DYNAMIC RISK MANAGEMENT IN ELECTRICITY PORTFOLIO OPTIMIZATION VIA POLYHEDRAL RISK FUNCTIONALS

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***Abstract*—We propose a methodology for combining risk management with optimal planning of power production and trading based on probabilistic knowledge about future uncertainties such as demands and spot prices. Typically, such a joint optimization of risk and (expected) revenue yields additional overall efficiency. Our approach is based on stochastic optimization (stochastic programming) with a risk functional as objective. The latter maps an uncertain cash flow to a real number. In particular, we employ so-called *polyhedral risk functionals* which, though being non-linear mappings, preserve linearity structures of optimization problems. Therefore, these are favorable to the numerical tractability of the optimization problems. The class of polyhedral risk functionals contains well-known risk functionals such as Average-Value-at-Risk and expected polyhedral utility. Moreover, it is also capable to model different dynamic risk mitigation strategies.**

I. Introduction

In medium term planning of electricity production and trading one is typically faced with uncertain parameters (such as future energy demands and market prices) that can be described reasonably by stochastic processes in discrete time. When time passes, additional information about the uncertain parameters may arrive (e.g., actual energy demands may be observed). Planning decisions can be made at each time stage based on the information available by then and on probabilistic information about the future (non-anticipativity). In terms of optimization, this situation is modeled by the framework of *multistage stochastic programming*; cf. Section II. This framework allows to anticipate the dynamic decision structure appropriately.

In energy risk management, which is typically carried out *ex post*, i.e., after power production planning, derivative products such as futures or options are traded in order to hedge a given power production plan. However, decisions about buying and selling derivative products can also be made at different time stages, i.e., the dynamics of the decisions process here is of the same type as in production and (physical) power trading. Hence, it is suggesting to integrate these two decision processes, i.e., to carry out simultaneously production planning, power trading, and trading of derivative products. E.g., in [3], [4] it has been demonstrated that such an integrated optimization approach (electricity portfolio optimization) yields additional overall efficiency.

However, for integrating risk management into a stochastic optimization framework, risk has to be quantified in a definite way. While in short term optimization simple risk functionals (risk measures) such as expected utility or Value-at-Risk might be appropriate, the dynamic nature of risk has to be taken into account if medium or long term time horizons are considered. Then, the partial information that is revealed gradually at different time stages may have a significant impact on the risk. Hence, in such situations it is necessary that risk functionals incorporate this information dynamics somehow [2], [17]. However, it turns out that there are numerous possibilities for doing so and that the adequacy of a certain risk functional depends strongly on the context (e.g., on the size of the company).

This paper is organized as follows: after brief reviews on multistage stochastic programming in Section II and risk measurement in Section III, our concept of polyhedral risk functionals from [6], [5] is presented in Section IV with regard to its employment in electricity portfolio optimization. The approach of polyhedral risk functionals, motivated through tractability issues in optimization, is a constructive framework providing particular flexibility with respect to dynamic aspects. We suggest various concrete multi-period polyhedral risk functionals and discuss the differences between them in Section V. Finally, we illustrate the effect of different polyhedral risk functionals with optimal cash flow curves from a medium term portfolio optimization model for a small power utility featuring a combined heat and power plant (CHP); cf. Fig. 1.

II. Multistage Stochastic Programming

For a broad presentation of stochastic programming we refer to [23]. Let the time stages of the planning horizon be denoted by $t = 1, ..., T$ and let, for each of these time steps, a d-dimensional random vector ξ_t be given. This random vector represents the uncertain planning parameters that become known at stage t, e.g., energy demands or market prices. We assume that ξ_1 is known from the beginning, i.e., a fixed vector

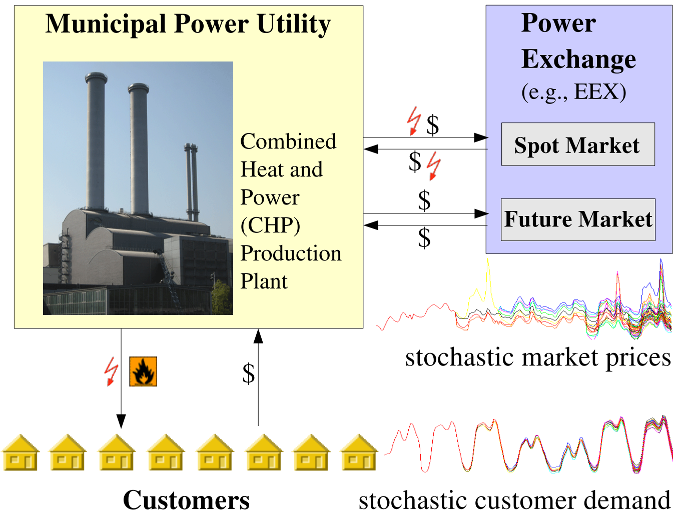

Fig. 1. Schematic diagram for a power production planning and trading model under demand and price uncertainty (portfolio optimization)

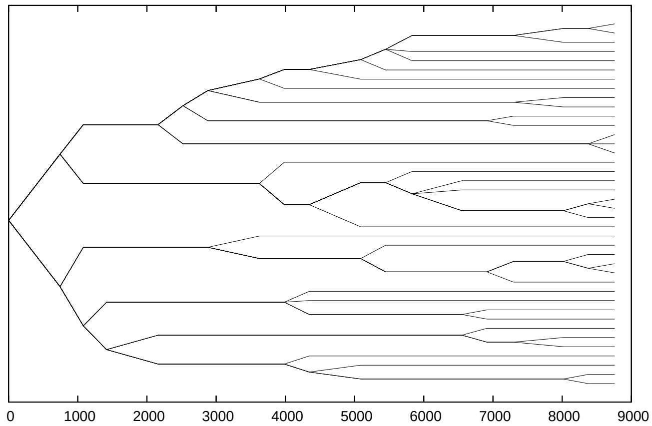

Fig. 2. Branching structure of a scenario tree wit 40 scenarios, $T = 8760$ time steps, and approx. $150,000$ nodes (there is a node at each time step for each scenario)

in $\mathbb{R}^d$. The collection $\xi := (\xi_1, ..., \xi_T)$ can be understood as multivariate discrete time stochastic process. Based on these notations a multistage stochastic program can be written as

$$\min_{x_1,\ldots,x_T} \left\{ \mathbb{F}(z_1, ..., z_T) \;\middle|\; \begin{array}{l} z_t := \sum_{s=1}^{t} b_s(\xi_s) \cdot x_s, \\ x_t = x_t(\xi_1, ..., \xi_t), \quad x_t \in X_t, \\ \sum_{s=0}^{t-1} A_{t,s}(\xi_t) x_{t-s} = h_t(\xi_t) \\ (t = 1, ..., T) \end{array} \right\} \quad (1)$$

where x_t is the decision vector for time stage t. The latter may depend and may only depend on the data observed until time t (non-anticipativity), i.e., on $\xi_1, ..., \xi_t$, respectively. In particular, the components of x_1 are *here and now* decisions since x_1 may only depend on ξ_1 which was assumed to be deterministic. The decisions are subject to constraints: each x_t has to be chosen within a given set X_t. Typically, each X_t is a polyhedron or even a box, potentially further constrained by integer requirements. Moreover, there are dynamic constraits involving matrices $A_{t,s}$ and right-hand sides h_t which may depend on ξ_t in an affinely linear way. For the objective, we introduce wealth values z_t (accumulated revenues) for each time stage defined by a scalar product of x_t and (negative) cost coefficients b_t. The latter may also depend on ξ_t in an affinely linear way. Hence, each z_t is a random variable ($t = 2, ..., T$).

The objective functional $\mathbb{F}$ maps the entire stochastic wealth process (cash flow) to a single real number. The classical choice in stochastic optimization is the *expected value* $\mathbb{E}$ (mean) of the overall revenue z_T, i.e.,

$$\mathbb{F}(z_1, ..., z_T) = -\mathbb{E}[z_T]$$

which is a *linear* functional. Linearity is a favorable property with respect to the numerical resolution of problem (1). However, if risk is a relevant issue in the planning process, then some sort of nonlinearity is required in the objective (or, alternatively, in the constraints). In this presentation, we will discuss *mean-risk* objectives of the form

$$\mathbb{F}(z_1, ..., z_T) = \gamma \cdot \rho(z_{t_1}, ..., z_{t_J}) - (1 - \gamma) \cdot \mathbb{E}[z_T]$$

with $\gamma \in [0, 1]$ and ρ being a *multi-period* risk functional applied to selected time steps $1 < t_1 < ... < t_J = T$ allowing for dynamic perspectives to risk.

If the stochastic input process ξ has infinite support (think of probability distributions with densities such as normal distributions), the stochastic program (1) is an infinite dimensional optimization problem. For such problems a solution can hardly be found in practice. Therefore, ξ has to be approximated by another process having finite support [15], [14]. Such an approximation must exhibit tree structure in order to reflect the monotone information structure of ξ; cf. Fig. 2. Note that appropriate *scenario tree* approximation schemes must rely on stability results for (1) (cf., e.g., [16], [8], [22]) that guarantee that the results of the approximate optimization problem are related to the (unknown) results of the original problem.

Though the framework (1) considers the dynamics of the decision process, typically only the first stage solution x_1 is used in practice since it is scenario independent whereas x_t is scenario dependent for $t \geq 2$. When the second time stage $t = 2$ is reached in reality one may solve a new problem instance of (1) such that the time stages are shifted one step ahead (rolling horizon). However, x_1 is a good decision in the sense that it anticipates future decisions and uncertainty.

III. Axiomatic Frameworks for Risk Functionals

The (multi-period) risk functional ρ in the objective of (1) is basically a mapping

$$z = (z_{t_1}, ..., z_{t_J}) \quad \mapsto \quad \rho(z) \in \mathbb{R}$$

i.e., a real number is assigned to each random wealth process from a certain class $\mathcal{Z}$. We assume that $\mathcal{Z}$ is a linear space of random processes $z = (z_{t_1}, ..., z_{t_J})$ which are *adapted* to the underlying information structure, i.e., $z_{t_j} = z_{t_j}(\xi_1, ..., \xi_{t_j})$ for $j = 1, ..., J$. Furthermore, one may require the existence of certain statistical moments for the random variables $z_{t_1}, ..., z_{t_J}$. The J time steps are denoted by $t_1, ..., t_J$ to indicate that they may be only a subset of the timesteps

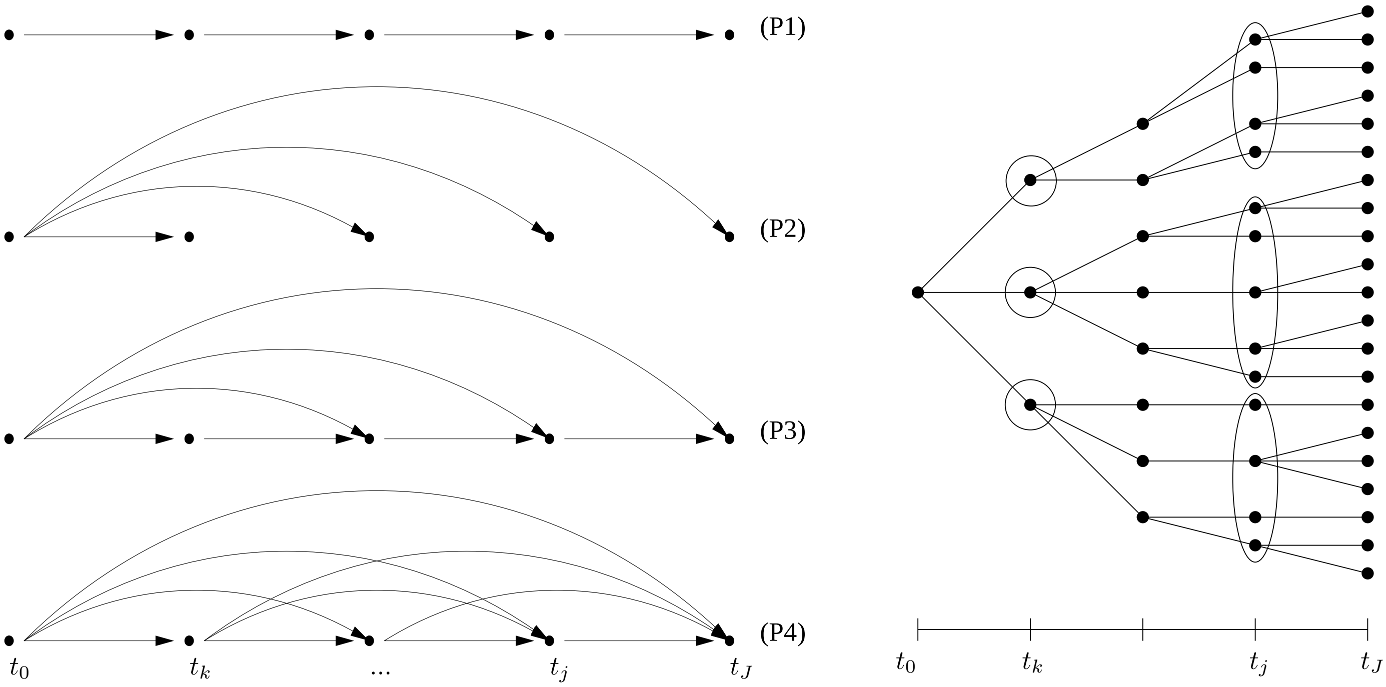

Fig. 3. Left: Different perspectives in the multi-period situation. A risk functional may focus on the transitions between two consecutive time-steps (P1), or it may only consider the distributions z_{t_j} seen from t_0 (P2). Hybrid forms are indicated by (P3) and (P4). Right: Illustration of the information structure of a stochastic wealth process $z_{t_1}, ..., z_{t_J}$ being discretely distributed. At each time stage t_k and in each scenario one can look at subsequent time steps $t_j > t_k$ and consider the discrete (sub-) distribution of z_{t_j} seen from this node. Each of these distributions may contribute to the risk $\rho(z_{t_1}, ..., z_{t_J})$.

$t = 1, ..., T$ of the underlying information structure. We assume $1 < t_1 < ... < t_J = T$ and set $t_0 = 1$ for convenience. The special case of *single-period* risk functionals occurs if only one time step is taken into account ($J = 1$, $t_J = T$).

Of course, ρ should exhibit certain properties that justify the term *risk functional*. A high number $\rho(z)$ should indicate a high risk of ending up at low wealth values z_{t_j}, a low (negative) number $\rho(z)$ indicates a small risk. Such and other intuitions have been formalized by various authors from economics and financial mathematics. For the single-period case ($J = 1$) there is a high degree of agreement about the relevant axioms; cf.,e.g., [1], [11], [17].

The multi-period case ($J > 1$) is a much more involved concern. As a start, we cite the first two axioms from [2], in addition to convexity as the third axiom. In this paper the number $\rho(z)$ is interpreted as the *minimal amount* μ *of additional risk-free capital* such that the process $z_{t_1} + \mu, ..., z_{t_J} + \mu$ is acceptable. This interpretation yields the axioms: A functional ρ is a multi-period *convex (capital) risk functional* if the following properties hold for all stochastic wealth processes $z = (z_{t_1}, ..., z_{t_J})$ and $\tilde{z} = (\tilde{z}_{t_1}, ..., \tilde{z}_{t_J})$ in $\mathcal{Z}$:

- *Monotonicity*: If $z_{t_j} \leq \tilde{z}_{t_j}$ in any case for $j = 1, ..., J$, then it holds that $\rho(z) \geq \rho(\tilde{z})$.
- *Cash invariance*: For each $\mu \in \mathbb{R}$ it holds that $\rho(z_{t_1} + \mu, ..., z_{t_J} + \mu) = \rho(z_{t_1}, ..., z_{t_J}) - \mu$.
- *Convexity*: For each $\mu \in [0, 1]$ it holds that $\rho(\mu z + (1-\mu)\tilde{z}) \leq \mu\rho(z) + (1-\mu)\rho(\tilde{z})$.

The convexity property is motivated by the idea that *diversification* might decrease risk but never increases it. Sometimes the following property is also required for all $z \in \mathcal{Z}$:

- *Positive homogeneity*: For each $\mu \geq 0$ it holds that $\rho(\mu z) = \mu\rho(z)$.

Note that, for the single-period case $J = 1$, the first three properties coincide with the classical axioms from [1], [10], [12]. A positively homogeneous convex risk functional is called *coherent* in [1]. We note, however, that other authors do not require positive homogeneity, but claim that risk should rather grow overproportionally, i.e., $\rho(\mu z) > \mu\rho(z)$ for $\mu > 1$; cf. [13], [11]. Clearly, the expectation functional $\mathbb{E}$ is a coherent risk functional, but the α-Value-at-Risk $\mathrm{VaR}_\alpha(z) = -\inf\{\mu \in \mathbb{R} : \mathbb{P}(z \leq \mu) > \alpha\}$, i.e., the negative α-quantile of z, is not.

For the multi-period case ($J > 1$) the three axioms of a convex risk functional are only a basis admitting many degrees of freedom. There are several aspects of risk that could be measured. First of all, one may want to measure the chance of very low values z_{t_j} at each time since very low values can mean bankruptcy (liquidity considerations). In addition, one may want to measure the degree of uncertainty one is faced with at each time step. If at some time t_k one can be sure about the future development of ones wealth z_{t_j} ($j > k$), this may be preferred to enduring uncertainty. E.g., low values z_{t_j} may be tolerable if one is sure that later the wealth is higher again.

Hence, in this multi-period situation, one may want to take into account not only the (marginal) distributions of $z_{t_1}, ..., z_{t_J}$ but also their chronological order and the *underlying information structure*. Therefore, a multi-period risk functional may

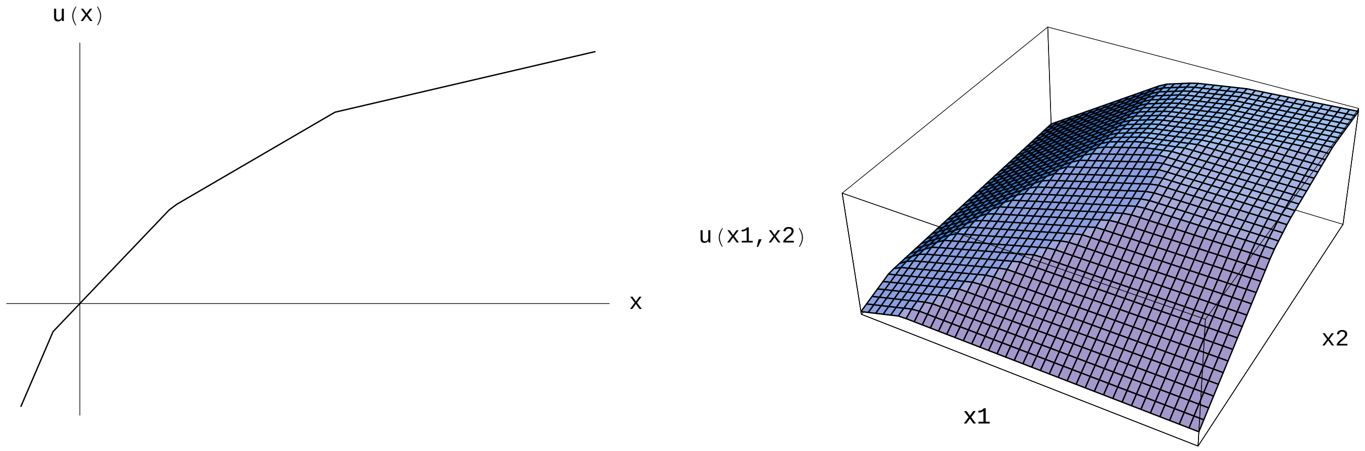

Fig. 4. Monotone and piecewise linear concave utility functions, single-period (left) and two-period ($J=2$) (right)

also take into account the conditional distributions of z_{t_j} given the information $z_{t_1},...,z_{t_k}$ with $k<j$ respectively $\xi_1,...,\xi_s$ with $s<t_j$ ($j=1,...,J$); cf. Fig. 3. Clearly, there are quite a lot of those conditional distributions and the question arises which ones are relevant and how to weight them reasonably. The above axioms leave this question open. In our opinion, a general answer can not be given, the requirements depend strongly on the application context, e.g., on the time horizon, on the size and capital reserves of the respective company, on the broadness of the model, etc.

Some authors, however, concretize the coherence axioms in this direction by suggesting stronger cash invariance properties, e.g.:

- $\rho(z_{t_1},...,z_{t_{j-1}},z_{t_j}+\mu,..,z_{t_J}+\mu)=\rho(z_{t_1},..,z_{t_J})-\mu$ for each $j\in\{1,...,J\}$ and $\mu\in\mathbb{R}$ [19]
- $\rho(z_{t_1},...,z_{t_{j-1}},z_{t_j}+z_0,..,z_{t_J}+z_0)=\rho(z_{t_1},..,z_{t_J})-\mathbb{E}[z_0]$ for each j and each random variable z_0 that is known at time t_{j-1}, i.e., $z_0=z_0(\xi_1,...,\xi_{t_{j-1}})$ [17]
- $\rho(z+\tilde{z})=\rho(z)+\tilde{z}_{t_J}$ if $\tilde{z}=(\tilde{z}_{t_1},...,\tilde{z}_{t_J})$ is such that $\tilde{z}_{t_J}$ is nonrandom (deterministic), i.e., $\tilde{z}_{t_J}=\tilde{z}_{t_J}(\xi_1)$ [13]

Each of these may be reasonable in some situations, but they seem to be too special for imposing them as general axioms. The framework of polyhedral risk functionals in the next section allows to model concrete instances with different perspectives to the dynamics.

IV. Polyhedral Risk Functionals

The basic motivation for polyhedral risk functionals is a technical, but important one. Consider the optimization problem (1). It is basically linear if the objective functional $\mathbb{F}$ is linear. In this case it is well tractable by various solution and decomposition methods. However, if $\mathbb{F}$ incorporates a risk functional ρ it is no longer linear since risk functionals are essentially nonlinear by nature. Decomposition structures may get lost and solution methods may take much longer or may even fail. To avoid the worst possible situation one should choose ρ to be convex. As discussed above, convexity is in accordance with economic considerations and axiomatic frameworks. Then (1) is at least a convex problem (except possible integer constraints contained in X_t), hence, any local optimum is automatically a global optimum.

Now, the framework of polyhedral risk functionals [6], [5] goes one step beyond convexity: polyhedral risk functionals maintain linearity structures even though they are nonlinear functionals. Namely, a polyhedral risk functional ρ is given by

$$\rho(z)=\inf\left\{\mathbb{E}\left[\sum_{j=0}^{J}c_j\cdot y_j\right]\;\middle|\;\begin{array}{l}y_j=y_j(\xi_1,...,\xi_{t_j})\in Y_j,\\ \sum_{k=0}^{j}V_{j,k}y_{j-k}=r_j\\ (j=0,...,J),\\ \sum_{k=0}^{j}w_{j,k}\cdot y_{j-k}=z_{t_j}\\ (j=1,...,J)\end{array}\right\} \quad (2)$$

where $z=(z_{t_1},...,z_{t_J})$ denotes a stochastic wealth process being non-anticipative with respect to ξ, i.e., $z_t=z_t(\xi_1,...,\xi_t)$. The notation $\inf\{\,.\,\}$ refers to the infimum. The definition includes fixed polyhedral cones Y_j (e.g., $\mathbb{R}_+\times...\times\mathbb{R}_+$) in some Euclidean spaces $\mathbb{R}^{k_j}$, fixed vectors c_j, h_j $w_{j,k}$, and matrices $V_{j,k}$, which have to be chosen appropriately. We will give examples for these parameters below. However, any functional ρ defined by (2) is always convex [6], [5].

Observe that problem (2) is more or less of the form (1), i.e., the risk of a stochastic wealth process is given by the optimal value of a stochastic program. Moreover, if (2) is inserted into the objective of (1) (i.e., $\mathbb{F}=\rho$), one is faced with two nested minimizations which, of course, can be carried out jointly. This yields the equivalent optimization problem

$$\min\left\{\mathbb{E}\left[\sum_{j=0}^{J}c_j\cdot y_j\right]\;\middle|\;\begin{array}{l}x_t=x_t(\xi_1,...,\xi_t)\in X_t,\\ y_j=y_j(\xi_1,...,\xi_{t_j})\in Y_j,\\ \sum_{s=0}^{t-1}A_{t,s}(\xi_t)x_{t-s}=h_t(\xi_t),\\ \sum_{k=0}^{j}V_{j,k}y_{j-k}=r_j,\\ \sum_{k=0}^{j}w_{j,k}\cdot y_{j-k}\\ \qquad=\sum_{s=1}^{t_j}b_s(\xi_s)\cdot x_s\end{array}\right\}$$

which is a stochastic program of the form (1) with *linear* objective. In other words: the nonlinearity of the risk functional ρ is transformed into additional variables and additional linear

No.	polyhedral representation (2)
ρ_1	$\inf\left\{ \frac{1}{J}\sum_{j=1}^{J}\left(y_{0,j}+\frac{1}{\alpha}\mathbb{E}\left[y_{j,2}\right]\right) \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R}^J,\ y_j=y_j(\xi_1,...\xi_{t_j})\in\mathbb{R}_+\times\mathbb{R}_+\ (j=1,...,J), \\ y_{j,1}-y_{j,2}=z_{t_j}+y_{0,j}\ (j=1,...,J) \end{array} \right\}$
ρ_2	$\inf\left\{ y_0+\frac{1}{J}\sum_{j=1}^{J}\frac{1}{\alpha}\mathbb{E}\left[y_{j,2}\right] \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R},\ y_j=y_j(\xi_1,...\xi_{t_j})\in\mathbb{R}_+\times\mathbb{R}_+\ (j=1,...,J), \\ y_{j,1}-y_{j,2}=z_{t_j}+y_{0,1}\ (j=1,...,J) \end{array} \right\}$
ρ_3	$\inf\left\{ y_0+\frac{1}{J}\sum_{j=1}^{J}\frac{1}{\alpha}\mathbb{E}\left[y_{j,2}\right] \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R},\ y_j=y_j(\xi_1,...\xi_{t_j})\in\mathbb{R}_+\times\mathbb{R}_+\ (j=1,...,J), \\ y_{1,1}-y_{1,2}=z_{t_1}+y_{0,1},\ y_{j,1}-y_{j,2}=z_{t_j}+y_{0,1}+y_{j-1,2}\ (j=2,...,J) \end{array} \right\}$
ρ_4	$\inf\left\{ \frac{1}{J}\left(y_0+\sum_{j=1}^{J}\frac{1}{\alpha}\mathbb{E}\left[y_{j,2}\right]\right) \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R},\ y_j=y_j(\xi_1,...\xi_{t_j})\in\mathbb{R}\times\mathbb{R}_+\ (j=1,...,J-1),\ y_J=y_J(\xi_1,...\xi_{t_J})\in\mathbb{R}_+\times\mathbb{R}_+ \\ y_{j,1}-y_{j,2}=z_{t_j}+y_{j-1,1}\ (j=1,...,J) \end{array} \right\}$
ρ_5	$\inf\left\{ \frac{1}{J}\sum_{j=1}^{J}\mathbb{E}\left[y_{j-1,1}+\frac{1}{\alpha}y_{j,3}\right] \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R},\ y_j=y_j(\xi_1,...\xi_{t_j})\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+\ (j=1,...,J), \\ \sum_{k=1}^{j}(y_{k,2}-y_{k,3}-y_{k-1,1})=z_{t_j}\ (j=1,...,J) \end{array} \right\}$
ρ_6	$\inf\left\{ y_0+\frac{1}{\alpha}\mathbb{E}\left[y_{J,2}\right] \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R},\ y_j=y_j(\xi_1,...\xi_{t_j})\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\ (j=1,...,J), \\ y_{1,2}-y_{1,3}=0,\ y_{j,2}-y_{j,3}-y_{j-1,2}=0\ (j=2,...,J),\ y_{j,1}-y_{j,2}-y_0=z_{t_j}\ (j=1,...,J) \end{array} \right\}$

TABLE I

REPRESENTATION (2) FOR EXEMPLARY MULTI-PERIOD POLYHEDRAL RISK MEASURES $\rho_1,...,\rho_6$

constraints in (1). This means that decomposition schemes and solution algorithms known for linear stochastic programs can also be used for (1) with $\mathbb{F}=\rho$. The case $\mathbb{F}=\gamma\cdot\rho-(1-\gamma)\cdot\mathbb{E}$ can be fully reduced to the case $\mathbb{F}=\rho$; cf. [5].

Another important advantage of polyhedral risk functionals is that they also behave favorable to stability with respect to (finite) approximation of the stochastic input process ξ [8]. Hence, there is a justification for the employment of the scenario tree approximation schemes from [15], [14].

It remains to discuss the issue of choosing the parameters c_j, h_j, $w_{j,k}$, $V_{j,k}$, Y_j in (2) such that the resulting functional ρ is indeed a convex risk functional satisfying, e.g., the coherence properties presented in the previous section. To this end, several criteria have been deduced in [6], [5] involving duality theory from convex analysis. However, here we restrict ourselves to the presentation of examples.

First, we consider the case $J=1$, i.e., the case of single-period risk functionals evaluating only the distribution of the final value z_T (total revenue). The starting point of the concept of polyhedral risk functionals was the well-known risk functional *Average-Value-at-Risk* AVaR_α at some probability level $\alpha\in(0,1)$. It is also known as Conditional-Value-at-Risk (cf. [21]), but as suggested in [11] we prefer the name Average-Value-at-Risk according to its definition

$$\mathrm{AVaR}_\alpha(z):=\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(z)d\beta$$

and avoid any conflict with the use of conditional distributions within VaR and AVaR. The Average-Value-at-Risk is a (single-period) coherent risk functional which is broadly accepted. $\mathrm{AVaR}_\alpha(z_T)$ can be interpreted as the mean (expectation) of the α-tail distribution of z_T, i.e., the mean of the distribution of z_T below the α-quantile of z_T. It has been observed in [21] that AVaR_α can be represented by

$$\begin{aligned} \mathrm{AVaR}_\alpha(z_T) &= \inf_{y_0\in\mathbb{R}}\left\{y_0+\tfrac{1}{\alpha}\mathbb{E}[(y_0+z_T)^-]\right\} \\ &= \inf\left\{ y_0+\tfrac{1}{\alpha}\mathbb{E}[y_{1,2}] \;\middle\vert\; \begin{array}{l} y_0\in\mathbb{R}, \\ y_1=y_1(\xi_1,...,\xi_T)\in\mathbb{R}^2_+, \\ y_0+z_T=y_{1,1}-y_{1,2} \end{array} \right\} \end{aligned}$$

where $(\,.\,)^-$ denotes the negative part of a real number, i.e., $a^-=\max\{0,-a\}$ for $a\in\mathbb{R}$. The second representation is deduced from the first one by introducing stochastic variables y_1 for the positive and the negative part of y_0+z_T. Hence, AVaR_α is of the form (2) with $J=1$, $c_0=1$, $c_1=(0,\frac{1}{\alpha})$, $w_{1,0}=(1,-1)$, $w_{1,1}=-1$, $Y_0=\mathbb{R}$, $Y_1=\mathbb{R}^2_+=\mathbb{R}_+\times\mathbb{R}_+$, and $h_0=h_1=V_{0,0}=V_{1,0}=V_{1,1}=0$. Thus, it is a (single-period) polyhedral risk functional.

Another single-period example for a polyhedral risk functional (satisfying monotonicity and convexity) is expected utility, i.e., $\rho_u(z_T):=-\mathbb{E}[u(z_T)]$ with a non-decreasing concave utility function $u:\mathbb{R}\to\mathbb{R}$; cf. [11]. Typically, nonlinear functions such as $u(x)=1-e^{-\beta x}$ with some fixed $\beta>0$ are used. Of course, in such cases ρ_u is not a polyhedral risk functional. However, in situations where the domain of z_T can be bounded a priori, it makes sense to use piecewise linear functions for u (see Fig. 4, left). Then, according to the infimum representation of piecewise linear convex functions [20, Corollary 19.1.2], it holds that

$$\rho_u(z_T)=\inf\left\{\mathbb{E}\left[c\cdot y_1\right] \;\middle\vert\; \begin{array}{l} y_1=y_1(\xi_1,...,\xi_T)\in\mathbb{R}^{n+2}_+, \\ w\cdot y_1=z_T,\ \sum_{i=1}^n y_{1,i}=1 \end{array}\right\}$$

where n is the number of cusps of u, $w_1,...,w_n$ are the x-coordinates of the cusps, and $c_i=-u(w_i)$ $(i=1,..,n)$. Thus, ρ_u is a polyhedral risk functional. This approach can also be generalized to the multi-period situation in an obvious way by specifying a (concave) utility function $u:\mathbb{R}^J\to\mathbb{R}$ (see Fig. 4, right). However, specifying an adequate utility function may be difficult in practice, in particular in the multi-period case. Furthermore, expected utility is not cash invariant, neither in the single-period nor in the multi-period case. Therefore we focus on generalizing AVaR to the multi-period case in the following.

V. MULTI-PERIOD POLYHEDRAL RISK FUNCTIONALS

In the multi-period case $J>1$, the framework of polyhedral risk functionals provides particular flexibility with respect to dynamic aspects, i.e., it allows to model different perspectives

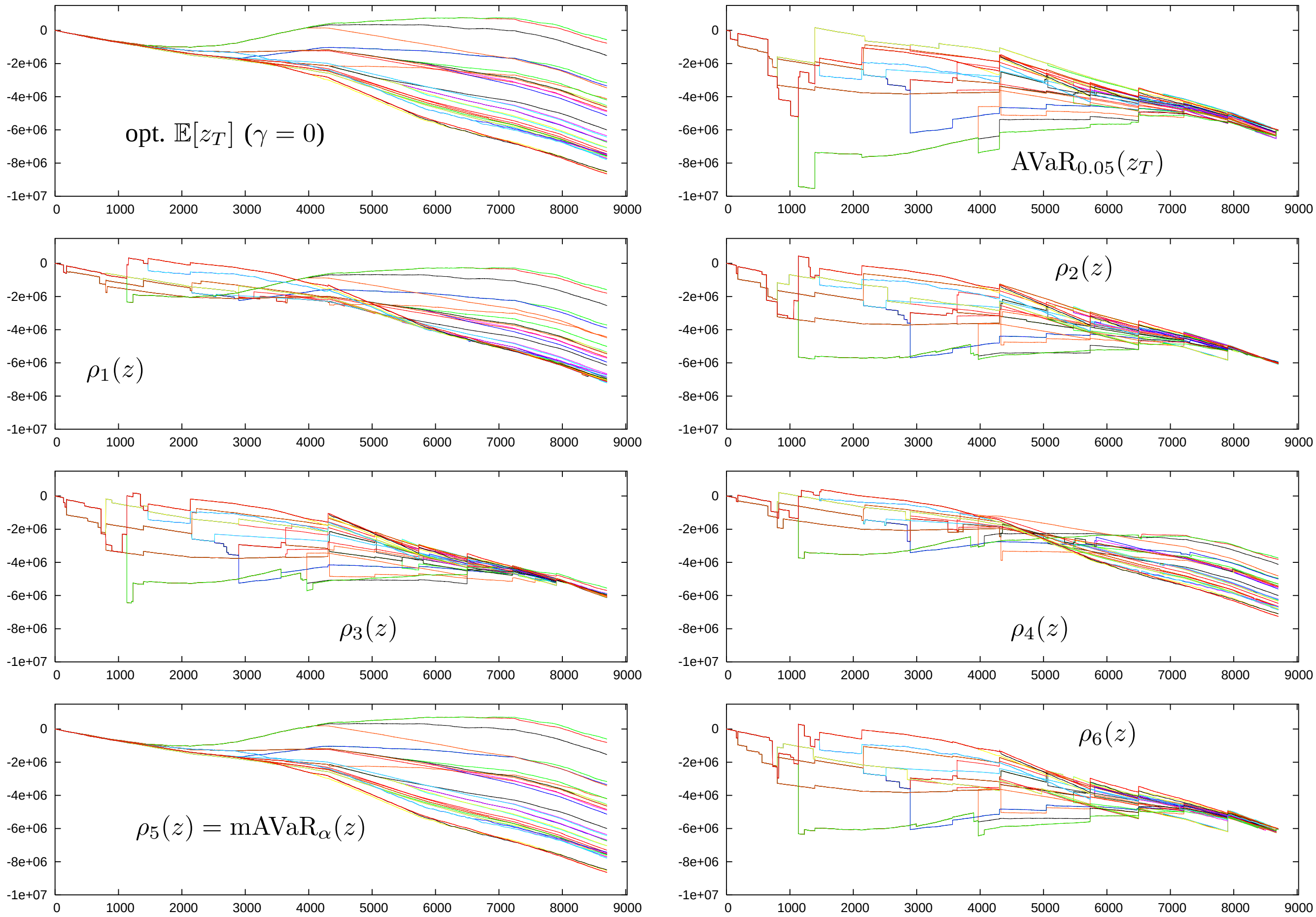

Fig. 5. Optimal cash values z_t (wealth) over time ($t = 1, ..., T$) for different risk functionals. Each curve in a graph represents one of the 40 scenarios.

to the relations between different time stages. In the following we introduce some examples which extend AVaR_α to the multi-period situation in different ways.

We start with defining ρ_1 as the simplest possible AVaR extension being defined as the sum of AVaR_α applied to different time stages, i.e., $\rho_1(z) := \frac{1}{J}\sum_{j=1}^J \mathrm{AVaR}_\alpha(z_{t_j})$. Next we deduce ρ_2 from ρ_1 by interchanging summation (Σ) and minimization (inf), i.e.,

$$\begin{aligned} \rho_1(z) &= \tfrac{1}{J}\sum_{j=1}^J \inf_{y_{0,j}\in\mathbb{R}} \left\{ y_{0,j} + \tfrac{1}{\alpha}\mathbb{E}\left[\left(z_{t_j} + y_{0,j} \right)^- \right] \right\} \\ \rho_2(z) &:= \inf_{y_0\in\mathbb{R}} \left\{ y_0 + \tfrac{1}{J}\sum_{j=1}^J \tfrac{1}{\alpha}\mathbb{E}\left[\left(z_{t_j} + y_0 \right)^- \right] \right\} \end{aligned}$$

for any wealth sequence z. The latter functional can be interpreted as $\mathrm{AVaR}_\alpha(z_\tau)$ with a randomly drawn time step $\tau \in \{t_1, ..., t_J\}$ being stochastically independent of z. Clearly, the random variable z_τ is a kind of mixture of $z_{t_1}, ..., z_{t_J}$. Note, however, that ρ_1 and ρ_2 do not consider the chronological order of $z_{t_1}, ..., z_{t_J}$. Hence, they only take into account the (marginal) distributions $z_{t_1}, ..., z_{t_J}$ seen from the present, i.e., from here and now ($t_0 = 1$). In Fig. 3 (left) this perspective is symbolized by (P2).

Applying AVaR_α to other possible mixtures yields, e.g.,

$$\begin{aligned} \rho_4(z) &:= \mathrm{AVaR}_\alpha\left(\tfrac{1}{J}\sum_{j=1}^J z_{t_j}\right) \\ \rho_6(z) &:= \mathrm{AVaR}_\alpha(\min\{z_{t_1}, ..., z_{t_J}\}) \end{aligned}$$

where the index numbers 4 and 6 are chosen compatibly to [6], [17], [5]. These risk functionals depend on the multivariate distribution of $(z_{t_1}, ..., z_{t_J})$. In our opinion, ρ_6 is the most reasonable multi-period extension of AVaR with regard to liquidity considerations, since AVaR is applied to the respectively lowest wealth values; see also [2, Section 4].

In [17], [18], *the multi-period* AVaR was suggested. This risk functional $\mathrm{mAVaR}_\alpha =: \rho_5$ takes up perspective (P1) in Fig. 3 (left), i.e., it focuses on variations between two consecutive time steps. It can be defined by

$$\rho_5(z) := \tfrac{1}{J}\sum_{j=1}^J \mathbb{E}\left[\mathrm{AVaR}_\alpha(z_{t_j} - z_{t_{j-1}} | \xi_1, ..., \xi_{t_{j-1}}) \right]$$

where $\mathrm{AVaR}_\alpha(\,.\,|\xi_1, ..., \xi_{t_k})$ refers to the *conditional* AVaR with respect to the information available at time t_k. The latter is easy to understand by considering the tree in Fig. 3 (right): The conditional AVaR is the collection of AVaR_α values calculated from the (sub-) distributions at t_j seen from each node at time $t_k < t_j$. Note that, in the mAVaR formula, these AVaR_α values are averaged in terms of the expected value $\mathbb{E}$.

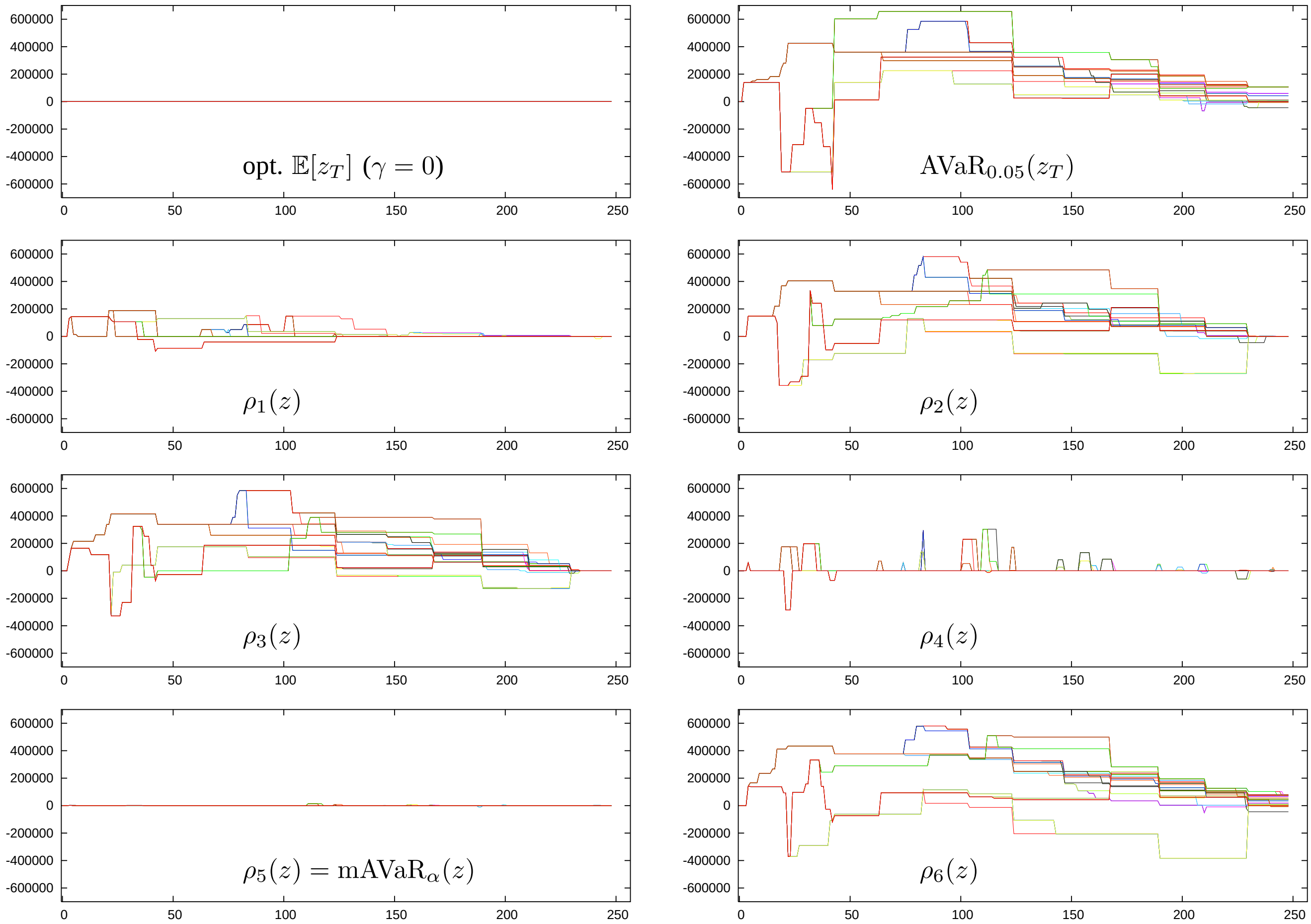

Fig. 6. Optimal future stock over time for different polyhedral risk functionals.

Now, all these examples can be represented through (2), i.e., they are *multi-period polyhedral risk measures*; cf. Table I. In addition, functional ρ_3 is a kind of mixture between ρ_2 and ρ_5 and, thus, models perspective (P3) from Fig. 3 (left). Note that $\rho_1, \ldots, \rho_6$ are multi-period coherent risk functionals according to Section III. These examples show that the concept of polyhedral risk functionals allows to model different perspectives to risk of processes. At this point, one might expect a risk functional that takes into account all possible (conditional) distributions as indicated in Fig. 3 (left), perspective (P4). However, modeling such an all-embracing approach raises the problem how to weight the risks of all these distribution reasonably. Therefore, we refrain from such examples here.

VI. Illustrative Simulation Results

Finally, we illustrate the effects of different polyhedral risk functionals by presenting some optimal wealth processes from an electricity portfolio optimization model [9], [7]. This model is of the form (1), it considers the one year planning problem of a *municipal power utility*, i.e., a price-taking retailer serving heat and power demands of a certain region; see Fig. 1. It is assumed that the utility features a combined heat and power (CHP) plant that can serve the heat demand completely but the power demand only in part. In addition, the utility can buy power at the day-ahead spot market of some power exchange, e.g., the European Energy Exchange EEX. Morover, the utility can trade monthly (purely financial) futures (e.g., Phelix futures at EEX).

The objective $\mathbb{F}$ of this model is a mean-risk objective as discussed in Section 1 incorporating a polyhedral risk functional ρ and the expected total revenue $\mathbb{E}[z_T]$; we use $\gamma = 0.9$ as weighting factor. Time horizon is one year in hourly discretization, i.e., $T = 8760$. For the risk time steps t_j we use 11 PM at the last trading day of each week ($j = 1, \ldots, J = 52$). Time series models for the uncertain input data (demands and prices) have been set up (see [9] for details) and approximated according to [15], [14] by a finite scenario tree consisting of 40 scenarios; see Fig. 2. Such approximations are justified for this optimization model through stability results for stochastic optimization problems and polyhedral risk functionals [8]. The resulting optimization problem is very large-scale, however, it is numerically tractable due to the favorable nature of polyhedral risk functionals. In particular, since we modeled the CHP plant without integer variables, it is a linear program (LP) which could be solved by ILOG CPLEX in about one hour.

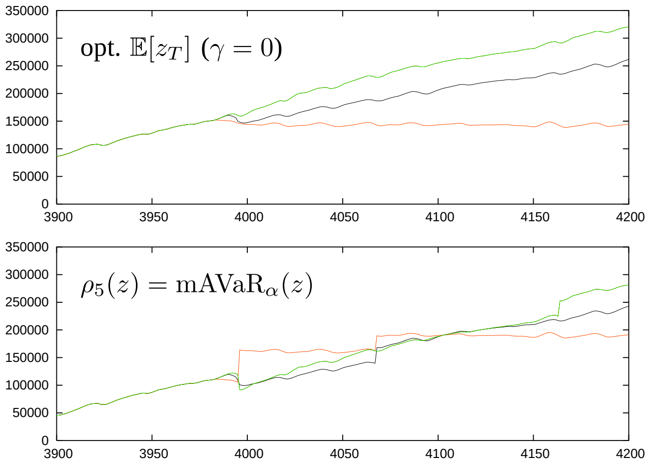

Fig. 7. Optimal cash values over time (excerpt)

In Fig. 5 the optimal cash flows are displayed, i.e., the wealth values z_t for each time step $t = 1, ..., T$ and each scenario, obtained from optimization with different mean-risk objectives. These families of curves differ in shape due to different policies of future trading induced by the different risk functionals; see Fig. 6. Setting $\gamma = 0$ (no risk functional at all) yield high spread for z_T. Using $\mathrm{AVaR}_\alpha(z_T)$ ($\gamma = 0.9$) yields low spread for z_T but low values and high spread at $t < T$. This shows that, for the situation here, single-period risk functionals are not appropriate means of risk aversion. The employment of multi-period polyhedral risk functionals yields spread that is better distributed over time. However, the way how this is achieved is different. The functionals ρ_2, ρ_3, and ρ_6 yield similar results: they aim at finding a level y_0 as high as possible such that the curves rarely fall below that level. The effect of ρ_4 and ρ_1 is different: these functionals aim at equal spread at all times (where the effect of ρ_1 is weaker). At the first glance, ρ_5 seems to have no effect at all, but zooming into the branching points (Fig. 7) reveals that its effect is a local one, namely branching is delayed only from t_j until t_{j+1} (perspective (P1) in Fig. 3). This effect may be more useful for long term models (e.g., pension fund models [17, Chapter 5]). Note that the displays in Fig. 5 illustrate primarily perspective (P2) from Fig. 3, i.e., everything is seen from here and now. Finally, we note that the effects of the risk functionals cost only less than 1% of the expected overall revenue $\mathbb{E}[z_T]$.

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