7.4.3) has to be replaced by the variational inequality problem of finding $u \in K_1 \subset V$ with

$$\int_\Omega \mathbb{C}(\varepsilon(u) - p) : \varepsilon(\tilde{u} - u)\, dx \geq l(\tilde{u} - u) \quad \forall\, \tilde{u} \in K_1, \tag{7.4.9}$$

where $l = l(t)$ is given by

$$l(\tilde{u}) = \int_\Omega f\tilde{u}\, dx + \int_{\Gamma_1} g(\tilde{u})\, d\mathcal{H}^{N-1}.$$

The formal equivalence to the strong formulation is discussed in [98, 76]. Adding (7.4.9) to the weak form (7.4.4) of the plastic flow law, we are now able to state the quasi-static elasto-plastic evolution problem with a rigid frictionless time-independent obstacle as a time-dependent variational inequality.

Problem 7.6. *Let $l \in H^1((0,T); V^*)$ with $l(0) = 0$. Find*

$$[u, p, \xi] : [0, T] \to K_1 \times Q_0 \times L^2(\Omega)^m,$$

with $[u, p, \xi](0) = 0$ such that for a.e. $t \in (0, T)$ it holds that

$$a_1([u,p,\xi], [\tilde{u},\tilde{p},\tilde{\xi}] - [u,\dot{p},\dot{\xi}]) + D(\tilde{p},\tilde{\xi}) - D(\dot{p},\dot{\xi}) \geq l(\tilde{u} - u) \tag{7.4.10}$$
$$\forall\, [\tilde{u},\tilde{p},\tilde{\xi}] \in K_1 \times Q_0 \times L^2(\Omega)^m$$

Note that in contrast to Problem 7.4 there is no time derivative on u involved. Instead, the non-penetration constraint has to be fulfilled. Existence results for this type of (time-dependent) contact problem are discussed, e.g., in [5].

8 The Time-Discretized Elasto-Plastic Contact Problem

8.1 Problem Formulation

In this section an appropriate time-discretization of Problem 7.6 is employed and it is shown that the time-incremental problem reduces to a variational inequality of the mixed, i.e., first and second kind. Furthermore, the unique solvability of the time-discretized problems corresponding to the case of combined linearly isotropic-kinematic hardening in conjunction with the von Mises yield criterion is established. Finally this gives rise to a numerical scheme to approximate the solution of Problem 7.6 by a well-defined time-stepping procedure.

8.1.1 The general case

To begin with, we assume that the time interval $[0, T]$ is partitioned into J subintervals

$$0 = t_0 < t_1 < \ldots < t_J = T, \quad t_n - t_{n-1} = \triangle t \quad \forall\, n = 1, \ldots, J,$$

of uniform length $\triangle t = T/J$ and we denote by

$$[u_n, p_n, \xi_n] \approx [u(t_n), p(t_n), \xi(t_n)],$$

an approximation of the state of the system at time $t = t_n$ which is defined as follows. Starting from $[u_0, p_0, \xi_0] = 0$ we replace the time derivatives in (7.4.10) by a backward Euler scheme,

$$\delta p_n := \tfrac{p_n - p_{n-1}}{\triangle t} \approx \dot{p}(t_n), \quad \delta\xi_n := \tfrac{\xi_n - \xi_{n-1}}{\triangle t} \approx \dot{\xi}(t_n)$$

where $[u_n, p_n, \xi_n] \in K_1 \times Q_0 \times L^2(\Omega)^m$ is defined as the solution of

$$a_1([u_n, p_n, \xi_n], [\tilde{u}, \tilde{p}, \tilde{\xi}] - [u_n, \delta p_n, \delta\xi_n]) + D(\tilde{p}, \tilde{\xi}) - D(\delta p_n, \delta\xi_n) \geq \langle l_n, \tilde{u} - u_n\rangle,$$
$$\forall\, [\tilde{u}, \tilde{p}, \tilde{\xi}] \in K_1 \times Q_0 \times L^2(\Omega)^m,$$

where $l_n := l(t_n)$. The existence and uniqueness of $[u_n, p_n, \xi_n]$, which depends on the ellipticity of a_1, is discussed below. Decoupling the variational inequality again by testing with $[\tilde{u}, \delta p_n, \delta\xi_n]$ and $[u_n, \tilde{p}, \tilde{\xi}]$, one obtains

$$a_1([u_n, p_n, \xi_n], [\tilde{u} - u_n, 0, 0]) \geq \langle l_n, \tilde{u} - u_n\rangle \quad \forall\, \tilde{u} \in K_1 \tag{8.1.1}$$

and

$$a_1([u_n, p_n, \xi_n], [0, \tilde{p} - \delta p_n, \tilde{\xi} - \delta\xi_n]) + D(\tilde{p}, \tilde{\xi}) - D(\delta p_n, \delta\xi_n) \geq 0$$
$$\forall\, [\tilde{p}, \tilde{\xi}] \in Q_0 \times L^2(\Omega)^m,$$

respectively. Multiplying by $\triangle t$ and using the positive homogeneity of D, the latter inequality is equivalent to

$$a_1([u_n, p_n, \xi_n], [0, \tilde{p} - \triangle p_n, \tilde{\xi} - \triangle \xi_n]) + D(\tilde{p}, \tilde{\xi}) - D(\triangle p_n, \triangle \xi_n) \geq 0 \qquad \forall\, [\tilde{p}, \tilde{\xi}] \in Q_0 \times L^2(\Omega)^m, \tag{8.1.2}$$

where $\triangle p_n = p_n - p_{n-1}, \triangle \xi_n = \xi_n - \xi_{n-1}$. Adding (8.1.1) to (8.1.2), one obtains

$$\begin{aligned} a_1([u_n, p_n, \xi_n], [\tilde{u} - u_n, \tilde{p} - \triangle p_n, \tilde{\xi} - \triangle \xi_n]) + D(\tilde{p}, \tilde{\xi}) - D(\triangle p_n, \triangle \xi_n) \\ \geq \langle l_n, \tilde{u} - u_n \rangle \quad \forall\, [\tilde{u}, \tilde{p}, \tilde{\xi}] \in K_1 \times Q_0 \times L^2(\Omega)^m, \end{aligned}$$

that is,

$$\begin{aligned} a_1([u_n, \triangle p_n, \triangle \xi_n], [\tilde{u} - u_n, \tilde{p} - \triangle p_n, \tilde{\xi} - \triangle \xi_n]) + D(\tilde{p}, \tilde{\xi}) - D(\triangle p_n, \triangle \xi_n) \\ \geq \langle l_n, \tilde{u} - u_n \rangle - a_1([0, p_{n-1}, \xi_{n-1}], [\tilde{u} - u_n, \tilde{p} - \triangle p_n, \tilde{\xi} - \triangle \xi_n]) \\ \forall\, [\tilde{u}, \tilde{p}, \tilde{\xi}] \in K_1 \times Q_0 \times L^2(\Omega)^m. \end{aligned}$$

Since a_1 is symmetric, this variational inequality problem is equivalent to the minimization problem of finding $[u_n, \triangle p_n, \triangle \xi_n]$ which solves

$$\begin{cases} \text{inf} & \frac{1}{2} a_1([u, \triangle p, \triangle \xi], [u, \triangle p, \triangle \xi]) + D(\triangle p, \triangle \xi) \\ & \qquad\qquad - l_n(u) + a_1([0, p_{n-1}, \xi_{n-1}], [u, \triangle p, \triangle \xi]) \\ \text{over} & [u, \triangle p, \triangle \xi] \in K_1 \times Q_0 \times L^2(\Omega)^m, \end{cases}$$

which means that $[u_n, p_n, \xi_n]$ solves

$$\begin{cases} \text{inf} & \frac{1}{2} a_1([u, p, \xi], [u, p, \xi]) + D(p - p_{n-1}, \xi - \xi_{n-1}) - l_n(u) \\ \text{over} & [u, p, \xi] \in K_1 \times Q_0 \times L^2(\Omega)^m. \end{cases} \tag{8.1.3}$$

One may alternatively derive the time-incremental problem (8.1.3) by considering the weak formulation of the elasto-plastic contact problem (7.3.5)-(7.3.11) with the time-derivative appearing in (7.3.9) replaced by backward divided differences.

8.1.2 The case of von Mises plasticity and linear hardening

From now on (and for the rest of Part III) we consider the case of combined linearly isotropic-kinematic hardening together with the von Mises yield criterion. The existence and uniqueness of $[u_n, p_n, \xi_n]$ can be seen as follows. Using the characteristic assumptions (7.4.7), one may easily show that

$$i^*_{\mathbb{K}}(p, \eta) = \begin{cases} \sigma_y |p|_F, & \text{if } |p|_F \leq \eta, \\ +\infty, & \text{else;} \end{cases}$$

see [61, Example 4.10]. As a result, (8.1.3) reads

$$\begin{cases} \text{inf} & \frac{1}{2} \int_\Omega \mathbb{C}(\varepsilon(u) - p) : (e(u) - p) + k_1 |p|_F^2 + k_2 \eta^2 \, dx \\ & \qquad\qquad + \int_\Omega \sigma_y |p - p_{n-1}|_F \, dx - l_n(u) \\ \text{s.t.} & |p - p_{n-1}|_F \leq \eta - \eta_{n-1}, \ u \in K_1 \\ \text{over} & [u, p, \eta] \in V \times Q_0 \times L^2(\Omega). \end{cases} \tag{8.1.4}$$

Hence, the isotropic hardening variable η may be eliminated from the optimization problem by setting

$$\eta_n = |p - p_{n-1}|_F + \eta_{n-1}.$$

It will further turn out to be convenient to make a variable transformation replacing $\triangle p_n = p - p_{n-1}$ by p. Setting

$$\beta := \sigma_y + k_2 \eta_{n-1}, \tag{8.1.5}$$

such that $\beta \in L^2(\Omega),\ \beta \geq \sigma_y$ a.e. in Ω, we obtain the following reduced optimization problem. For notational convenience, we do not explicitly indicate the time-dependence of $\beta = \beta(t_{n-1})$.

Problem 8.1.

$$\begin{cases} \inf & J(u,p) \quad \text{over } (u,p) \in V \times Q_0 \\ s.t. & u \in K_1, \end{cases}$$

where

$$J(u,p) := \tfrac{1}{2} \int_\Omega \mathbb{C}(\varepsilon(u) - p) : (\varepsilon(u) - p) + \bar{k}\, p : p \, dx + \int_\Omega \beta |p|_F \, dx + \tilde{l}_n(u,p),$$

with $\bar{k} := k_1 + k_2$, and the linear functional $\tilde{l}_n \in (V \times Q_0)^*$ is given by

$$\begin{aligned} \tilde{l}_n(u,p) := &-(g(t_n), u)_{L^2(\Gamma_1;\mathbb{R}^N)} - (f(t_n), u)_{L^2(\Omega;\mathbb{R}^N)} \\ &+ (k_1\, p_{n-1}, p)_Q - (\mathbb{C}(\varepsilon(u) - p), p_{n-1})_Q. \end{aligned}$$

Note that Problem 8.1 is equivalent to an elliptic variational inequality of the mixed kind. Writing

$$\begin{aligned} y &:= (u,p) \in Y := V \times Q_0, \\ \pi_{Q_0}(u,p) &:= p, \ \pi_{Q_0} \in \mathcal{L}(Y, Q_0), \\ a([u,p],[\tilde{u},\tilde{p}]) &:= \int_\Omega \mathbb{C}(\varepsilon(u) - p) : (\varepsilon(\tilde{u}) - \tilde{p}) + \bar{k}\, p : \tilde{p} \, dx, \end{aligned}$$

yields a more compact form of $J : Y \to \mathbb{R}$:

$$J(y) = \tfrac{1}{2} \langle Ay, y \rangle_{(Y^*,Y)} + \tilde{l}_n(y) + \int_\Omega \beta \cdot |\pi_{Q_0} y|_F \, dx, \tag{8.1.6}$$

where $A \in \mathcal{L}(Y, Y^*)$ is the linear and continuous operator associated to the bilinear form $a : Y \times Y \to \mathbb{R}$ given by

$$A = \begin{bmatrix} \varepsilon^* \mathbb{C} \varepsilon & -\varepsilon^* \mathbb{C} \\ -\mathbb{C}\varepsilon & \mathbb{C} + \bar{k}\, \mathrm{id}_{Q_0} \end{bmatrix}. \tag{8.1.7}$$

We note that a is Y-elliptic since $\operatorname{ess\,inf}_\Omega \bar{k} > 0$, cf. [60], and standard arguments show that Problem 8.1 admits a unique solution $\bar{y} = [\bar{u}, \bar{p}] \in Y$. Consequently, the time-incremental problem (8.1.4) has a unique solution determined by

$$u_n = \bar{u}, \quad p_n = \bar{p} + p_{n-1}, \quad \eta_n = \eta_{n-1} + |\bar{p}|_F, \tag{8.1.8}$$

and $[u_n, p_n, \xi_n]$ is well-defined in the case of linearly isotropic-kinematic hardening. Again, the cases of isotropic ($k_1 = 0$) and kinematic hardening ($k_2 = 0$) are implicitly contained in Problem 8.1.

In the absence of the contact constraint, which corresponds to $K_1 = V$, and without higher regularity assumptions, one can show that the solutions $[u_n, p_n, \eta_n] = [u_n^{\triangle t}, p_n^{\triangle t}, \eta_n^{\triangle t}]$, converge to the solution $[u, p, \eta]$ of Problem 7.6 in the sense that, for $\triangle t \to 0$,

$$\max_{n\in\{1,\dots,J(\triangle t)\}} \|[u_n^{\triangle t}, p_n^{\triangle t}, \eta_n^{\triangle t}] - [u(t_n^{\triangle t}), p(t_n^{\triangle t}), \eta(t_n^{\triangle t})]\| \to 0;$$

see [61, Theorem 11.9].

Without loss of generality, we will henceforth assume that (1.2.20) is fulfilled, i.e.,

$$\Gamma_c \subset \partial\Omega \text{ is open with Lipschitz boundary } \partial\Gamma_c,\ \overline{\Gamma}_c \subset \Sigma,$$

such that the constraint set K_1 with respect to the displacement is given by (1.2.21); cf. Section 1.2.4. For ease of notation, we designate by

$$Z := H^{1/2}(\Gamma_c)$$

the space of traces of $H^1(\Omega)$-functions restricted to Γ_c. Consequently, the non-penetration constraint on the displacement u reads

$$u_\nu - \psi \in Z_-, \quad Z_- := \{z \in Z : z \le 0 \text{ a.e. on } \Gamma_c\},$$

i.e., Z_- denotes the standard negative cone in $H^{1/2}(\Gamma_c)$. Moreover, Corollary 1.5 ensures that the normal trace mapping restricted to Γ_c,

$$\tau_\nu^{\Gamma_c} : V \to Z, \quad u \mapsto \tau_\nu^{\Gamma_c}(u) = (\tau(u)\cdot\nu)|_{\Gamma_c},$$

is a well-defined, surjective, bounded linear operator. For further reference we restate Problem 8.1 of the n-th time-step of the time-discretized elasto-plastic contact problem with linearly isotropic-kinematic hardening and the von Mises yield criterion in compact form:

Problem (EPC). *Find the solution $[\bar{u}, \bar{p}]$ of*

$$\begin{cases} \min & J(u,p) \quad \textit{over } [u,p] \in V \times Q_0, \\ \textit{s.t.} & u_\nu \le \psi \quad \textit{a.e. on } \Gamma_1, \end{cases} \tag{8.1.9}$$

where

$$J(u,p) = J(y) = \tfrac{1}{2}\langle Ay, y\rangle_{(Y^*,Y)} + \tilde{l}_n(y) + \int_\Omega \beta \cdot |\pi_{Q_0} y|_F \, dx.$$

This is the problem of interest in the forthcoming sections.

8.1.3 A reduced formulation

With the help of Moreau's theorem, (8.1.4) can be further reduced to a (Fréchet) differentiable problem in the displacement only, cf. [58]: To see this, one may endow Q with the alternative scalar product

$$(\tilde{q}, q)_{\mathbb{C}} := (\mathbb{C}q, \tilde{q})_Q, \quad \|q\|_{\mathbb{C}} := \sqrt{(q,q)_{\mathbb{C}}},$$

such that (8.1.4) takes the compact form

$$\begin{cases} \inf & \tilde{J}(u,p) := \tfrac{1}{2}\|\varepsilon(u) - p\|_{\mathbb{C}}^2 + \tilde{G}(p) - l_n(u), \\ \text{over} & [u,p] \in K_1 \times Q, \end{cases}$$

with a convex, proper and l.s.c. function $\tilde{G} : Q \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{G}(p) := k_1 |p|_F^2 + k_2(\eta_{n-1} + |p - p_{n-1}|_F)^2 + \int_\Omega \sigma_y |p - p_{n-1}|_F \, dx, \quad p \in Q_0,$$

and $\tilde{G}(p) = +\infty$ if $p \notin Q_0$. Observe that the term

$$\inf_{p \in Q} \quad \tfrac{1}{2} \|\varepsilon(u) - p\|_{\mathbb{C}}^2 + \tilde{G}(p) \tag{8.1.10}$$

represents the Moreau-Yosida regularization (with regularization parameter equal to one) of the function $\tilde{G}$ at $\varepsilon(u)$ in the space $(Q, \| \,.\, \|_{\mathbb{C}})$. The standard result for the Moreau-Yosida regularization ensures that, for fixed $u \in V$, the problem (8.1.10) admits a unique solution $p = \hat{p}_n(\varepsilon(u))$ and that the reduced problem in the displacement only,

$$\inf \quad \hat{J}(u) := \tilde{J}(u, \hat{p}_n(\varepsilon(u))), \quad \text{over } u \in K_1, \tag{8.1.11}$$

has a Fréchet differentiable objective function $\hat{J} : V \to \mathbb{R}$. However, the resulting optimality condition is not eligible to Newton differentiation (in the sense of Definition 2.7) in infinite dimensions: For the sake of illustration, assume that there is no contact constraint and consider the case of isotropic hardening. Then, the optimality condition for the solution u_n to (8.1.11) with $K_1 = V$ is given by

$$\langle \hat{J}'(u_n), \tilde{u} \rangle = (\varepsilon(u_n) - \hat{p}_n(\varepsilon(u_n)), \varepsilon(\tilde{u}))_{\mathbb{C}} - l_n(\tilde{u}) = 0, \quad \forall\, \tilde{u} \in V,$$

and $\hat{p}_n : Q \to Q$ is given by

$$\hat{p}_n(\varepsilon(u)) = c\, \mathfrak{m}(\operatorname{dev} \mathbb{C}(\varepsilon(u) - p_{n-1})) + p_{n-1}, \tag{8.1.12}$$

where,

$$\mathfrak{m}(\tilde{p}) = [|\tilde{p}|_F - c_n]^+ \mathfrak{q}(\tilde{p}), \; \tilde{p} \in Q_0,$$

with a material-dependent constant $c > 0$ and a nonnegative function $c_n \in L^2(\Omega)$, see [58, Theorem 3.8]. Here we follow the notation of Lemma 2.10 except that $\mathfrak{q}$ is defined with respect to the $| \,.\, |_F$-norm. The Newton differentiability of the mapping $\hat{J}' : V \to V^*$ hinges on the Newton differentiability of $u \mapsto \hat{p}_n(\varepsilon(u))$ as a mapping from $V \to Q$. From the discussion of Lemma 2.10 it follows that the latter requires

$$\operatorname{dev} \mathbb{C}(\varepsilon(u) - p_{n-1}) \in L^6(\Omega; \mathbb{M}_0^{N \times N}), \quad c_n \in L^\infty(\Omega),$$

which is certainly not fulfilled even if the data is more regular. In the discrete setting, this issue is of no relevance such that the analogous discrete semismooth Newton algorithm is well-defined and locally superlinearly convergent. However, the lack of a well-defined Newton iteration in the continuous setting usually results in mesh-dependent convergence of the associated generalized Newton scheme.

8.2 A Fenchel Dual Problem

8.2.1 Fenchel duality set-up

For numerical purposes it turns out that the Fenchel dual problem to Problem (EPC) is favorable in the sense that, upon regularization, it can be solved efficiently by semismooth Newton techniques.

In order to suit the application of the Fenchel duality theory set forth in Section 2.2, Problem (EPC) is rewritten in the form

$$\min \quad F(y) + G(\Lambda y), \qquad \text{over } y \in Y,$$

with a Gâteaux-differentiable function F, a l.s.c., proper, and convex function G and a linear and continuous operator Λ. In fact, we define $F : Y \to \mathbb{R}$ by

$$F(y) := \tfrac{1}{2}\langle Ay, y\rangle_{(Y^*,Y)} + \tilde{I}_n(y).$$

and $G : Z \times L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ by

$$G(z,q) := G_1(z) + G_2(q) := i_{\psi+Z_-}(z) + \int_\Omega \beta |q|_2 \, dx.$$

Furthermore, we set

$$\Lambda := \begin{bmatrix} \tau_\nu^{\Gamma_c} & 0 \\ 0 & M_F^{1/2} P^{-1} \end{bmatrix} \in \mathcal{L}(Y, Z \times L^2(\Omega)^d),$$

where $i_{\psi+Z_-}$ is the indicator function of the set $\psi + Z_-$, and

$$P : (L^2(\Omega)^d, \|\,.\,\|_{L^2(\Omega)^d}) \to (Q_0, \|\,.\,\|_{Q_0}),$$

with $d = \frac{N(N+1)}{2} - 1$, denotes the canonical parametrization of Q_0 given by

$$[q_1, q_2] \overset{P}{\mapsto} \begin{bmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{bmatrix}; \quad [q_1, q_2, q_3, q_4, q_5] \overset{P}{\mapsto} \begin{bmatrix} q_1 & q_3 & q_4 \\ q_3 & q_2 & q_5 \\ q_4 & q_5 & -(q_1+q_2) \end{bmatrix}; \tag{8.2.1}$$

for $N = 2, 3$, respectively. The symmetric positive definite matrix M_F is determined by the condition

$$|M_F^{1/2} P^{-1} p|_2 = |p|_F, \quad \forall\, p \in \mathbb{M}_0^{N\times N},$$

i.e., we require $Pp : Pq = M_F p \cdot q$ for all $p, q \in \mathbb{R}^d$. For example, it holds that

$$M_F = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}; \quad M_F = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

for $N = 2, 3$, respectively. Comparing with the Fenchel duality set-up presented in [115] for elastic contact problems, the novel definition of the operator Λ entails a partially different interpretation of the dual variable q, cf. (8.2.10). We next compute and analyze the dual problem to (EPC).

8.2.2 Computation of the Fenchel conjugates

For the adjoint operator of Λ we just note that, identifying $Q_0^* \simeq Q_0$ and $[L^2(\Omega)^d]^* \simeq L^2(\Omega)^d$, P^* is given by $P^* = M_F P^{-1}$ since

$$(Pq, \tilde{q})_Q = (Pq, PP^{-1}\tilde{q})_Q = (M_F q, P^{-1}\tilde{q})_{L^2(\Omega)^d} = (q, M_F P^{-1}\tilde{q})_{L^2(\Omega)^d},$$

for all $q \in L^2(\Omega)^d, \tilde{q} \in Q_0$, such that

$$\Lambda^* = \begin{bmatrix} \tau_\nu^{\Gamma_c *} & 0 \\ 0 & PM_F^{-1/2} \end{bmatrix} \in \mathcal{L}(Z^* \times L^2(\Omega)^d, V^* \times Q_0^*).$$

Since F is just a linear-quadratic form, the convex conjugate $F^* : Y^* \to \mathbb{R}$ can be computed in a straightforward fashion;

$$F^*(y^*) = \tfrac{1}{2}\langle y^* - \tilde{l}_n, A^{-1}(y^* - \tilde{l}_n)\rangle_{(Y^*,Y)},$$

where A is defined in (8.1.7).

Upon identifying $[Z \times L^2(\Omega)^d]^* \simeq Z^* \times L^2(\Omega)^d$, we first observe that the Fenchel conjugate for the nondifferentiable part G,

$$G^* : Z^* \times L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\},$$

can be computed separately in z and q, i.e.,

$$G^*(z^*, q) = G_1^*(z^*) + G_2^*(q),$$

where $G_2^* : L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ is given by $G_2^*(q) = i_K(q)$ for

$$K := \{q \in L^2(\Omega)^d : |q|_2 \le \beta \text{ a.e. in } \Omega\};$$

cf. Lemma 2.4. The definition of Fenchel conjugation implies

$$G_1^* : Z^* \to \mathbb{R} \cup \{+\infty\}, \quad G_1^*(z^*) = \sup_{z \in \psi + Z_-} \langle z^*, z\rangle = i_{Z_-^*}(z^*) + \langle z^*, \psi\rangle,$$

where the polar cone of Z_- is given by

$$\begin{aligned} Z_-^* &= \{z^* \in Z^* : \langle z^*, z\rangle \le 0 \;\forall z \in Z_-\} \\ &= \{z^* \in Z^* : \langle z^*, z\rangle \ge 0 \;\; \forall z \in Z \text{ with } z \ge 0 \text{ a.e. on } \Gamma_c\}. \end{aligned}$$

In abstract form, the dual problem to (EPC) is defined by

$$\inf F^*(-\Lambda^*[z^*, q]) + G^*(z^*, q) \quad \text{over } [z^*, q] \in Z^* \times L^2(\Omega)^d.$$

Applying the above calculations and changing the sign of the dual variables we may equivalently state the dual problem as follows.

Problem (D). *The Fenchel dual problem to Problem (EPC) is given by*

$$\begin{cases} \inf & \frac{1}{2}\langle [\tau_\nu^{\Gamma_c *} z^*, PM_F^{-1/2} q] - \tilde{l}_n, A^{-1}([\tau_\nu^{\Gamma_c *} z^*, PM_F^{-1/2} q] - \tilde{l}_n)\rangle - \langle z^*, \psi\rangle \\ \textit{s.t.} & z^* \le 0, \\ & |q|_2 \le \beta \textit{ a.e. in } \Omega, \\ \textit{over} & [z^*, q] \in Z^* \times L^2(\Omega)^d. \end{cases}$$

The first inequality constraint has to be understood in the sense that $z^* \in Z_+^*$ where Z_+^* is the polar cone to the set $Z_+ := -Z_-$.

Since the constraints in Problem (D) do not contain interior points, the generalized Slater condition (2.2.3) fails to hold. Hence, the hypothesis of the Fenchel Duality Theorem in its usual version is not satisfied, see, for instance, [46, III, Theorem 4.1]. However, in this special situation it is still possible to exclude the presence of a duality gap.

Proposition 8.2 (Duality). *There is no duality gap, i.e., it holds that*

$$\inf \text{ (EPC)} = -\inf \text{ (D)}.$$

Moreover, there exists a unique solution $[\bar{z}^*, \bar{q}] \in Z^* \times L^2(\Omega)^d$ to the dual problem.

Proof. (i) inf (EPC) = − inf (D): We make use of the constraint qualification (2.2.5) which reads

$$0 \in \operatorname{int}\left(\Lambda^* \operatorname{dom} G^* + \operatorname{dom} F^*\right). \tag{8.2.2}$$

As F^* is finite everywhere, we have $\operatorname{dom} F^* = Y^*$. Further, $\operatorname{dom} G^* \neq \emptyset$ implies $\Lambda^* \operatorname{dom} G^* + \operatorname{dom} F^* = Y^*$ such that (8.2.2) is always satisfied. It follows that no duality gap occurs.

(ii) Existence and uniqueness of solutions to (D): We first note that the surjectivity of $\tau_\nu^{\Gamma_c}$ implies that Λ is surjective, too. Hence, Λ^* is injective. The continuity and the strong convexity of F^* indicate that the dual objective function is strictly convex and continuous. Moreover, the coercivity of the objective function follows from the ellipticity of the bilinear form associated to A^{-1}. Indeed, with $\kappa > 0$ denoting the corresponding ellipticity constant, it follows that

$$\begin{aligned}
&F^*(\Lambda^*[z^*, q]) - \langle z^*, \psi\rangle \\
&= \tfrac{1}{2}\langle \Lambda^*[z^*, q] - \tilde{l}_n, A^{-1}(\Lambda^*[z^*, q] - \tilde{l}_n)\rangle_{(Y^*,Y)} - \langle z^*, \psi\rangle \\
&\geq \tfrac{\kappa}{2}\|\Lambda^*[z^*, q]\|_{Y^*}^2 - \|\Lambda A^{-1}\tilde{l}_n + [\psi, 0]\|\,\|[z^*, q]\|_{Z^*\times L^2(\Omega)^d} + \tfrac{\kappa}{2}\|\tilde{l}_n\|^2 \\
&\geq \tfrac{\kappa}{2\|\Lambda^{-*}\|^2}\|[z^*, q]\|_{Z^*\times L^2(\Omega)^d}^2 - \|\Lambda A^{-1}\tilde{l}_n + [\psi, 0]\|\,\|[z^*, q]\|_{Z^*\times L^2(\Omega)^d} + \tfrac{\kappa}{2}\|\tilde{l}_n\|^2,
\end{aligned}$$

where the last estimate follows from the fact that Λ^* has a bounded inverse on its (closed) range owing to the closed range theorem. With these properties of the objective function and the closedness of the constraint set, the assertion follows from standard arguments. □

8.2.3 Primal-dual optimality conditions

By the absence of a duality gap (Proposition 8.2), the solution $\bar{y} = [\bar{u}, \bar{p}]$ of the primal problem (EPC) is related to the solution $[\bar{z}^*, \bar{q}]$ of (D) by the primal-dual optimality system

$$\Lambda^*[\bar{z}^*, \bar{q}] = A\bar{y} + \tilde{l}_n, \tag{8.2.3}$$
$$-[\bar{z}^*, \bar{q}] \in \partial G(\Lambda\bar{y}),; \tag{8.2.4}$$

see (2.2.7). This is equivalent to

$$\begin{cases}
\Lambda^*[\bar{z}^*, \bar{q}] = A\bar{y} + \tilde{l}_n, \\
\quad \tau_\nu^{\Gamma_c}\bar{u} \leq \psi \quad \text{a.e. on } \Gamma_c, \\
\quad -\bar{z}^* \in N_{Z_-}(\tau_\nu^{\Gamma_c}\bar{u} - \psi), \\
\quad -\bar{q}(x) \in \begin{cases} \left\{\beta(x)\dfrac{M_F^{1/2}P^{-1}\bar{p}(x)}{|\bar{p}(x)|_F}\right\} & \text{, if } \bar{p}(x) \neq 0, \\ \overline{B}_{\beta(x)}(0) & \text{, else,} \end{cases} \qquad \text{a.e. in } \Omega,
\end{cases}$$

where $\overline{B}_{\beta(x)}(0) := \{q \in \mathbb{R}^d : |q|_2 \leq \beta(x)\}$.

Due to the constraint qualification (2.2.5), the necessary and sufficient optimality conditions for a solution $[\bar{z}^*, \bar{q}] \in Z^* \times L^2(\Omega)^d$ to (D) are characterized by the existence of $\bar{\lambda} = [\bar{\mu}, \bar{v}] \in Z \times L^2(\Omega)^d$ satisfying

$$\Lambda A^{-1}\Lambda^*[\bar{z}^*, \bar{q}] - \Lambda A^{-1}\tilde{l}_n - [\psi, 0] + \bar{\lambda} = 0, \tag{OC1}$$

$$\bar{z}^* \leq 0 \text{ in } Z^*, \quad |\bar{q}|_2 \leq \beta \text{ a.e. in } \Omega, \tag{OC2}$$

$$\langle \bar{\mu}, z^* - \bar{z}^* \rangle \leq 0, \ (\bar{v}, q - \bar{q}) \leq 0 \ \ \forall z^* \leq 0, \ \forall |q|_2 \leq \beta \text{ a.e. in } \Omega, \tag{OC3}$$

where (OC3) determines $\bar{\lambda}$ as an element of the normal cone to $Z^*_+ \times K$ at $[\bar{z}^*, \bar{q}]$. Note that the conditions $|\bar{q}|_2 \leq \beta$ and $\bar{v} \in N_K(\bar{q})$ can be characterized with the help of a pointwise NCP-function, i.e., a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ with the property

$$a \geq 0, \ b \geq 0, \ ab = 0 \Longleftrightarrow \phi(a, b) = 0.$$

For solving complementarity problems with the semismooth Newton method, it is convenient to use

$$\phi(a, b) = a - \max(0, a - cb), \quad c > 0, \tag{8.2.5}$$

as an NCP-function. Applied to this context, (OC1)-(OC3) is equivalent to the existence of $[\bar{\mu}, \bar{\zeta}]$ with

$$\Lambda A^{-1}\Lambda^*[\bar{z}^*, \bar{q}] - \Lambda A^{-1}\tilde{l}_n - [\psi, 0] + [\bar{\mu}, \bar{\zeta}\bar{q}] = 0, \tag{8.2.6}$$

$$\bar{\zeta} - \max(0, \bar{\zeta} + c(|\bar{q}|_2 - \beta)) = 0, \tag{8.2.7}$$

$$\bar{z}^* \leq 0, \tag{8.2.8}$$

$$\langle \bar{\mu}, z^* - \bar{z}^* \rangle \leq 0 \ \ \forall z^* \leq 0, \tag{8.2.9}$$

where $c > 0$ is fixed. In general, these conditions are not directly eligible to the semismooth Newton method in the sense of Equation (2.3.2): Firstly, for generalized differentiation of the mapping associated with the left hand side of (8.2.7) in infinite dimensions, the setting lacks a suitable norm gap, see Section 2.3. Secondly, (8.2.9) cannot be reformulated with the help of an pointwise NCP-function such as (8.2.5). This is due to the fact that elements of Z^* in general do not allow for a pointwise interpretation. Note that these issues are absent if a direct discretization is applied, which may, however, be at the cost of mesh dependent convergence rates. For these reasons, we employ a penalization-regularization approach in the next sections.

8.2.4 Interpretation of the dual variables

Considering the second component in (8.2.3) and using $P^* = M_F P^{-1}$, we obtain a direct relation between $\bar{q}$ and the stress $\sigma_n = \mathbb{C}(\varepsilon(u_n) - p_n)$;

$$P(M_F^{-1/2}\bar{q}) = -\mathbb{C}(\varepsilon(\bar{u}) - \bar{p}) + \bar{k}\bar{p} + k_1 p_{n-1} + \mathbb{C}p_{n-1} \quad \text{in } Q_0^*.$$

Further using (8.1.8), it follows that

$$P(M_F^{-1/2}\bar{q}) = -\sigma_n + k_1 p_n + k_2(p_n - p_{n-1}) \quad \text{in } Q_0^*.$$

This implies

$$P(M_F^{-1/2}\bar{q}) = \operatorname{dev}(-\sigma_n + k_1 p_n) + k_2(p_n - p_{n-1}) \quad \text{in } Q_0, \tag{8.2.10}$$

such that $|\bar{q}|_2$ determines the value of the von Mises yield function in the case of kinematic hardening, cf. (7.4.8). Thus, the norm of $\bar{q}$ characterizes the elasto-plastic material behavior. Moreover, by multiplying (8.2.3) by $[u, 0], u \in V$, it can be shown that

$$\bar{z}^* = (\sigma_n \nu)_\nu, \quad \text{in } Z^*,$$

i.e., $\bar{z}^*$ corresponds to the normal component of the stress at the contact boundary. We refer to [114, p.63] for the analogous derivation of this result in the elastic case.

9 A Duality-Based Path-Following Strategy

9.1 The Regularized Problem

As explained in the preceding section, it is not possible to apply the semismooth Newton method to the optimality conditions (OC1)-(OC3) in the infinite-dimensional setting. To overcome this drawback, a standard Moreau-Yosida-regularization is applied to algorithmically handle the constraints in the dual problem (D). This modification is combined with a Tikhonov regularization governed by an appropriate dense subspace of $Z^* \times L^2(\Omega)^d$ in order to fulfill the norm gap requirement of the semismooth Newton method, see Section 2.3. This approach has been proposed in a similar context in [41]. To set the stage for the Tikhonov regularization we first choose a separable Hilbert subspace $\mathcal{H} = H_1 \times H_2 \subset L^2(\Gamma_c) \times L^2(\Omega)^d$ with dense embedding

$$\mathcal{H} = H_1 \times H_2 \hookrightarrow L^2(\Gamma_c) \times L^2(\Omega)^d,$$

together with a symmetric, continuous and elliptic bilinear form $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ represented by the operator $B \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$ with ellipticity constant $\kappa_b > 0$. We are now ready to state the regularized version of Problem (D).

Problem (D$_\gamma$). *Let $\gamma > 0$. Find the solution $[z_\gamma, q_\gamma]$ of*

$$\min \quad J^*_\gamma(z, q) \qquad \text{over } [z, q] \in \mathcal{H}$$

with

$$J^*_\gamma(z, q) := F^*(\Lambda^* \iota^*[z, q]) - (z, \psi)_{L^2(\Gamma_c)} + M^1_\gamma(z) + M^2_\gamma(q) + T_\gamma(z, q),$$

where we employ the following Moreau-Yosida-type regularizations of the indicator function associated with the inequality constraints in (D):

$$M^1_\gamma(z) := \tfrac{1}{2\gamma} \|[\hat{\mu} + \gamma z]^+\|^2_{L^2(\Gamma_c)},$$
$$M^2_\gamma(q) := \tfrac{1}{2\gamma} \|[\hat{\nu} + \gamma(|q|_2 - \beta_\gamma)]^+\|^2_{L^2(\Omega)},$$

as well as a regularization term of Tikhonov type:

$$T_\gamma([z, q]) := \tfrac{1}{2\gamma} b([z, q], [z, q]), \tag{9.1.1}$$

A few explanations are still in order. For algorithmic reasons, a nonnegative shift parameter

$$[\hat{\mu}, \hat{\nu}] \in Z_+ \times L^\infty_+(\Omega),$$

has been included in the Moreau-Yosida regularization. This is motivated for example in [66]. With regard to the assumptions on the Newton differentiability result of Lemma 2.10, the pointwise upper bound $\beta = \sigma_y + k_2 \eta_{n-1}$ is replaced by an approximation $\beta_\gamma \in L^\infty(\Omega)$ with the additional properties

$$\sigma_y \le \beta_\gamma \le \beta \text{ a.e.}, \quad \|\beta_\gamma - \beta\|_{L^2(\Omega)} \le \tfrac{1}{\gamma}, \tag{9.1.2}$$

for all γ. Note that this additional modification is only necessary in the case of isotropic hardening with $\eta_{n-1} \notin L^\infty(\Omega)$. Under these assumptions it is standard to show that Problem (D$_\gamma$) has a unique solution $[z_\gamma, q_\gamma]$. It should also be remarked that the dependence on the state of the system at the preceding time step is hidden in the definition of F^* and β, cf. Section 8.2.

In the following it is sometimes useful to specify the different canonical injections arising from the problem statement and the Tikhonov regularization. We define the pivot space $\mathcal{L}^2 := L^2(\Gamma_c) \times L^2(\Omega)^d$ and denote by

$$\iota = [\iota_1, \iota_2] : Z \times L^2(\Omega)^d \to \mathcal{L}^2$$

the canonical injection into the pivot space. The corresponding adjoint operator

$$\iota^* = [\iota_1^*, \iota_2^*] : \mathcal{L}^2 \hookrightarrow [Z \times L^2(\Omega)^d]^* \simeq Z^* \times L^2(\Omega)^d$$

is given by

$$[z, q] \mapsto [(z, \,.\,)_{L^2(\Gamma_c)}|_Z, q]. \tag{9.1.3}$$

Likewise, the embedding $\tilde{\imath} : \mathcal{H} \hookrightarrow \mathcal{L}^2$ is given by

$$[z, q] \mapsto ([z, q], \,.\,)_{\mathcal{L}^2}|_{\mathcal{H}} \in \mathcal{H}^*.$$

Building upon these canonical injections, the Tikhonov regularization is based on the dense embedding

$$\tilde{\imath} : \mathcal{H} \to Z^* \times L^2(\Omega)^d, \quad [z, q] \mapsto \iota^* \tilde{\imath}[z, q]$$

of $\mathcal{H}$ into $Z^* \times L^2(\Omega)^d$. The embedding framework together with the different Gelfand triples are illustrated in Figure 1. In this section only ι and ι^* will be mentioned explicitly whereas the other

$$\begin{tikzcd}
Z \times L^2(\Omega)^d \arrow[dr, hook, "\iota"] & & Z^* \times L^2(\Omega)^d \\
& \mathcal{L}^2 = L^2(\Gamma_c) \times L^2(\Omega)^d \arrow[ur, hook, "\iota^*"] \arrow[dr, hook, "\tilde{\imath}^*"] & \\
\mathcal{H} = H_1 \times H_2 \arrow[ur, hook, "\tilde{\imath}"] & & \mathcal{H}^*
\end{tikzcd}$$

Figure 1: Gelfand triple framework for the regularization

injections are employed tacitly. Since J_γ^* is strictly convex and Fréchet differentiable, the unique solution $v_\gamma = [z_\gamma, q_\gamma] \in \mathcal{H}$ of (D_γ) is characterized by

$$0 = N_\gamma v_\gamma - \iota \hat{l}_n + ([\mu_\gamma, \nu_\gamma], \,.\,)_{\mathcal{L}^2} \quad \text{in } \mathcal{H}^*, \tag{OC1$_\gamma$}$$

with

$$\begin{cases} \hat{l}_n := \Lambda A^{-1} \tilde{l}_n + [\psi, 0], \\ \mu_\gamma := [\hat{\mu} + \gamma z_\gamma]^+ \in L^2(\Gamma_c), \\ \nu_\gamma := [\hat{\nu} + \gamma(|q_\gamma|_2 - \beta_\gamma)]^+ \mathfrak{q}(q_\gamma) \in L^2(\Omega)^d, \end{cases} \tag{OC2$_\gamma$}$$

where $\mathfrak{q}$ is defined in (2.3.3) and the homeomorphism $N_\gamma \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$ is defined as

$$N_\gamma := \iota \Lambda A^{-1} \Lambda^* \iota^* + \tfrac{1}{\gamma} B.$$

Finally, we want to study the consistency of the regularized problems (D_γ) with respect to the original problem (D). In view of the results from Section 4.1, H_1 and H_2 are expected to satisfy the following density property.

Assumption 9.1 (Density of convex intersections)**.** *The following density assertions are supposed to hold:*

$$\overline{\iota_1^*(\{z \in H_1 : z \leq 0 \text{ a.e. on } \Gamma_c\})}^{Z^*} = Z^*_+, \tag{9.1.4}$$

$$\overline{\{q \in H_2 : |q|_2 \leq \beta \text{ a.e. in } \Omega\}}^{L^2(\Omega)^d} = \{q \in L^2(\Omega)^d : |q|_2 \leq \beta \text{ a.e. in } \Omega\}, \tag{9.1.5}$$

where $Z^*_+ := \{z^* \in Z^* : \langle z^*, z\rangle_{(Z^*,Z)} \leq 0 \ \ \forall z \geq 0\}$ *and* ι_1^* *is given by* (9.1.3).

Density properties of this type are extensively studied in Chapter 5 and we emphasize that Assumption 9.1 is satisfied in relevant cases. Several suitable examples for H_1 and H_2 with regard to Assumption 9.1, possibly depending on the smoothness of Γ_c, are provided in Section 9.3. The section is closed with an important consistency result concerning the convergence of the solutions of the regularized problems as $\gamma \to +\infty$. The result suggests a path-following-type method to approximate the solution of (D) and the associated primal-dual-path is induced by an appropriate sequence (γ_n) with $\gamma_n > 0$. For notational convenience, we only consider one sequence of positive parameters (γ_n) and we omit the subscript n by abuse of notation. The generalization of the subsequent results to different regularization-penalization parameter sequences does not pose any difficulty, see Section 4.1. The statement is obtained by verifying that the regularization defines a quasi-monotone perturbation (Definition 2.18) of the convex constraint set in (D).

Theorem 9.2 (Convergence of regularized dual solutions)**.** *Let* $(\gamma) \subset \mathbb{R}^+, \gamma \to +\infty$. *Under Assumption 9.1 it holds that*

(i) $v_\gamma = [z_\gamma, q_\gamma] \to [\bar{z}^*, \bar{q}]$ *in* $Z^* \times L^2(\Omega)^d$,

(ii) $\lambda_\gamma = [\mu_\gamma, \nu_\gamma] \rightharpoonup [\bar{\mu}, \bar{\nu}]$ *in* $\mathcal{H}^* = H_1^* \times H_2^*$,

as $\gamma \to +\infty$.

Proof. (i) First observe that Problem (D) is equivalent to the variational inequality problem of the first kind of finding $\bar{v} = [\bar{z}^*, \bar{q}] \in Z^* \times L^2(\Omega)^d$ such that

$$a(\bar{v}, v - \bar{v}) + i_{\mathcal{K}}(v) - i_{\mathcal{K}}(\bar{v}) \geq l(v - \bar{v}) \qquad \forall\, v = [z^*, q] \in Z^* \times L^2(\Omega)^d,$$

where

$$a([z^*, q], [\tilde{z}^*, \tilde{q}]) := \langle \Lambda^*[z^*, q], A^{-1}\Lambda^*[\tilde{z}^*, \tilde{q}]\rangle, \quad \mathcal{K} := Z^*_+ \times K, \quad l := \hat{l}_n.$$

For the definition of $\hat{l}_n$ see (OC2$_\gamma$). Similarly, $v_\gamma = [z_\gamma, q_\gamma]$ is the unique solution of (D$_\gamma$) characterized by

$$a(v_\gamma, v - v_\gamma) + j_\gamma(v) - j_\gamma(v_\gamma) \geq l(v - v_\gamma) \qquad \forall\, v \in Z \times L^2(\Omega)^d,$$

with the perturbed functionals $j_\gamma : Z^* \times L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$,

$$j_\gamma([z, q]) = \begin{cases} M^1_\gamma(z) + M^2_\gamma(q) + \frac{1}{2\gamma} b([z, q][z, q]), & \text{if } [z, q] \in \mathcal{H}, \\ +\infty, & \text{else.} \end{cases}$$

It can further be proven that (j_γ) is a quasi-monotone perturbation (Definition 2.18) of $i_{\mathcal{K}}$ with respect to the dense subspace $\mathcal{H}$ in $Z^* \times L^2(\Omega)^d$. In fact, set

$$\bar{j}_\gamma(z, q) := i_{\mathcal{K}}([z, q]) + \tfrac{\|\hat{\mu}\|^2}{2\gamma} + \tfrac{\|\hat{\nu} + \gamma(\beta - \beta_\gamma)\|^2}{2\gamma} + \tfrac{1}{2\gamma} b([z, q], [z, q]).$$

It is easily seen that $j_\gamma \leq \overline{j_\gamma}$ for all γ. The assumptions on β_γ from (9.1.2) imply that $\overline{j}_\gamma$ converges pointwise to $i_{\mathcal{K}\cap\mathcal{H}}$. Thus, (2.4.5) is fulfilled. Moreover, we set

$$\underline{j_\gamma}([z^*, q]) := \tfrac{\gamma}{2} r(z^*) + \tfrac{\gamma}{2} \|[(|q|_2 - \beta)]^+\|^2_{L^2(\Omega)},$$

where

$$r(z^*) = (\max\{ \sup_{\substack{z \in Z_+ \\ \|z\|_Z = 1}} \langle z^*, z\rangle, 0\})^2.$$

The functional $r : Z^* \to \mathbb{R}$ is weakly l.s.c. and fulfills

(a) $r(z^*) = 0$ for all $z^* \in Z^*_+$,

(b) $r(z^*) > 0$ for all $z^* \notin Z^*_+$,

(c) $r(z) \leq \|z^+\|^2_{L^2(\Gamma_c)}$ for all $z \in L^2(\Gamma_c)$.

In fact, as the composition of a convex, continuous and monotone function with a supremum of l.s.c. and convex functions, $r : Z^* \to \mathbb{R}$ is weakly l.s.c.. Assertions (a) and (b) are direct consequences of the definition of Z^*_+. For

$$z \in L^2_-(\Gamma_c) = \{z \in L^2(\Gamma_c) : z \leq 0 \text{ a.e. in } \Omega\},$$

it holds that $g(z) = 0$ and (c) is always satisfied. Assume now $z \notin L^2_-(\Gamma_c)$. By the density of $Z_+ = H^{1/2}_+(\Gamma_c)$ in $L^2_+(\Gamma_c)$, it holds that

$$\sup_{\substack{\tilde{z} \in Z_+ \\ \|\tilde{z}\|_Z = 1}} \langle z, \tilde{z}\rangle = \sup_{\substack{\tilde{z} \in Z_+ \\ \|\tilde{z}\|_Z = 1}} (z, \tilde{z})) > 0.$$

Moreover, one obtains

$$\begin{aligned} \|z^+\|_{L^2(\Gamma_c)} &= \sup_{\substack{\tilde{z} \in L^2(\Gamma_c) \\ \tilde{z} \neq 0}} \tfrac{1}{\|\tilde{z}\|_{L^2(\Gamma_c)}} (z^+, \tilde{z}) \\ &\geq \sup_{\substack{\tilde{z} \in L^2(\Gamma_c) \\ \tilde{z} \neq 0, \tilde{z} \geq 0 \text{ a.e.}}} \tfrac{1}{\|\tilde{z}\|_{L^2(\Gamma_c)}} (z, \tilde{z}) \geq \sup_{\substack{\tilde{z} \in Z_+ \\ \tilde{z} \neq 0}} \tfrac{1}{\|\tilde{z}\|_Z} (z, \tilde{z}) = r(z)^{1/2}, \end{aligned}$$

which implies (c). For the q-component we observe that

$$M^2_\gamma(q) = \tfrac{\gamma}{2} \|[\tfrac{\hat{v}}{\gamma} + (|q|_2 - \beta_\gamma)]^+\|^2_{L^2(\Omega)} \geq \tfrac{\gamma}{2} \|[|q|_2 - \beta]^+\|^2_{L^2(\Omega)}.$$

This implies that $\underline{j_\gamma}$ is a convex l.s.c. function which satisfies the lower bound assumptions (2.4.4). Consequently, $(\overline{j_\gamma})$ is a quasi-monotone perturbation with weakly l.s.c. lower bound $\underline{j_\gamma}$. Under Assumption 9.1, Proposition 2.19 implies that (j_γ) Mosco-converges to $i_{\mathcal{K}}$. An application of Theorem 3.1 to the above setting yields the assertion.

(ii) Testing (OC1$_\gamma$) with $v_\gamma = [z_\gamma, q_\gamma]$ yields

$$\langle \Lambda^* \iota^* v_\gamma, A^{-1} \Lambda^* \iota^* v_\gamma \rangle_{(Y^*, Y)} + \tfrac{1}{\gamma} b(v_\gamma, v_\gamma) - (\hat{l}_n, v_\gamma)_{\mathcal{L}^2} + (\lambda_\gamma, v_\gamma)_{\mathcal{L}^2} = 0,$$

which implies

$$\|\tfrac{1}{\sqrt{\gamma}} v_\gamma\|^2_{\mathcal{H}} \leq (\iota \hat{l}_n, v_\gamma)_{\mathcal{L}^2} - (\lambda_\gamma, v_\gamma)_{\mathcal{L}^2}. \tag{9.1.6}$$

Next we consider the boundary term $(\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)}$;

$$\begin{aligned}(\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)} &= (\mu_\gamma, \tfrac{1}{\gamma}\hat{\mu} + z_\gamma - \tfrac{1}{\gamma}\hat{\mu})_{L^2(\Gamma_c)} \\ &= \tfrac{1}{\gamma}\|\mu_\gamma\|^2_{L^2(\Gamma_c)} - \tfrac{1}{\gamma}(\mu_\gamma, \hat{\mu})_{L^2(\Gamma_c)} \\ &= \tfrac{1}{2\gamma}\|\mu_\gamma - \hat{\mu}\|^2_{L^2(\Gamma_c)} + \tfrac{1}{2\gamma}\|\mu_\gamma\|^2_{L^2(\Gamma_c)} - \tfrac{1}{2\gamma}\|\hat{\mu}\|^2_{L^2(\Gamma_c)} \\ &\geq -\tfrac{1}{2\gamma}\|\hat{\mu}\|^2_{L^2(\Gamma_c)}.\end{aligned}$$

Since $(\nu_\gamma, q_\gamma)_{L^2(\Omega)^d}$ is nonnegative, this entails that $(\lambda_\gamma, v_\gamma)_{\mathcal{L}^2}$ is bounded below. Together with the estimate (9.1.6) and the regularity of $\hat{l}_n$, we find that

$$\|\tfrac{1}{\sqrt{\gamma}} v_\gamma\|^2_{\mathcal{H}} \leq \|\hat{l}_n\|_{Z\times L^2(\Omega)^d}\|v_\gamma\|_{Z^*\times L^2(\Omega)^d} + c \quad (c > 0).$$

From the boundedness of (v_γ) in $Z^* \times L^2(\Omega)^d$ (cf. part (i)), one deduces that $(\sqrt{\gamma}^{-1} v_\gamma)$ is bounded in $\mathcal{H}$.

Taking the $\|.\|_{H^*}$-norm in (OC1$_\gamma$) yields

$$\begin{aligned}\|[\mu_\gamma, \nu_\gamma]\|_{H^*} &\leq \|\iota\hat{l}_n\|_{H^*} + \|\iota\Lambda A^{-1}\Lambda^*\iota^* v_\gamma\|_{H^*} + \tfrac{1}{\gamma}\|Bv_\gamma\|_{H^*} \\ &\leq \|\iota\hat{l}_n\|_{H^*} + \|\iota\Lambda A^{-1}\Lambda^*\|_{\mathcal{L}(Z^*\times L^2(\Omega)^d, \mathcal{H}^*)}\|\iota^* v_\gamma\|_{Z^*\times L^2(\Omega)^d} + \tfrac{1}{\gamma}\|B\|\|v_\gamma\|_H,\end{aligned}$$

which proves that $[\mu_\gamma, \nu_\gamma]$ is bounded in $\mathcal{H}^*$. Consequently, there exists $[\tilde{\mu}, \tilde{\nu}] \in \mathcal{H}^*$ such that

$$[\mu_\gamma, \nu_\gamma] \rightharpoonup [\tilde{\mu}, \tilde{\nu}] \quad \text{in } \mathcal{H}^*,$$

along a subsequence.

Next, testing (OC1$_\gamma$) with an arbitrary $v \in \mathcal{H}$ yields

$$\begin{aligned}0 = &\langle \iota^* v_\gamma, \Lambda A^{-1}\Lambda^*\iota^* v\rangle_{(Z^*\times L^2(\Omega)^d, Z\times L^2(\Omega)^d)} + \tfrac{1}{\gamma} b(v_\gamma, v) \\ &- (\iota\hat{l}_n, v)_{\mathcal{L}^2} + ([\mu_\gamma, \nu_\gamma], v)_{\mathcal{L}^2}.\end{aligned} \tag{9.1.7}$$

With the help of the above results, we may now pass to the limit as $\gamma \to +\infty$ in (9.1.7), such that

$$\begin{aligned}0 &= \langle \Lambda A^{-1}\Lambda^*[\bar{z}^*, \bar{q}], \iota^* v\rangle_{(Z\times L^2(\Omega)^d, Z^*\times L^2(\Omega)^d)} - (\iota\hat{l}_n, v)_{\mathcal{L}^2} + \langle [\tilde{\mu}, \tilde{\nu}], v\rangle_{(\mathcal{H}^*, \mathcal{H})} \\ &= (\Lambda A^{-1}\Lambda^*[\bar{z}^*, \bar{q}], v)_{\mathcal{L}^2} - (\iota\hat{l}_n, v)_{\mathcal{L}^2} + \langle [\tilde{\mu}, \tilde{\nu}], v\rangle_{(\mathcal{H}^*, \mathcal{H})}.\end{aligned}$$

From the density of $\mathcal{H}$ in $\mathcal{L}^2$ we infer that

$$-[\tilde{\mu}, \tilde{\nu}] = \Lambda A^{-1}\Lambda^*[\bar{z}^*, \bar{q}] - \Lambda A^{-1} l_n - [\psi, 0],$$

which corresponds to (OC1). Hence, it holds that $[\tilde{\mu}, \tilde{\nu}] = [\bar{\mu}, \bar{\nu}]$ and by uniqueness the entire sequence $[\mu_\gamma, \nu_\gamma]$ weakly converges to $[\bar{\mu}, \bar{\nu}]$. □

As an immediate consequence of the previous theorem and the primal-dual optimality conditions, the sequence of approximations of the optimal displacement-strain pair and the sequence of trial stresses converge strongly to the corresponding solution of the original elasto-plastic contact problem (EPC).

Corollary 9.3 (Convergence of primal solutions)**.** *Under Assumption 9.1, the following assertions hold true:*

(i) *For* $y_\gamma := A^{-1}(\Lambda^* \iota^* [z_\gamma, q_\gamma] - \tilde{l}_n)$ *it holds that* $y_\gamma \to \bar{y}$ *in* Y *as* $\gamma \to +\infty$.

(ii) *For* $\sigma_\gamma := \mathbb{C}(\varepsilon(u_\gamma) - p_\gamma)$ *it holds that* $\sigma_\gamma \to \bar{\sigma}$ *in* Q *as* $\gamma \to +\infty$.

Proof.

(i) The statement follows from the continuity of the operator A and (8.2.3).

(ii) The assertion follows from (i).

□

9.2 A Semismooth Newton Method for the Regularized Problem

The goal of this section is to apply the semismooth Newton method to solve the dual regularized problems (D_γ) using the necessary and sufficient optimality conditions ($OC1_\gamma$) - ($OC2_\gamma$). For that reason, the optimality conditions are reformulated with the help of a suitable semismooth operator equation. For the definition and main properties of the semismooth Newton method we refer to Section 2.3 and the references therein.

To begin with, observe that given a solution $v_\gamma = [z_\gamma, q_\gamma]$ of (D_γ), $\lambda_\gamma = [\mu_\gamma, \nu_\gamma]$ defined in ($OC2_\gamma$) is a solution of the nonsmooth operator equation

$$\Psi_\gamma(\lambda) = 0 \tag{9.2.1}$$

where the operator $\Psi_\gamma : \mathcal{H}^* \to \mathcal{H}^*$ is defined by

$$\Psi_\gamma \begin{bmatrix} \mu \\ \nu \end{bmatrix} := \begin{bmatrix} \mu \\ \nu \end{bmatrix} - \iota^* \begin{bmatrix} [\hat{\mu} + \gamma z(\lambda)]^+ \\ [\hat{\nu} + \gamma(|q(\lambda)|_2 - \beta_\gamma)]^+ \mathfrak{q}(q(\lambda)) \end{bmatrix},$$

where $v(\lambda) := [z(\lambda), q(\lambda)] := N_\gamma^{-1}(\iota \hat{l}_n - \lambda) \in \mathcal{H}$ denotes the solution to ($OC1_\gamma$) given some candidate λ for λ_γ. Consider the generalized Newton method

$$\lambda^{(j+1)} = \lambda^{(j)} - G_{\Psi_\gamma}(\lambda^{(j)})^{-1} \Psi_\gamma(\lambda^{(j)}) \tag{9.2.2}$$

to solve (9.2.1). The convergence of this iteration (at a superlinear rate) depends, among others, on the Newton differentiability of Ψ_γ in the sense of Definition 2.7. In order to comply with the norm gap requirements for the calculus rules in Section 2.3, additional restrictions on the choice of the spaces H_1 and H_2 have to be taken into account to ensure the Newton differentiability of Ψ_γ.

Assumption 9.4 (Norm gap). *The space* $\mathcal{H}$ *satisfies the continuous embedding*

$$\mathcal{H} \hookrightarrow L^{2+\varepsilon}(\Gamma_c) \times [L^6(\Omega)]^d,$$

for some $\varepsilon > 0$.

We emphasize that Assumption 9.4 is satisfied for relevant candidates of H_1 and H_2, see Section 9.3 below for specific examples. From now on, it is assumed that the regularization space $\mathcal{H}$ is selected in such a way that Assumption 9.4 is fulfilled. Under this premise, one may invoke Lemma 2.9 and Lemma 2.10 to infer the Newton differentiability of the involved pointwise (generalized) maximum functions. As a result, the operator $\Psi_\gamma : \mathcal{H}^* \to \mathcal{H}^*$ is Newton differentiable and we proceed by computing a particular Newton derivative using the aforementioned calculus rules and the chain rule for the composition with affine continuous functions. A Newton derivative of Ψ_γ is given by

$$G_{\Psi_\gamma}(\lambda)(\,.\,) = \mathrm{id}_{\mathcal{H}^*}(\,.\,) + \gamma \tilde{\imath}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda)) \end{bmatrix} \circ N_\gamma^{-1}(\,.\,),$$

which includes the following quantities:

$$\begin{aligned} \rho(q) &:= [|q|_2 + \tfrac{\hat{v}}{\gamma} - \beta_\gamma]^+ \tfrac{1}{|q|_2}, \\ \mathfrak{M}(q(\lambda))(\,.\,) &= \rho(q(\lambda))(\,.\,) + (1-\rho(q(\lambda)))\tfrac{q(\lambda)q(\lambda)^\top(\,.\,)}{|q(\lambda)|_2^2}, \end{aligned}$$

as well as the active set approximations

$$\mathcal{Z}_\gamma(z) := \{x \in \Gamma_c : (z + \tfrac{\hat{\mu}}{\gamma})(x) > 0\}, \quad \mathcal{Q}_\gamma(q) := \{x \in \Omega : (|q|_2 + \tfrac{\hat{v}}{\gamma} - \beta_\gamma)(x) > 0\}.$$

We begin the analysis of the generalized Newton iteration with the following lemma.

Lemma 9.5 (Uniform invertibility). *The operator*

$$G_{\Psi_\gamma}(\lambda) \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*)$$

is uniformly invertible with respect to λ, i.e., $G_{\Psi_\gamma}(\lambda)$ is bijective for all $\lambda \in \mathcal{H}^*$ and there exists a constant $c_1(\gamma) > 0$ independent of λ such that

$$\|\delta\|_{\mathcal{H}^*} \le c_1(\gamma)\|G_{\Psi_\gamma}(\lambda)\delta\|_{\mathcal{H}^*},$$

for all $\delta \in \mathcal{H}^*, \lambda \in \mathcal{H}^*$.

Proof. Similarly to [41] we write $G_{\Psi_\gamma}(\lambda) = \tilde{N}_\gamma(\lambda) \circ N_\gamma^{-1}$ with

$$\tilde{N}_\gamma(\lambda) = \left(N_\gamma + \gamma\tilde{\imath}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda)) \end{bmatrix}\right).$$

Since N_γ is independent of λ, it suffices to prove that the operator $\tilde{N}_\gamma(\lambda) \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$ has a uniform ellipticity constant, i.e., independent of λ. Therefore let $[z, q] \in \mathcal{H}$ and note that

$$\begin{aligned} &\langle \tilde{\imath}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda)) \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix}, \begin{bmatrix} z \\ q \end{bmatrix} \rangle_{(\mathcal{H}^*,\mathcal{H})} \\ &\qquad = (\chi_{\mathcal{Z}_\gamma(z(\lambda))} z, z)_{L^2(\Gamma_c)} + (\chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda))q, q)_{L^2(\Omega)^d} \\ &\qquad \ge \int_{\mathcal{Q}_\gamma(q(\lambda))} \rho(q(\lambda)) \left(|q|_2^2 - \tfrac{(q(\lambda)\cdot q)^2}{|q(\lambda)|_2^2}\right) \ge 0. \end{aligned}$$

This implies that

$$\langle \tilde{N}_\gamma(\lambda)v, v\rangle_{(\mathcal{H}^*,\mathcal{H})} \ge \langle N_\gamma v, v\rangle_{(\mathcal{H}^*,\mathcal{H})} \ge \tfrac{\kappa_b}{\gamma}\|v\|_{\mathcal{H}}^2 \quad \forall v \in \mathcal{H},$$

which proves the assertion. □

Lemma 9.5 guarantees that the iteration (9.2.2) and the subsequent algorithm are well-defined. The local superlinear convergence of the Newton iteration follows immediately from Theorem 2.8.

Algorithm SSN$^\lambda(\gamma)$: SSN algorithm in λ

input: $\lambda^{(0)} := (\mu^{(0)}, \nu^{(0)}) \in \mathcal{H}^* = H_1^* \times H_2^*$

set $j := 0$;
while some stopping rule is not satisfied **do**
  compute the solution $\delta_\lambda^{(j)} \in \mathcal{H}^*$ of $G_{\Psi_\gamma}(\lambda^{(j)})\delta_\lambda^{(j)} = -\Psi_\gamma(\lambda^{(j)})$;
  set $\lambda^{(j+1)} := \lambda^{(j)} + \delta^{(j)}$ and $j := j+1$;

Corollary 9.6 (Semismooth Newton algorithm). *If $\lambda^{(0)} \in \mathcal{H}^*$ is sufficiently close to λ_γ, then the following assertions hold true:*

(i) The Newton iterates $(\lambda^{(j)}) \subset \mathcal{H}^*$ generated by Algorithm SSN$^\lambda(\gamma)$ converge superlinearly to λ_γ in $\mathcal{H}^*$.

(ii) The Newton iterates $(v^{(j)}) \subset \mathcal{H}$ defined by $v^{(j)} = N_\gamma^{-1}(\imath \hat{l}_n - \lambda^{(j)})$ converge superlinearly to v_γ in $\mathcal{H}$.

Moreover, if $\lambda^{(0)} \in \mathcal{L}^2$, then $(\lambda^{(j)})_{j \in \mathbb{N}} \subset \mathcal{L}^2$.

Proof.

(i) The assertion follows directly from Theorem 2.8.

(ii) The assertion is a consequence of the fact that superlinear convergence is preserved by the topological isomorphism N_γ.
If $\lambda^{(j)} \in \mathcal{L}^2$, then we have $\Psi_\gamma(\lambda^{(j)}) \in \mathcal{L}^2$.
The definition of the Newton step (9.2.2) yields

$$G_{\Psi_\gamma}(\lambda^{(j)})\delta_\lambda^{(j)} = -\Psi_\gamma(\lambda^{(j)}) \iff$$
$$\delta_\lambda^{(j)} + \underbrace{\gamma\, \tilde{\imath}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda^{(j)}))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda^{(j)}))}\mathfrak{M}(q(\lambda^{(j)})) \end{bmatrix} \circ N_\gamma^{-1}\delta_\lambda^{(j)}}_{\in \mathcal{L}^2} = \underbrace{-\Psi_\gamma(\lambda^{(j)})}_{\in \mathcal{L}^2}$$

which proves the assertion.

□

Up to now, the Newton algorithm, whose local properties are analyzed in Corollary 9.6, is only guaranteed to converge if the starting point is chosen appropriately. To achieve a globalization of the iterative approach, the Newton-scheme may be embedded into a line search procedure. For this purpose it is convenient to formulate the infinite-dimensional semismooth Newton algorithm in v (rather than in λ). The convergence properties of the globalized algorithm depend on the descent property of the search directions in Algorithm 1, say $\delta_v^{(j)}$, with respect to the objective function J_γ^*. Therefore, we study the relation between $\delta_v^{(j)}$ and the gradient of J_γ^*. The subsequent proposition serves to verify the gradient-relatedness of the search directions.

Algorithm SSN(γ): Globalized SSN algorithm in v

input: $v^{(0)} \in \mathcal{H}$

set $j := 0$;
while some stopping rule is not satisfied **do**
  compute $\lambda^{(j)} := -N_\gamma v^{(j)} + \iota \hat{l}_n$;
  compute the solution $\delta_v^{(j)} \in \mathcal{H}$ of $\tilde{N}_\gamma(\lambda^{(j)})(-\delta_v^{(j)}) = -\Psi_\gamma(\lambda^{(j)})$;
  determine $\alpha^{(j)} > 0$ by a line search method based on $\alpha \mapsto J_\gamma^*(v^{(j)} + \alpha\delta_v^{(j)})$;
  set $v^{(j+1)} := v^{(j)} + \alpha^{(j)}\delta_v^{(j)}$ and $j := j + 1$;

Proposition 9.7 (Gradient-relatedness)**.** *The search directions* $(\delta_v^{(j)})$ *generated by Algorithm SSN(γ) satisfy*

$$\langle J_\gamma^{*\prime}(v^{(j)}), \delta_v^{(j)} \rangle_{(\mathcal{H}^*,\mathcal{H})} \leq -\tfrac{\kappa_b}{\gamma c_2(\gamma)^2} \|J_\gamma^{*\prime}(v^{(j)})\|^2_{\mathcal{H}^*},$$

where $c_2(\gamma) = \sup_\lambda \|\tilde{N}_\gamma(\lambda)\| \in (0, +\infty)$.

Proof. Note that $J_\gamma^{*\prime}(v^{(j)}) = -\Psi_\gamma(\lambda^{(j)})$. Using the definition of $\delta_v^{(j)}$ we conclude that

$$\begin{aligned}\langle J_\gamma^{*\prime}(v^{(j)}), \delta_v^{(j)} \rangle_{(\mathcal{H}^*,\mathcal{H})} &= \langle J_\gamma^{*\prime}(v^{(j)}), -\tilde{N}_\gamma(\lambda^{(j)})^{-1}(J_\gamma^{*\prime}(v^{(j)})) \rangle_{(\mathcal{H}^*,\mathcal{H})} \\ &\leq -\tfrac{\kappa_b}{\gamma\|\tilde{N}_\gamma(\lambda^{(j)})\|} \|J_\gamma^{*\prime}(v^{(j)})\|^2_{\mathcal{H}^*},\end{aligned}$$

since it holds for arbitrary $v^* = \tilde{N}_\gamma(\lambda)v \in \mathcal{H}^*, v \in \mathcal{H}$, that

$$\langle \tilde{N}_\gamma(\lambda)^{-1} v^*, v^* \rangle = \langle \tilde{N}_\gamma(\lambda)v, v \rangle \geq \tfrac{\kappa_b}{\gamma}\|v\|^2_{\mathcal{H}} \geq \tfrac{\kappa_b}{\gamma} \tfrac{1}{\|\tilde{N}_\gamma(\lambda)\|^2} \|v^*\|^2_{\mathcal{H}^*}.$$

Besides, the definition of $\mathfrak{M}$, cf. (2.3.4), yields

$$\begin{aligned}\|\tilde{N}_\gamma(\lambda)v\|_{\mathcal{H}^*} &\leq \|N_\gamma v\|_{\mathcal{H}^*} + \gamma \left\| \tilde{\iota}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} z \\ \chi_{\mathcal{Q}_\gamma(q(\lambda))} \mathfrak{M}(q(\lambda))q \end{bmatrix} \right\|_{\mathcal{H}^*} \\ &\leq \|N_\gamma\| \|v\|_{\mathcal{H}} + \gamma c \|v\|_{\mathcal{L}^2} \leq (\|N_\gamma\| + \gamma c)\|v\|_{\mathcal{H}},\end{aligned}$$

for all $v = [z, q] \in \mathcal{H}$ where $c > 0$ is a constant. This prove that

$$\sup_\lambda \|\tilde{N}_\gamma(\lambda)\| < +\infty.$$

$\square$

With the help of Proposition 9.7 and the strong convexity of the objective function J_γ^*, it is standard to infer that the sequence $(v^{(j)})$ generated by SSN(γ) equipped with an Armijo line search is globally convergent in the following sense.

Corollary 9.8 (Global convergence)**.** *For any starting point* $v^{(0)} \in \mathcal{H}$*, the entire sequence of Newton iterates* $(v^{(j)})$ *generated by the semismooth Newton algorithm SSN(γ) endowed with an Armijo line search* *converges strongly to the solution of* (D_γ) *in the norm of* $\mathcal{H}$*;*

$$v^{(j)} \to [z_\gamma, q_\gamma] \quad \text{in } \mathcal{H}.$$

We refer, e.g., to [20] for details.

9.3 The Discrete Solver

9.3.1 Regularization setting

In the previous sections, a solid theoretical algorithmic framework for an infinite-dimensional semismooth Newton solver for the elasto-plastic contact problem (EPC) has been developed. The purpose of this section is to formulate a stable discrete counterpart and to verify the theoretical property of mesh-independent superlinear convergence by suitable numerical tests. To start with, a proper choice of the Tikhonov regularization setting $[\mathcal{H}, b]$ should be both, computationally practical and admissible with respect to the theoretical requirements for the regularization framework. Recall that $[\mathcal{H}, b]$ is supposed to meet two assumptions. First, the density property Assumption 9.1 has to be fulfilled in order to guarantee the consistency of the regularization approach, cf. Theorem 9.2. Secondly, the Newton differentiability of the operator Ψ_γ representing the optimality conditions hinges on Assumption 9.4 about the norm gap, cf. (9.2.1). We propose two choices for the Tikhonov regularization pair $[\mathcal{H}, b]$.

(R1) If Γ_c is C^∞-smooth, we set $\mathcal{H} := H^1(\Gamma_c) \times H^1(\Omega)^d$ and define

$$b([z,q],[\tilde{z},\tilde{q}]) := (z,\tilde{z})_{H^1(\Gamma_c)} + (q,\tilde{q})_{H^1(\Omega)^d}.$$

(R2) Setting $\mathcal{H} := H^{1/2}(\Gamma_c) \times H^1(\Omega)^d$, we define

$$b([z,q],[\tilde{z},\tilde{q}]) := (z,\tilde{z})_{H^{1/2}(\Gamma_c)} + (q,\tilde{q})_{H^1(\Omega)^d}.$$

The H^1-inner product on Γ_c is defined analogously as for the usual domain case, i.e.,

$$(z,\tilde{z})_{H^1(\Gamma_c)} := (z,\tilde{z})_{L^2(\Gamma_c)} + (\nabla z, \nabla \tilde{z})_{L^2(\vec{\Gamma}_c)},$$

where the Hilbert space $L^2(\vec{\Gamma}_c)$ is the space of equivalence classes of measurable vector fields on Γ_c with integrable Riemannian product, cf. (1.2.22). It should also be noted that the additional regularity requirement on Γ_c in $(R1)$ can be alleviated if the definition of the space $H^1(\Gamma_c)$ from [63, 56] is properly adapted to the regularity of Γ_c by establishing a definition of the space $H^1(\Gamma_c)$ (and the distributional gradient) based on lower order distributions. We refer to Section 1.2.4 for further details about this issue.

Whereas (R2) is primarily of theoretical interest, alternative choices such as H^1_0-regularizations are also possible in view of the Poincaré-Friedrichs inequality on manifolds [118] and the results of Chapter 5. However, due to the stress-like nature of the dual variables, cf. (8.2.10), we prefer not to impose additional boundary conditions.

Since $N \in \{2,3\}$, the Sobolev Imbedding Theorem ensures that Assumption 9.4 on the norm gap is satisfied for both choices [1]. From the analysis of Section 5.1, it follows that the density property (9.1.4) is fulfilled in both cases. The validity of (9.1.5) reduces to the condition

$$\overline{K(H^1(\Omega)^d)}^{L^2(\Omega)^d} = L^2(\Omega)^d, \tag{9.3.1}$$

which depends on the upper bound $\beta = \sigma_y + k_2\eta_{n-1}$, cf. Chapter 5. For this purpose, only little extra regularity is required, e.g., $\beta \in \mathbb{LC}(\Omega)$ is sufficient (Theorem 5.17). In particular, the case of kinematic hardening ($k_2 = 0$) is always covered. In fact, the weaker closure property (9.3.1) is sufficient for the validity of Assumption 9.1.

In view of Theorem 9.2 and Corollary 9.3, Algorithm SSN(γ) is embedded into an update scheme for γ, i.e., once Algorithm SSN(γ) terminates successfully for a given γ, the (set of) penalty/regularization parameter(s) is increased and Algorithm SSN(γ) is restarted. In order to avoid the inverse of A in the computation of the Newton step, we explicitly involve the primal variable y by solving the coupled elliptic second-order system

$$\begin{bmatrix} A & -\Lambda^*\iota^* \\ \iota\Lambda & \frac{1}{\gamma}B + \tilde{\iota}^* G_{\mathcal{M}}(v) \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_v \end{bmatrix} = \begin{bmatrix} 0 \\ -\iota\Lambda y + \iota[\psi, 0] - \frac{1}{\gamma}Bv - \tilde{\iota}^*\mathcal{M}(v) \end{bmatrix}, \tag{9.3.2}$$

where

$$\mathcal{M}(z,q) := \begin{bmatrix} [\hat{\mu} + \gamma z]^+ \\ [\hat{\nu} + \gamma(|q|_2 - \beta_\gamma)]^+ \mathfrak{q}(q) \end{bmatrix}, \quad G_{\mathcal{M}}(z,q) := \gamma \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z)} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q)}\mathfrak{M}(q) \end{bmatrix}.$$

9.3.2 The discrete semismooth Newton algorithm

In the following numerical examples $\Omega \subset \mathbb{R}^2$ is polygonal, Γ_c is a line segment and we choose option (R1) for the Tikhonov regularization. We employ a conforming finite element method to solve (9.3.2) numerically: let $(\mathcal{T}_h)$ be a sequence of geometrically conformal shape-regular triangulations of Ω with $|\mathcal{T}_h|$ elements and mesh width h [47]. Denote by $(\mathcal{S}_h)$ the sequence of partitions of Γ_c into $|\mathcal{S}_h|$ line segments induced by the triangulation of Ω, i.e., $\mathcal{S}_h$ is defined by those mesh nodes of $\mathcal{T}_h$ that lie on the contact boundary Γ_c. The discrete counterparts of Y and $\mathcal{H}$ are given by

$$Y_h := [P_{1,h}^{\Gamma_0}(\Omega)]^2 \times [P_{0,h}(\Omega)]^2, \quad \mathcal{H}_h = H_{1,h} \times H_{2,h} := P_{1,h}(\Gamma_c) \times [P_{1,h}(\Omega)]^2, \tag{9.3.3}$$

with the usual P_0- and P_1−finite element spaces

$$\begin{aligned} P_{1,h}^{\Gamma}(\Omega) &= \{u \in L^\infty(\Omega) : u|_T \in \mathbb{P}_1 \ \forall\, T \in \mathcal{T}_h, u|_\Gamma = 0 \text{ a.e. }\} \cap C(\overline{\Omega}), \\ P_{0,h}(\Omega) &= \{u \in L^\infty(\Omega) : u|_T \in \mathbb{P}_0 \ \forall\, T \in \mathcal{T}_h\}, \\ P_{1,h}(\Gamma_c) &= \{u \in L^\infty(\Gamma_c) : u|_S \in \mathbb{P}_1 \ \forall\, S \in \mathcal{S}_h\} \cap C(\overline{\Gamma}_c), \end{aligned}$$

for $\Gamma \subset \partial\Omega$. Here, $\mathbb{P}_k$ denotes the set of polynomials of total degree less than or equal to k and we omit the superscript Γ whenever Γ has vanishing surface measure. The discretization $[P_{0,h}(\Omega)]^2$ of the space Q_0 is realized using the parametrization P defined in (8.2.1). The superscript h stands for the discrete version of a given linear operator corresponding to the discrete spaces (9.3.3). In order to simplify the numerical realization of the constraints, we assume that ψ and β are continuous on Ω. For the time being, we also set $\hat{\mu} = \hat{\nu} = 0$.

In the discretized setting, computational cost is kept as low as possible by approximating the L^2-norm-penalty terms in the definition of the objective in (D$_\gamma$) by the standard (barycentric) midpoint quadrature rule. The analogous node-based approach is readily derived. Moreover, in the definition of F^*, we replace the operator A by its discrete counterpart A^h, which yields

$$F_h^*(y^*) = \tfrac{1}{2}\langle y^* - \tilde{l}_n^h, (A^h)^{-1}(y^* - \tilde{l}_n^h)\rangle, \quad y^* \in Y_h^*.$$

Employing the $P_{0,h}(\Gamma_c)$-midpoint interpolant ψ_h of ψ, as well as the midpoint evaluation maps

$$\pi^h_{\Gamma_c} : H_{1,h} \to \mathbb{R}^{|\mathcal{S}_h|}, \quad \pi^h_\Omega = [\pi^h_{\Omega,1}, \pi^h_{\Omega,2}] : H_{2,h} \to \mathbb{R}^{|\mathcal{T}_h|\times 2} \simeq \mathbb{R}^{2|\mathcal{T}_h|},$$

one finally obtains the discretized-regularized problems

$$\min \quad J^*_{\gamma,h}(z,q) \quad \text{over } [z,q] \in \mathcal{H}_h, \tag{D$_{\gamma,h}$}$$

with

$$\begin{aligned} J^*_{\gamma,h}(z,q) := F^*_h(\Lambda^*\iota^*[z,q]) - (z,\psi_h)_{L^2(\Gamma_c)} + \tfrac{1}{\gamma}(z,z)_{H^1(\Gamma_c)} + \tfrac{1}{\gamma}(q,q)_{H^1(\Omega)^d} \\ + \tfrac{\gamma}{2}\sum_{i=1}^{|\mathcal{S}_h|} s_{h,i}([\pi^h_{\Gamma_c} z]_i^+)^2 + \tfrac{\gamma}{2}\sum_{i=1}^{|\mathcal{T}_h|} a_{h,i}([|[\pi^h_\Omega q]_i|_2 - \beta_{h,i}]^+)^2, \end{aligned}$$

where $\beta_{h,i}$ is the value of β at the midpoint of the i-th element of the triangulation $\mathcal{T}_h$, and $s_h = [s_{h,1}, \dots, s_{h,|\mathcal{S}_h|}] \in \mathbb{R}^{|\mathcal{S}_h|}$ and $a_h = [a_{h,1}, \dots, a_{h,|\mathcal{T}_h|}] \in \mathbb{R}^{|\mathcal{T}_h|}$ denote the vectors of side lengths and element areas corresponding to the partitions $\mathcal{S}_h$ and $\mathcal{T}_h$, respectively. The discrete counterparts of μ_γ and ν_γ are given by

$$\mu^h_\gamma(z) := \gamma \operatorname{diag}(s_h)[\pi^h_{\Gamma_c} z]^+ \in \mathbb{R}^{|\mathcal{S}_h|}, \tag{9.3.4}$$

$$\nu^h_\gamma(q) := \operatorname{diag}(\operatorname{kron}(\begin{bmatrix}1 & 1\end{bmatrix}^\top, \zeta^h_\gamma(q)))\pi^h_\Omega(q) \in \mathbb{R}^{2|\mathcal{T}_h|}, \tag{9.3.5}$$

with

$$\zeta^h_{\gamma,i}(q) := \gamma a_{h,i}[|[\pi^h_\Omega q]_i|_2 - \beta]^+ \frac{1}{|[\pi^h_\Omega q]_i|_2}, \quad i = 1, \dots, |\mathcal{T}_h|.$$

The discrete optimality system to problem (D$_{\gamma,h}$) reads

$$\Psi^h_\gamma([z^h_\gamma, q^h_\gamma]) = 0, \tag{9.3.6}$$

where the operator $\Psi^h_\gamma : \mathcal{H}_h = H_{1,h} \times H_{2,h} \to \mathcal{H}^*_h$ is defined by

$$\Psi^h_\gamma([z,q]) := N^h_\gamma[z,q] - \iota\,\hat{l}^h_n + [\pi^{h*}_{\Gamma_c}\mu^h_\gamma(z), \pi^{h*}_\Omega \nu^h_\gamma(q)],$$

for $\hat{l}^h_n := \Lambda(A^h)^{-1}\tilde{l}^h_n - [\psi_h, 0]$ and $N^h_\gamma := \iota\Lambda(A^h)^{-1}\Lambda^*\iota^* + \frac{1}{\gamma}B^h$. Each step computation of the finite-dimensional semismooth Newton iteration applied to (9.3.6) requires solving the discretized version of (9.3.2). The discrete analogue $\mathcal{M}^h$ to $\mathcal{M}$ corresponding to the approximation by the midpoint quadrature rule is given by $\mathcal{M}^h(v) = [\mu^h_\gamma(z), \nu^h_\gamma(q)]$, and its Newton derivative is denoted by $G_{\mathcal{M}^h}$. Consequently, the resulting semismooth Newton system at $[y,v] \in Y_h \times \mathcal{H}_h$ reads

$$\begin{bmatrix} A^h & -\Lambda^*\iota^* \\ \iota\Lambda & \frac{1}{\gamma}B^h + G_{\mathcal{M}^h}(v) \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_v \end{bmatrix} = \begin{bmatrix} 0 \\ -\iota\Lambda y + \iota[\psi^h, 0] - \frac{1}{\gamma}B^h v - \mathcal{M}^h(v) \end{bmatrix}, \tag{9.3.7}$$

which is posed in the space $Y^*_h \times \mathcal{H}^*_h$. For a given Newton differentiable operator $\Psi_\gamma = [\Psi_{\gamma,1}, \Psi_{\gamma,2}]$ we summarize the following discrete version of Algorithm SSN(γ) for fixed regularization-penalization parameter γ, mesh width h, starting point $v^{(0)}$ and tolerance ε_{in} to solve (9.3.6):

Algorithm SSN(γ, h): Globalized discrete semismooth Newton algorithm

input: $\varepsilon_{\text{in}} > 0,\ v^{(0)} \in \mathcal{H}_h$

1 initialize primal variables: $y^{(0)} \in Y_h$ by solving $A^h y^{(0)} = \Lambda^* \iota^* v^{(0)} - \tilde{l}_n^h$;
2 set $j := 0$;
3 **while** $(\|\Psi_\gamma^h(v^{(j)})\|_{\mathcal{H}_h^*} < \varepsilon_{\text{in}})$ not fulfilled **do**
4   compute the solution $[\delta_y^{(j)}, \delta_v^{(j)}]$ of (9.3.7);
5   determine $\alpha^{(j)} > 0$ by Armijo line search based on $\alpha \mapsto J_{\gamma,h}^*(v^{(j)} + \alpha \delta_v^{(j)})$;
6   update $[y^{(j+1)}, v^{(j+1)}] := [y^{(j)} + \alpha^{(j)} \delta_y^{(j)}, v^{(j)} + \alpha^{(j)} \delta_v^{(j)}]$;
7   set $j := j + 1$;

The discrete norm $\|\,.\,\|_{\mathcal{H}_h^*}$ in step 3 of Algorithm SSN(γ, h) is computed by solving the corresponding homogeneous coercive Neumann problems. For the implementation of the operator A^h we incorporate the zero-trace condition in the definition of the space Q_0 using the parametrization P defined in (8.2.1). In our numerical tests, the stopping criterion for Algorithm SSN(γ, h) is usually set to $\varepsilon_{\text{in}} = 10^{-10}$.

9.3.3 An inexact path-following algorithm

In order to study convergence with regard to the regularization-penalization-parameter γ we implement a heuristic version of the inexact path-following (IPF) approach designed for the obstacle problem [66]. In contrast to the foregoing sections, we assume that the penalization-regularization parameters are not equal, that is, we assume $\gamma = [\gamma_1, \gamma_2, \gamma_3, \gamma_4] \in \mathbb{R}_+^4$ where the objective functional in ($D_{\gamma,h}$) is given by

$$\begin{aligned} J_{\gamma,h}^*(z,q) := F_h^*(\Lambda^* \iota^*[z,q]) - (z, \psi_h)_{L^2(\Gamma_c)} + \tfrac{1}{\gamma_1}(z,z)_{H^1(\Gamma_c)} + \tfrac{1}{\gamma_2}(q,q)_{H^1(\Omega)^d} \\ + \tfrac{\gamma_3}{2} \sum_{i=1}^{|\mathcal{S}_h|} s_{h,i}([\pi_{\Gamma_c} z]_i^+)^2 + \tfrac{\gamma_4}{2} \sum_{i=1}^{|\mathcal{T}_h|} a_{h,i}([|[\pi_\Omega q]_i|_2 - \beta_{h,i}]^+)^2, \end{aligned} \tag{9.3.8}$$

and the semismooth Newton algorithm is embedded into an outer loop which determines a positive parameter set

$$\gamma^{(k)} = [\gamma_1^{(k)}, \gamma_2^{(k)}, \gamma_3^{(k)}, \gamma_4^{(k)}] \in \mathbb{R}_+^4$$

at the k-th outer iteration. The paradigm of inexact path-following consists in the idea that each subproblem $(D_{\gamma^{(k)},h})$ is only solved approximately with increasing precision using Algorithm SSN(γ, h) with $\gamma := \gamma^{(k)}$. In fact, for fixed $\tau_{\text{in}} > 0$, the modified (inexact) stopping criterion

$$\left(\|\Psi_{\gamma,1}^h(\tilde{v})\|_{H_{1,h}^*} < \max(\tfrac{\tau_{\text{in}}}{\gamma_3}, 0.1 \cdot \varepsilon_{\text{out}})\right) \wedge \left(\|\Psi_{\gamma,2}^h(\tilde{v})\|_{H_{2,h}^*} < \max(\tfrac{\tau_{\text{in}}}{\gamma_4}, 0.1 \cdot \varepsilon_{\text{out}})\right), \tag{9.3.9}$$

replaces line 3 of SSN(γ, h). Here, $\Psi_\gamma^h = [\Psi_{\gamma,1}^h, \Psi_{\gamma,2}^h]$ denotes the splitting into z- and q-component, respectively. After a suitable update of the parameter set $\gamma^{(k)}$, which is based on the individual residuals (lines 5 and 9), the computed approximate solution $\tilde{v} \approx v_{\gamma^{(k)}}^h$ is accepted as the next outer iterate $\tilde{v}^{(k)} := \tilde{v}$ and $\tilde{v}^{(k)}$ is used as a starting point for the solution of the subsequent problem

$(D_{\gamma^{(k+1)},h})$. As for the path-parameter update, it proved to be efficient to start with comparably large Tikhonov regularization parameters $\gamma_1^{(0)}, \gamma_2^{(0)}$ so that line 5 of IPF(h) is rarely executed. The selection of an appropriate initial parameter set is guided by tests on very coarse meshes, cf. Tables 2 and 3. In this way the effort of approximatively solving the subproblems can be expected to be kept rather low. Differently from [66] we are testing a constant augmentation of $\gamma^{(k)}$ driven by a factor $\theta > 0$. For the outer stopping criterion we consider the optimality conditions for the solution $[z^h, q^h]$ of the discrete limit problem

$$\begin{cases} \min & F_h^*(\Lambda^*\iota^*[z,q]) - (z,\psi_h) \quad \text{over } [z,q] \in \mathcal{H}_h \\ \text{s.t.} & [\pi_{\Gamma_c} z]_i \leq 0, \quad i = 1,\dots,|\mathcal{S}_h|, \\ & |[\pi_\Omega q]_i|_2 \leq \beta_{h,i}, \quad i = 1,\dots,|\mathcal{T}_h|, \end{cases}$$

which are given by

$$\begin{aligned} \Psi^h(z^h,q^h) := \iota \Lambda {A^h}^{-1} \Lambda^* \iota^* [z^h,q^h] - \iota\, \hat{l}_n^h + [\pi_{\Gamma_c}^{h*}\mu^h, \pi_\Omega^{h*}\nu^h] &= 0 \quad \text{in } \mathcal{H}_h^*, \\ \mu^h - \max(0, \mu^h + \pi_{\Gamma_c}^h z^h) &= 0 \quad \text{in } \mathbb{R}^{|\mathcal{S}_h|}, \\ \nu^h - \operatorname{diag}(\operatorname{kron}(\begin{bmatrix}1 & 1\end{bmatrix}^\top, \zeta^h))\pi_\Omega^h(q^h) &= 0 \quad \text{in } \mathbb{R}^{2|\mathcal{T}_h|} \\ \zeta^h - \max(0, \zeta^h + |\pi_\Omega^h q^h|_2 - \beta_h) &= 0 \quad \text{in } \mathbb{R}^{|\mathcal{T}_h|}. \end{aligned}$$

For given iterates $[z,q]$ and associated multipliers $[\mu(z), \nu(z)]$ we define the associated residuals $r_h \in \mathbb{R}^4, r_h = [r_{h,1},\dots,r_{h,4}]$, by

$$\begin{aligned} r_{h,1}(z,q) &:= \|\Psi_1^h(z,q)\|_{H_{1,h}^*}, \\ r_{h,2}(z,q) &:= \|\Psi_2^h(z,q)\|_{H_{2,h}^*}, \\ r_{h,3}(z,q) &:= \|\mu(z) - \max(0, \mu(z) + \pi_{\Gamma_c}^h z)\|_{L_h^2(\Gamma_c)}, \\ r_{h,4}(z,q) &:= \|\zeta(q) - \max(0, \zeta(q) + |\pi_\Omega^h q^h|_2 - \beta_h)\|_{L_h^2(\Omega)}, \end{aligned}$$

where $\|\,.\,\|_{L_h^2(\,.\,)}$ denotes the L^2-norm of the corresponding piecewise constant midpoint interpolant. In Step 2 of Algorithm IPF(h), the Lagrange multiplier candidates for μ, ν are chosen as $\mu_\gamma^h(\tilde{z}^{(k)})$ and $\nu_\gamma^h(\tilde{q}^{(k)})$ which have been defined in (9.3.4) and (9.3.5).

Algorithm IPF(h): Inexact path-following algorithm

input: $\gamma^{(0)} \in \mathbb{R}^4_+, \theta > 1, \tau_{\text{in}} > 0, \varepsilon_{\text{out}} > 0,\ \tilde{v}^{(0)} = [\tilde{z}^{(0)}, \tilde{q}^{(0)}] \in H_{1,h} \times H_{2,h}$

set $k := 0$;
while $(|r_h(\tilde{v}^{(k)})|_\infty < \varepsilon_{\text{out}})$ not fulfilled **do**
  apply Algorithm SSN(γ, h) with $\gamma = \gamma^{(k)}$, $v^{(0)} = \tilde{v}^{(k)}$ to find $\tilde{v} \in H_{1,h} \times H_{2,h}$ satisfying (9.3.9) ;
  if $\max(r_{h,3}(\tilde{v}), r_{h,4}(\tilde{v})) < \varepsilon_{out}$ **then** `// i.e., feasibility achieved`
    $[\gamma_1^{(k+1)}, \gamma_2^{(k+1)}] = \theta \cdot [\gamma_1^{(k)}, \gamma_2^{(k)}]$ `// update regularization parameters`
  else
    for $i = 3, 4$ **do**
      if $r_{h,i}(\tilde{v}) > \varepsilon_{out}$ **then**
        $\gamma_i^{(k+1)} := \gamma_i^{(k)} \cdot \theta$; `// update penalization parameters`
  update $\tilde{v}^{(k+1)} := \tilde{v}$ set $k := k+1$;

9.3.4 Numerical tests

In all our numerical tests we assume that we compute the first step $(n = 1)$ of the time-incremental problems (*EPC*). In particular, the initial conditions are given by $p_0 \equiv 0$, $u_0 \equiv 0$ and, in the presence of isotropic hardening, $\eta_0 \equiv 0$.

Example (a) - Screw wrench

In this example we consider an elasto-plastic screw wrench, whose geometry can be extracted from Figure 3. The elastic behavior is described by $\mathbb{C}\varepsilon = \mu_1 \operatorname{tr}(\varepsilon) I + 2\mu_2\varepsilon$ with $\mu_1 \equiv$ 1.15e01, $\mu_2 \equiv$ 7.69e00, and the material is assumed to satisfy the isotropic hardening law ($k_1 \equiv 0$) with $k_2 \equiv$ 4.0e-01 and $\sigma_y =$ 2e-01. The deformation is caused by a pressure force $g(t_1, x) = -$6.0e-03 $\cdot\nu(x)$ on $\Gamma_1 = \operatorname{conv}(\{(5, 2.6), (8, 2)\})$. Volume forces are assumed to be absent; $f(t_1) \equiv 0$. The domain is split into the Dirichlet part $\Gamma_0 := ((0,1) \times \{2\}) \cup ((0,1) \times \{3\})$, and the potential contact zone $\Gamma_c := (0,1) \times \{4\}$ with $\psi \equiv$ 1.0e00, such that the contact constraint can be expected to be inactive at the solution and only plasticity effects have to be taken account of. The results obtained by Algorithm SSN(γ, h) are summarized in Table 1. To verify mesh-independent convergence, we compute the solution for various fixed parameters γ on meshes with decreasing mesh width starting from approximately $1.25 \cdot 10^4$ nodes to about $1.6 \cdot 10^6$ nodes, cf. Table 1, using uniform mesh refinement. The solution on a given mesh is prolongated to the next finer mesh to serve as a starting point $v^{(0)}$ of Algorithm SSN(γ, h) on the refined triangulation. For validation purposes a restart strategy using the zero function as a starting point on each mesh is also tested. It is observed that the iterations count for the restart strategy stays bounded as the number of nodes are increased. Variations may be caused by the necessity for globalization in SSN(γ, h) for higher values of γ . The iteration numbers for the nested strategy even tend to decrease with decreasing mesh width. The theoretical property of local mesh-independent superlinear rate of convergence is verified experimentally by investigating the convergence quotients Q_j associated with the iterates $(v^{(j)})$ generated for fixed (γ, h),

$$Q_j := \frac{\|v^{(\omega-5+j)} - v^\star\|_H}{\|v^{(\omega-6+j)} - v^\star\|_H}, \quad j = 1 \ldots, 5,$$

where ω denotes the iteration count for Algorithm SSN(γ, h) and $v^\star$ denotes the solution obtained by applying the same algorithm with higher precision $\varepsilon_{\text{in}} = 10^{-14}$. As predicted by the theory,

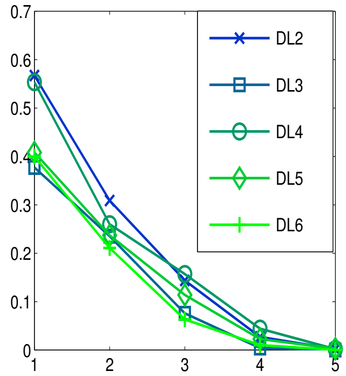

Figure 2: Example (a): $Q_j, j = 1, \ldots, 5$, for $\gamma = 1.0e05$ and various discretization levels (DL)

the convergence quotients Q_j tend to zero and rest stable under decreasing mesh width even for large γ, cf. Figure 2. This clearly indicates mesh-independent convergence behavior for each fixed γ. Applying the heuristic inexact path-following approach IPF(h) with regard to the penalty parameter set γ, we display in Figure 4 the resulting approximate optimal plastic strain as well the regions of extensive plastic straining on the deformed configuration. In Figure 3, relation (8.2.10) is employed to plot the approximate yield function.

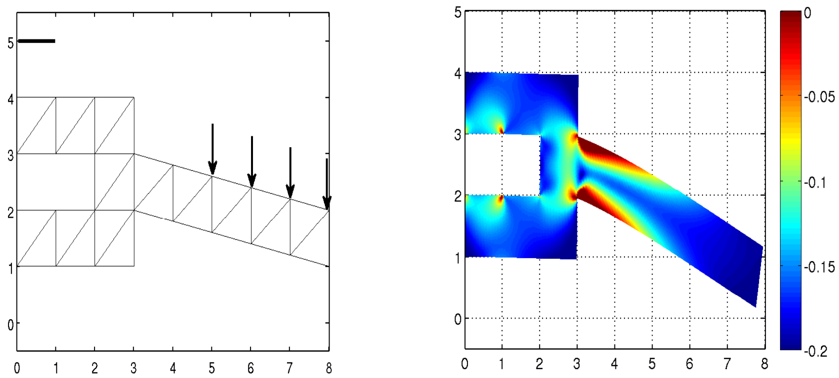

Figure 3: Example (a): initial configuration *(left)*, yield functional *(right)*

Example (b) - L-shape

We consider an L-shaped domain $\Omega = (0, 0.5] \times (0.5, 1) \cup (0.5, 1) \times (0, 1)$ and assume that the elastic behavior of the material is described by $\mathbb{C}\varepsilon = \mu_1 \operatorname{tr}(\varepsilon) I + 2\mu_2 \varepsilon$ with $\mu_1 = \mu_2 \equiv 5.0\text{e}02$. It is further assumed that the material obeys the kinematic hardening law, i.e., $k_2 \equiv 0$. The plastic material parameters are given as follows: $\sigma_y = 2.0\text{e}01$, $k_1 \equiv 5.0\text{e}01$. The body shall be fixed at $\Gamma_0 = (0.5, 1) \times \{0\}$. We set $\psi \equiv 4.0\text{e-}02$ on $\Gamma_c = (0, 1) \times \{1\}$ and apply a pressure force

γ / # nodes	12,5k	25k	50k	100k	200k	400k	800k	1.6M
1.0e01	1	1	1	1	1	1	1	1
1.0e02	4	4	3	3	2	2	2	2
1.0e03	7	7	6	5	5	4	5	3
1.0e04	32	18	21	16	15	11	10	9
1.0e04*	22	29	28	22	22	24	22	24
1.0e05	79	64	54	66	67	60	51	30
1.0e05*	62	66	61	57	63	71	63	58

Table 1: Algorithm SSN(γ, h), Example (a), ε_{in}=1.0e-10: no. of iterations w.r.t. mesh size and γ, * fixed starting point

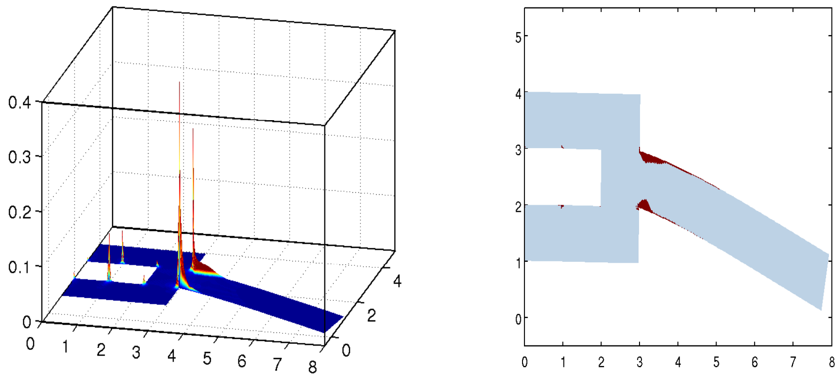

Figure 4: Example (a): plastic strain $|p|_F$ *(left)*, dominant plastic zones *(dark)*, i.e. $|p|_F >$ 1e-02 *(right)*

$g(t_1, x) = -2.0\text{e}01 \cdot \nu(x)$ on $\Gamma_1 = (0, 0.5) \times \{0.5\}$ which leads to a nonempty contact region at the solution. Volume forces are assumed to be absent; $f(t_1) \equiv 0$. To verify mesh-independent convergence of Algorithm SSN(γ, h), we compute the solution for each fixed γ on meshes with decreasing mesh width starting from approximately $1.6 \cdot 10^3$ nodes to about $1.6 \cdot 10^6$ nodes, cf. Table 2, using uniform mesh refinement. The solution on a given mesh is prolongated to the next finer mesh to serve as a starting point $v^{(0)}$ of Algorithm SSN(γ, h) on the refined triangulation. It is observed that below $\gamma \approx$ 1.0e04, both active set approximations of contact and plasticity constraints are empty. For γ between 1.0e04 and 1.0e05, only the contact constraint has a nonempty active set. Starting from $\gamma \approx$ 1.0e05, both, plastic and contact effects need to be dealt with. For validation purposes, a restart strategy using the zero function as a starting point on each mesh is also tested. The iterations count for the restart strategy stays bounded as the number of nodes are increased whereas the iteration numbers for the nested strategy even tend to decrease with decreasing mesh width.

As the result of the application of the inexact path-following approach IPF(h) with regard to the penalty parameter set γ, we display in Figure 5 the approximate optimal plastic strain as well as the regions of extensive plastic straining in the deformed configuration. Employing relation (8.2.10), we also plot the approximate yield function in the deformed configuration and the normal stress component on the initial configuration in Figure 6.

Table 2: Algorithm SSN(γ, h), Example (b), ε_{in}=1.0e-10: no. of iterations w.r.t. mesh size and fixed γ, * fixed starting point, † for vector-valued γ cf. (9.3.8)

γ / # nodes	1.6k	6k	25k	100k	400k	1.6M
1.0e03	1	1	1	1	1	1
1.0e04	4	4	3	3	1	1
5.0e04	4	9	8	4	3	5
1.0e05	22	24	25	16	13	9
[2.0e07, 2.0e07, 1.0e00, 1.0e00]†	15	13	8	6	6	5
[2.0e07, 2.0e07, 1.0e00, 1.0e00]*	15	14	21	25	23	22

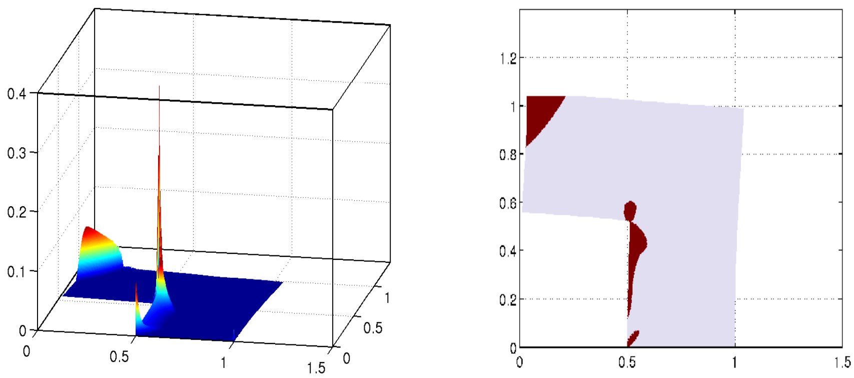

Figure 5: Example (b): $|p|_F$ *(left)*, dominant plastic zones *(dark)*, i.e. $|p|_F > 0.1$ *(right)*

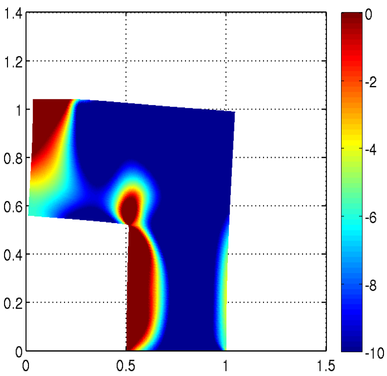

Figure 6: Example (b): yield functional

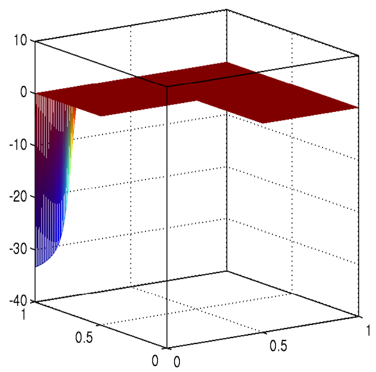

Figure 7: Example (b): normal stress approximation $(\sigma\nu)_\nu$ on Γ_c

Example (c) - Sine-shaped obstacle

In this example we consider a rectangular domain $\Omega = (0, 5) \times (0, 1)$. The elastic behavior is described by $\mathbb{C}\varepsilon = \mu_1 \operatorname{tr}(\varepsilon)I + 2\mu_2\varepsilon$ with $\mu_1 \equiv$ 8.0e01, $\mu_2 \equiv$ 5.3e01. The material is assumed to satisfy the isotropic hardening law ($k_1 \equiv 0$) with $k_2 \equiv$ 1.0e02 and $\sigma_y =$ 8.0e00. We apply a pressure $g(t_1, x) = -8.0\text{e}00 \cdot \nu(x)$ on $\Gamma_1 = (1, 4) \times \{0\}$. We further admit a vanishing volume force $f \equiv 0$. Moreover, $\Gamma_0 = (\{0\} \times (0, 1)) \cup (\{1\} \times (0, 1))$, and $\Gamma_c = (1, 4) \times \{1\}$ with $\Psi(x, 1) =$

$0.2 + \sin(5\pi(x - 1.5))$ for $x \in (1.5, 3.5)$ and $\Psi(x, 1) \equiv 0.2$, else. The results obtained by Algorithm SSN(γ, h) are summarized in Table 3. As in the previous examples, we verify mesh-independent convergence by computing the solution for each fixed γ on meshes with decreasing mesh width. Again, we choose the (prolongated) solution of the preceding coarser mesh as a starting point for Algorithm 1 . It is observed that, both, plastic and contact effects need to be taken account of starting from $\gamma \approx$ 5.0e03. Considering Table 3 we observe that the number of iterations even tends to decrease with smaller mesh width. This clearly indicates mesh-independent convergence for fixed γ as the mesh width tends to zero. As for the preceding examples, the results of the inexact path-following approach IPF(h) (cf. below), are displayed in Figure 8,9 and 10.

Table 3: Algorithm SSN(γ, h), Example (c), ε_{in}=1.0e-10: no. of iterations w.r.t. mesh size and fixed γ, † for vector-valued γ cf. (9.3.8), * fixed starting point

γ / # nodes	1.3k	5k	21k	83k	300k	1.3M
1.0e03	1	1	1	1	1	1
5.0e03	11	11	8	7	5	5
1.0e04	17	23	17	11	8	9
[1.0e06,1.0e06,1.0e00,1.0e00]†	13	10	8	6	5	4
[1.0e06,1.0e06,1.0e00,1.0e00]*	13	12	13	15	17	15

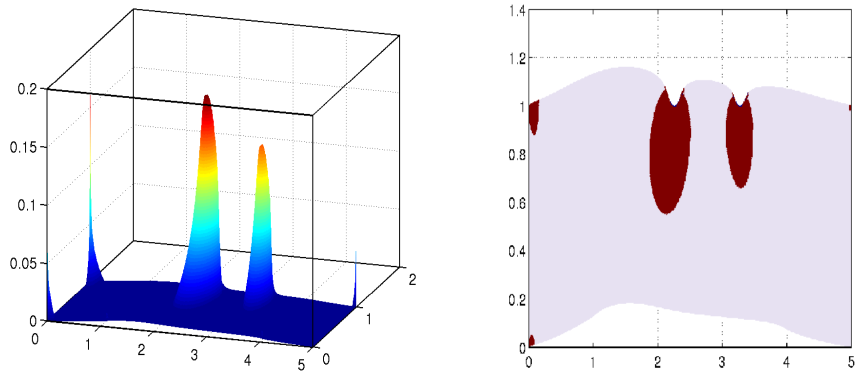

Figure 8: Example (c): $|p|_F$ *(left)*, dominant plastic zones *(dark)*, i.e. $|p|_F > 0.01$ *(right)*

Performance of the path-following approach

Tables 4-6 show the results for the application of Algorithm IPF(h) to the previous test examples for fixed outer stopping criterion $\varepsilon_{\text{out}} = 10^{-5}$. For validation purposes, we first test IPF(h) on various meshes using for each mesh the zero function as a starting outer iterate $\tilde{v}^{(0)}$. This restart strategy is observed to converge with a constant number of outer iterations. This indicates that for fixed required precision ε_{out} in the outer loop of IPF(h), the appropriate path-parameter set does not depend on the mesh width. Moreover, the total number of inner iterations stays bounded as the mesh width goes to zero, which indicates that even for this heuristic the inexact path-following algorithm behaves almost mesh-independent. It should be pointed out that automated path-following methods from variational inequalities of the first kind bear a high potential to further improve the performance of the path-following algorithm, see [66]. To keep high-dimensional

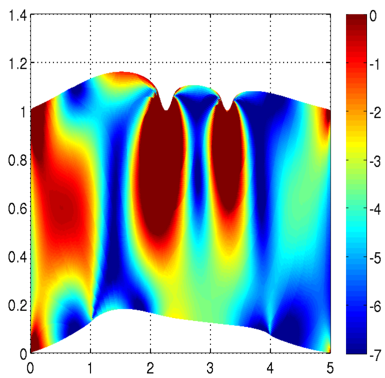

Figure 9: Example (c): yield functional

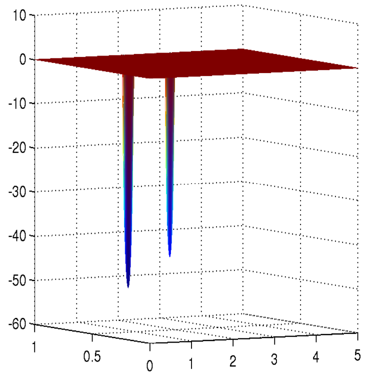

Figure 10: Example (c): normal stress approximation $(\sigma\nu)_\nu$ on Γ_c

calculations as low as possible we also test a nested iteration. In this approach the solution on a given mesh is prolongated to the next finer mesh to serve as a starting point $\tilde{v}^{(0)}$ of Algorithm IPF(h) on the refined mesh together with the final parameter set of the coarser mesh. In this way, the major part of the computations related to the identification of the appropriate parameter set is transferred to the smallest mesh and the nested approach proves to be particularly efficient for Example (a) and (b). With this strategy, no further γ-updates are necessary after the computations on the coarsest mesh and the total number of inner Newton-iterations decreases significantly as the number of nodes increases.

The efficiency of our path-following approach can be numerically verified as follows. On a fixed mesh associated with a mesh width h, denote by $\gamma^{(\text{end})}$ the final parameter set from the application of IPF(h). A straightforward application of Algorithm SSN(γ, h) to $(D_{\gamma^{(\text{end})},h})$, where the stopping criterion is replaced by the respective inexact version (9.3.9), typically requires a multiple of the iterations. This shows the advantage of our path-following approach.

# nodes	12,5k	25k	50k	100k	200k	400k	800k	1.6M
restart	9(40)	9(38)	9(40)	9(41)	9(32)	9(30)	9(30)	9(26)
nested	1(21)	1(13)	1(23)	1(19)	1(14)	1(15)	1(14)	1(9)

Table 4: No. of outer(total inner) iterations IPF(h), $\gamma^{(0)} = 1.0\text{e}03 \cdot [1,1,1,1]$, $\varepsilon_{\text{out}} = 1.0\text{e-}05$, $\theta = 2$ and $\tau_{\text{in}} = 1.0\text{e}00$ for Example (a)

# nodes	1.5k	6k	25k	100k	400k	1.6M
restart	21(41)	21(37)	21(49)	21(52)	21(52)	21(57)
nested	21(41)	1(79)	1(63)	1(48)	1(38)	1(20)

Table 5: No. of outer(total inner) iterations for IPF(h), $\gamma^{(0)} = [2.0\text{e}07, 2.0\text{e}07, 1.0\text{e}00, 1.0\text{e}00]$, $\varepsilon_{\text{out}} = 1.0\text{e-}05$, $\theta = 1.5$ and $\tau_{\text{in}} = 1.0\text{e}00$ for Example (b)

# nodes	1.3k	5k	21k	83k	320k	1.3M
restart $\theta = 1.5$	26(29)	26(48)	26(71)	26(65)	26(51)	26(57)
restart $\theta = 2.0$	16(23)	16(40)	16(54)	16(52)	16(40)	16(53)
nested $\theta = 1.5$	16(21)	1(221)	1(172)	1(141)	1(141)	1(90)
nested $\theta = 2.0$	16(23)	1(239)	1(209)	1(142)	1(164)	1(138)

Table 6: No. of outer(total inner) iterations IPF(h), $\gamma^{(0)} = [1e08,1e08,1,1]$, $\varepsilon_{\text{out}} = 1.0e\text{-}05$, $\tau_{\text{in}} = 1.0e00$ for Example (c)

Part IV

Duality Results and Regularization Schemes for Perfect Plasticity

10 Perfect Plasticity

10.1 Introduction

The analysis of the time-dependent problem of quasi-static small strain associative perfect plasticity or *Prandtl-Reuss plasticity* goes back to [44, 78], where [78] includes the first existence result for the time-dependent case based on a suitable stress-based dual formulation. In [92], existence as well as approximation results are given for time-varying yield criteria. The understanding of the appropriate functional analytic setting of the weak formulation of perfect plasticity has been considerably improved in [117], which builds on the functional analytic framework for the corresponding static problem usually referred to as *Hencky plasticity* [116, 119]. The requirement for a more involved functional analytic setting is physically justified by the possible presence of strain localization. Simulations of this phenomenon are carried out in [88, 97]. In the static case, which itself is of limited practical use, existence results for the appropriate primal formulation in terms of the displacement, also called strain problem, have been obtained on the basis of relaxation principles. The relation of the strain problem to the stress problem is discussed within the theory of Fenchel duality [7, 119]. These developments build upon a proper weak formulation of the flow law which depends on a suitable pairing of stresses and strains. The pairing cannot be derived in a straightforward way, owing to the fact that the strain in perfect plasticity is just a measure [82].

Although the literature on mathematical plasticity and related issues is rather vast, it was not until the relatively recent work [38] that the corresponding primal problem of quasi-static perfect plasticity has been formulated and studied in a satisfying way. The appropriate setting of this problem is gained from the abstract theory of energetic formulations for a very general class of rate-independent systems [91, 94]. One of the key points of the approach in [38] consists in a proper extension of the stress-strain duality from Hencky plasticity to the time-dependent case. Moreover, the equivalence to the stress-based weak formulation from [78] is set forth and the existence of a quasi-static evolution is proven by an appropriate time-discretization. Important extensions, for example, to heterogeneous materials [50, 113] and with respect to regularity theory [42] are also established.

The numerical analysis of finite element methods in perfect plasticity is mainly governed by regularization techniques, and a convergence result for the discretized stresses for a suitable coupling of discretization and regularization parameter is known [101]. In the quasi-static case, perfect plasticity can be characterized as a limit of plasticity problems with vanishing hardening. The approach can be coupled with a standard finite element discretization and convergence of displacement, stresses and strains, as the mesh size goes to zero, can be proven under minimal regularity [15]. For a discussion of the static case in the context of adaptive methods, we also refer to [100], [109] and [25]. To the best of the author's knowledge, there is no convergence result for a direct finite element approximation of the Prandtl-Reuss model under minimal regularity.

As for algorithmic approaches to the discrete problems of perfect plasticity, we mention the standard *return mapping* algorithm from [112]. The superlinear convergence of this generalized Newton method is explained by the semismoothness of the plastic response function [107]. Other approaches, like SQP [124] and multigrid techniques [123], typically depend on the smoothness of the yield surface. The convergence usually displays a high degree of mesh-dependence whereas the convergence of infinite-dimensional Augmented Lagrangian methods hinges on the higher regularity of the strain which is often not given [106]. We also refer to [45] for a survey on the various complications in both theoretical and algorithmic Prandtl-Reuss plasticity.

The outline of this part is as follows. In the remainder of this chapter, the system of equations of the Prandtl-Reuss model of perfect plasticity is given. Thereupon, the properties of the different weak formulations, their interrelation and the generalized stress-strain duality is reviewed. In Chapter 11 we consider the time-discretized problem of quasi-static evolution in perfect plasticity which represents a convex minimization problem over the space of functions with bounded deformation and the space of Borel measures. An equivalent inf-sup problem formulation which is posed in a usual separable and reflexive Lebesgue space is derived. With the help of this reduced formulation we prove that the classical incremental stress problem from [78] is a Fenchel dual problem to the primal problem. As a result, we can derive new necessary and sufficient optimality conditions for the time-discrete problem.

In the subsequent chapter we propose a primal-dual regularization scheme which combines the visco-plastic regularization with a penalty type approach with respect to the mechanical equilibrium condition. The approach can be considered as an alternative to techniques that are based on the approximation by plasticity problems with vanishing hardening; see [15] and Part III. The regularization is shown to be consistent with the initial problem in that displacements, stresses and strains are shown to converge to a solution of the initial problem in suitable topologies. The scheme gives rise to a well-defined Fenchel dual problem, which is a modification of the usual stress problem in perfect plasticity. Moreover, the regularized dual problem has a simple structure, which appears to be well-suited for numerical purposes. In order to design a mesh-independent solver for the corresponding subproblems, we propose an algorithmic approach in the infinite-dimensional setting based on the semismooth Newton method, and we include a convergence result for the regularized problems. Finally, we give an outlook on the finite-dimensional counterpart of the solver, and we discuss some of the difficulties related to the stability of finite-dimensional approximations.

10.2 The Prandtl-Reuss Model of Perfect Plasticity

In this chapter, the fundamental set of conditions that describes the evolution of an elasto-plastic material subject to time-dependent applied forces given by the densities $f = f(t,x)$ and $g = g(t,x)$ is set forth. We assume that the loading process takes place in the time interval $[0,T]$ and that the body is represented by a bounded domain $\Omega \subset \mathbb{R}^N, N \in \{2,3\}$. As in Chapter 7, we focus on the quasi-static regime under the small strain assumption, and the flow rule is assumed to be associative. Again we assume that the body is fixed on a given portion Γ_0 of the boundary $\partial\Omega$ whereas g acts on the complement $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}_0$. The mechanical quantities of interest, $u = u(t,x), \sigma = \sigma(t,x)$ and $p = p(t,x)$, are introduced in Section 7.2 to which we refer for details. The elastic behavior of Ω is described by a fourth order elasticity tensor $\mathbb{C} = \mathbb{C}(x) \in (\mathbb{M}^{N\times N})^2$ which is assumed to have the properties specified in (7.2.2). In particular, $\mathbb{C}$ is assumed to be symmetric,

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl},$$

and pointwise stable, i.e., there exists $\kappa_1 > 0$ with

$$\mathbb{C}(x)\sigma : \sigma \geq \kappa_1 |\sigma|_F^2 \quad \text{for all } \sigma \in \mathbb{M}^{N\times N} \text{ and a.e. } x \in \Omega.$$

In contrast to Chapter 7, the Prandtl-Reuss model of perfect plasticity does not account for hardening effects. In the absence of hardening, the loading procedure may cause the material to form shear bands which are surfaces along which shear strains concentrate and discontinuities in the displacement may occur. The standard example is the formation of a slip surfaces. These are surfaces in the material along which the tangential component of the displacement exhibits a jump-type discontinuity. On the mathematical level, this is reflected by the observation that optimal displacements may exhibit discontinuities on $(N-1)$-dimensional submanifolds which rules out the usual Sobolev setting. Thus, perfect plasticity requires a different functional analytic framework. Without hardening effects, the set of admissible stresses has to be adapted in a suitable way. In accordance with the notation from Section 7.2 on hardening plasticity, we assume that the set of admissible stresses $\mathbb{K} \subset \mathbb{M}^{N\times N}$ is a nonempty, convex and closed set determined by some yield function $\phi : \mathbb{M}^{N\times N} \to \mathbb{R} \cup \{+\infty\}$. Together with an initial condition at time $t = 0$, the classical set of conditions for the elasto-perfectly plastic evolution of a body subject to the external forces f and g is given as follows.

Problem 10.1 (Prandtl-Reuss plasticity). *Given $f = f(t,x)$ and $g = g(t,x)$ with $f(0,x) = 0$ in Ω and $g(0,x) = 0$ on Γ_1, find $[u,p,\sigma] = [u,p,\sigma](t,x)$, with*

$$[u,p,\sigma](0,x) = 0 \quad \text{in } \Omega$$

such that

$$u(t,x) = 0 \quad \text{on } \Gamma_0, \tag{10.2.1}$$

$$\sigma\nu(t,x) = g(t,x) \quad \text{on } \Gamma_1, \tag{10.2.2}$$

$$-\operatorname{Div}\sigma(t,x) = f(t,x) \quad \text{in } \Omega, \tag{10.2.3}$$

$$\varepsilon(u)(t,x) = \mathbb{C}^{-1}(x)\sigma(t,x) + p(t,x) \quad \text{in } \Omega, \tag{10.2.4}$$

$$\sigma(t,x) \in \mathbb{K} \quad \text{in } \Omega \tag{10.2.5}$$

$$\dot p(t,x)) \in N_{\mathbb{K}}(\sigma(t,x)) \quad \text{in } \Omega, \tag{10.2.6}$$

for all $t \in [0,T]$.

Here, $N_{\mathbb{K}}$ denotes the normal cone of $\mathbb{K}$ in $\mathbb{M}^{N\times N}$, i.e.,

$$N_{\mathbb{K}}(\sigma) = \{p \in \mathbb{M}^{N\times N} \colon\ p : (\tilde\sigma - \sigma) \le 0 \ \ \forall \tilde\sigma \in \mathbb{K}\}.$$

The plastic flow rule $\dot p(t,x) \in N_{\mathbb{K}}(\sigma(t,x))$ implies that the plastic strain can only evolve if $\sigma(t,x)$ is an element of the yield surface $\partial\mathbb{K}$. In comparison with hardening plasticity, the set of admissible stresses, and in particular the yield surface, does not change in time. At this point we emphasize that the yield surface is not necessarily smooth. Discrete plasticity solvers sometimes require the smoothness of the yield surfaces in order to work with a multiplier-based reformulation of the flow rule; cf., for instance, [124].

If the yield criterion of the considered material is pressure-insensitive then the set $\mathbb{K}$ that determines the yield criterion is given by

$$\mathbb{K} = \mathbb{K}_0 + \mathbb{R}I_N, \tag{10.2.7}$$

where $\mathbb{K}_0$ is a nonempty, compact and convex neighborhood of the origin of the space of trace-free symmetric matrices $\mathbb{M}_0^{N\times N}$, and I_N is the identity matrix of dimension N. In particular, (10.2.7) together with the flow rule (10.2.6) implies that $p(t,x) \in \mathbb{M}_0^{N\times N}$ and (10.2.5), (10.2.6) are equivalent to

$$\operatorname{dev}\sigma \in \mathbb{K}_0 \quad \text{and} \quad \dot p(t,x) \in N_{\mathbb{K}_0}(\operatorname{dev}\sigma(t,x)),$$

where $\operatorname{dev}\sigma = \sigma - \frac{\operatorname{tr}\sigma}{N} I_N$ and $N_{\mathbb{K}_0}(\operatorname{dev}\sigma(t,x))$ denotes the normal cone of $\mathbb{K}_0$ at $\operatorname{dev}\sigma(t,x)$ in

$\mathbb{M}_0^{N\times N}$. Using the support function of $\mathbb{K}_0$,

$$i^*_{\mathbb{K}_0} : \mathbb{M}_0^{N\times N} \to \mathbb{R} \cup \{+\infty\}, \quad i^*_{\mathbb{K}_0}(\sigma) = \sup_{\tau\in\mathbb{K}_0} \tau : \sigma,$$

(10.2.6) can be equivalently written as

$$i^*_{\mathbb{K}_0}(\dot{p}) = \dot{p} : \operatorname{dev}\sigma. \tag{10.2.8}$$

10.3 Function Space Setting and Variational Formulations

We begin the analysis of the conditions (10.2.1)-(10.2.6) by reviewing several notions of weak solutions and their relation. In the following, we will assume that $\Omega \subset \mathbb{R}^N, N \in \{2,3\}$, is a bounded Lipschitz domain with nonempty open Dirichlet boundary portion $\Gamma_0 \subset \partial\Omega$. For a fixed subspace $X(\Omega) \subset Q$, the set of admissible stresses in $X(\Omega)$ is denoted by

$$S_{\mathrm{ad}}(X(\Omega)) := \{\sigma \in X(\Omega) : \sigma(x) \in \mathbb{K} \text{ a.e. in } \Omega\},$$

and for $X(\Omega) = Q$ we simply write $S_{\mathrm{ad}} = S_{\mathrm{ad}}(Q)$. We also define the space

$$\Sigma(\mathrm{Div};\Omega) := \{\sigma \in Q : \operatorname{Div}\sigma \in L^N(\Omega)^N\}.$$

The applied forces are given by $f = f(t) \in L^N(\Omega)^N$ and $g = g(t) \in L^\infty(\Gamma_1)^N$ and we set

$$l(\tilde{u}) = \langle l(t), \tilde{u}\rangle := \int_\Omega f\cdot\tilde{u}\,dx + \int_{\Gamma_1} g\cdot\tilde{u}\,d\mathcal{H}^{N-1}, \quad \tilde{u} \in V.$$

The correct functional analytic setting requires that the displacement is sought in the space of functions with bounded deformation

$$BD(\Omega) = \{u \in L^1(\Omega)^N : \varepsilon(u) \in M(\Omega;\mathbb{M}^{N\times N})\}.$$

Consequently, the plastic strains are only measures which may also be supported on the boundary. Hence, the plastic strains are expected to lie in the space of Borel measures $M(\Omega\cup\Gamma_0;\mathbb{M}_0^{N\times N})$. This setting sharply contrasts with the case of hardening plasticity. We refer to Section 1.2 for important properties of these spaces.

10.3.1 Johnson's weak formulation

Dating back to the two seminal works of [44] and [78], a weak formulation of (10.2.1)-(10.2.6) in terms of the stress and the velocity can been formulated by replacing $\dot{p}$ in the flow law (10.2.6) using the strain decomposition (10.2.4). In this way we obtain

$$(\varepsilon(\dot{u}) - \mathbb{C}^{-1}\dot{\sigma}, \tilde{\sigma} - \sigma) \le 0, \quad \forall\,\tilde{\sigma} \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)),\ \tilde{\sigma}\nu = g \text{ on } \Gamma_1.$$

We also note that each function $\tilde{\sigma} \in \Sigma(\mathrm{Div};\Omega)$ has a normal trace on Γ_1 according to (1.2.5), such that the equality $\tilde{\sigma}\nu = g$ has to be understood in the sense of $H_{00}^{-1/2}(\Gamma_1)^N$. By a (formal) application of Green's formula, one obtains the variational inequality

$$\langle \dot{u}, \operatorname{Div}\tilde{\sigma} - \operatorname{Div}\sigma\rangle + (\mathbb{C}^{-1}\dot{\sigma}, \tilde{\sigma} - \sigma) \ge 0, \quad \forall\,\tilde{\sigma} \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)), \tilde{\sigma}\nu = g \text{ on } \Gamma_1,$$

where $\langle \,.\,,\,.\,\rangle$ stands for the duality pairing of $L^{N/(N-1)}(\Omega)^N$ with $L^N(\Omega)^N$. Together with the standard weak formulation of the equilibrium condition (10.2.3) using the adjoint of the operator

$$\varepsilon : V \to Q, \quad \varepsilon(u) = \tfrac{1}{2}(\nabla u + \nabla u^\top),$$

this leads to the following coupled variational inequality problem for the displacement rate and the stress.

Problem 10.2 (Johnson's weak formulation). *Let $f \in C([0,T]; L^N(\Omega)^N)$, and $g \in C([0,T]; L^\infty(\Gamma_1)^N)$ with $f(0) = 0$, $g(0) = 0$. Find*

$$[\dot{u}, \sigma] : [0,T] \to BD(\Omega) \times Q, \quad \textit{with } \sigma(0) = 0,$$

such that $\sigma \in S_{ad}(\Sigma(\mathrm{Div};\Omega))$ and

$$(\sigma, \varepsilon(\tilde{u})) = \langle l(t), \tilde{u}\rangle \quad \forall\, \tilde{u} \in V, \tag{10.3.1}$$

$$\langle \dot{u}, \mathrm{Div}\,\tilde{\sigma} - \mathrm{Div}\,\sigma\rangle + (\mathbb{C}^{-1}\dot{\sigma}, \tilde{\sigma} - \sigma) \geq 0 \quad \forall\, \tilde{\sigma} \in S_{ad}(\Sigma(\mathrm{Div};\Omega)),\ \tilde{\sigma}\nu = g(t) \text{ on } \Gamma_1, \tag{10.3.2}$$

for a.e. $t \in (0,T)$.

In the initial analysis of Johnson [78], the formulation is oblivious of the precise spatial regularity of u. Under a suitable assumption on the load, existence of a solution to Problem 10.2 in $L^2_w(0,T; BD(\Omega)) \times L^\infty(0,T;Q)$ is proven in the latter reference by a combination of a time-discretization with a Moreau-Yosida regularization of $i_{\mathbb{K}}$. The choice of the space $BD(\Omega)$ and the refined regularity statements for the displacement are attributed to Suquet [117]. The latter reference also provides one-dimensional examples exhibiting non-unique and discontinuous velocity solutions $\dot{u}$. For a further discussion of the possible types of discontinuities of the displacement or the velocity, we refer to [45] and the references therein. By contrast, the stress solution is uniquely determined by Problem 10.2. Indeed, upon testing (10.3.2) with $\tilde{\sigma} \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega))$, where $\tilde{\sigma}\nu = g$ on Γ_1 and $-\mathrm{Div}\,\tilde{\sigma} = f$, it can be observed that any stress solution also solves the following problem, which is uniquely solvable owing to the properties of $\mathbb{C}$.

Problem 10.3 (Johnson's stress problem). *Let $f \in C([0,T]; L^N(\Omega)^N)$ and $g \in C([0,T]; L^\infty(\Gamma_1)^N)$ with $f(0) = 0$, $g(0) = 0$. Find*

$$\sigma : [0,T] \to Q, \quad \textit{with } \sigma(0) = 0,$$

such that $\sigma \in S_{ad}(\Sigma(\mathrm{Div};\Omega))$ and

$$(\sigma, \varepsilon(\tilde{u})) = \langle l(t), \tilde{u}\rangle \quad \forall\, \tilde{u} \in V,$$
$$(\mathbb{C}^{-1}\dot{\sigma}, \tilde{\sigma} - \sigma) \geq 0 \quad \forall\, \tilde{\sigma} \in S_{ad}(\Sigma(\mathrm{Div};\Omega)),\ \tilde{\sigma}\nu = g(t) \text{ on } \Gamma_1, -\mathrm{Div}\,\tilde{\sigma} = f(t),$$

for a.e. $t \in (0,T)$.

At this point, many theoretical questions still remain open. This primarily concerns the interpretation of the flow law, since the equality $p = \varepsilon(u) - \mathbb{C}^{-1}\sigma$ implies that p does not have a pointwise (a.e.) interpretation. Also, it is not clear in what way solutions u to Problem 10.2 fulfill the Dirichlet boundary condition. These questions are answered by the study of an appropriate primal problem in u and p. For an overview of the various approaches to the stress problem we also refer to [45] and the references therein.

10.3.2 Quasi-static evolution

In [38], the problem of Prandtl-Reuss perfect plasticity is studied within the general context of energetic formulations for rate-independent systems which are defined by the axioms of energy stability and energy balance. Establishing a suitable time-discretization, problems in this class are approximated consistently as the time step goes to zero by the corresponding time-incremental procedure. For details we refer to [91] and the recent monograph [94].

Assumptions and notation

In the remainder of this chapter, we collect some important results from [38]. In doing so, we focus on a homogeneous boundary condition on Γ_0 and zero initial conditions, i.e.,

$$[u,e,p](0) = 0, \quad l(0) = 0,$$

but the definition and the subsequent results can be extended to time-dependent prescribed boundary displacements and nontrivial initial conditions. To begin with, we introduce some additional notation. The elastic part of the strain is denoted by

$$e := \mathbb{C}^{-1}\sigma.$$

The strain decomposition (10.2.4) motivates the definition of the set of admissible states

$$\begin{aligned} W_{\text{ad}} := \{(u,e,p) \in BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N}) : \\ \varepsilon(u) = p\lfloor\Omega + e,\ p\lfloor\Gamma_0 = -(u \odot \nu)\mathcal{H}^{n-1}\}, \end{aligned} \tag{10.3.3}$$

where $p\lfloor\Omega$ and $p\lfloor\Gamma_0$ designate the restriction of the measure p to Ω and Γ_0, respectively. With the symmetrized outer product

$$a \odot b := \tfrac{1}{2}ab^\top + ba^\top, \quad a,b \in \mathbb{R}^N,$$

the boundary condition on Γ_0 is the appropriate relaxation of the Dirichlet boundary condition $u = 0$ on Γ_0 in plasticity theory, cf. [119].

Another technical assumption concerns the smoothness of the bounded domain Ω and the interface joining the parts of the different boundary conditions;

$$\partial\Omega \in C^2, \quad \partial\Gamma_0 = \partial\Gamma_1 \text{ is } C^2\text{-regular [82]}. \tag{10.3.4}$$

The elasticity tensor $\mathbb{C} \in (\mathbb{M}^{N\times N})^2$ is assumed to be positive definite and invariant with respect to the orthogonal subspaces $\mathbb{M}_0^{N\times N}$ and $\{cI_N : c \in \mathbb{R}\}$. Consequently, there exists a positive definite tensor $\mathbb{C}_{\text{dev}} \in (\mathbb{M}_0^{N\times N})^2$ and a scalar $\lambda_0 > 0$ such that

$$\mathbb{C}\sigma = \mathbb{C}_{\text{dev}} \operatorname{dev}\sigma + \lambda_0 \operatorname{tr}\sigma I_N, \quad \forall\, \sigma \in \mathbb{M}^{N\times N}. \tag{10.3.5}$$

We also assume that the yield criterion is pressure-insensitive, such that the flow law (10.2.6) can be equivalently expressed by (10.2.8). In order to derive a weak primal formulation, the mapping D defined in (7.4.5),

$$D(\tilde{p}) = \int_\Omega i^*_{\mathbb{K}_0}(\tilde{p})\, dx, \quad \tilde{p} \in Q,$$

has to be extended to the measure space $M(\Omega \cup \Gamma_0, \mathbb{M}_0^{N\times N})$. This is achieved with the help of the theory of convex functions of measures [55, 119] by setting

$$i^*_{\mathbb{K}_0}(\tilde{p}) := i^*_{\mathbb{K}_0}(\tilde{p}/|\tilde{p}|)|\tilde{p}|, \quad \tilde{p} \in M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N}),$$

where $\tilde{p}/|\tilde{p}|$ denotes the Radon-Nikodým derivative of $\tilde{p}$ with respect to its total variation $|\tilde{p}|$. We note that $\tilde{p}/|\tilde{p}| \in L^1_{|\tilde{p}|}(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N})$, i.e., $\tilde{p}/|\tilde{p}|$ is Lebesgue integrable on $\Omega \cup \Gamma_0$ with respect to the measure $|\tilde{p}|$. Consequently, $i^*_{\mathbb{K}_0}(\tilde{p}) \in M_+(\Omega \cup \Gamma_0)$. The mapping

$$D : M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N}) \to \mathbb{R}$$

is then defined by

$$D(\tilde{p}) := \int_{\Omega\cup\Gamma_0} i^*_{\mathbb{K}_0}(\tilde{p}/|\tilde{p}|)\, d|\tilde{p}| = i^*_{\mathbb{K}_0}(\tilde{p})(\Omega \cup \Gamma_0).$$

Note that the properties of $\mathbb{K}_0$ ensure that D is nonnegative and finite. The dissipation in the time interval $[0, t]$, $t \le T$, is measured by the generalized total variation functional with respect to D,

$$\mathcal{D}(p; 0, t) := \sup\{\sum_{j=1}^{\tilde{J}} D(p(t_j) - p(t_{j-1})) : \tilde{J} \in \mathbb{N}, 0 = t_0 \le t_1 \le \ldots \le t_{\tilde{J}} = t\}.$$

The notion of quasi-static evolution turns out to be essential for a weak formulation of perfect plasticity. We recall that we assume the system to be initially at rest.

Definition 10.4 (Quasi-static evolution). A function $[0, T] \ni t \mapsto [u(t), e(t), p(t)] \in BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N})$ with $[u, e, p](0) = 0$ is called *quasi-static evolution* if the following conditions are fulfilled.

(i) Stability: For every $t \in [0, T]$, it holds that $[u(t), e(t), p(t)] \in W_{\text{ad}}$ and

$$\tfrac{1}{2}(\mathbb{C}e(t), e(t)) - \langle l(t), u(t)\rangle \le \tfrac{1}{2}(\mathbb{C}\tilde{e}, \tilde{e}) + D(\tilde{p} - p(t)) - \langle l(t), \tilde{u}\rangle$$

for all $[\tilde{u}, \tilde{e}, \tilde{p}] \in W_{\text{ad}}$.

(ii) Energy equality: It holds that $p \in BV([0, T], M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N}))$, and for every $t \in [0, T]$ the equation

$$\tfrac{1}{2}(\mathbb{C}e(t), e(t)) - \langle l(t), u(t)\rangle + \mathcal{D}(p; 0, t) = -\int_0^t \langle \dot{l}(s), u(s)\rangle ds$$

is valid.

Problem 10.5 (Existence of quasi-static evolutions). *Given*

$$f \in AC([0, T]; L^N(\Omega)^N), \quad g \in AC([0, T]; L^\infty(\Gamma_1)^N) \tag{10.3.6}$$

with $f(0) = 0$ and $g(0) = 0$, find

$$[u, e, p] : [0, T] \to BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N})$$

with $[u, e, p](0) = 0$ such that $t \mapsto [u(t), e(t), p(t)]$ is a quasi-static evolution.

It is well-known that in order to establish the existence of weak solutions in perfect plasticity only certain qualified f and g are admissible.

Assumption 10.6 (Safe-load condition)**.** *There exists* $\hat{\sigma} \in AC([0,T];Q)$ *and* $\rho > 0$ *such that*

(i) $\operatorname{dev}\hat{\sigma} \in AC([0,T];L^\infty(\Omega;\mathbb{M}_0^{N\times N}))$,

(ii) for every $t \in [0,T]$ *it holds that*

$$\begin{aligned}
&\operatorname{Div}\hat{\sigma}(t) = -f(t) \text{ in } \Omega, \quad \hat{\sigma}(t)\nu = g(t) \text{ on } \Gamma_1,\\
&\operatorname{dev}\hat{\sigma}(t) + B(0;\rho) \subset \mathbb{K}_0 \text{ a.e. in } \Omega,
\end{aligned}$$

where $B_\rho(0) := \{\tau \in \mathbb{M}_0^{N\times N} : |\tau|_F < \rho\}$.

Note that Assumption 10.6(ii) ensures that there exists an element in the feasible set to Problem 10.2 that has a Slater-type property. In other words, the condition essentially requires the applied forces f and g to be nondegenerate in the sense that they allow for an admissible stress state corresponding to a purely elastic material response. For instance, consider the following practically relevant situation.

Example 10.7. Let

$$f(t) := 0, \ \forall\, t \in [0,T], \quad g(t,x) := c(t)\nu(x),$$

where $c(t,x) \equiv c(t)$, $c \in AC([0,T])$, describes a time-dependent homogeneous traction or pressure. In this case,

$$\hat{\sigma}(t,x) := c(t)I_N$$

represents the desired element to ensure that f and g comply with Assumption 10.6.

If, additionally, the (mild) assumptions on the boundary regularity (10.3.4), the pressure insensitivity of the yield criterion (10.2.7) and the standard assumptions on $\mathbb{C}$ (10.3.5) are given, then there exists a solution $[u,e,p] \in AC([0,T];BD(\Omega)\times Q \times M(\Omega\cup\Gamma_0,\mathbb{M}_0^{N\times N}))$ of Problem 10.5. Moreover, $e : [0,T] \to Q$ (and thus $\sigma : [0,T] \to Q$) is uniquely determined by its initial condition [38, Theorem 4.5, 5.2 and 5.9]. The solutions are obtained by a time-discretization process, which is defined in the subsequent section.

The connection between the two types of weak solutions to the system (10.2.1)-(10.2.6) relies on a suitable extension of the meaning of the flow law (10.2.6) to linearized strains $\varepsilon(u)$ that are only measures, and which reduces to the conventional (pointwise a.e.) meaning if $\dot{p} \in Q$. For that reason, a duality pairing between admissible stresses and strains can be defined, which extends earlier approaches within the context of Hencky plasticity set forth by Kohn and Temam [82].

Stress-strain duality

Let $\mathbb{K}$ be pressure-insensitive, i.e., (10.2.7) holds true. For $\sigma \in \Sigma(\operatorname{Div};\Omega)$ with $\operatorname{dev}\sigma \in L^\infty(\Omega;\mathbb{M}_0^{N\times N})$ and $u \in BD(\Omega)$ with $\operatorname{div} u \in L^{N/(N-1)}(\Omega)$, one may then define the distribution $[\operatorname{dev}\sigma, \operatorname{dev}\varepsilon(u)]$ by

$$\langle[\operatorname{dev}\sigma,\operatorname{dev}\varepsilon(u)],\varphi\rangle := -\langle\operatorname{Div}\sigma,\varphi u\rangle - \tfrac{1}{N}\langle\operatorname{tr}\sigma,\varphi\operatorname{div}u\rangle - \langle\sigma, u\odot\nabla\varphi\rangle, \tag{10.3.7}$$

for all $\varphi \in C_c^\infty(\Omega)$. Note that all $(L^N, L^{N/(N-1)})$-pairings on the right hand side of (10.3.7) are well-defined since any $\sigma \in \Sigma(\operatorname{Div};\Omega)$ with $\operatorname{dev}\sigma \in L^\infty(\Omega;\mathbb{M}_0^{N\times N})$ fulfills the integrability condition

$$\sigma \in L^r(\Omega;\mathbb{M}^{N\times N}), \quad \forall\, r \in [1,+\infty);$$

see [82]. It even holds that $[\operatorname{dev}\sigma,\operatorname{dev}\varepsilon(u)] \in M(\Omega)$ as shown in [82, 119]. With the help of this generalized duality pairing, one may use the additive strain decomposition (10.2.4) to define a pairing between admissible stresses and plastic strains. In fact, for $\sigma \in \Sigma(\mathrm{Div};\Omega)$ with $\operatorname{dev}\sigma \in L^\infty(\Omega;\mathbb{M}_0^{N\times N})$ and $[u,e,p] \in W_{\mathrm{ad}}$, define the measure $[\operatorname{dev}\sigma, p] \in M(\Omega \cup \Gamma_0)$ by

$$[\operatorname{dev}\sigma, p] := \begin{cases} [\operatorname{dev}\sigma, \operatorname{dev}\varepsilon(u)] - \operatorname{dev}\sigma : \operatorname{dev} e, & \text{in } \Omega, \\ -(\sigma\nu)_T \cdot u\, \mathcal{H}^{N-1}, & \text{on } \Gamma_0, \end{cases} \tag{10.3.8}$$

where $(\sigma\nu)_T := \sigma\nu - (\sigma\nu)_\nu \nu$ is the tangential component of $\sigma\nu$. Note that $(\sigma\nu)_T \in L^\infty(\partial\Omega;\mathbb{R}^N)$ such that $[\operatorname{dev}\sigma, p]$ is well-defined, [82, Lemma 2.4]. With the help of this duality pairing, it can be shown that Problem 10.5 is essentially equivalent to Problem 10.2. We recall the corresponding result from [38, Theorem 6.1].

Theorem 10.8. *Let (10.2.7), (10.3.4), (10.3.5), (10.3.6) and Assumption 10.6 hold true. The following assertions are equivalent.*

(i) $[u,e,p] : [0,T] \to BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0;\mathbb{M}_0^{N\times N})$ *is a quasi-static evolution (Problem 10.5).*

(ii) $[\dot u, \sigma]$ *solves Johnson's weak formulation (Problem 10.2) and*

a) $[u,e] \in AC([0,T];BD(\Omega) \times Q)$,

b) $p(t)\lfloor_\Omega = \varepsilon(u)(t) - e(t)$, $p(t)\lfloor_{\Gamma_0} = -u(t) \odot \nu\, \mathcal{H}^{N-1}$ *for all* $t \in [0,T]$.

(iii) *a)* $[u,e,p] \in AC([0,T];BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0;\mathbb{M}_0^{N\times N}))$,

b) $[u,e,p](t) \in W_{ad}$ *for all* $t \in [0,T]$,

c) for all $t \in [0,T]$ *it holds that*

$$\sigma(t) \in S_{ad}(\Sigma(\mathrm{Div};\Omega)), \ -\operatorname{Div}\sigma(t) = f(t) \text{ a.e. in } \Omega, \ \sigma(t)\nu = g(t) \text{ a.e. on } \Gamma_1,$$

d) $D(\dot p(t)) = [\operatorname{dev}\sigma(t) : \dot p(t)](\Omega \cup \Gamma_0)$ *for a.e.* $t \in (0,T)$.

Thus, under slightly improved regularity in time, quasi-static evolutions correspond to solutions of Problem 10.2. Moreover, (iii)(d) represents the appropriate weak form of the flow rule (10.2.8). We proceed by considering a time-discretized version of Problem 10.5.

11 The Time-Incremental Problem

11.1 Problem Statement

In this section we formulate the incremental problem of quasi-static evolution in perfect plasticity. Therefore we assume from now on and for the rest of Part IV that the assumptions (10.2.7),(10.3.4),(10.3.5), (10.3.6) required for the theory of [38] are fulfilled. The dependence on the safe-load condition will be explicited whenever necessary. To begin with, let

$$0 = t_0 < t_1 < \cdots < t_J = T,$$

denote a partition of the time interval $[0, T]$ with $\triangle t = \max_{n\in\{1,\ldots,J\}}(t_n - t_{n-1})$. We define $[u_n, e_n, p_n], n = 1, \ldots, J$, inductively as follows. Starting from $[u_0, e_0, p_0] = 0$, at each fixed point in time we are given the state of the system $[u_{n-1}, e_{n-1}, p_{n-1}] \in W_{\mathrm{ad}}$ from the preceding time instance and the current applied forces $f_n := f(t_n) \in L^N(\Omega; \mathbb{R}^N), g_n := g(t_n) \in L^\infty(\Gamma_1; \mathbb{R}^N)$, which define the total load $l_n = l(t_n) \in BD(\Omega)^*$,

$$l_n(\tilde{u}) := \int_\Omega f_n \cdot \tilde{u}\, dx + \int_{\Gamma_1} g_n \cdot \tilde{u}\, d\mathcal{H}^{N-1}, \quad \tilde{u} \in BD(\Omega). \tag{11.1.1}$$

The triple $[u_n, e_n, p_n]$ is defined as a solution to the following problem.

Problem (P).

$$\begin{cases} \inf & J(u,e,p) \quad \textit{over } [u,e,p] \in BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0, \mathbb{M}_0^{N\times N}) \\ s.t. & [u,e,p] \in W_{ad}, \end{cases}$$

where the objective functional J is defined by

$$J(u,e,p) := \tfrac{1}{2}(\mathbb{C}e, e) + D(p - p_{n-1}) - \langle l_n, u\rangle.$$

Under the safe-load condition (Assumption 10.6), Problem (P) has a solution, which is in general only unique in the elastic strain e. Following the existence proof from [38, Theorem 3.3], one may reformulate Problem (P) by eliminating the dependence on u from the objective function using a suitable integration by parts formula for the generalized pairing of stresses and strains (10.3.8) and an element $\hat{\sigma} \in \Sigma(\mathrm{Div}; \Omega)$ which meets the requirements of the safe-load condition. Thereupon we obtain the equivalent problem

$$\begin{cases} \inf & \tilde{J}(u,e,p) \quad \text{over } [u,e,p] \in BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N}), \\ \text{s.t.} & [u,e,p] \in W_{\mathrm{ad}} \end{cases} \tag{11.1.2}$$

with

$$\tilde{J}(u,e,p) := \tfrac{1}{2}\langle \mathbb{C}e, e\rangle - (\hat{\sigma}, e) + D(p - p_{n-1}) - [\operatorname{dev}\hat{\sigma}, p - p_{n-1}](\Omega \cup \Gamma_0),$$

where Assumption 10.6 guarantees the coercivity of the mapping

$$p \mapsto D(p) - [\operatorname{dev}\hat{\sigma}, p](\Omega \cup \Gamma_0)$$

on $M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N})$; see [38, Lemma 3.2]. Below, we provide an alternative existence proof based on another reformulation of Problem (P).

The time-incremental problems allow for a consistent approximation of Problem 10.5 in that the piecewise constant time interpolates

$$[u^{\triangle t}, e^{\triangle t}, p^{\triangle t}] : (0, T) \to BD(\Omega) \times Q \times M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N})$$

constructed from the time-incremental solutions (the superscript $\triangle t$ indicating the dependence on the step size)

$$[u_n, e_n, p_n] = [u_n^{\triangle t}, e_n^{\triangle t}, p_n^{\triangle t}], \quad n = 1, \ldots, J(\triangle t),$$

converge for $\triangle t \to 0$ (along a subsequence) to a quasi-static evolution $[u, e, p]$ in the sense that

$$\begin{aligned} u^{\triangle t}(t) &\overset{*}{\rightharpoonup} u(t) && \text{in } BD(\Omega), \\ p^{\triangle t}(t) &\overset{*}{\rightharpoonup} p(t) && \text{in } M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N\times N}), \\ e^{\triangle t}(t) &\to e(t) && \text{in } Q, \end{aligned}$$

for all $t \in [0, T]$, see [38, Theorem 4.8].

11.2 Inf-Sup Problem Formulation

As a result of the nonsmoothness of the objective function and the structure of the constraint set in conjunction with the non-reflexive Banach space setting, the convex Problem (P) poses a variety of complexities, which complicates a direct numerical approach to this problem. The goal of this section is to establish a suitable problem reduction, which yields an unconstrained reformulation posed in a conventional Lebesgue space. This reformulation is the main step to establish a Fenchel duality result that relates the primal formulation (P) to the incremental version of the stress problem (Problem 10.3).

With the help of the results from [38] on the various characterizations of the generalized stress-strain duality, the plastic strain p can be eliminated from the optimization problem using the definition of W_{ad}; in fact, for given $[u, e] \in BD(\Omega) \times Q$, $p = p(u, e)$ is uniquely determined by

$$p\lfloor_{\Omega} = \varepsilon(u) - e, \quad p\lfloor_{\Gamma_0} = -u \odot \nu\, \mathcal{H}^{N-1}, \tag{11.2.1}$$

such that only the plastic incompressibility condition $\operatorname{tr} p = 0$ is left to be taken into account in order to ensure $[u, e, p] \in W_{\text{ad}}$. The resulting reduced objective function, which is derived in the subsequent Lemma 11.1, turns out to be given by

$$\hat{J}(u, e) := \tfrac{1}{2}(\mathbb{C}e, e) + \sup_{\substack{\sigma \in S_{\text{ad}}(\Sigma(\operatorname{Div};\Omega)) \\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \operatorname{Div} \sigma, u\rangle\} - \langle f_n, u\rangle, \tag{11.2.2}$$

where $\hat{p}_{n-1}$ is understood as an element of $\Sigma(\operatorname{Div}; \Omega)^*$ defined by

$$\langle \hat{p}_{n-1}, \sigma\rangle = -(\sigma, e_{n-1}) - \langle \operatorname{Div} \sigma, u_{n-1}\rangle, \quad \sigma \in \Sigma(\operatorname{Div}; \Omega). \tag{11.2.3}$$

Note that $\hat{p}_{n-1} = p_{n-1}$ if $u_{n-1} \in H_0^1(\Omega; \mathbb{R}^N)$. The elimination of p from the initial problem (P) is detailed in the following statement.

Lemma 11.1. *Let* $\hat{J} : BD(\Omega) \times Q \to \mathbb{R}$ *be defined by* (11.2.2). *Then Problem (P) is equivalent to the* *problem*

$$\begin{cases} \inf & \hat{J}(u,e) \quad \textit{over } [u,e] \in BD(\Omega) \times Q, \\ \textit{s.t.} & \operatorname{div} u = \operatorname{tr} e \quad \textit{in } \Omega, \\ & u \cdot \nu = 0 \qquad \textit{on } \Gamma_0, \end{cases} \tag{11.2.4}$$

in the following sense.

(i) If $[\bar{u}, \bar{e}, \bar{p}]$ is a solution of Problem (P) then $[\bar{u}, \bar{e}]$ solves (11.2.4).

(ii) For each solution $[\bar{u}, \bar{e}]$ to (11.2.4), *it holds that $[\bar{u}, \bar{e}, p(\bar{u}, \bar{e})]$ is a solution to Problem (P), where $p(\bar{u}, \bar{e})$ is defined by* (11.2.1).

Proof. Let $[u, e, p] \in W_{\mathrm{ad}}$. As the safe-load condition is assumed to hold, we have [38, Prop. 2.4]

$$D(p) = \sup\{[\operatorname{dev} \sigma, p](\Omega \cup \Gamma_0) : \sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div}; \Omega)), \sigma\nu = g_n \text{ on } \Gamma_1\},$$

where the measure $[\operatorname{dev} \sigma, p] \in M(\Omega \cup \Gamma_0)$ is defined by the generalized duality pairing between admissible stresses and strains (10.3.8). The integration by parts formula from [38, Prop. 2.2] provides a useful characterization of the generalized duality. In fact, for any $\sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div}; \Omega))$ with $\sigma\nu \in L^\infty(\Gamma_1; \mathbb{R}^N)$, it holds that

$$[\operatorname{dev} \sigma, p](\Omega \cup \Gamma_0) = -(\sigma, e) - \langle \mathrm{Div}\, \sigma, u \rangle + \langle \sigma\nu, u \rangle_{\Gamma_1}. \tag{11.2.5}$$

Here, we use the shorthand notation $\langle\, .\, ,\, .\, \rangle_{\Gamma_1}$ for the duality pairing of $L^\infty(\Gamma_1; \mathbb{R}^N)$ and $L^1(\Gamma_1; \mathbb{R}^N)$. By (11.2.5) we obtain for all $\sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div}; \Omega))$ with $\sigma\nu = g_n$ on Γ_1,

$$\begin{aligned}[\operatorname{dev} \sigma, p - p_{n-1}](\Omega \cup \Gamma_0) = &- (\sigma, e) - \langle \mathrm{Div}\, \sigma, u \rangle + \langle g_n, u \rangle_{\Gamma_1} \\ &- \langle \hat{p}_{n-1}, \sigma \rangle - \langle g_n, u_{n-1} \rangle,\end{aligned}$$

where $\hat{p}_{n-1}$ is defined in (11.2.3). Hence, we may remove the dependence on p of the objective functional;

$$J(u, e, p) = \hat{J}(u, e) \qquad \forall\, [u, e, p] \in W_{\mathrm{ad}}. \tag{11.2.6}$$

Now let $[\bar{u}, \bar{e}, \bar{p}] \in W_{\mathrm{ad}}$ be a solution of Problem (P) and $[u, e] \in \tilde{W}_{\mathrm{ad}}$, where

$$\tilde{W}_{\mathrm{ad}} := \{[u, e] \in BD(\Omega) \times Q : \operatorname{div} u = \operatorname{tr} e \text{ in } L^2(\Omega),\ u \cdot \nu = 0 \text{ a.e. on } \Gamma_0\}.$$

By taking the trace in the two conditions (10.3.3) of the definition of W_{ad}, one may observe that (11.2.1) defines an element $p \in M(\Omega \cup \Gamma_0; \mathbb{M}_0^{N \times N})$ such that $[u, e, p] \in W_{\mathrm{ad}}$ if and only if $[u, e] \in \tilde{W}_{\mathrm{ad}}$. Using (11.2.6), one deduces that

$$\hat{J}(\bar{u}, \bar{e}) = J(\bar{u}, \bar{e}, \bar{p}) \leq J(u, e, p(u, e)) = \hat{J}(u, e),$$

for all $[u, e] \in \tilde{W}_{\mathrm{ad}}$. This proves assertion (i).

Let $[\bar{u}, \bar{e}] \in \tilde{W}_{\mathrm{ad}}$ be a solution of (11.2.4). Following the above discussion, we find that for any $[u, e, p] \in W_{\mathrm{ad}}$ it holds that $[u, e] \in \tilde{W}_{\mathrm{ad}}$. Hence, (11.2.6) implies that

$$J(\bar{u}, \bar{e}, p(\bar{u}, \bar{e})) = \hat{J}(\bar{u}, \bar{e}) \leq \hat{J}(u, e) = J(u, e, p),$$

for all $[u, e, p] \in W_{\mathrm{ad}}$, which accomplishes the proof of assertion (ii).

□

As usual in plasticity problems with pressure-insensitive yield criteria, it can be expected that there is no need to explicitly take account of the plastic incompressibility constraint $\operatorname{tr} p = 0$ as it is already contained in the variational formulation. For this aspect we refer to the discussion following (10.2.7) and [24]. In fact, it can be shown that the plastic incompressibility constraints in (11.2.4) are redundant.

Lemma 11.2. *Let $\hat{J}$ be given by* (11.2.2)*. Assume the safe-load condition (Assumption 10.6) is fulfilled. Then the problem*

$$\inf \quad \hat{J}(u,e) \quad \text{over } [u,e] \in BD(\Omega) \times Q \tag{11.2.7}$$

is equivalent to Problem (P) in the sense of Lemma 11.1.

Proof. Let $[u,e] \in BD(\Omega) \times Q$. For arbitrary $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on Γ_1 we define $\sigma_\varphi := \hat{\sigma} + \varphi I_N$, where $\hat{\sigma} \in \Sigma(\mathrm{Div};\Omega)$ denotes an element fulfilling the safe-load assumption. Thus, it holds that $\sigma_\varphi \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega))$ with $\sigma_\varphi \nu = g_n$ on Γ_1. Consequently, one may derive the following estimate;

$$\begin{aligned}
&\sup_{\substack{\sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)),\\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \mathrm{Div}\,\sigma, u\rangle\} - \langle f_n, u\rangle \\
&\geq \sup_{\varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_1} \{-\langle \hat{p}_{n-1}, \sigma_\varphi\rangle - (\sigma_\varphi, e) - \langle \mathrm{Div}\,\varphi I_N, u\rangle\} \\
&= -\langle \hat{p}_{n-1}, \hat{\sigma}\rangle - (\hat{\sigma}, e) + \sup_{\substack{\varphi \in C^1(\overline{\Omega}),\\ \varphi = 0 \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \varphi I_N\rangle - (\varphi, \operatorname{tr} e) - \langle \nabla\varphi, u\rangle\}.
\end{aligned}$$

Taking the trace in the Green's formula (1.2.10) implies that

$$\int_\Omega u \cdot \nabla\varphi \, dx = -\int_\Omega \varphi \; d(\operatorname{div} u) + \int_{\partial\Omega} u\nu \, \varphi \, d\mathcal{H}^{N-1} \tag{11.2.8}$$

for all $\varphi \in C^1(\overline{\Omega})$, such that

$$-\langle \hat{p}_{n-1}, \varphi I_N\rangle = (\varphi, \operatorname{tr} e_{n-1}) + \langle \nabla\varphi, u_{n-1}\rangle = 0. \tag{11.2.9}$$

The latter term vanishes since $[u_{n-1}, e_{n-1}, p_{n-1}] \in W_{\mathrm{ad}}$ implies that

$$\operatorname{div} u_{n-1} = \operatorname{tr} e_{n-1}, \quad u_{n-1} \cdot \nu = 0 \;\text{ a.e. on } \Gamma_0.$$

By (11.2.9) and (11.2.8), one obtains

$$\begin{aligned}
&\sup_{\substack{\sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)),\\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \mathrm{Div}\,\sigma, u\rangle\} - \langle f_n, u\rangle \\
&\geq -\langle \hat{p}_{n-1}, \hat{\sigma}\rangle - (\hat{\sigma}, e) \\
&\quad + \sup_{\substack{\varphi \in C^1(\overline{\Omega}),\\ \varphi = 0 \text{ on } \Gamma_1}} \left\{ \int_\Omega \varphi \; (d(\operatorname{div} u) - \operatorname{tr} e \; dx) - \int_{\Gamma_0} u \cdot \nu \, \varphi \, d\mathcal{H}^{N-1} \right\},
\end{aligned} \tag{11.2.10}$$

which implies that $\hat{J}(u,e) = +\infty$ unless

$$\operatorname{div} u - \operatorname{tr} e = 0 \;\text{ in } \Omega. \tag{11.2.11}$$

The redundancy of the boundary condition can be derived as follows. It can be verified that the density property

$$\overline{\{\varphi|_{\Gamma_0} : \varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_1\}}^{C_0(\Gamma_0)} = C_0(\Gamma_0) \tag{11.2.12}$$

is fulfilled; in fact, let $g \in C_c(\Gamma_0)$ and choose an extension $\tilde{g} \in C_c(\omega)$ of g to a nonempty open set $\omega \subset \mathbb{R}^N$ with $\omega \cap \overline{\Gamma}_1 = \emptyset$, $\operatorname{supp} g \subset \omega$ and $\tilde{g}|_{\omega \cap \Gamma_0} = g$. Let $g_n := \rho_n * \tilde{g} \in C_c(\mathbb{R}^N)$ be the standard mollification of $\tilde{g}$; cf. (5.2.9). As $\tilde{g} \in C_c(\omega)$, (g_n) converges uniformly to $\tilde{g}$ in ω. For sufficiently large n, it further holds that $\operatorname{supp} g_n \subset \omega$, and in particular, the sequence of restrictions $(g_n|_{\Gamma_0})$ of (g_n) to Γ_0 represents a feasible approximating sequence in the sense of the left hand side of (11.2.12). Taking account of the fact that $(g_n|_{\Gamma_0})$ converges uniformly to g on Γ_0, the density property (11.2.12) is verified.

Consequently, it holds that

$$\int_{\Gamma_0} u \cdot \nu \, \varphi \; d\mathcal{H}^{N-1} = 0,$$

for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on Γ_1, if and only if,

$$\|u \cdot \nu \, \mathcal{H}^{N-1}\|_{M(\Gamma_0)} = \|u \cdot \nu\|_{L^1(\Gamma_0)} = 0. \tag{11.2.13}$$

Finally, (11.2.10) together with (11.2.11) and (11.2.13) imply that $\hat{J}(u,e) < +\infty$ requires that $u \cdot \nu$ vanishes on Γ_0.

As a conclusion, the constraints in problem (11.2.4) are redundant and the assertion follows from Lemma 11.1. □

Observe also that the reformulation comes at the loss of the finiteness of the objective function. The goal of the subsequent lemma is to show that the objective functional in (11.2.4) can be extended to displacements u in the space $L^{N/(N-1)}(\Omega; \mathbb{R}^N)$.

Lemma 11.3. *Assume the safe-load condition is satisfied. Then the objective function $\hat{J} = \hat{J}(u,e)$ from (11.2.2) is coercive in $BD(\Omega) \times Q$. More precisely, there exist constants $c_0 \in \mathbb{R}, c_1 > 0$ such that*

$$\begin{aligned} &\sup_{\substack{\sigma \in S_{ad}(\Sigma(\operatorname{Div};\Omega)) \\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \operatorname{Div}\sigma, u\rangle\} - \langle f_n, u\rangle \\ &\quad \geq c_0 - c_1\|e\|_Q + \rho \max(\|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})}, -\|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})} + \tfrac{1}{\sqrt{2}}\|u\|_{L^1(\Gamma_0;\mathbb{R}^N)}) \end{aligned} \tag{11.2.14}$$

for all $[u,e] \in BD(\Omega) \times Q$. Here, $\rho > 0$ is the constant from Assumption 10.6.

Proof. First, we state the elementary result

$$|\operatorname{dev}\tau|_F \leq |\tau|_F \quad \text{for all } \tau \in \mathbb{M}^{N\times N}. \tag{11.2.15}$$

Making use of Assumption 10.6 and (11.2.15), it holds that

$$\begin{aligned} &\sup_{\substack{\sigma \in S_{ad}(\Sigma(\operatorname{Div};\Omega)), \\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \operatorname{Div}\sigma, u\rangle\} - \langle f_n, u\rangle \\ &\geq \sup_{\substack{\tau \in C^1(\overline{\Omega}, \mathbb{M}^{N\times N}), \tau = 0 \text{ on } \Gamma_1, \\ \|\tau\|_{C(\overline{\Omega};\mathbb{M}^{N\times N})} \leq \rho}} \{-\langle \hat{p}_{n-1}, \hat{\sigma} + \tau\rangle - (\hat{\sigma} + \tau, e) - \langle \operatorname{Div}\tau, u\rangle\} \\ &\geq c + \sup_{\substack{\tau \in C^1(\overline{\Omega};\mathbb{M}^{N\times N}), \tau = 0 \text{ on } \Gamma_1, \\ \|\tau\|_{C(\overline{\Omega};\mathbb{M}^{N\times N})} \leq \rho}} \{-\langle \hat{p}_{n-1}, \tau\rangle - (\hat{\sigma} + \tau, e) - \langle \operatorname{Div}\tau, u\rangle\}, \end{aligned}$$

for all $e \in Q$ and $u \in BD(\Omega)$, where $c \in \mathbb{R}$ denotes a constant which may take different values on different occasions. Using Green's formula for $BD(\Omega)$-functions (1.2.10), one obtains

$$\begin{aligned} -\langle \hat{p}_{n-1}, \tau \rangle &= (e_{n-1}, \tau) + \langle u_{n-1}, \operatorname{Div} \tau \rangle \\ &\geq -c\|e_{n-1}\|_Q - \int_\Omega \tau : \varepsilon(u_{n-1}) + \int_{\Gamma_0} (u_{n-1} \odot \nu) : \tau \, d\mathcal{H}^{N-1} \\ &\geq -c\|e_{n-1}\|_Q - \rho(|\varepsilon(u_{n-1})|_F(\Omega) + \|u_{n-1} \odot \nu\|_{L^1(\Gamma_0;\mathbb{M}^{N\times N})}) \end{aligned}$$

and

$$-(\hat{\sigma} + \tau, e) \geq -(\|\hat{\sigma}\|_Q + \rho|\Omega|^{1/2})\|e\|_Q$$

for all $\tau \in C^1(\overline{\Omega}, \mathbb{M}^{N\times N})$ with $\|\tau\|_{C(\overline{\Omega}, \mathbb{M}^{N\times N})} \leq \rho$ and $\tau|_{\Gamma_1} = 0$. This implies that

$$\begin{aligned} \sup_{\substack{\sigma \in S_{\mathrm{ad}}(\Sigma(\operatorname{Div};\Omega)),\\ \sigma\nu = g_n \text{ on } \Gamma_1}} &\{-\langle \hat{p}_{n-1}, \sigma \rangle - (\sigma, e) - \langle \operatorname{Div} \sigma, u \rangle\} - \langle f_n, u \rangle \\ &\geq c_0 - c_1\|e\|_Q + \sup_{\substack{\tau \in C^1(\overline{\Omega};\mathbb{M}^{N\times N}), \tau = 0 \text{ on } \Gamma_1,\\ \|\tau\|_{C(\overline{\Omega};\mathbb{M}^{N\times N})} \leq \rho}} \{-\langle \operatorname{Div} \tau, u \rangle\}, \end{aligned} \tag{11.2.16}$$

where

$$\begin{aligned} \sup_{\substack{\tau \in C^1(\overline{\Omega}, \mathbb{M}^{N\times N}), \tau = 0 \text{ on } \Gamma_1,\\ \|\tau\|_{C(\overline{\Omega};\mathbb{M}^{N\times N})} \leq \rho}} \{-\langle \operatorname{Div} \tau, u \rangle\} &\geq \sup_{\substack{\tau \in C_0^1(\Omega;\mathbb{M}^{N\times N}),\\ \|\tau\|_{C_0(\Omega;\mathbb{M}^{N\times N})} \leq \rho}} \{-\langle \operatorname{Div} \tau, u \rangle\} \\ &= \rho\|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})}. \end{aligned} \tag{11.2.17}$$

Furthermore, it is well known that for $\partial\Omega \in C^2$ each $\tau \in C^1(\partial\Omega)$ may be extended to a function $T_\tau \in C^1(\overline{\Omega})$; see [51]. This can be achieved by

$$T_\tau(x) := \theta(r \operatorname{dist}(x, \partial\Omega))\tau(\pi(x)),$$

where π denotes the locally uniquely determined projection of x onto the boundary $\partial\Omega$, $r \in \mathbb{R}$ is sufficiently large, and $\theta \in C^\infty(\mathbb{R})$ denotes a smooth function with

$$\theta \in [0, 1], \theta(t) = 0 \text{ for } t \geq 2 \text{ and } \theta(t) = 1 \text{ for } t \leq 1.$$

Again using (1.2.10), one obtains

$$
\begin{aligned}
&\sup_{\substack{\tau\in C^1(\overline{\Omega};\mathbb{M}^{N\times N}),\tau=0 \text{ on } \Gamma_1,\\ \|\tau\|_{C(\overline{\Omega};\mathbb{M}^{N\times N})}\le\rho}} \{-\langle \operatorname{Div}\tau, u\rangle\} \\
&\ge \sup_{\substack{\tau\in C_0^1(\Gamma_0;\mathbb{M}^{N\times N}),\tau=0 \text{ on } \Gamma_1\\ \|\tau\|_{C_0(\Gamma_0;\mathbb{M}^{N\times N})}\le\rho}} \{-\langle \operatorname{Div}T_\tau, u\rangle\} \\
&= \sup_{\substack{\tau\in C_0^1(\Gamma_0;\mathbb{M}^{N\times N}),\\ \|\tau\|_{C_0(\Gamma_0;\mathbb{M}^{N\times N})}\le\rho}} \left(\int_\Omega T_\tau : \varepsilon(u) - \int_{\Gamma_0} (u\odot\nu) : \tau \, d\mathcal{H}^{N-1}\right) \\
&\ge \rho(-\|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})} + \|u\odot\nu\|_{L^1(\Gamma_0;\mathbb{M}^{N\times N})}) \\
&\ge \rho\left(-\|\varepsilon(u)\|_{M(\Omega;\mathbb{M}^{N\times N})} + \tfrac{1}{\sqrt{2}}\|u\|_{L^1(\Gamma_0;\mathbb{R}^N)}\right).
\end{aligned}
$$

In the last estimate we use the elementary property

$$|a\odot b|_F \ge \tfrac{1}{\sqrt{2}}|a|_2|b|_2,$$

and together with (11.2.16), (11.2.17), the proof of (11.2.14) is accomplished. The coercivity of the objective function $\hat{J}$ in $BD(\Omega)\times Q$ now follows from (11.2.14), the ellipticity property

$$(\mathbb{C}e, e) \ge \kappa_1\|e\|_Q^2$$

and the fact that

$$u \mapsto \|u\|_{L^1(\Gamma_0;\mathbb{R}^N)} + \|u\|_{M(\Omega;\mathbb{M}^{N\times N})}$$

defines an equivalent norm on $BD(\Omega)$, cf. Section 1.2.3. □

The significance of the preceding lemma is twofold. First, note that the objective function $\hat{J}(u,e)$ has a natural extension to $L^{N/(N-1)}(\Omega)^N\times Q$. Indeed, by definition, $\hat{J}$ is well-defined as an extended real-valued function on $L^{N/(N-1)}(\Omega)^N\times Q$; cf. (11.2.2). Moreover, from the proof of the preceding lemma one obtains that

$$u\in L^{N/(N-1)}(\Omega)^N\setminus BD(\Omega) \Longrightarrow \hat{J}(u,e) = +\infty, \tag{11.2.18}$$

for all $e\in Q$, since the regularity constraint $\varepsilon(u)\in M(\Omega)$ is implicitly contained in the objective function owing to the estimates (11.2.16) and (11.2.17). Consequently, we obtain the following equivalent problem.

Problem ($\mathbf{P_{red}}$).

$$\inf \quad \hat{J}(u,e) \qquad \textit{over } [u,e]\in L^{N/(N-1)}(\Omega)^N\times Q,$$

where $\hat{J} : L^{N/(N-1)}(\Omega)^N\times Q\to\mathbb{R}\cup\{+\infty\}$ *is given by* (11.2.2).

A similar situation arises in the context of total bounded variation regularization in image restoration; see Section 6.2. However, the argument here additionally relies on the validity of the safe-load condition.

Secondly, Lemma 11.3 gives rise to an alternative existence proof to Problem (P). These results are summarized in the following theorem.

Theorem 11.4. *Let $\hat{J}$ be given by (11.2.2). Assume the safe-load condition (Assumption 10.6) is fulfilled. Then Problem (P_{red}) is equivalent to Problem (P) in the sense of Lemma 11.1, and Problem (P) has a solution $[\bar{u}, \bar{e}, \bar{p}]$, which is unique in $\bar{e}$.*

Proof. The equivalence of the problems (P) and (P_{red}) is induced by Lemma 11.1, Lemma 11.2 and (11.2.18). For the existence proof, we use the problem formulation (11.2.7). As a pointwise limit of affine continuous functions, the mapping

$$[u,e] \mapsto \sup_{\substack{\sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)),\\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \mathrm{Div}\,\sigma, u\rangle\} - \langle f_n, u\rangle \tag{11.2.19}$$

is sequentially l.s.c. in $L^{N/(N-1)}(\Omega)^N \times Q$ equipped with the weak$\times$weak topology. If $u_k \overset{*}{\rightharpoonup} u$ in $BD(\Omega)$ then (u_k) is bounded in $BD(\Omega)$ and fulfills $u_k \to u \in L^1(\Omega)^N$. By the continuous embedding (1.2.9), each subsequence of (u_k) has a subsequence converging weakly in $L^{N/(N-1)}(\Omega)^N$ to u. Urysohn's principle implies that the entire sequence (u_k) weakly converges to u in $L^{N/(N-1)}(\Omega)^N$. Consequently, the mapping from (11.2.19) is also sequentially l.s.c. in $BD(\Omega) \times Q$ endowed with the weak$^*\times$weak topology. Together with the coercivity property in $BD(\Omega) \times Q$ given by Lemma 11.3, the direct method can be applied to prove the existence of a solution $[\bar{u}, \bar{e}]$ to (11.2.7). The existence of a solution to (P) follows by Lemma 11.2 and the uniqueness of $\bar{e}$ is an immediate consequence of the strict convexity of the mapping $e \mapsto (\mathbb{C}e, e)_Q$. □

In contrast to the original problem, (P_{red}) is an unconstrained minimization problem in a reflexive Banach space (even Hilbert space for $N = 2$). This seems to be more attractive from a numerical point of view, and it facilitates the analysis of the primal problem (P) within Fenchel duality theory. In fact, a Fenchel duality result can be derived based on the alternative functional analytic setting provided by the reduced problem formulation (P_{red}). This is precisely the purpose of the following section.

11.3 The Incremental Stress Problem as a Fenchel Dual Problem

In Hencky plasticity, equality of the extremal values between the stress problem, and the initial strain problem posed in a Sobolev space and its relaxation in $BD(\Omega)$ is well known; cf. [119, p.251 ff.], and it is expected that a similar result is true for perfect plasticity. Indeed, the goal of this paragraph is to demonstrate that the standard incremental stress problem of perfect plasticity can be derived from the primal problem (P) within the theory of Fenchel duality (see Section 2.2) using the reduced formulation problem (P_{red}). In this regard, the subsequent developments justify the formal duality approaches to perfect plasticity; cf. [106].

For further reference, we introduce the set of admissible stresses which fulfill a boundary condition for a given function $\tilde{g}$ on Γ_1;

$$S_{\mathrm{ad}}(\tilde{g}) := \{\sigma \in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)) : \sigma\nu = \tilde{g} \text{ in } [H_{00}^{-1/2}(\Gamma_1)]^N\}, \quad \tilde{g} \in [H_{00}^{-1/2}(\Gamma_1)]^N. \tag{11.3.1}$$

Note that the regularity of the boundary trace is ensured by the property $S_{\mathrm{ad}}(\tilde{g}) \subset H(\mathrm{div};\Omega)$. Under Assumption 10.6, $S_{\mathrm{ad}}(g_n)$ is nonempty, such that the indicator function

$$i_{S_{\mathrm{ad}}(g_n)} : \Sigma(\mathrm{Div};\Omega) \to \mathbb{R} \cup \{+\infty\} \tag{11.3.2}$$

of the convex set $S_{\mathrm{ad}}(g_n)$ in the space $\Sigma(\mathrm{Div};\Omega)$ is proper. We also define the bounded linear operator

$$\Lambda \in \mathcal{L}(L^{N/(N-1)}(\Omega)^N \times Q, \Sigma(\mathrm{Div};\Omega)^*), \quad \Lambda(u,e) := -\mathrm{Div}^*\, u - e. \tag{11.3.3}$$

Furthermore, we set

$$F(u,e) := -\langle f_n, u\rangle + \tfrac{1}{2}(\mathbb{C}e,e), \quad G(\sigma^*) := i^*_{S_{\mathrm{ad}}(g_n)}(\sigma^*), \tag{11.3.4}$$

for $[u,e] \in L^{N/(N-1)}(\Omega)^N \times Q$ and $\sigma^* \in \Sigma(\mathrm{Div};\Omega)^*$. The Fenchel conjugate function of $i_{S_{\mathrm{ad}}(g_n)}$ in $\Sigma(\mathrm{Div};\Omega)$ is denoted by

$$i^*_{S_{\mathrm{ad}}(g_n)} : \Sigma(\mathrm{Div};\Omega)^* \to \mathbb{R} \cup \{+\infty\}.$$

With these definitions, (P$_{\mathrm{red}}$) takes the equivalent compact form

$$\begin{cases} \min & F(u,e) + G(\Lambda[u,e] - \langle \hat{p}_{n-1}, \,.\,\rangle) \\ \text{over} & [u,e] \in L^{N/(N-1)}(\Omega)^N \times Q. \end{cases}$$

A straightforward computation leads to

$$F^*(u^*,e^*) = i_{\{-f_n\}}(u^*) + \tfrac{1}{2}(\mathbb{C}^{-1}e^*,e^*), \quad G^*(\sigma) = i^{**}_{S_{\mathrm{ad}}(g_n)}(\sigma),$$

for $[u^*,e^*] \in [L^N(\Omega)]^N \times Q$ and $\sigma \in \Sigma(\mathrm{Div};\Omega)$. The adjoint of Λ is given by

$$\Lambda^*\sigma = [-\mathrm{Div}\,\sigma, -\sigma] \;\in [L^N(\Omega)]^N \times Q. \tag{11.3.5}$$

Since $S_{\mathrm{ad}}(g_n) \subset \Sigma(\mathrm{Div};\Omega)$ is convex and closed, it holds that $i^{**}_{S_{\mathrm{ad}}(g_n)} = i_{S_{\mathrm{ad}}(g_n)}$. According to (2.2.4), the Fenchel dual problem of (P$_{\mathrm{red}}$) corresponding to the above setting is given by

Problem (DP).

$$\begin{cases} \inf & \tfrac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) + \langle \hat{p}_{n-1},\sigma\rangle \\ s.t. & -\mathrm{Div}\,\sigma = f_n,\ \sigma\nu = g_n \ on\ \Gamma_1,\ \sigma \in S_{ad} \\ over & \sigma \in \Sigma(\mathrm{Div};\Omega). \end{cases}$$

Problem (DP) is exactly the stress problem (Problem 10.3) of perfect plasticity in incremental form resulting from an implicit Euler time discretization;

$$\dot{\sigma}(t_n) \approx \tfrac{\sigma(t_n)-\sigma(t_{n-1})}{t_n - t_{n-1}}.$$

We summarize the result in the following theorem.

Theorem 11.5. *Suppose that Assumption 10.6 is satisfied. A Fenchel dual problem of the time-incremental problem of quasi-static evolution in perfect plasticity in reduced form (Problem (P$_{red}$)) is given by Problem (DP), which is the stress problem in incremental form. There is no duality gap between primal and dual problem, i.e., it holds that*

$$\inf(P_{red}) = -\inf(DP). \tag{11.3.6}$$

Proof. To show that a duality gap between (P$_{\mathrm{red}}$) and (DP) can be precluded, it suffices that the following constraint qualification is fulfilled;

$$-\hat{p}_{n-1} \in \mathrm{int}(\mathrm{dom}\,G - \Lambda\,\mathrm{dom}\,F); \tag{11.3.7}$$

cf. (2.2.2). The validity of (11.3.7) can be seen as follows: From the definition of the adjoint (11.3.5), it follows directly that Λ^* is injective, which implies that $\mathrm{ran}\,\Lambda$ is dense in $\Sigma(\mathrm{Div};\Omega)^*$. By the Closed Range Theorem, Λ is surjective if and only if the range of Λ^* is closed. The latter is obvious from the definition of Λ^*. Together with $\mathrm{dom}\,G \neq \emptyset$ and $\mathrm{dom}\,F = L^{N/(N-1)}(\Omega)^N \times Q$, the surjectivity of Λ implies that

$$\operatorname{dom} G - \Lambda \operatorname{dom} F = \Sigma(\operatorname{Div};\Omega)^*,$$

such that the constraint qualification (11.3.7) is satisfied. $\square$

Under Assumption 10.6, the direct method allows to derive that (DP) has a solution $\bar{\sigma} \in \Sigma(\operatorname{Div};\Omega)$, which is unique owing to the strict convexity of the mapping $\sigma \mapsto (\mathbb{C}^{-1}\sigma,\sigma)$. By virtue of (2.2.2) and Lemma 2.6, $\bar{\sigma}$ can be linked to solutions $[\bar{u},\bar{e}]$ of the primal problem (P) by the following primal-dual optimality system;

$$\bar{\sigma} \in S_{\mathrm{ad}}(g_n), \quad \operatorname{Div}\bar{\sigma} = -f_n, \quad \mathbb{C}\bar{e} = \bar{\sigma}, \tag{11.3.8}$$

$$-\hat{p}_{n-1} - \operatorname{Div}^* \bar{u} - \bar{e} \in N_{S_{\mathrm{ad}}(g_n)}(\bar{\sigma}), \tag{11.3.9}$$

with $N_{S_{\mathrm{ad}}(g_n)}(\bar{\sigma}) = \partial i_{S_{\mathrm{ad}}(g_n)}(\bar{\sigma})$, where $\partial i_{S_{\mathrm{ad}}(g_n)}$ denotes the usual (convex) subdifferential of the function $i_{S_{\mathrm{ad}}(g_n)}$ defined in (11.3.2) in the space $\Sigma(\operatorname{Div};\Omega)$. Note that (11.3.9) is equivalent to

$$\langle \bar{u} - u_{n-1}, \operatorname{Div}\tilde{\sigma} - \operatorname{Div}\bar{\sigma}\rangle + (\bar{e} - e_{n-1}, \tilde{\sigma} - \bar{\sigma}) \geq 0 \quad \forall\, \tilde{\sigma} \in S_{\mathrm{ad}}(g_n); \tag{11.3.10}$$

that is, the optimality system (11.3.8)-(11.3.9) represents precisely the time-discretized version of Johnson's weak formulation (Problem 10.2). Moreover, our result shows that the necessary optimality conditions for the time-discretized primal problem (P) given in [38, Theorem 3.6(c)] can be supplemented by the normal cone condition (11.3.10) to obtain necessary and sufficient optimality conditions for the solution of the time-incremental problem in quasi-static perfect plasticity. A rigorous Fenchel duality result for the time-discrete primal problem of perfect plasticity and the dual stress problem has thus been established. We stress that the proof of the Fenchel duality result requires the correct choice of the topology in which primal and dual problem are set.

Finally, one may use the definition (11.2.3) of $\hat{p}_{n-1}$ as an element of $\Sigma(\operatorname{Div})^*$ together with the weak form of the equality constraints in (DP) in order to derive the following equivalent problem to (DP).

Problem 11.6.

$$\begin{cases} \inf & \frac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) - (\mathbb{C}^{-1}\sigma_{n-1},\sigma) \\ \textit{s.t.} & \varepsilon^*\sigma = l_n \;\; \textit{in } V^*,\; \sigma \in S_{ad}, \\ \textit{over} & \sigma \in Q. \end{cases}$$

Here, we make use of the adjoint ε^* of the operator $\varepsilon \in \mathcal{L}(V,Q)$ to pose Problem 11.6 in the less regular space Q. The linear functional l_n is defined in (11.1.1).

12 A Modified Visco-Plastic Regularization for the Time-Incremental Problems

12.1 Visco-Plastic Regularization

In perfect plasticity, the flow law is given by the nonsmooth inclusion

$$\dot{p} \in N_{\mathbb{K}}(\sigma) = \partial i_{\mathbb{K}}(\sigma), \tag{12.1.1}$$

where admissible stresses are supposed to lie in the set $\mathbb{K}$; cf. Problem 10.1. A classical approach to the problem of perfect plasticity is the visco-plastic regularization. The idea of this approach is to replace the indicator function $i_{\mathbb{K}}$ associated with the constraint $\sigma(x) \in \mathbb{K}$ by a smooth approximation $i_{\mathbb{K}}^{\gamma}$ such as the Moreau-Yosida regularization. In this way, (12.1.1) is transformed into a smooth equation where the stresses may lie outside the feasible set $\mathbb{K}$, and the Fréchet derivative of the regularization of $i_{\mathbb{K}}$ serves as an approximation of the plastic strain rate. In this way, perfect plasticity can be seen as the limit of visco-plasticity as $\gamma \to +\infty$, which is the basis for the existence proofs in [78, 117]. Replacing $\dot{p}$ by an implicit Euler scheme, the time-incremental version of the flow law in visco-plasticity is then given by

$$p_n = p_{n-1} + \triangle t_n \, {i_{\mathbb{K}}^{\gamma}}'(\sigma_n), \quad \triangle t_n := t_n - t_{n-1}. \tag{12.1.2}$$

On the level of the weak formulation in terms of the stress (Problem 11.6), we employ a Moreau-Yosida regularization corresponding to the constraint $\sigma \in S_{\mathrm{ad}}$. For fixed parameter $\gamma > 0$, this leads to the stress problem of visco-plasticity.

Problem (VP$_\gamma$).

$$\begin{cases} \inf & \frac{1}{2}(\mathbb{C}^{-1}\sigma, \sigma) - (\mathbb{C}^{-1}\sigma_{n-1}, \sigma) + i_{S_{ad}}^{\gamma}(\sigma) \\ s.t. & \varepsilon^*\sigma = l_n \in V^*, \\ over & \sigma \in Q. \end{cases} \tag{12.1.3}$$

In order to find a practical characterization of the projection onto the set of admissible stresses (cf. equation (12.1.6) below), one usually employs the Moreau-Yosida regularization $i_{S_{\mathrm{ad}}}^{\gamma}(\sigma)$ of $i_{S_{\mathrm{ad}}} : Q \to \mathbb{R} \cup \{+\infty\}$ with respect to the scalar product

$$(\sigma, \tilde{\sigma})_{\mathbb{C}^{-1}} := (\mathbb{C}^{-1}\sigma, \tilde{\sigma})_Q = \int_\Omega \mathbb{C}^{-1}\sigma : \tilde{\sigma} \, dx,$$

on Q. By the properties of $\mathbb{C}$ according to (10.3.5), $\sqrt{(\,.\,,\,.\,)_{\mathbb{C}^{-1}}}$ yields an equivalent norm on Q. Consequently, we obtain that

$$i_{S_{\mathrm{ad}}}^{\gamma}(\sigma) = \tfrac{\gamma}{2}\|\sigma - \pi_{S_{\mathrm{ad}}}\sigma\|_{\mathbb{C}^{-1}}^2, \quad q \in Q,$$

where $\pi_{S_{\mathrm{ad}}} : Q \to S_{\mathrm{ad}}$ denotes the projection onto S_{ad} with respect to $(Q, (\,.\,,\,.\,)_{\mathbb{C}^{-1}})$. At this point, we mention that in the literature on plasticity, the stress problem in incremental visco-plasticity is usually derived by considering the weak form of (12.1.2). This results in Problem (VP$_\gamma$), where γ is replaced by the time-dependent parameter $\triangle t_n \gamma$; cf. [78, p.436]. In this approach, the magnitude of the product $\triangle t_n \gamma$ determines whether the approximative material behavior is dominated by perfectly plastic ($\triangle t_n \gamma \to +\infty$) or elastic material behavior ($\triangle t_n \gamma \to 0$). For a more detailed discussion we refer to [106, p.79].

In order to derive the corresponding predual formulation of incremental visco-plasticity, we apply Fenchel duality theory according to (2.2.4) with

$$\begin{aligned} F(\sigma) &:= \tfrac{1}{2}(\sigma,\sigma)_{\mathbb{C}^{-1}} - (\sigma_{n-1},\sigma)_{\mathbb{C}^{-1}} + i^{\gamma}_{S_{\text{ad}}}(\sigma), \quad \sigma \in Q; \\ G(u^*) &:= i_{\{-l_n\}}(u^*); \quad u^* \in V^*, \quad \Lambda := -\varepsilon^*. \end{aligned}$$

As a result of this setting, the primal problem of visco-plasticity in the Sobolev space V is given by

$$\begin{cases} \inf & \sup_{\sigma \in Q}\{(\varepsilon(u),\sigma) - \frac{1}{2}(\sigma,\sigma)_{\mathbb{C}^{-1}} + (\sigma_{n-1},\sigma)_{\mathbb{C}^{-1}} - i^{\gamma}_{S_{\text{ad}}}(\sigma)\} - l_n(u) \\ \text{over} & u \in V. \end{cases} \tag{12.1.4}$$

Further note that the constraint qualification (2.2.2) is satisfied since $\varepsilon^* \in \mathcal{L}(Q, V^*)$ is surjective by Korn's inequality, which states that there exists a $c > 0$ such that

$$\|\varepsilon(u)\|_Q \geq c\|u\|_{H^1(\Omega;\mathbb{R}^N)}, \quad \forall\, u \in V;$$

see, e.g., [61, p. 147]. Consequently, there is no duality gap, i.e.,

$$\inf(VP_\gamma) = \inf(12.1.4).$$

In order to eliminate the inner sup-problem in (12.1.4), one may use the corresponding optimality conditions. To begin with, the derivative of the Moreau-Yosida term is given by

$$i^{\gamma}_{S_{\text{ad}}}{}'(\sigma) = \gamma\mathbb{C}^{-1}(\sigma - \pi_{S_{\text{ad}}}(\sigma)),$$

such that the unique solution $\sigma = \sigma(u)$ of the inner optimization problem in (12.1.4) is characterized by

$$\sigma = \mathbb{C}\varepsilon(u) + \sigma_{n-1} - \gamma(\sigma - \pi_{S_{\text{ad}}}(\sigma)). \tag{12.1.5}$$

Consequently, $(\mathbb{C}\varepsilon(u) + \sigma_{n-1})$ is on the line joining σ and the projection $\pi_{S_{\text{ad}}}(\sigma)$, which entails that

$$\pi_{S_{\text{ad}}}(\sigma) = \pi_{S_{\text{ad}}}(\mathbb{C}\varepsilon(u) + \sigma_{n-1}); \tag{12.1.6}$$

see [106, Lemma 3.2]. Using (12.1.6) in (12.1.5), one finds that

$$\sigma = \tfrac{1}{1+\gamma}(\mathbb{C}\varepsilon(u) + \sigma_{n-1} + \gamma\pi_{S_{\text{ad}}}(\mathbb{C}\varepsilon(u) + \sigma_{n-1})). \tag{12.1.7}$$

Using (12.1.7), one may eliminate the inner sup-problem, such that problem (12.1.4) can be given a closed form in u. Furthermore, it can be shown (see, e.g., [106, Theorem 5.2]) that (12.1.4) has a unique solution $u_\gamma \in V$, which is linked to the unique solution $\sigma_\gamma \in Q$ of Problem (VP$_\gamma$) by

$$\begin{aligned} \varepsilon(u_\gamma) &= \mathbb{C}^{-1}((1+\gamma)\sigma_\gamma - \sigma_{n-1} - \gamma\pi_{S_{\text{ad}}}(\sigma)) \quad \text{in } Q, \\ \varepsilon^*\sigma &= l \quad \text{in } V^*. \end{aligned}$$

Note the fact that in visco-plasticity the optimal displacement u_γ lies in the Sobolev space V which sharply contrasts with the case of perfect plasticity. In [106, Lemma 3.8], it is shown that the visco-plastic regularization is equivalent to a problem of plasticity with kinematic hardening where the hardening modulus depends on the regularization parameter γ. From the discussion of Chapter 9, it follows that this problem class requires itself further regularization techniques to design an efficient algorithm in function space. Moreover, the convergence of related Augmented Lagrangian methods hinges on the pointwise interpretation of the flow rule (10.2.6) which requires the L^2-regularity of the plastic strain; see, for instance, [112, 106]. In perfect plasticity however, p is in general only a measure and as already stated in Theorem 10.8, (10.2.6) holds only in a measure space sense.

For these reasons, it appears to be worthwhile to consider an alternative regularization scheme which is different from a vanishing hardening approach, and which maintains the original function space setting of the primal problems (P) and ($\mathrm{P}_{\mathrm{red}}$).

12.2 A Modified Visco-Plastic Regularization

In this section we propose a primal modification which combines the usual visco-plastic regularization of the flow law with a Tikhonov regularization of the objective functional in ($\mathrm{P}_{\mathrm{red}}$). As it turns out, this approach allows to recover a one-to-one relation between the approximations of the primal variable pair $[u, p]$ and the solution of a suitably modified version of the incremental stress problem (DP) in the original infinite-dimensional setting. In particular, the approximations of u are not assumed to be elements of the Sobolev space V. For $N' := N/(N-1)$, recall that $BD(\Omega) \hookrightarrow L^{N'}(\Omega)^N$ (see (1.2.9)) and consider the following family of regularized problems induced by a sequence of positive parameters $\mu > 0$.

Problem (MVP$_\mu$).

$$\begin{cases} \inf & \hat{J}_\mu(u,e) \\ \mathit{over} & [u,e] \in L^{N'}(\Omega)^N \times Q, \end{cases}$$

where

$$\begin{aligned} \hat{J}_\mu(u,e) := \tfrac{1}{\mu N'} \|u\|^{N'}_{L^{N'}(\Omega)^N} - \langle f_n, u\rangle + \tfrac{1}{2}(\mathbb{C}e, e) \\ + \sup_{\substack{\sigma\in\Sigma(\mathrm{Div};\Omega),\\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \operatorname{Div}\sigma, u\rangle - i^\mu_{S_{ad}}(\sigma)\}. \end{aligned}$$

Existence and uniqueness of a solution to (MVP$_\mu$) then follows by standard arguments from convex analysis as summarized in the following proposition.

Proposition 12.1. *Assume that the safe-load condition (Assumption 10.6) is fulfilled. Then Problem (MVP$_\mu$) admits a unique solution $[u_\mu, e_\mu]$ which satisfies $u_\mu \in BD(\Omega)$, $u_\mu\nu = 0$ on Γ_0 and $\operatorname{div} u_\mu = \operatorname{tr} e_\mu$ in Ω.*

Proof. The function

$$[u,e] \mapsto \sup_{\substack{\sigma\in\Sigma(\mathrm{Div};\Omega),\\ \sigma\nu = g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1}, \sigma\rangle - (\sigma, e) - \langle \operatorname{Div}\sigma, u\rangle - i^\mu_{S_{\mathrm{ad}}}(\sigma)\}$$

represents the pointwise supremum of a sequence of affine functions on $L^{N'}(\Omega)^N \times Q$ and as such, it is convex and weakly-l.s.c. in $L^{N'}(\Omega)^N \times Q$. Under Assumption 10.6 it is also proper. The additional strictly convex term

$$\tfrac{1}{N'\mu}\|u\|^{N'}_{L^{N'}(\Omega)^N} \tag{12.2.1}$$

yields the coercivity of $\hat{J}_\mu$ on $L^{N'}(\Omega)^N \times Q$. Existence and uniqueness of a solution now follows by the direct method. The regularity statement $\varepsilon(u) \in M(\Omega; \mathbb{M}^{N\times N})$ follows under Assumption 10.6

by

$$\begin{aligned}
&\sup_{\substack{\sigma\in\Sigma(\mathrm{Div};\Omega),\\ \sigma\nu=g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1},\sigma\rangle - (\sigma,e) - \langle \operatorname{Div}\sigma, u\rangle - i^{\mu}_{S_{\mathrm{ad}}}(\sigma)\} - \langle f_n, u\rangle \\
&\quad \geq \sup_{\substack{\sigma\in S_{\mathrm{ad}}(\Sigma(\mathrm{Div};\Omega)),\\ \sigma\nu=g_n \text{ on } \Gamma_1}} \{-\langle \hat{p}_{n-1},\sigma\rangle - (\sigma,e) - \langle \operatorname{Div}\sigma, u\rangle\} - \langle f_n, u\rangle,
\end{aligned} \tag{12.2.2}$$

together with the estimate (11.2.14). Since $u_\mu \in BD(\Omega)$, the validity of the plastic incompressibility conditions $u_\mu \cdot \nu = 0$ on Γ_0 and $\operatorname{div} u_\mu = \operatorname{tr} e_\mu$ can be deduced from (12.2.2) as in the proof of Lemma 11.2. □

Unlike the case of the visco-plastic regularization, we do neither dispose of an explicit problem formulation of Problem (MVP$_\mu$) in terms of u nor is it possible to prove that the optimal displacement in Problem (MVP$_\mu$) is an element of the Sobolev space V. Therefore Problem (MVP$_\mu$) does not fall into the realm of hardening plasticity. However, the above regularization seems to be useful for devising stable algorithmic schemes to solve (P) via its approximation (MVP$_\mu$), which has the advantage of being uniquely solvable. It can also be expected that (MVP_μ) yields a close approximation of (P$_{\mathrm{red}}$), at least for large μ. Before discussing this issue, we proceed by computing the associated Fenchel dual problem. This problem turns out to be a penalized version of the incremental stress problem, which is unconstrained apart from the Neumann boundary condition.

Problem (D$_\mu$).

$$\begin{cases} \inf & J^*_\mu(\sigma) \\ s.t. & \sigma\nu = g_n \text{ on } \Gamma_1 \\ over & \sigma \in \Sigma(\mathrm{Div};\Omega). \end{cases}$$

with

$$J^*_\mu(\sigma) := \tfrac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) + \langle \hat{p}_{n-1},\sigma\rangle + \tfrac{\mu^{N-1}}{N}\|\operatorname{Div}\sigma + f_n\|^N_{L^N(\Omega)^N} + i^{\mu}_{S_{ad}}(\sigma).$$

Proposition 12.2. *Under the safe-load condition (Assumption 10.6), a Fenchel dual problem to (MVP_μ) is given by the modified stress problem (D$_\mu$). Moreover, (D$_\mu$) has a unique solution σ_μ and there is no duality gap, i.e.,*

$$\min(MVP_\mu) = -\min(D_\mu). \tag{12.2.3}$$

Proof. Existence and uniqueness of a solution σ_μ to (D$_\mu$) follows by the direct method noting that

$$\sigma \mapsto \tfrac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) + \tfrac{\mu^{N-1}}{N}\|\operatorname{Div}\sigma + f_n\|^N_{L^N(\Omega)^N}$$

defines a strictly convex and coercive functional on $\Sigma(\mathrm{Div};\Omega)$. Similar to (11.3.5), we employ the linear operator Λ from (11.3.3) and we rewrite (MVP_μ) in compact form as

$$\min \quad F(u,e) + G(\Lambda[u,e] - \hat{p}_{n-1}) \quad \text{over } [u,e] \in L^{N'}(\Omega)^N \times Q, \tag{12.2.4}$$

with slightly altered definitions of the functionals F and G;

$$\begin{aligned}
F : L^{N'}(\Omega)^N \times Q \to \mathbb{R}\cup\{\infty\}, \quad & F(u,e) := \tfrac{1}{\mu N'}\|u\|^{N'}_{L^{N'}(\Omega)^N} - \langle f_n, u\rangle + \tfrac{1}{2}(\mathbb{C}e,e), \\
G : \Sigma(\mathrm{Div};\Omega)^* \to \mathbb{R}\cup\{\infty\}, \quad & G(\sigma^*) := \sup_{\substack{\sigma\in\Sigma(\mathrm{Div};\Omega),\\ \sigma\nu=g_n \text{ on } \Gamma_1}} \{\langle \sigma^*,\sigma\rangle - i^{\mu}_{S_{\mathrm{ad}}}(\sigma)\}.
\end{aligned}$$

An application of [46, I, Remark 4.1] leads to

$$F^*(u^*, e^*) = \frac{\mu^{N-1}}{N} \|u^* + f_n\|^N_{L^N(\Omega)^N} + \tfrac{1}{2}(\mathbb{C}^{-1}e^*, e^*),$$

for all $[u^*, e^*] \in L^N(\Omega)^N \times Q$. Moreover, it holds that $G(\sigma^*) = \tilde{G}^*(\sigma^*)$ for

$$\tilde{G}(\sigma) := i_{\Sigma_{g_n}(\mathrm{Div};\Omega)}(\sigma) + i^{\mu}_{S_{\mathrm{ad}}}(\sigma), \quad \sigma \in \Sigma(\mathrm{Div};\Omega),$$

where

$$\Sigma_{\tilde{g}}(\mathrm{Div};\Omega) := \{\sigma \in \Sigma(\mathrm{Div};\Omega) : \sigma\nu = \tilde{g} \text{ on } \Gamma_1\}, \quad \tilde{g} \in H^{-1/2}_{00}(\Gamma_1).$$

Since $\tilde{G}$ is convex, l.s.c. and proper, one obtains

$$G^* = \tilde{G} = i_{\Sigma_{g_n}(\mathrm{Div};\Omega)} + i^{\mu}_{S_{\mathrm{ad}}}.$$

The Fenchel dual problem of (MVP$_\mu$) corresponding to this setting is given by

$$-\inf \quad F^*(-\Lambda^*\sigma) + G^*(\sigma) + \langle \hat{p}_{n-1}, \sigma\rangle, \tag{12.2.5}$$

which is exactly problem (D$_\mu$). Since $\operatorname{dom} F^* = L^N(\Omega)^N \times Q$ and $\operatorname{dom} G^* \neq \emptyset$, we infer that (2.2.5) is valid and thus

$$\inf(MVP_\mu) = -\inf(D_\mu).$$

□

Hence, adding the strictly convex term (12.2.1) to (P$_{\mathrm{red}}$) results in a penalty approach to the mechanical equilibrium constraint $-\operatorname{Div}\sigma = f_n$ in the space $L^N(\Omega)^N$. This type of penalization is also useful for a posteriori error estimation in adaptive strategies [109]. Since both problems are uniquely solvable, we retrieve a one-to-one relation between regularized stresses and strains via the primal-dual optimality conditions (2.2.7) for the saddle point $[u_\mu, e_\mu; \sigma_\mu] \in BD(\Omega) \times Q \times \Sigma(\mathrm{Div};\Omega)$. In fact, $[u_\mu, e_\mu; \sigma_\mu]$ is characterized by the existence of $\lambda_\mu \in \Sigma(\mathrm{Div};\Omega)^*$ such that

$$\mathbb{C}e_\mu = \sigma_\mu \text{ in } Q, \quad \sigma_\mu\nu = g_n \text{ on } \Gamma_1 \tag{12.2.6}$$

$$|u_\mu|^{1/(N-1)} \star \operatorname{sign}(u_\mu) = \mu(f_n + \operatorname{Div}\sigma_\mu) \text{ in } \Omega \tag{12.2.7}$$

$$-\hat{p}_{n-1} - \operatorname{Div}^* u_\mu - (1+\mu)\mathbb{C}^{-1}\sigma_\mu + \mu\mathbb{C}^{-1}\pi_{S_{\mathrm{ad}}}(\sigma_\mu) - \lambda_\mu = 0, \tag{12.2.8}$$

$$\lambda_\mu \in N_{\Sigma_{g_n}(\mathrm{Div};\Omega)}(\sigma_\mu), \tag{12.2.9}$$

where $\pi_{S_{\mathrm{ad}}}$ denotes the projection on S_{ad} from Section 12.1. The application of the absolute value and the sign operation in equation (12.2.7) has to be understood componentwise, and

$$|a|^p := [|a_1|^p, \dots, |a_d|^p], \quad a \star b := [a_1b_1, \dots, a_db_d]$$

denotes the Hadamard product for vectors $a, b \in \mathbb{R}^d$.

This shows that the displacement can be easily computed from the solution σ_μ of the dual problem using (12.2.7). In contrast to the primal problem (MVP$_\mu$), which is only given in inf-sup-form, the dual problem is again given explicitly. This facilitates the analysis of the consistency of the regularization with regard to the limit problems (P) and (DP).

Theorem 12.3 (Consistency)**.** *Under the safe-load condition (Assumption 10.6), the following assertions about the solutions to Problem (MVP$_\mu$) and Problem (D$_\mu$) hold true.*

(i) The sequence of approximate elastic strains (e_μ) *fulfills*

$$e_\mu \to \bar{e} \quad \text{in } Q, \quad \text{for } \mu \to \infty.$$

The sequence of approximate displacements (u_μ) *is bounded in* $BD(\Omega)$ *and for any limit* $\bar{u} \in BD(\Omega)$ *of a weakly*-convergent subsequence of* $(u_\mu) \subset BD(\Omega)$*, it holds that* $[\bar{u}, \bar{e}]$ *is a solution to* (P_{red})*.*

(ii) The sequence of approximate stresses (σ_μ) *fulfills*

$$\sigma_\mu \rightharpoonup \bar{\sigma} \quad \text{in } \Sigma(\mathrm{Div};\Omega), \quad \sigma_\mu \to \bar{\sigma} \quad \text{in } Q, \quad \text{for } \mu \to \infty.$$

Proof. Step 1 (primal problem). We first show that any sequence of minimizers $[u_\mu, e_\mu]$ is uniformly bounded. Since

$$\begin{aligned} c &\geq \hat{J}_\mu(u_{n-1}, e_{n-1}) \geq \hat{J}_\mu(u_\mu, e_\mu) \geq \hat{J}(u_\mu, e_\mu) \\ &\geq c_0 - c_1\|e_\mu\| + \kappa_1\|e_\mu\|^2 + \rho \max(\|\varepsilon(u_\mu)\|_{M(\Omega)}, -\|\varepsilon(u_\mu)\|_{M(\Omega)} + \tfrac{1}{\sqrt{2}}\|u_\mu\|_{L^1(\Gamma_0;\mathbb{R}^N)}) \end{aligned}$$

by Lemma 11.3, (e_μ) is bounded in Q and (u_μ) is bounded in $BD(\Omega)$, uniformly in μ. Thus, along an appropriate subsequence, we have

$$u_\mu \overset{*}{\rightharpoonup} \bar{u} \text{ in } BD(\Omega), \quad e_\mu \rightharpoonup \bar{e} \text{ in } Q.$$

Using the sequential weak* $\times$weak lower semicontinuity of $\hat{J}$, (cf. the proof of Theorem 11.4), one obtains

$$\begin{aligned} \hat{J}(\bar{u}, \bar{e}) \leq \liminf_{\mu\to\infty} \hat{J}(u_\mu, e_\mu) &\leq \liminf_{\mu\to\infty} \hat{J}_\mu(u_\mu, e_\mu) \\ &= \liminf_{\mu\to\infty} \min(MVP_\mu) = -\limsup_{\mu\to\infty} \min(D_\mu), \end{aligned} \tag{12.2.10}$$

where the last equality follows from (12.2.3). Under the safe-load condition (Assumption 10.6), the objective function J_μ^* of the dual problem (D_μ) is proper, l.s.c. and convex. Moreover, J_μ^* is weakly l.s.c. and pointwise monotonically increasing, where the pointwise limit is given by

$$\lim_{\mu\to\infty} (J_\mu^*(\sigma) + i_{\Sigma_{g_n}(\mathrm{Div};\Omega)}(\sigma)) = \tfrac{1}{2}(\mathbb{C}^{-1}\sigma, \sigma) + \langle \hat{p}_{n-1}, \sigma \rangle + i_{S_{\mathrm{ad}}(g_n)}(\sigma). \tag{12.2.11}$$

An application of Proposition 2.15(i) yields that (12.2.11) also holds as the Γ-limit in the space $\Sigma(\mathrm{Div};\Omega)$ endowed with the weak topology. Moreover, it is easy to show that the sequence of minimizers (σ_μ) of problem (D_μ) is bounded in $\Sigma(\mathrm{Div};\Omega)$ such that a subsequence of (σ_μ) converges weakly in $\Sigma(\mathrm{Div};\Omega)$ to the solution $\bar{\sigma}$ of (DP) (Theorem 2.14). By uniqueness, this also holds for the entire sequence (σ_μ). According to Theorem 2.14, it further holds that

$$\limsup_{\mu\to\infty} \min(D_\mu) = \lim_{\mu\to\infty} \min(D_\mu) = \min(DP).$$

With the help of (11.3.6), the above estimate (12.2.10) then implies that

$$\hat{J}(\bar{u}, \bar{e}) \leq -\min(DP) = \min(P_{\mathrm{red}}),$$

i.e., $[\bar{u}, \bar{e}]$ solves (P_{red}).

Step 2 (dual problem) First observe that J_μ^* defines a quasi-monotone perturbation of the indicator function $i_{S_{\mathrm{ad}}(g_n)} : \Sigma(\mathrm{Div};\Omega) \to \mathbb{R} \cup \{+\infty\}$ via

$$R_\mu(\sigma) := \tfrac{\mu^{N-1}}{N} \| \operatorname{Div} \sigma + f_n \|^N_{L^N(\Omega)^N} + i^\mu_{S_{\mathrm{ad}}}(\sigma) + i_{\Sigma_{g_n}(\mathrm{Div};\Omega)}(\sigma), \quad \sigma \in \Sigma(\mathrm{Div};\Omega). \tag{12.2.12}$$

In fact, setting

$$K := S_{\mathrm{ad}}(g_n),\ Y := X := \Sigma(\mathrm{Div};\Omega),\ \underline{R_\mu} := R_\mu,\ \overline{R}_\mu := i_{\Sigma_{g_n}(\mathrm{Div};\Omega)},$$

the conditions of Proposition 2.19 are satisfied and (R_μ) Mosco-converges to $i_{S_{\mathrm{ad}}(g_n)}$ in $\Sigma(\mathrm{Div};\Omega)$. In particular, there exists a sequence $(\tilde{\sigma}_\mu)$ with $\tilde{\sigma}_\mu \to \bar{\sigma}$ in $\Sigma(\mathrm{Div};\Omega)$ and $R_\mu(\sigma_\mu) \to i_{S_{\mathrm{ad}}(g_n)}(\bar{\sigma}) = 0$.

Further note that by convexity, the unique solution σ_μ of (D$_\mu$) is characterized by the variational inequality

$$(\mathbb{C}^{-1}\sigma_\mu, \tilde{\sigma} - \sigma_\mu) + R_\mu(\tilde{\sigma}) - R_\mu(\sigma_\mu) \geq \langle -\hat{p}_{n-1}, \tilde{\sigma} - \sigma_\mu \rangle \quad \forall\, \tilde{\sigma} \in \Sigma(\mathrm{Div};\Omega),$$

which is equivalent to

$$R_\mu(\sigma_\mu) + (\mathbb{C}^{-1}\sigma_\mu, \sigma_\mu) \leq (\mathbb{C}^{-1}\sigma_\mu, \tilde{\sigma}) + R_\mu(\tilde{\sigma}) + \langle \hat{p}_{n-1}, \tilde{\sigma} - \sigma_\mu \rangle \tag{12.2.13}$$

for all $\tilde{\sigma} \in \Sigma(\mathrm{Div};\Omega)$. Arguing as in the proof of Theorem 3.1, we obtain using (12.2.13) with $\tilde{\sigma} = \tilde{\sigma}_\mu$,

$$\begin{aligned}
0 &\leq \liminf_{\mu\to\infty} R_\mu(\sigma_\mu) \leq \limsup_{\mu\to\infty} (\tilde{\kappa}_1 \|\sigma_\mu - \bar{\sigma}\|^2_Q + R_\mu(\sigma_\mu)) \\
&\leq -2(\mathbb{C}^{-1}\bar{\sigma}, \sigma_\mu) + (\mathbb{C}^{-1}\bar{\sigma}, \bar{\sigma}) + (\mathbb{C}^{-1}\sigma_\mu, \sigma_\mu) + R_\mu(\sigma_\mu) \\
&\leq -2(\mathbb{C}^{-1}\bar{\sigma}, \sigma_\mu) + (\mathbb{C}^{-1}\bar{\sigma}, \bar{\sigma}) + (\mathbb{C}^{-1}\sigma_\mu, \tilde{\sigma}_\mu) + R_\mu(\tilde{\sigma}_\mu) + \langle \hat{p}_{n-1}, \tilde{\sigma}_\mu - \sigma_\mu \rangle.
\end{aligned}$$

Here, $\tilde{\kappa}_1$ denotes an ellipticity constant of $\mathbb{C}^{-1}$;

$$(\mathbb{C}^{-1}\sigma, \sigma) \geq \tilde{\kappa}_1 \|\sigma\|_Q, \quad \forall\, \sigma \in Q.$$

With the properties $\sigma_\mu \rightharpoonup \bar{\sigma}$ in $\Sigma(\mathrm{Div};\Omega)$ (see step 1), $\tilde{\sigma}_\mu \to \bar{\sigma}$ in $\Sigma(\mathrm{Div};\Omega)$ and $R_\mu(\tilde{\sigma}_\mu) \to 0$, the latter inequality entails

$$\limsup_{\mu\to\infty} (\tilde{\kappa}_1 \|\sigma_\mu - \bar{\sigma}\|^2_Q + R_\mu(\sigma_\mu)) = 0,$$

which implies that $\sigma_\mu \to \bar{\sigma}$ in Q for $\mu \to \infty$. Since $\mathbb{C} e_\mu = \sigma_\mu$, it also holds that $e_\mu \to \bar{e}$ in Q for $\mu \to \infty$. Hence, the proof is accomplished. □

12.3 An Infinite-Dimensional Dual Solver for the Regularized Time-Incremental Stress Problem

This section aims to provide a theoretical framework for an efficient infinite-dimensional algorithmic scheme to solve the initial problem (P) via its reduced formulation (P$_{\mathrm{red}}$). For that reason, we rely on the consistency properties of the sequence of problems (MVP$_\mu$) with regard to the limit problem (P) established in the preceding section. In particular, Theorem 12.3 justifies to assume that (MVP$_\mu$) is a given acceptable approximation of (P) for some $\mu \gg 1$, which is considered to be a fixed parameter in this section. In contrast to standard methods in plasticity, our approach is based on the modified (incremental) stress problems (D$_\mu$), and as such it is a purely stress-based dual method. From the solution of the modified dual problem, it is possible to retrieve corresponding primal approximations $[u_\mu, p_\mu]$ using the primal-dual optimality condition (12.2.7). For simplicity, we focus on the case $N = 2$ which entails that the incremental stress problem as well as its regularization is posed in the well-known Hilbert space $H(\mathrm{Div};\Omega)$.

12.3.1 The shifted problem

Let $N = 2$. On account of the safe-load condition (Assumption 10.6), there exists an element $\hat{\sigma} \in H(\mathrm{Div};\Omega)$ that fulfills

$$-\operatorname{Div}\hat{\sigma} = f_n,\ \hat{\sigma}\nu = g_n \text{ on } \Gamma_1,\ \operatorname{dev}\hat{\sigma} \in L^\infty(\Omega;\mathbb{M}_0^{N\times N}).$$

Employing the element $\hat{\sigma}$, we may shift Problem (D_μ) to obtain the equivalent homogeneous problem

$$\begin{cases} \inf & \hat{J}_\mu^*(\sigma) \\ \text{s.t.} & \sigma\nu = 0 \text{ on } \Gamma_1 \\ \text{over} & \sigma \in H(\mathrm{Div};\Omega). \end{cases} \tag{$\hat{\mathrm{D}}_\mu$}$$

with

$$\hat{J}_\mu^*(\sigma) := \tfrac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) + \langle \hat{l}_{n-1},\sigma\rangle + \tfrac{\mu}{2}\|\operatorname{Div}\sigma\|^2_{L^2(\Omega)^2} + i^\mu_{S_{\mathrm{ad}}}(\sigma+\hat{\sigma})$$

and $\langle \hat{l}_{n-1},\sigma\rangle := \langle \hat{p}_{n-1},\sigma\rangle + (\mathbb{C}^{-1}\hat{\sigma},\sigma)$.

Solving the optimality conditions associated with the discrete formulation of (D_μ) by a semismooth Newton method in the sense of [31, 75] usually results in a mesh-dependent solver. This is a result of the fact that mesh-independent convergence requires the Newton differentiability of the operator defined by the optimality conditions from the continuous problem ($\hat{\mathrm{D}}_\mu$). However, being posed in $H(\mathrm{Div};\Omega)$, which does not embed into a more regular L^p-space for $p > 2$, the problem lacks the necessary norm gap; cf. Lemma 2.10. On the discrete level, a conformal discretization of the space $H(\mathrm{Div};\Omega)$ requires that the symmetry property as well as the regularity of the divergence is incorporated in an appropriate way. This problem already emerges in elasticity. In fact, the dual problem of elasticity may formally be considered as a special case of Problem (DP) by setting $S_{\mathrm{ad}} := Q$. In this approach, the displacement is considered as a Lagrange multiplier to the divergence constraint. *Mixed finite element methods* are characterized by the reformulation of the (two-dimensional) elasticity problem as a saddle point system involving both, displacement and stress, as unknown variables. To achieve a stable approximation in the sense of the LBB condition [22], it is necessary to simultaneously deal with symmetry and divergence constraints in the definition of the space $H(\mathrm{Div};\Omega)$. Therefore, relatively complex elements such as those of Arnold and Winther [8], where the symmetry constraint is imposed in the strong sense, are required. We also refer to the overview on mixed finite element methods for elasticity [28].

12.3.2 Tikhonov regularization

In order to overcome these drawbacks associated to an immediate discretization of ($\hat{\mathrm{D}}_\mu$), we suggest to replace problem (D_μ) by a Tikhonov-type regularized problem in the dense Hilbert subspace

$$H^1(\Omega;\mathbb{M}^{2\times 2}) \hookrightarrow H(\mathrm{Div};\Omega).$$

Note that by approximating the stress in the Sobolev space $H^1(\Omega;\mathbb{M}^{2\times 2})$, the divergence condition in $H(\mathrm{Div};\Omega)$ is automatically fulfilled and the symmetry condition can be easily imposed using a parametrization. Together with the space $H^1(\Omega;\mathbb{M}^{2\times 2})$, we consider a continuous and elliptic symmetric bilinear form

$$b(\,.\,,\,.\,) : H^1(\Omega;\mathbb{M}^{2\times 2}) \times H^1(\Omega;\mathbb{M}^{2\times 2}) \to \mathbb{R}$$

with associated bounded linear operator $B \in \mathcal{L}(H^1(\Omega;\mathbb{M}^{2\times 2}), H^1(\Omega;\mathbb{M}^{2\times 2})^*)$. Induced by a sequence of positive parameters (γ), we now contemplate the approximation of the modified stress problem $(\hat{\mathrm{D}}_\mu)$ by the regularized problems

$$\begin{cases} \inf & \hat{J}^*_{\mu,\gamma}(\sigma) \\ \text{s.t.} & \sigma\nu = 0 \text{ on } \Gamma_1, \\ \text{over} & \sigma \in H^1(\Omega, \mathbb{M}^{2\times 2}), \end{cases} \tag{$\hat{\mathrm{D}}_{\mu,\gamma}$}$$

where

$$\hat{J}^*_{\mu,\gamma}(\sigma) := \tfrac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) + \langle \hat{l}_{n-1},\sigma\rangle + \tfrac{\mu}{2}\|\operatorname{Div}\sigma\|^2_{L^2(\Omega)^2} + i^{\mu}_{S_{\mathrm{ad}}}(\sigma+\hat{\sigma}) + \tfrac{1}{2\gamma} b(\sigma,\sigma).$$

The assumptions on b ensure that the problems $(\hat{\mathrm{D}}_{\mu,\gamma})$ have a unique solution which is henceforth denoted by $\sigma_{\mu,\gamma}$. The problem $(\hat{\mathrm{D}}_{\mu,\gamma})$ further promises a good approximation of (D_μ) at least for large γ. In fact, in order to relate the problems $(\hat{\mathrm{D}}_{\mu,\gamma})$ to (D_μ) we need the following assumption.

Assumption 12.4. *The splitting of $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \partial\Gamma_0$ is regular enough to ensure that the density result*

$$\overline{C^\infty_{0,\Gamma_1}(\overline{\Omega})}^{H^1(\Omega)} = H^1_{0,\Gamma_1}(\Omega) \tag{12.3.1}$$

for $H^1_{0,\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$ and

$$C^\infty_{0,\Gamma_1}(\overline{\Omega}) := \{\varphi \in C^\infty(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_1\} \tag{12.3.2}$$

holds true.

Assumption 12.4 does not represent a restriction from a practical point of view since only very irregular boundaries are critical with respect to (12.3.1); see [43, 17]. As the following lemma shows, the density property (12.3.1) suffices to extend the density result

$$\overline{C^\infty_c(\Omega;\mathbb{M}^{N\times N})}^{H(\operatorname{Div};\Omega)} = H_0(\operatorname{Div};\Omega),$$

see [52, I, Theorem 2.6], to problems with mixed boundary conditions. For this purpose we define the appropriate subspace

$$H_{0,\Gamma_1}(\operatorname{Div};\Omega) := \{\sigma \in H(\operatorname{Div};\Omega) : \sigma\nu = 0 \text{ on } \Gamma_1\}$$

of $H(\operatorname{Div};\Omega)$-functions whose normal component vanishes on Γ_1 in the sense of the space $H^{-1/2}_{00}(\Gamma_1)$; cf. (1.2.16).

Lemma 12.5. *Let $N \in \mathbb{N}$ and suppose Assumption 12.4 holds true. Then the density property*

$$\overline{C^\infty_{0,\Gamma_1}(\overline{\Omega};\mathbb{M}^{N\times N})}^{H(\operatorname{Div};\Omega)} = H_{0,\Gamma_1}(\operatorname{Div};\Omega)$$

is satisfied, where

$$C^\infty_{0,\Gamma_1}(\overline{\Omega};\mathbb{M}^{N\times N}) := \{\varphi \in C^\infty(\overline{\Omega};\mathbb{M}^{N\times N}) : \varphi|_{\Gamma_1} = 0\},$$

Proof. The continuity of the normal trace operator restricted to Γ_1 [14],

$$\tau_\nu^{\Gamma_1} : H(\mathrm{Div};\Omega) \to [H_{00}^{-1/2}(\Gamma_1)]^N, \quad \tau_\nu^{\Gamma_1}(\sigma) := \tau_\nu(\sigma)\big|_{\Gamma_1},$$

shows that $H_{0,\Gamma_1}(\mathrm{Div};\Omega) = \ker \tau_\nu^{\Gamma_1}$ is a closed subspace of $H(\mathrm{Div};\Omega)$. Hence, the inclusion

$$\overline{C_{0,\Gamma_1}^\infty(\overline{\Omega};\mathbb{M}^{N\times N})}^{H(\mathrm{Div};\Omega)} \subset \ker \tau_\nu^{\Gamma_1}$$

is valid. To show equality, we adapt the strategy of the proof of [52, I, Theorem 2.6], which deals with the case $\Gamma_1 = \partial\Omega$, by showing that any linear form $\sigma^* \in (\ker \tau_\nu^{\Gamma_1})^*$ that vanishes on $C_{0,\Gamma_1}^\infty(\overline{\Omega};\mathbb{M}^{N\times N})$ is identical to zero. By the Riesz Representation Theorem, σ^* can be identified with an element $\sigma_0 \in \ker \tau_\nu^{\Gamma_1}$ such that

$$\langle \sigma^*, \sigma \rangle = (\sigma_0, \sigma)_Q + (q_0, \mathrm{Div}\,\sigma)_{L^2(\Omega)^N}, \quad \forall\, \sigma \in \ker \tau_\nu^{\Gamma_1}, \tag{12.3.3}$$

where $q_0 := \mathrm{Div}\,\sigma_0$. Since, by assumption, σ^* vanishes on $C_c^\infty(\Omega;\mathbb{M}^{N\times N})$, one deduces that $\varepsilon(q_0) = \sigma_0$ and thus

$$q_0 \in H^1(\Omega)^N. \tag{12.3.4}$$

We further prove that $q_0 = 0$ on Γ_0. Together with the hypothesis on σ^*, Green's formula for $H(\mathrm{Div};\Omega)$-function (1.2.3) and (12.3.3) imply that

$$\begin{aligned}\langle \sigma^*, \sigma \rangle &= (\varepsilon(q_0), \sigma)_Q + (q_0, \mathrm{Div}\,\sigma)_{L^2(\Omega)^N} \\ &= \langle \sigma\nu, q_0 \rangle_{(H^{-1/2}(\partial\Omega)^N, H^{1/2}(\partial\Omega)^N)} \\ &= \int_{\Gamma_0} (\sigma\nu) q_0 \, d\mathcal{H}^{N-1} = 0, \end{aligned} \tag{12.3.5}$$

for all $\sigma \in C_{0,\Gamma_1}^\infty(\overline{\Omega};\mathbb{M}^{N\times N})$. By the density property (12.3.1) and the continuity of

$$\tau_\nu^{\Gamma_0} : H_{0,\Gamma_1}^1(\Omega, \mathbb{M}^{N\times N}) \to [H_{00}^{1/2}(\Gamma_0)]^N, \quad \tau_\nu^{\Gamma_0}(\sigma) := \tau_\nu(\sigma)|_{\Gamma_0},$$

(12.3.5) also holds for all $\sigma \in H_{0,\Gamma_1}^1(\Omega;\mathbb{M}^{N\times N})$. As the operator $\tau_\nu^{\Gamma_0}$ is surjective (cf. Corollary 1.7), one obtains that

$$\int_{\Gamma_0} z q_0 \, d\mathcal{H}^{N-1} = 0 \quad \forall\, z \in H_{00}^{1/2}(\Gamma_0)^N,$$

and by the density of $H_{00}^{1/2}(\Gamma_0)$ in $L^2(\Gamma_0)$, we have that $q_0 = 0$ on Γ_0. It follows from (12.3.4) that $q_0 \in H_{0,\Gamma_0}^1(\Omega)^N$ and, by definition, also $q_0\big|_{\Gamma_1} \in H_{00}^{1/2}(\Gamma_1)^N$. Let $\sigma \in \ker \tau_\nu^{\Gamma_1}$. Using $q_0 \in H_{0,\Gamma_0}^1(\Omega)^N$, we infer that

$$\langle \sigma^*, \sigma \rangle = (\varepsilon(q_0), \sigma) + (q_0, \mathrm{Div}\,\sigma)_{L^2(\Omega)^N} = \langle \sigma\nu, q_0 \rangle_{(H_{00}^{-1/2}(\Gamma_1)^N, H_{00}^{1/2}(\Gamma_1)^N)} = 0,$$

which shows that σ^* is the zero functional on $\ker \tau_\nu^{\Gamma_1}$. $\square$

With the help of the density property provided by Lemma 12.5, the main consistency result for $\gamma \to \infty$ can be derived on the basis of the general results from Section 4.1.

Theorem 12.6. *Let $\mu > 0$ be fixed and assume that Assumption 12.4 is fulfilled. For a sequence of positive* *parameters* $(\gamma) \subset \mathbb{R}^+$ *with* $\gamma \to \infty$*, the solutions* $\sigma_{\mu,\gamma} \in H^1(\Omega; \mathbb{M}^{2\times 2})$ *to* $(\hat{\mathrm{D}}_{\mu,\gamma})$ *fulfill*

$$\hat{\sigma} + \sigma_{\mu,\gamma} \to \sigma_\mu \ \text{ in } H(\mathrm{Div}; \Omega), \text{ as } \gamma \to \infty,$$

where σ_μ *is the solution to* (D_μ)*.*

Proof. By convexity, the solution $\hat{\sigma}_\mu := \sigma_\mu - \hat{\sigma}$ of the shifted problem $(\hat{\mathrm{D}}_\mu)$ is characterized by the variational inequality

$$a(\hat{\sigma}_\mu, \tilde{\sigma} - \hat{\sigma}_\mu) + j_\mu(\tilde{\sigma}) - j_\mu(\hat{\sigma}_\mu) \geq \langle -\hat{l}_{n-1}, \tilde{\sigma} - \hat{\sigma}_\mu \rangle, \quad \forall\, \tilde{\sigma} \in H(\mathrm{Div}; \Omega),$$

where

$$a(\sigma, \tilde{\sigma}) := (\mathbb{C}^{-1}\sigma, \tilde{\sigma}) + \mu(\mathrm{Div}\,\sigma, \mathrm{Div}\,\tilde{\sigma})_{L^2(\Omega)^2},$$
$$j_\mu(\tilde{\sigma}) := i^\mu_{S_{\mathrm{ad}}}(\tilde{\sigma} + \hat{\sigma}) + i_{H_{0,\Gamma_1}(\mathrm{Div};\Omega)}(\tilde{\sigma}).$$

On the other hand, the solution $\sigma_{\mu,\gamma}$ of $(\hat{\mathrm{D}}_{\mu,\gamma})$ is characterized by the variational inequality

$$a(\sigma_{\mu,\gamma}, \tilde{\sigma} - \sigma_{\mu,\gamma}) + j_{\mu,\gamma}(\tilde{\sigma}) - j_{\mu,\gamma}(\sigma_{\mu,\gamma}) \geq \langle -\hat{l}_{n-1}, \tilde{\sigma} - \sigma_{\gamma,\mu} \rangle, \quad \forall\, \tilde{\sigma} \in H(\mathrm{Div}; \Omega),$$

where

$$j_{\mu,\gamma}(\tilde{\sigma}) := i^\mu_{S_{\mathrm{ad}}}(\tilde{\sigma} + \hat{\sigma}) + i_{H_{0,\Gamma_1}(\mathrm{Div};\Omega)}(\tilde{\sigma}) + \tfrac{1}{\gamma}\|\tilde{\sigma}\|_{H^1(\Omega;\mathbb{M}^{2\times 2})}.$$

Here, it is understood that $j_{\mu,\gamma}(\tilde{\sigma}) = +\infty$ for $\tilde{\sigma} \notin H^1(\Omega; \mathbb{M}^{2\times 2})$. It is further easy to see that the functional

$$R_\gamma(\tilde{\sigma}) := i_{H_{0,\Gamma_1}(\mathrm{Div};\Omega)}(\tilde{\sigma}) + \tfrac{1}{\gamma}\|\tilde{\sigma}\|_{H^1(\Omega;\mathbb{M}^{2\times 2})}, \quad R_\gamma(\tilde{\sigma}) = +\infty \text{ for } \tilde{\sigma} \notin H^1(\Omega; \mathbb{M}^{2\times 2}),$$

defines a quasi-monotone perturbation of the indicator function $i_{H_{0,\Gamma_1}(\mathrm{Div};\Omega)}$ in $H(\mathrm{Div}; \Omega)$ with respect to the dense subspace $H^1(\Omega; \mathbb{M}^{2\times 2})$. Indeed, the premises of Definition 2.18 are satisfied with $\underline{R}_\gamma := i_{H_{0,\Gamma_1}(\mathrm{Div};\Omega)}$ and $\overline{R}_\gamma := R_\gamma$.

Further note that Proposition 4.3 applies to $(j_{\mu,\gamma})$ despite the fact that $i^\mu_{S_{\mathrm{ad}}}(\,.\,+\hat{\sigma})$ is not coercive. However, $i^\mu_{S_{\mathrm{ad}}}(\,.\,+\hat{\sigma})$ is convex and continuous on Q for any $\gamma > 0$. As a result, $i^\mu_{S_{\mathrm{ad}}}(\,.\,+\hat{\sigma})$ is weakly l.s.c. in $H(\mathrm{Div}; \Omega)$, and thus, one may retrace the proof of Proposition 4.3 to find that $(j_{\mu,\gamma})$ Mosco-converges in $H(\mathrm{Div}; \Omega)$ to

$$i^\mu_{S_{\mathrm{ad}}}(\,.\,+\hat{\sigma}) + i_{\overline{H_{0,\Gamma_1}(\mathrm{Div};\Omega) \cap H^1(\Omega;\mathbb{M}^{2\times 2})}^{H(\mathrm{Div};\Omega)}}.$$

From Assumption 12.4 and Lemma 12.5, it follows that

$$\overline{H_{0,\Gamma_1}(\mathrm{Div}; \Omega) \cap H^1(\Omega; \mathbb{M}^{2\times 2})}^{H(\mathrm{Div};\Omega)} = H_{0,\Gamma_1}(\mathrm{Div}; \Omega),$$

which entails that $(j_{\mu,\gamma})$ Mosco-converges to j_μ in $H(\mathrm{Div}; \Omega)$ for $\gamma \to \infty$. An application of Theorem 3.1 concludes the proof. $\square$

12.3.3 An infinite-dimensional semismooth Newton method

The von Mises yield criterion

In this section, we assume that the set of admissible stresses is determined by the von Mises yield criterion, i.e,

$$\mathbb{K} := \{\sigma \in \mathbb{M}^{2\times 2} : |\operatorname{dev}\sigma|_F \leq \sigma_y\}, \quad \sigma_y > 0 \text{ fixed},$$

which is one of the most frequently used yield criteria in practice. It is obvious that this criterion is pressure-insensitive, such that the theory of Section 10.3 applies. In this case the projection onto the feasible set S_{ad} in Q, where Q is equipped with the inner product $(\,.\,,\,.\,)_{\mathbb{C}^{-1}}$, is known to be given by

$$\pi_{S_{\mathrm{ad}}}(\sigma) = \sigma - [|\operatorname{dev}\sigma|_{\mathrm{F}} - \sigma_y]^+ \tfrac{\operatorname{dev}\sigma}{|\operatorname{dev}\sigma|_F}, \tag{12.3.6}$$

provided the elastic behavior is isotropic, cf. [112]. If the elastic behavior is not isotropic, one may replace $\pi_{S_{\mathrm{ad}}}$ by the projection with respect to the standard inner product on Q to retrieve (12.3.6).

Under these premises, the problem ($\hat{\mathrm{D}}_{\mu,\gamma}$) takes the form

$$\begin{cases} \inf & \tfrac{1}{2}(\mathbb{C}^{-1}\sigma,\sigma) + \langle \hat{l}_{n-1},\sigma\rangle + \tfrac{\mu}{2}\|\operatorname{Div}\sigma\|^2_{L^2(\Omega)^2} \\ & \quad + \tfrac{\mu}{2}\|[|\operatorname{dev}(\sigma+\hat{\sigma})|_F - \sigma_y]^+\|^2_{L^2(\Omega)} + \tfrac{1}{2\gamma}b(\sigma,\sigma). \\ \text{over} & \sigma \in H^1_{0,\nu}(\Omega;\mathbb{M}^{2\times 2}), \end{cases} \tag{12.3.7}$$

where

$$H^1_{0,\nu}(\Omega;\mathbb{M}^{2\times 2}) := \{\sigma \in H(\Omega;\mathbb{M}^{2\times 2}) : \sigma\nu = 0 \text{ on } \Gamma_1\}.$$

We proceed by the derivation of a reformulation of problem ($\hat{\mathrm{D}}_\mu$) by means of a Newton differentiable operator equation.

Semismooth reformulation

By abuse of notation, we denote the solution $\sigma_{\mu,\gamma}$ to ($\hat{\mathrm{D}}_{\mu,\gamma}$) by σ_γ. The convexity of problem ($\hat{\mathrm{D}}_{\mu,\gamma}$) implies that the necessary and sufficient optimality condition for a solution σ_γ to ($\hat{\mathrm{D}}_{\mu,\gamma}$), is characterized by the nonsmooth operator equation

$$\Psi_\gamma(\sigma_\gamma) = 0, \tag{12.3.8}$$

where the operator $\Psi_\gamma : H^1_{0,\nu}(\Omega;\mathbb{M}^{2\times 2}) \to H^1_{0,\nu}(\Omega;\mathbb{M}^{2\times 2})^*$ is defined by

$$\Psi_\gamma(\sigma) := \mathbb{C}^{-1}\sigma + \hat{l}_{n-1} + \mu\operatorname{Div}^*\operatorname{Div}\sigma + \mu\operatorname{dev}^*\mathfrak{m}(\operatorname{dev}(\hat{\sigma}+\sigma)) + \tfrac{1}{\gamma}B\sigma. \tag{12.3.9}$$

Here,

$$\mathfrak{m}(\sigma) := [(|\sigma|_F - \sigma_y)]^+\mathfrak{q}(\sigma), \text{where } \mathfrak{q}(\sigma) = \begin{cases} \sigma/|\sigma|_F, & \text{if } \sigma \neq 0, \\ 0, & \text{else;} \end{cases}$$

denotes the nonlinear operator associated with the Fréchet derivative of the Moreau-Yosida regularization. We proceed by showing that this equation can be solved efficiently by a generalized Newton scheme which relies on the notion of Newton differentiability (Section 2.3).

Using Lemma 2.10, the Sobolev imbedding theorem and the fact that the composition with affine continuous operators preserves the Newton differentiability, one may infer that the mapping Ψ defined in (12.3.9) is Newton differentiable. Using the chain rule, one infers that a Newton

Algorithm SSN(μ, γ): Globalized SSN algorithm

input: $\sigma^{(0)} \in H^1_{0,\nu}(\Omega; \mathbb{M}^{2\times 2})$

set $j := 0$;
while some stopping rule is not satisfied **do**
  compute the solution $\delta^{(j)} \in$ of $G_{\Psi_\gamma}(\sigma^{(j)})\delta^{(j)} = -\Psi_\gamma(\sigma^{(j)})$;
  determine $\alpha^{(j)} > 0$ by an Armijo line search based on $\alpha \mapsto \hat{J}^*_{\mu,\gamma}(\sigma^{(j)} + \alpha\delta^{(j)})$;
  set $\sigma^{(j+1)} := \sigma^{(j)} + \alpha^{(j)}\delta^{(j)}$ and $j := j+1$;

derivative

$$G_{\Psi_\gamma}(\sigma) \in \mathcal{L}(H^1_{0,\nu}(\Omega; \mathbb{M}^{2\times 2}), H^1_{0,\nu}(\Omega; \mathbb{M}^{2\times 2})^*)$$

of Ψ_γ at σ is given by

$$\langle G_{\Psi_\gamma}(\sigma)\tilde{\sigma}, \, . \, \rangle := (\mathbb{C}^{-1}\tilde{\sigma}, \, . \,) + \mu \operatorname{Div}^* \operatorname{Div} \tilde{\sigma} + \mu \operatorname{dev}^* G_{\mathfrak{m}}(\operatorname{dev}(\hat{\sigma} + \sigma))[\operatorname{dev} \tilde{\sigma}] + \tfrac{1}{\gamma} B\tilde{\sigma},$$

for all $\tilde{\sigma} \in H^1_{0,\nu}(\Omega; \mathbb{M}^{2\times 2})$. Here, $G_{\mathfrak{m}}$ denotes the Newton derivative of $\mathfrak{m}$ according to Lemma 2.10. Analogously to Lemma 9.5, one may show that G_{Ψ_γ} is uniformly invertible, i.e., independent of σ. As a result of Theorem 2.8, it can be inferred that the corresponding Newton iteration

$$\sigma^{(j+1)} = \sigma^{(j)} - G_{\Psi_\gamma}(\sigma^{(j)})\Psi_\gamma(\sigma^{(j)})$$

is well-defined provided the starting point $\sigma^{(0)}$ is sufficiently close to σ_γ. Moreover, the iterates $(\sigma^{(j)})$ converge locally at a superlinear rate, which is mesh-independent upon discretization. To enforce global convergence, one may equip the search directions

$$\delta^{(j)} := -G_{\Psi_\gamma}(\sigma^{(j)})\Psi_\gamma(\sigma^{(j)})$$

with a step size determined by the Armijo line search procedure. The resulting method is summarized in Algorithm (SSN(μ, γ)). In a manner entirely analogous to Section 9.2, it can be shown that the semismooth Newton solver (SSN(μ, γ)) is a globally convergent method in the sense of Corollary 9.8.

12.3.4 Outlook on a discrete solver

While the semismooth reformulation of Problem (D$_\mu$) based on a Tikhonov regularization resembles the approach for hardening plasticity from Chapter 9, cf. Problem (D$_\gamma$), the construction of a stable discrete counterpart necessitates a more involved inspection. In fact, consider a discretization of the dual problem ($\hat{\mathrm{D}}_{\mu,\gamma}$) in terms of conformal P_1-finite elements in the spirit of Problem (D$_{\gamma,h}$) from Part III . For the inspection of the limiting case as $h \to 0$ and $\mu, \gamma \to +\infty$, one may resign to the stability analysis of Part II. However, the presence of the additional equality constraints

$$-\operatorname{Div} \sigma = f_n, \quad \sigma\nu = g_n \text{ on } \Gamma_1,$$

defining the feasible set of the limit problem (DP) complicates the verification of the Mosco-convergence of the perturbed indicator function, which is necessary to infer the convergence of the discretized-regularized problems with the help of Theorem 3.1. For example, the extension of the required density result (cf., e.g., Example 4.7 in Part II) to problems with equality constraints does not seem to be possible. Moreover, a numerical realization of the boundary condition $\sigma\nu = 0$ is only obvious if ν is (at least locally) constant, and even in this case, one obtains another side restriction for the discrete approximations.

Another aspect concerns the convergence of the discrete primal solutions. Even if the convergence of the discrete regularized stresses arising from a finite element discretization of ($\hat{\mathrm{D}}_{\mu,\gamma}$) can be shown, it is still necessary to pass to the limit in the resulting discretized version of the primal-dual optimality conditions (12.2.6)-(12.2.9) in order to make statements about the convergence of the discretized-regularized displacements and plastic strains. In the case of elasticity, which formally corresponds to Problem (DP) with the pointwise constraint $\sigma \in S_{\mathrm{ad}}$ being absent, the convergence of the discrete stress-displacement pair in mixed finite element methods hinges on the validity of the LBB condition for saddle point problems [22]. In this case, the LBB condition necessitates the usage of rather sophisticated finite elements for a conformal discretization of the stress space $H(\mathrm{Div};\Omega)$, for example those of Arnold and Winther [8]. The resulting finite-dimensional approximation involves a large number of (local) degrees of freedom. As a consequence, this type of discretization entails a considerable amount of computational complexity, especially when applied in the context of a path-following semismooth Newton approach to solve the problem Problem (P) via the regularized problems (D_μ) or ($\hat{\mathrm{D}}_{\mu,\gamma}$). In the latter case, the discretization with the elements of Arnold and Winther represents a non-conforming discretization.

Conclusion, Outlook and Some Related Open Problems

This thesis is devoted to variational inequality and constrained optimization problems over a convex subset K of a Banach space X with applications to elasto-plasticity. The focus of Part II is on the significance of density properties of K with respect to the consistency of various perturbation methods. As discussed in Chapter 4, many approaches, including finite-dimensional approximations as well as Tikhonov regularizations, lead to a limit problem which is defined over the closure of the intersection of the convex constraint set K with respect to certain dense subspaces of X. In order to be consistent with the original problem, it is therefore of fundamental importance to study whether this closure corresponds to the initial constraint set K. If this is the case, one may prove the unconditional consistency of various penalization/regularization schemes in the sense that the solutions of the perturbed problems converge to the solution of the original problem without any special coupling of regularization or discretization parameters. These arguments are rigorously set forth in Chapter 4 and their derivation relies on the theory of Γ-convergence. In this regard, the introduction of the class of quasi-monotone perturbations (Definition 2.18) provides an abstract framework to unify the analysis of a large amount of approximation methods.

Chapter 4 represents the basic motivation to prove density properties for specific convex subsets in Lebesgue and Sobolev spaces with respect to the subspace of continuous or smooth functions defined over a Lipschitz domain. In Chapter 5 we consider pointwise constraint sets which are either defined by a sign condition on the function value or an upper bound on the norm of the function. Whereas the former case gives rise to a cone constraint which yields a couple of positive results (Section 5.1), the latter case demands further attention as a simple mollification is not sufficient to produce a feasible approximating sequence of smooth functions. However, if the upper bound is uniformly continuous and strictly positive, further scaling techniques allow to derive density properties in $L^p, W^{1,p}, W_0^{1,p}$ and $H_0(\mathrm{div})$; see Section 5.2. Using the invariance of convolution and differentiation, some results extend to pointwise constraints on the partial derivatives [69]. This does not concern the gradient-constrained case in $W^{1,p}$. Due to the lack of a suitable extension operator, also the case $X = H(\mathrm{div};\Omega)$ remains open. The ensuing question of whether these results can be extended to discontinuous obstacles is analyzed in Section 5.3. Notably, we prove that the density result is not valid in general in case the obstacle is just a Sobolev function (Theorem 5.14). However, one may enlarge the admissible set of obstacles to functions which fulfill a generalized lower semicontinuity condition as long as the respective set is an appropriate limit of a sequence of sets for which the density property holds; cf. Theorem 5.17 and Theorem 5.22. For supersolutions of elliptic PDEs, a different strategy is the smoothing approach via elliptic PDEs with vanishing coefficient. In this case, one may even drop the condition that the upper bound is bounded away from zero. On the other hand, the regularity of the dense subspace is limited by elliptic regularity theory; see Theorem 5.24.

Even if the perturbation lacks a generalized monotonicity property in the sense of Definition 2.18, density results are still useful to prove convergence of finite element schemes under minimal regularity (Chapter 6). More precisely, density properties allow to deduce the Mosco-convergence of the discretized constraint sets. In this respect, the construction of a suitable recovery sequence is achieved by resigning to a dense subset of K on which the respective interpolation operator is well-defined. Several results for piecewise affine and Raviart-Thomas elements are achieved in various function spaces. Before embarking on the application of density results in elasto-plasticity, we also consider the case of total variation based image restoration (Section 6.2), where a density result in $H_0(\mathrm{div};\Omega)$ is required to prove a Fenchel duality result. We also propose an alternative Raviart-Thomas finite element scheme to solve the dual problem. This approach necessitates the design of an efficient solver for the discrete problems. Moreover, the effect of the discretization for the image restoration problem remains to be investigated.

In Part III we establish an infinite-dimensional solver for the incremental contact problem of quasi-static elasto-plasticity under the small strain assumption and combined linear kinematic-isotropic hardening. Since the original problem (EPC) as well as the displacement-only reformulation from [58] does not allow for a Newton differentiable reformulation in the sense of Definition 2.7, one cannot ensure mesh-independent convergence of the associated solvers; see Chapter 8. As an alternative, we consider the special Fenchel dual problem (D) to the primal problem of quasi-static plasticity and prove that there is no duality gap (Proposition 8.2). Employing the yield criterion of von Mises, the dual problem (D) is a smooth and uniformly convex minimization problem subject to pointwise constraints of the type discussed in Part II. Again, the optimality conditions are not Newton differentiable in infinite dimensions. As a remedy, we replace the dual problem by the combined Moreau-Yosida/Tikhonov regularized problem (D_γ) which is set in a dense subspace. Resigning to the density conditions from the abstract perturbation analysis from Chapter 4, we prove the consistency of the approximation in that regularized displacement, stresses and strains converge strongly to the solution of the original elasto-plastic contact problem; cf. Theorem 9.2. In this respect, it would also be of interest whether a similar approach can be established for other yield criteria. Another condition on the regularization subspace arises from the norm gap requirement for the Newton differentiability of the mapping associated with the optimality conditions of the regularized problem. Under this condition, the semismooth Newton method is shown to converge globally in the original infinite-dimensional setting giving rise to a locally superlinearly convergent solver, which converges mesh-independently upon discretization. This is studied in detail in Section 9.2. The approach suggests a path-following strategy with respect to the penalization-regularization parameters which is set forth in Section 9.3 together with the choice of a suitable dense subspace for the Tikhonov regularization. In fact, the weaker density condition (9.3.1) is sufficient to ensure the consistency of the regularization. But it remains an open issue whether the conditions on the obstacle can be considerably alleviated in comparison to the conditions stated in Chapter 5. Furthermore, three two-dimensional numerical tests are given to corroborate the theoretical results. Indeed, for each path problem, the semismooth Newton method is observed to converge locally superlinearly and the convergence is mesh-independent. The heuristic path-following strategy $\mathrm{IPF}(h)$ is proposed to solve the limit problem and we observe (almost) mesh-independent convergence. A suitable path-following strategy based on a reliable model of the path-value functional

$$\gamma \mapsto J^*_\gamma([z_\gamma, q_\gamma])$$

leading to an automated regularization-discretization update procedure promises a higher efficiency. For some classes of variational inequalities of the first kind, these methods are already well established and prove to be remarkably efficient; see e.g. [66]. Furthermore, the usage of adaptive strategies is strongly recommended in view of the singularities of the solutions corresponding to Examples (a),(b) and (c) from Section 9.3; we refer to [26] for adaptive methods in elasto-plasticity. It should be emphasized that the approach presented in this paper can be extended to contact problems with Tresca friction. These problems are characterized by an additional weighted L^1-norm functional on the contact zone resulting in an additional inequality in the dual problem. Tresca friction problems serve as a high-level substep of the standard fixed point approach to the quasi-variational inequality problem of Coulomb friction [98].

Part IV is dedicated to the time-incremental problem of quasi-static evolution in perfect plasticity stated in [38]. The yield criterion is assumed to be pressure insensitive but the yield surface is not necessarily smooth. It is a convex nonsmooth constrained minimization problem with respect to the displacement, the elastic strain and the plastic strain. Moreover, the problem is posed in a non-reflexive Banach space since the plastic strain is just a Borel measure and the displacement is a function of bounded deformation. The constraint is given by the additive split of the linearized total strain and the relaxed form of the Dirichlet boundary condition. With the help of this constraint, one may either eliminate the elastic strain as described in [15], or, and this is our approach, one eliminates the plastic strain. The resulting minimization problem must incorporate the plastic incompressibility condition (Lemma 11.1). However, it can be shown that any solution of the minimization problem already fulfills this restriction (Lemma 11.2). Furthermore, Lemma 11.3 states that the regularity of the linearized strain is also implicitly contained in the minimization process, such that the displacement may equivalently be sought in a suitable Lebesgue space. The resulting new inf-sup formulation gives rise to an alternative existence proof; see Theorem 11.4. More importantly, one may retrieve the standard incremental stress problem as a Fenchel dual problem, which entails new optimality conditions for the time-incremental problem of quasi-static perfect plasticity: this is examined in Theorem 11.5 and the subsequent discussion. We also point out that, unlike the case of linear hardening plasticity [58], an explicit (primal) problem formulation only in terms of the displacement is still not available.

In Chapter 12 we discuss alternatives to vanishing hardening approaches to perfect plasticity; see e.g. [15]. A modified version of the visco-plastic regularization using an additional Tikhonov regularization for the displacements is proposed. The resulting regularized problems (MVP$_\mu$) are uniquely solvable and do not enforce higher regularity of displacements and strains. In terms of the stress problem, the regularization induces an additional penalty term with respect to the equilibrium condition (see Proposition 12.2), and Theorem 12.3 guarantees the consistency of the approximation with respect to the original time-incremental problem of perfect plasticity.

In the two-dimensional case and under the von Mises yield criterion, we propose the additional Tikhonov regularization ($\hat{\mathrm{D}}_{\mu,\gamma}$) in the subspace H^1 to solve each subproblem (D$_\mu$) by an infinite-dimensional semismooth Newton method (Algorithm SSN(μ, γ)). Upon discretization, this would lead to a mesh-independent solver for each subproblem. In order to efficiently approximate the limit case ($\mu = \gamma = +\infty$), one may embed SSN(μ, γ) into a path-following procedure with respect to the parameters γ and μ along the lines of Section 9.3 in Part III. The construction of a suitable discretization for the (regularized) dual problem in conjunction with an efficient path-following approach, which is stable in the limit, can be considered as a natural follow-up project to this thesis. In this regard, it should be remarked that even the purely elastic case requires a considerable amount of complexity; we refer to the *mixed finite element methods* from [8] and the discussion at the end of Chapter 12 for this matter.

Moreover, the extension of ($\hat{\mathrm{D}}_{\mu,\gamma}$) to three-dimensional problems is not straightforward since $H^1(\Omega; \mathbb{M}^{3\times 3})$ does not embed into $\Sigma(\mathrm{Div}; \Omega)$. Naturally, imposing the H^1-regularity of the stresses in ($\hat{\mathrm{D}}_{\mu,\gamma}$) is not necessary to ensure the norm gap requirement for the Newton differentiability of the term

$$\sigma \mapsto \mathrm{dev}^* \mathfrak{m}(\mathrm{dev}(\hat{\sigma} + \sigma))$$

in (12.3.9). Instead, it is sufficient to impose a higher regularity just on the deviatoric part, for instance by means of a Tikhonov regularization

$$\sigma \mapsto \tfrac{\gamma}{r} \| \mathrm{dev}\, \sigma \|^r_{L^r(\Omega; \mathbb{M}_0^{N\times N})}, \quad \gamma \gg 0,$$

where $r > 2$ is sufficiently large to guarantee the premises of Lemma 2.10. The resulting regularization can be expected to be consistent with the limit problem (DP) as $\gamma, \mu \to +\infty$. However, a proof of (uniform) invertibility of the resulting Newton derivative seems to be beyond reach. A remedy consists of employing a so-called *lifting step*. For related issues in the case of pointwise gradient constraints in Sobolev spaces we refer to the discussion in [68, Section 7].

We conclude the thesis with a summary of the main results.

Central Results

- Part II: The significance of density properties of constraint sets for the consistency of a very general class of perturbation approaches to nonsmooth optimization problems and variational inequalities is investigated.
- Part II: Several density results for pointwise constraint sets in Sobolev spaces for continuous as well as large classes of discontinuous obstacles are proven. The study of a suitable counterexample shows the limits of the closure property in terms of the regularity of the upper bound as a Sobolev function.
- Part III: We propose a path-following method for the contact problem of quasi-static elasto-plasticity based on a novel Fenchel dual problem to the primal problem. The convergence of this scheme is proven based on conditions set forth in Part II.
- Part III: Each subproblem can be solved by the semismooth Newton method in the continuous setting. Upon discretization, the efficiency of the resulting solver is verified by several numerical tests.
- Part IV: The primal problem of incremental quasi-static evolution in Prandtl-Reuss plasticity is equivalently reformulated as an inf-sup problem. The latter problem can be characterized as a Fenchel predual to the classical incremental stress problem. As a consequence, we obtain necessary and sufficient optimality conditions for the time-discretized problem.
- Part IV: We study a modified visco-plastic regularization which may be attractive from a numerical point of view.

Bibliography

[1] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces*. Pure and Applied Mathematics. Elsevier Science, 2003.

[2] J. Alberty and C. Carstensen. Numerical analysis of time-depending primal elastoplasticity with hardening. *SIAM Journal on Numerical Analysis*, 37(4):1271–1294, 2000.

[3] J. Alberty, C. Carstensen, and D. Zarrabi. Adaptive numerical analysis in primal elastoplasticity with hardening. *Computer Methods in Applied Mechanics and Engineering*, 171:175–204, 1999.

[4] H.W. Alt. *Lineare Funktionalanalysis*. Springer, 2012.

[5] A. Amassad, M. Shillor, and M. Sofonea. A quasistatic contact problem for an elastic perfectly plastic body with Tresca's friction. *Nonlinear Analysis: Theory, Methods & Applications*, 35(1): 95–109, 1999.

[6] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Clarendon Press Oxford, 2000.

[7] G. Anzellotti and M. Giaquinta. Existence of the displacements field for an elasto-plastic body subject to Hencky's law and Von Mises yield condition. *manuscripta mathematica*, 32(1): 101–136, 1980.

[8] D.N. Arnold and R. Winther. Mixed finite elements for elasticity. *Numerische Mathematik*, 92 (3):401–419, 2002.

[9] R.B. Ash. *Real Variable with Basic Metric Space Topology*. IEEE Press, 1993.

[10] H. Attouch, G. Buttazzo, and G. Michaille. *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*. SIAM, 2nd edition, 2014.

[11] J.P. Aubin and I. Ekeland. *Applied Nonlinear Analysis*. Dover Books on Mathematics Series. Dover Publications, 1984.

[12] L. Badea and R. Krause. One- and two-level Schwarz methods for variational inequalities of the second kind and their application to frictional contact. *Numerische Mathematik*, 120(4): 573–599, 2012.

[13] C. Bahriawati and C. Carstensen. Three MATLAB implementations of the lowest-order Raviart-Thomas MFEM with a posteriori error control. *Computational Methods in Applied Mathematics*, 5(4):333–361, 2005.

[14] C. Baiocchi and A. Capelo. *Variational and Quasivariational Inequalities*. John Wiley and Sons, 1984.

[15] S. Bartels, A. Mielke, and T. Roubíček. Quasi-static small-strain plasticity in the limit of vanishing hardening and its numerical approximation. *SIAM J. Numer. Anal.*, 50(2):951–976, 2012.

[16] M. Bergounioux, M Haddou, M. Hintermüller, and K. Kunisch. A Comparison of a Moreau–Yosida-Based Active Set Strategy and Interior Point Methods for Constrained Optimal Control Problems. *SIAM Journal on Optimization*, 11(2):495–521, 2000.

[17] J.-M. E. Bernard. Density results in Sobolev spaces whose elements vanish on a part of the boundary. *Chinese Annals of Mathematics, Series B*, 32(6):823–846, 2011.

[18] L. Boccardo and F. Murat. Nouveaux résultats de convergence dans des problèmes unilatéraux. In *Non-linear Partial Differential Equations and their Applications (Collège de France Seminar)*, volume II, pages 68–85. Pitman Research Notes in Math., 1982.

[19] L. Boccardo and F. Murat. Homogenization of nonlinear unilateral problems. In *Composite Media and Homogenization Theory*, pages 81–105. Springer, 1991.

[20] J.-F. Bonnans, J.C. Gilbert, C. Lemaréchal, and C.A. Sagastizábal. *Numerical Optimization*. Springer, 2nd edition, 2006.

[21] A. Braides. *Γ-convergence for Beginners*, volume 22. Oxford University Press, 2002.

[22] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Spinger Series in Computational Mathematics, 1991.

[23] F. Brezzi, W.W. Hager, and P.A. Raviart. Error Estimates for the Finite Element Solution of Variational Inequalities. Part I. Primal Theory. *Numerische Mathematik*, 28:431–444, 1977.

[24] C. Carstensen. Domain decomposition for a non-smooth convex minimization problem and its application to plasticity. *Numerical Linear Algebra with Applications*, 4(3):177–190, 1997.

[25] C. Carstensen. Convergence of adaptive finite element methods in computational mechanics. *Appl. Numer. Math.*, 59(9):2119–2130, 2009.

[26] C. Carstensen, A. Orlando, and J. Valdman. A convergent adaptive finite element method for the primal problem of elastoplasticity. *J. Numer. Meth. Engrg*, pages pp. 1851–1887, 2005.

[27] C. Carstensen, R. Klose, and A. Orlando. Reliable and efficient equilibrated a posteriori error finite element error control in elastoplasticity and elastoviscoplasticity with hardening. *Comput. Methods Appl. Mech. Engrg.*, 195(19-22):2574–2598, 2006.

[28] C. Carstensen, M. Eigel, and J. Gedicke. Computational competition of symmetric mixed FEM in linear elasticity. *Comput. Methods Appl. Mech. Engrg.*, 200(41–44):2903–2915, 2011.

[29] J. Cerdà. *Linear Functional Analysis*. American Mathematical Society, 2010.

[30] A. Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, 20(1):89–97, 2004.

[31] X. Chen, Z. Nashed, and L. Qi. Smoothing methods and semismooth methods for nondifferentiable operator equations. *SIAM Journal on Numerical Analysis*, 38(4):pp. 1200–1216, 2001.

[32] P. W. Christensen. A semi-smooth Newton method for elasto-plastic contact problems. *International Journal of Solids and Structures*, 39(8):2323 – 2341, 2002.

[33] P.G. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications*. SIAM, 2013.

[34] M. Cocu, E. Pratt, and M. Raous. Formulation and approximation of quasistatic frictional contact. *International Journal of Engineering Science*, 34(7):783 – 798, 1996.

[35] B. Dacorogna. *Direct Methods in the Calculus of Variations*. Springer, New York, 2nd edition, 2010.

[36] G. Dal Maso. Some necessary and sufficient conditions for the convergence of sequences of unilateral convex sets. *Journal of Functional Analysis*, 62(2):119 – 159, 1985.

[37] G. Dal Maso. *An Introduction to Γ-convergence*. Birkhäuser, Boston, 1993.

[38] G. Dal Maso, A. DeSimone, and M.G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Archive for Rational Mechanics and Analysis*, 180(2):237–291, 2006.

[39] R. Dautray and J.-L Lions. *Mathematical Analysis and Numerical Methods for Science and Technology*, volume 2. Springer, 2000.

[40] E. De Giorgi and T. Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, 58(6):842–850, 1975.

[41] J.C. De Los Reyes and M. Hintermüller. A duality based semismooth Newton framework for solving variational inequalities of the second kind. *Interfaces and Free Boundaries*, 13:437–462, 2011.

[42] A. Demyanov. Regularity of stresses in Prandtl-Reuss perfect plasticity. *Calculus of Variations and Partial Differential Equations*, 34(1):23–72, 2009.

[43] P. Doktor and A. Ženíšek. The density of infinitely differentiable functions in Sobolev spaces with mixed boundary conditions. *Appl. Math., Praha*, 51(5):517–547, 2006.

[44] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en phyisque*. Dunod, Paris, 1972.

[45] F. Ebobisse and B.D. Reddy. Some mathematical problems in perfect plasticity. *Computer Methods in Applied Mechanics and Engineering*, 193(48):5071 – 5094, 2004.

[46] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1987.

[47] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. Springer, 2004.

[48] L.C. Evans. *Partial Differential Equations*. American Mathematical Society, 1998.

[49] R.S. Falk. Error estimates for the approximation of a class of variational inequalities. *Mathematics of Computation*, 28(128):963–971, 1974.

[50] G.A. Francfort and A. Giacomini. Small-strain heterogeneous elastoplasticity revisited. *Communications on Pure and Applied Mathematics*, 65(9):1185–1241, 2012.

[51] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Berlin: Springer, 1998.

[52] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes equations*. Springer-Verlag, 1986.

[53] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems*. Springer, 1984.

[54] R. Glowinski. *Variational Methods for the Numerical Solution of Nonlinear Elliptic Problems*. SIAM, 2015.

[55] C. Goffman and J. Serrin. Sublinear functions of measures and variational integrals. *Duke Math. J.*, 31(1):159–178, 03 1964.

[56] A. Grigor'yan. *Heat Kernel and Analysis on Manifolds*. American Mathematical Society - International Press, 2009.

[57] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman Publishing, 1985.

[58] P. Gruber and J. Valdman. Solution of one-time-step problems in elastoplasticity by a slant Newton method. *SIAM Journal on Scientific Computing*, 31(2):1558–1580, 2009.

[59] C. Hager and B. Wohlmuth. Nonlinear complementarity functions for plasticity problems with frictional contact. *Comput. Methods Appl. Mech. Engrg.*, 198:3411–3427, 2009.

[60] W. Han and B.D. Reddy. Computational plasticity: The variational basis and numerical analysis. *Computational Mechanics Advances*, 2:283–400, 1995.

[61] W. Han and B.D. Reddy. *Plasticity: Mathematical Theory and Numerical Analysis*. Springer, New York, 2nd edition, 2013.

[62] B. Hanouzet and J.L. Joly. Méthodes d'ordre dans l'interprétation de certaines inéquations variationnelles et applications. *Journal of Functional Analysis*, 34(2):217 – 249, 1979.

[63] E. Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*. American Mathematical Society, 1999.

[64] M. Hintermüller. Mesh independence and fast local convergence of a primal-dual active-set method for mixed control-state constrained elliptic control problems. *ANZIAM J.*, 49(1):1–38, 2007.

[65] M. Hintermüller and K. Kunisch. Total bounded variation regularization as bilaterally constrained optimization problem. *SIAM J. Applied Mathematics*, 64:1311–1333, 2004.

[66] M. Hintermüller and K. Kunisch. Path-following methods for a class of constrained minimization problems in function space. *SIAM Journal on Optimization*, 17(1):159–187, 2006.

[67] M. Hintermüller and K. Kunisch. PDE-constrained optimization subject to pointwise constraints on the control, the state and its derivative. *SIAM Journal on Optimization*, 20(3): 1133–1156, 2009.

[68] M. Hintermüller and C.N. Rautenberg. A sequential minimization technique for elliptic quasi-variational inequalities with gradient constraints. *SIAM Journal on Optimization*, 22(4): 1224–1257, 2012.

[69] M. Hintermüller and C.N. Rautenberg. On the density of classes of closed convex sets with pointwise constraints in Sobolev spaces. *Journal of Mathematical Analysis and Applications*, 426 (1):585 – 593, 2015.

[70] M. Hintermüller and C.N. Rautenberg. Optimal selection of the regularization function in a generalized total variation model. Part I: Modelling and theory. Technical report, WIAS Berlin, 2016.

[71] M. Hintermüller and S. Rösel. An infinite-dimensional semismooth Newton method for elasto-plastic contact problems. *PAMM*, 13(1):387–388, 2013.

[72] M. Hintermüller and S. Rösel. A duality-based path-following semismooth Newton method for elasto-plastic contact problems. *Journal of Computational and Applied Mathematics*, 292:150 – 173, 2016.

[73] M. Hintermüller and G. Stadler. An infeasible primal-dual algorithm for total bounded variation–based inf-convolution-type image restoration. *SIAM Journal on Scientific Computing*, 28(1):1–23, 2006.

[74] M. Hintermüller and M. Ulbrich. A mesh-independence result for semismooth Newton methods. *Math. Program., Ser. B*, 171:151–184, 2004.

[75] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set strategy as a semi-smooth Newton method. *SIAM J. Optimization*, 13(3):865–888, 2003.

[76] I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek. *Solution of Variational Inequalities in Mechanics*. Springer, New York, 1988.

[77] S. Hüeber, G. Stadler, and B. Wohlmuth. A primal-dual active set algorithm for three-dimensional contact problems with Coulomb friction. *SIAM J. Sci. Comput.*, 30:572–596, 2008.

[78] C. Johnson. Existence theorems for plasticity problems. *J. Math. pures et appl.*, 55:431–444, 1976.

[79] C. Johnson. On plasticity with hardening. *Journal of Math. Anal. and Appl.*, 62:325–336, 1978.

[80] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and their Applications*. Academic Press, Inc., New York, 1980.

[81] M. Kirszbraun. Über die zusammenziehende und Lipschitzsche Transformationen. *Fundamenta Mathematicae*, 22:77–108, 1934.

[82] R. Kohn and R. Temam. Dual spaces of stresses and strains, with applications to Hencky plasticity. *Applied Mathematics and Optimization*, 10(1):1–35, 1983. ISSN 0095-4616.

[83] R. Kornhuber and R. Krause. Adaptive multigrid methods for Signorini's problem in linear elasticity. *Computing and Visualization in Science*, 4(1):9–20, 2001.

[84] R. Kornhuber, R. Krause, O. Sander, P. Deuflhard, and S. Ertel. A monotone multigrid solver for two body contact problems in biomechanics. *Computing and Visualization in Science*, 11(1): 3–15, 2008.

[85] K. Krabbenhoft, A. V. Lyamin, S. W. Sloan, and P. Wriggers. An interior-point algorithm for elastoplasticity. *International Journal for Numerical Methods in Engineering*, 69(3):592–626, 2007.

[86] R. Krause. A nonsmooth multiscale method for solving frictional two-body contact problems in 2D and 3D with multigrid efficiency. *SIAM J. Sci. Comput.*, 31(2):1399–1423, 2009.

[87] K. Kunisch and G. Stadler. Generalized Newton methods for the 2D-Signorini contact problem with friction in function space. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39:827–854, 2005.

[88] S. Li and W.K. Liu. Numerical simulations of strain localization in inelastic solids using mesh-free methods. *International Journal for Numerical Methods in Engineering*, 48(9):1285–1309, 2000.

[89] J.-L. Lions, R. Glowinski, and R. Trémolières. *Numerical Analysis of Variational Inequalities*. North-Holland, Amsterdam, 1981.

[90] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*, volume 1. Dunod, Paris, 1968.

[91] A. Mainik and A. Mielke. Existence results for energetic models for rate-independent systems. *Calculus of Variations and Partial Differential Equations*, 22(1):73–99, 2005.

[92] H. Matthies. Finite element approximations in thermo-plasticity. *Numerical Functional Analysis and Optimization*, 1(2):145–160, 1979.

[93] W.C.H. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000.

[94] A. Mielke and T. Roubíček. *Rate-independent Systems*. Springer, 2015.

[95] U. Mosco. Convergence of convex sets and of solutions of variational inequalities. *Advances in Mathematics*, 3(4):510 – 585, 1969.

[96] J. Nečas. *Les méthodes directes en théorie des équations elliptiques*. Academia, Prague, 1967.

[97] A. Needleman and V. Tvergaard. Analyses of plastic flow localization in metals. *Applied Mechanics Reviews*, 45(3S):S3–S18, 1992.

[98] J. T. Oden and N. Kikuchi. *Contact Problems in Elasticity*. Society for Industrial and Applied Mathematics, 1988.

[99] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*, volume 23. Springer Science & Business Media, 2008.

[100] R. Rannacher and F.-T. Suttmeier. A posteriori error control in finite element methods via duality techniques: Application to perfect plasticity. *Computational mechanics*, 21(2):123–133, 1998.

[101] S. Repin. Errors of finite element method for perfectly elasto-plastic problems. *Mathematical Models and Methods in Applied Sciences*, 06(05):587–604, 1996.

[102] R.T. Rockafellar. *Convex Analysis*, volume 46. Princeton University Press, 1970.

[103] J.F. Rodrigues. *Obstacle Problems in Mathematical Physics*. North-Holland, 1987.

[104] J.F. Rodrigues. On hyperbolic variational inequalities of first order and some applications. *Monatshefte für Mathematik*, 142(1-2):157–177, 2004.

[105] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1):259 – 268, 1992.

[106] M. Sauter. *Numerical Analysis of Algorithms for Infinitesimal Associated and Non-Associated Elasto-Plasticity*. PhD thesis, Karlsruher Institut für Technologie, 2010.

[107] M. Sauter and C. Wieners. On the superlinear convergence in computational elasto-plasticity. *Computer Methods in Applied Mechanics and Engineering*, 200(49-52):3646 – 3658, 2011.

[108] A. Schröder and S. Wiedemann. Error estimates in elastoplasticity using a mixed method. *Appl. Numer. Math.*, 61(10):1031–1045, 2011.

[109] A. Schwarz, J. Schröder, and G. Starke. Least-squares mixed finite elements for small strain elasto-viscoplasticity. *International journal for numerical methods in engineering*, 77(10): 1351–1370, 2009.

[110] K. G. Siebert. A convergence proof for adaptive finite elements without lower bound. *IMA Journal of Numerical Analysis*, 31(3):947–970, 2011.

[111] K. G. Siebert and A. Veeser. A unilaterally constrained quadratic minimization with adaptive finite elements. *SIAM Journal on Optimization*, 18(1):260–289, 2007.

[112] J.C. Simo and T.J.R. Hughes. *Computational Inelasticity*. Springer-Verlag Berlin, 1998.

[113] F. Solombrino. Quasistatic Evolution in Perfect Plasticity for General Heterogeneous Materials. *Archive for Rational Mechanics and Analysis*, 212(1):283–330, 2014.

[114] G. Stadler. *Infinite-Dimensional Semismooth Newton and Augmented Lagrangian Methods for Friction and Contact Problem in Elasticity*. PhD thesis, Karls-Franzens-Universität Graz, 2004.

[115] G. Stadler. Path-following and augmented Lagrangian methods for contact problems in linear elasticity. *J. Comput. Appl. Math.*, 203:533–547, June 2007.

[116] G. Strang and R. Temam. Functions of Bounded Deformation. *Archive for Rational Mechanics and Analysis*, 75(1):7–21, 1980.

[117] P.-M. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mec., Paris*, 20:3–39, 1981.

[118] M.E. Taylor. *Partial Differential Equations I: Basic Theory*. Springer-Verlag New York, 1996.

[119] R. Temam. *Mathematical Problems in Plasticity*. Gauthier-Villars, 1985.

[120] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*, volume 343. American Mathematical Soc., 2001.

[121] H. Toyoizumi. Continuous dependence on obstacles in variational inequalities. *Funkcialaj Ekvacioj*, 34:103–115, 1991.

[122] N. S. Trudinger. On the regularity of generalized solutions of linear, non-uniformly elliptic equations. *Archive for Rational Mechanics and Analysis*, 42(1):50–62, 1971.

[123] C. Wieners. Multigrid methods for Prandtl-Reuss plasticity. *Numerical Linear Algebra with Applications*, 6(6):457–478, 1999.

[124] C. Wieners. Nonlinear solution methods for infinitesimal perfect plasticity. *Zeitschrift für Angewandte Mathematik und Mechanik*, 87(8-9):643–660, 2007.

[125] C. Wieners. SQP methods for incremental plasticity with kinematic hardening. In B.Daya Reddy, editor, *IUTAM Symposium on Theoretical, Computational and Modelling Aspects of Inelastic Media*, volume 11 of *IUTAM BookSeries*, pages 143–153. Springer Netherlands, 2008.

[126] J. Wloka. *Partielle Differentialgleichungen*. B.G. Teubner, Stuttgart, 1982.

[127] K. Yosida. *Functional Analysis*. Springer-Verlag, Berlin, 1980.

[128] E. Zeidler. *Nonlinear Functional Analysis II/B: Nonlinear Monotone Operator*. Springer, 1990.

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 08.06.2016 Simon Rösel