Quasi-Monte Carlo methods applied to stochastic energy optimization models

H. Leövey und W. Römisch

Humboldt-University Berlin
Institute of Mathematics
http://www.math.hu-berlin.de/~romisch

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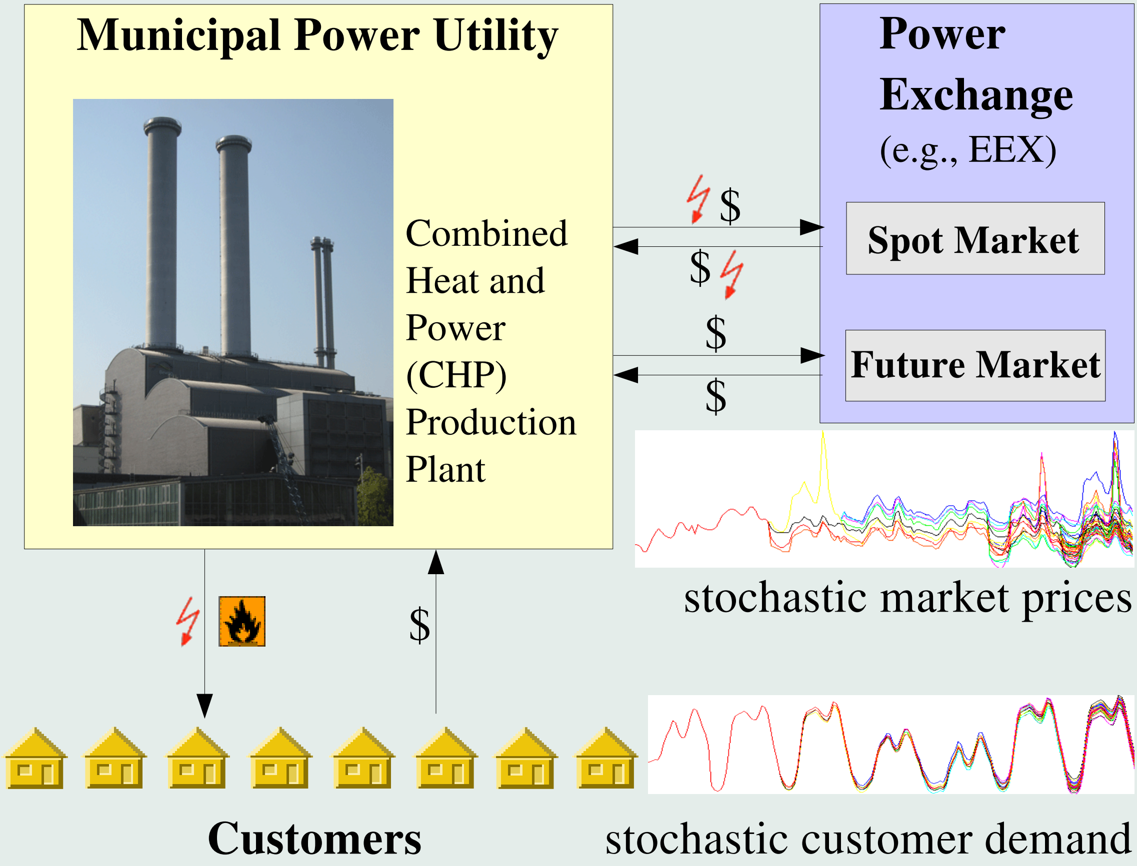
Introduction

- Electricity management models often contain uncertain parameters (market prices, electrical load, inflows, wind speed), for which historical data and/or statistical models are often available.
- Here, we consider models which are relevant for smaller market participants, since their decisions do not influence market prices.
- During the last 15 years a breakthrough was obtained for the numerical computation of high-dimensional integrals by means of new randomized Quasi-Monte Carlo methods for integrands with mixed first partial derivatives in the sense of Sobolev (Kuo-Sloan 05).
- Stochastic two-stage mixed-integer optimization models lead even to discontinuous integrands. For integrands of the form $g(\cdot)\mathbf{1}_B(\cdot)$ on $[0,1]^d$ with g smooth and B convex polyhedral, the following convergence rate for RQMC methods is derived:

$$O(n^{-\frac{1}{2}-\frac{1}{4d_*-2}+\delta}) \qquad \text{(He-Wang 15)},$$

where d_* is the number of coordinate axes which are not parallel to some discontinuity boundary.

Electricity Portfolio Management



The optimization model contains the electrical load ξ_δ and the electricity price ξ_c as stochastic parameters. Both are components of the random vector

$$\xi = (\xi_{\delta,1}, \xi_{c,1}, \ldots, \xi_{\delta,T}, \xi_{c,T})^\top.$$

At each period $t \in \{1, \ldots, T\}$ of the T time intervals the company has to cover the load. During peak load periods load covering requires electricity trading based on bilateral contracts with fixed prices and/or day-ahead trading with stochastic prices. Peak/offpeak load periods may require to switch on/off cycling units.

A two-stage electricity production and trading model is of the form

$$\min \Big\{ \sum_{t=1}^{T} \langle c_t, x_t \rangle + \int_{\mathbb{R}^T} \Phi(q(\xi), h(\xi) - Vx) P(d\xi) : x \in X \Big\}$$

$$\Phi(q,h) = \inf \Big\{ \sum_{t=1}^{T} \langle q_t, y_t \rangle : Wy + Vx \geq h,\, y \in Y \Big\},$$

where x_t denotes the outputs of the base load units and c_t their costs at t. The set X contains capacity limits and eventual ramping constraints at each t. Φ denotes the second-stage optimal value function.

The vector y_t of second-stage decisions contains the 0-1 variables and outputs of cycling units, and the amounts of trading.
The constraints $Wy + Vx \geq h(\xi)$ describe load covering at any t and minimum up/down times of the cycling units. The constraint $y \in Y$ describes capacity limits, ramping constraints and integer requirements. P denotes the probability distribution of ξ on $\mathbb{R}^{2T}$.

We assume that the centered stochastic load-price process $\{\bar{\xi}_t = (\bar{\xi}_{\delta,t}, \bar{\xi}_{c,t})\}_{t=1}^{T}$ may be modeled as linear multivariate time series ARMA(p,q)

$$\bar{\xi}_t + \sum_{i=1}^{p} A_i \bar{\xi}_{t-i} = \sum_{i=0}^{q} B_i \eta_{t-i}\,,\ t = 1, \ldots, T,$$

with independent standard normal innovations η_t, $t = 1, \ldots, T$, and suitable matrices A_i and B_i (Eichhorn-Römisch-Wegner 05).

Let m and Σ denote mean and covariance matrix of ξ, respectively.

Quasi-Monte Carlo methods

We consider the approximate computation of

$$I_d(g) = \int_{[0,1]^d} g(x)dx$$

by a Quasi-Monte Carlo (QMC) method

$$Q_n(g) = \frac{1}{n}\sum_{j=1}^{n} g(x^j)$$

with (deterministic) points x^j, $j = 1, \ldots, n$, from $[0,1]^d$.

There exist two main groups of QMC methods:

(Dick-Pillichshammer 10, Dick-Kuo-Sloan 13, Leobacher-Pillichshammer 14)

(1) Digital nets and sequences,

(2) Lattice rules.

Examples of digital sequences:

Sobol' sequences (Sobol' 67);

Faure sequences (Faure 82);

Niederreiter sequences (Niederreiter 87);

generalized Niederreiter sequences (Niederreiter 05)

Rank-1 lattices:

$$\left\{\frac{(j-1)}{n}\mathbf{g}\right\} \in [0,1]^d, \ j = 1, \ldots, n,$$

where $\mathbf{g} \in \mathbb{Z}_+^d$ is the generator of the lattice, the braces $\{\cdot\}$ mean taking componentwise the fractional part.

Classical convergence rate: $|Q_n(g) - I_d(g)| = O(n^{-1}(\log n)^d)$ if g has bounded variation on $[0,1]^d$ (in the Hardy and Krause sense).

Notice that the sequence $(n^{-1}(\log n)^d)$ increases until $n \leq \exp d$ and decreases only for $n > \exp d$.

Quasi-Monte Carlo methods often have good convergence properties if the integrands have low effective dimension.

Randomized QMC methods

A randomized version of a QMC point set has the properties that
(i) each point in the randomized point set has a uniform distribution over $[0,1)^d$ (uniformity),
(ii) the QMC properties are preserved under the randomization with probability one (equidistribution).

Randomization procedures for digital sequences, in particular, for Sobol' sequences, were first considered in (Owen 95). For an overview on randomization techniques see (L'Ecuyer-Lemieux 02, Dick-Pillichshammer 10).

The properties (i) and (ii) allow for error estimates and may lead to improved convergence properties compared to the original QMC method.

Examples of such techniques are
(a) random shifts of lattice rules,
(b) scrambling, i.e., random permutations of $\mathbb{Z}_b = \{0, 1, \ldots, b-1\}$ applied to the digits in b-adic representations,
(c) affine matrix scrambling which generates random digits by random linear transformations of the original digits, where the elements of all matrices and vectors are chosen randomly, independently and uniformly over $\mathbb{Z}_b$.

Weighted tensor product Sobolev spaces

$$\mathbb{G}_d = \mathcal{W}^{(1,\ldots,1)}_{2,\gamma,\mathrm{mix}}([0,1]^d) = \bigotimes_{i=1}^{d} W^1_{2,\gamma_i}([0,1]),$$

where $W^1_{2,\gamma_i}([0,1])$ is the Sobolev space of absolutely continuous functions h on $[0,1]$ with derivative $h' \in L_2([0,1])$. Its inner product is

$$\langle h, \tilde{h} \rangle = \Big(\int_0^1 h(t)dt \Big) \Big(\int_0^1 \tilde{h}(t)dt \Big) + \gamma_i^{-1} \int_0^1 h'(t)\tilde{h}'(t)dt\,.$$

The weighted norm $\|g\|_\gamma = \sqrt{\langle g, g \rangle_\gamma}$ and inner product of $\mathbb{G}_d$ are given by

$$\langle g, \tilde{g} \rangle_\gamma = \sum_{u \subseteq D} \gamma_u^{-1} \int_{[0,1]^{|u|}} I_u g(t^u) I_u \tilde{g}(t^u) dt^u,$$

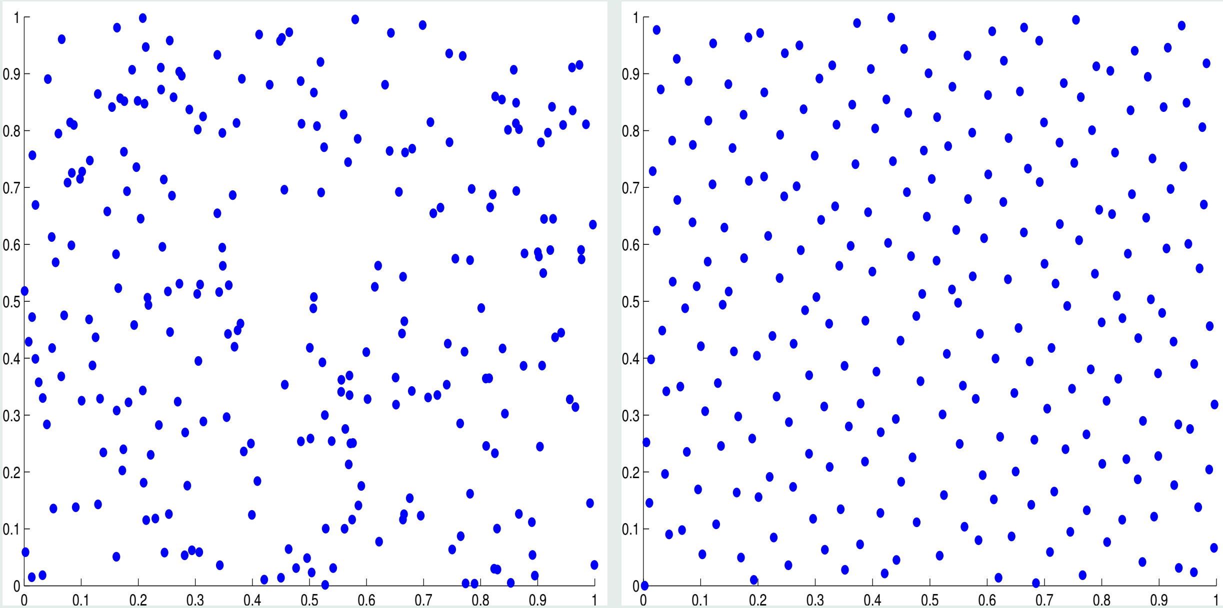
where $D = \{1, \ldots, d\}$, the weights γ_i are positive nonincreasing, and

$$I_u g(t^u) = \int_{[0,1]^{d-|u|}} \frac{\partial^{|u|}}{\partial t^u} g(t) dt^{-u} \quad \text{and} \quad \gamma_u = \prod_{i \in u} \gamma_i$$

for $u \subseteq D$, where $\gamma_\emptyset = 1$. For $u \subseteq D$ we use the notation $|u|$ for its cardinality, $-u$ for $D \setminus u$ and t^u for the $|u|$-dimensional vector with components t_j for $j \in u$. Moreover, $\mathbb{G}_d$ is a reproducing kernel Hilbert space with the kernel

$$K_{d,\gamma}(t,s) = \prod_{i=1}^{d} \big(1 + \gamma_i(0.5 B_2(|t_i - s_i|) + B_1(t_i)B_1(s_i))\big) \ (t, s \in [0,1]^d),$$

where $B_1(t) = t - \frac{1}{2}$ and $B_2(t) = t^2 - t + \frac{1}{6}$.



Comparison of $n = 2^7$ MC Mersenne Twister points and randomly binary shifted Sobol' points in dimension $d = 500$, projection onto the 8. and 9. components

Randomly scrambled Sobol' sequences admits the following convergence rate of the RMSE on $\mathcal{W}^{(1,\dots,1)}_{2,\gamma,\mathrm{mix}}([0,1]^d)$

$$\sup_{\|g\|_\gamma \leq 1} \sqrt{\mathbb{E}|Q_n(\omega)(g) - I_d(g)|^2} \leq C_d\, n^{-\frac{3}{2}} (\log n)^{\frac{d-1}{2}} .$$

(Dick-Pillichshammer 10, Chapter 13)

Usually a rate close to $O(n^{-1})$ is observable unless the sample sizes become huge.

Randomly shifted lattice rules

With a random vector $\triangle$ which is uniformly distributed on $[0,1]^d$, we consider the randomly shifted lattice rule

$$Q_n(\omega)(g) = \frac{1}{n}\sum_{j=1}^{n} g\Big(\Big\{\frac{(j-1)}{n}\mathbf{g} + \triangle(\omega)\Big\}\Big).$$

Theorem: Let $n \in \mathbb{N}$ be prime and $g \in \mathcal{W}_{2,\gamma,\mathrm{mix}}^{(1,\ldots,1)}([0,1]^d)$.
Then $\mathbf{g} \in \mathbb{Z}_+^d$ can be constructed componentwise such that for each $\delta \in (0, \frac{1}{2}]$ there exists a constant $C(\delta) > 0$ with

$$\sup_{\|g\|_\gamma \le 1} \sqrt{\mathbb{E}|Q_n(\omega)(g) - I_d(g)|^2} \le C(\delta)\, n^{-1+\delta},$$

where the constant $C(\delta)$ increases if δ decreases, but does not depend on d if the sequence (γ_j) satisfies

$$\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{2(1-\delta)}} < \infty \qquad (\text{e.g. } \gamma_j = \tfrac{1}{j^3}).$$

(Sloan-Kuo-Joe 02, Kuo 03, Nuyens-Cools 06)

Transformation of integrals for general densities ρ

We mostly consider functions $f : \mathbb{R}^d \to \mathbb{R}$ and computing the integral

$$\int_{\mathbb{R}^d} f(\xi)\rho(\xi)d\xi \,.$$

Step 1: Transformation of the multivariate density function ρ on $\mathbb{R}^d$ into a product-density $\rho(\xi) = \prod_{k=1}^{d} \rho_k(\xi_k)$ with d independent one-dimensional marginal densities ρ_k.

Example: If P is normal with mean 0 and nonsingular covariance matrix Σ, then for any matrix A with $\Sigma = A\,A^\top$ the density of $P \circ A$ has product form.

Step 2: Let ρ_k denote the independent marginal densities and ϕ_k the marginal distribution functions of the probability distribution P. With the transformations $x_k = \phi_k(\xi_k)$, $k = 1, \ldots, d$, one obtains

$$\int_{\mathbb{R}^d} f(\xi) \prod_{k=1}^{d} \rho_k(\xi_k) d\xi = \int_{[0,1]^d} f(\phi_1^{-1}(x_1), \ldots, \phi_d^{-1}(x_d)) dx_1 \cdots dx_d$$

ANOVA decomposition and effective dimension

We consider a multivariate function $f : \mathbb{R}^d \to \mathbb{R}$ and intend to compute the mean of $f(\xi)$, i.e.

$$\mathbb{E}[f(\xi)] = I_{d,\rho}(f) = \int_{\mathbb{R}^d} f(\xi_1, \dots, \xi_d)\rho(\xi_1, \dots, \xi_d) d\xi_1 \cdots d\xi_d \,,$$

where ξ is a d-dimensional random vector with density

$$\rho(\xi) = \prod_{k=1}^{d} \rho_k(\xi_k) \quad (\xi \in \mathbb{R}^d).$$

We are interested in a representation of f consisting of 2^d terms

$$f(\xi) = f_0 + \sum_{i=1}^{d} f_i(\xi_i) + \sum_{\substack{i,j=1 \\ i<j}}^{d} f_{ij}(\xi_i, \xi_j) + \cdots + f_{12\cdots d}(\xi_1, \dots, \xi_d).$$

The previous representation can be more compactly written as

$$(*) \qquad f(\xi) = \sum_{u \subseteq D} f_u(\xi^u) \,,$$

where $D = \{1, \dots, d\}$ and ξ^u contains only the components ξ_j with $j \in u$ and belongs to $\mathbb{R}^{|u|}$. Here, $|u|$ denotes the cardinality of u.

Next we make use of the space $L_{2,\rho}(\mathbb{R}^d)$ of all square integrable functions with inner product

$$\langle f, \tilde{f} \rangle_\rho = \int_{\mathbb{R}^d} f(\xi)\tilde{f}(\xi)\rho(\xi)d\xi \,.$$

A representation of the form $(*)$ of $f \in L_{2,\rho}(\mathbb{R}^d)$ is called ANOVA decomposition of f if

$$\int_{\mathbb{R}} f_u(\xi^u)\rho_k(\xi_k)d\xi_k = 0 \quad (\text{for all } k \in u \text{ and } u \subseteq D).$$

The ANOVA terms f_u, $\emptyset \neq u \subseteq D$, are orthogonal in $L_{2,\rho}(\mathbb{R}^d)$, i.e.

$$\langle f_u, f_v \rangle_\rho = \int_{\mathbb{R}^d} f_u(\xi) f_v(\xi)\rho(\xi)d\xi = 0 \quad \text{if and only if} \quad u \neq v,$$

The ANOVA terms f_u allow a representation in terms of (so-called) (ANOVA) projections, i.e.

$$(P_k f)(\xi) = \int_{-\infty}^{\infty} f(\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d)\rho_k(s)ds \; (\xi \in \mathbb{R}^d; k \in D)$$

and

$$P_u f = \Big(\prod_{k \in u} P_k\Big)(f) \quad (u \subseteq D).$$

Then it holds (Kuo-Sloan-Wasilkowski-Woźniakowski 10):

$$f_u = \Big(\prod_{j\in u}(I - P_j)\Big)P_{-u}(f) = P_{-u}(f) + \sum_{v\subsetneq u}(-1)^{|u|-|v|}P_{-v}(f)\,,$$

($-u$ bezeichnet das Komplement von u bzgl. D).

We consider the variances of f and f_u

$$\sigma^2(f) = \|f - I_{d,\rho}(f)\|_{2,\rho}^2 \quad \text{und} \quad \sigma_u^2(f) = \|f_u\|_{2,\rho}^2$$

and obtain

$$\sigma^2(f) = \|f\|_{L_2}^2 - (I_{d,\rho}(f))^2 = \sum_{\emptyset\neq u\subseteq D} \sigma_u^2(f)\,.$$

The quotients

$$\frac{\sigma_u^2(f)}{\sigma^2(f)} \qquad (u \subseteq D)$$

are called global sensitivity indices for the importance of the group ξ_j, $j \in u$, of variables of f. For small $\varepsilon \in (0,1)$ (e.g. $\varepsilon = 0.01$)

$$d_S(\varepsilon) = \min\Big\{s \in D : \sum_{|u|\leq s}\frac{\sigma_u^2(f)}{\sigma^2(f)} \geq 1 - \varepsilon\Big\}$$

is called effective (superposition) dimension of f.

The following estimate is valid

$$(+) \qquad \Big\| f - \sum_{|u| \le d_S(\varepsilon)} f_u \Big\|_{2,\rho} \le \sqrt{\varepsilon}\,\sigma(f)\,,$$

i.e., the function f is approximated by a truncated ANOVA decomposition which contains all ANOVA terms f_u such that $|u| \le d_S(\varepsilon)$.

If f is nonsmooth and the ANOVA terms f_u, $|u| \le d_S(\varepsilon)$, are smoother than f, the estimate $(+)$ means an approximate smoothing of f.

Unfortunately, the effective dimension is hardly computable in general, but an upper bound can be computed by finding the smallest $s \in D$ such that

$$\sum_{v \subseteq \{1,\dots,s\}} \sigma_v^2(f) \ge (1-\varepsilon)\sigma^2(f)\,.$$

This relies on a particular integral representation of the left-hand side, where the occuring integrals are computed approximately by means of Monte Carlo or Quasi-Monte Carlo methods based on large samples.

QMC error analysis

Assume $f \in \mathbb{F}_d = \bigotimes_{i=1}^d W^1_{2,\gamma_i,\psi_i}(\mathbb{R})$, where $W^1_{2,\gamma_i,\psi_i}(\mathbb{R})$ is the Sobolev space of functions $h \in L_{2,\rho_i}(\mathbb{R})$ which is absolutely continuous with derivative $h' \in L_{2,\psi_i}(\mathbb{R})$ and norm

$$\|h\|^2_{\gamma_i,\psi_i} = \Big(\int_{\mathbb{R}} h(\xi)\rho_i(\xi)d\xi\Big)^2 + \frac{1}{\gamma_i}\int_{\mathbb{R}} (h'(\xi)\psi_i(\xi))^2 d\xi\,.$$

The functions ψ_i, $i = 1, \ldots, d$, are selected such that the function $g = f(\phi_1^{-1}(\cdot), \ldots, \phi_d^{-1}(\cdot))$ belongs to $\mathbb{G}_d$ and $\mathbb{F}_d$ is a complete tensor product Sobolev space (Kuo-Sloan-Wasilkowski-Waterhouse 10, Nichols-Kuo 14).

The QMC error may be estimated as follows:

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} f(\xi)\rho(\xi)d\xi - n^{-1}\sum_{j=1}^n f(\xi^j)\Big| &= \Big|\int_{[0,1]^d} g(x)dx - n^{-1}\sum_{j=1}^n g(x^j)\Big| \\ &\le \sum_{0<|u|\le d}\Big|\int_{[0,1]^{|u|}} g_u(x^u)dx^u - n^{-1}\sum_{j=1}^n g_u(x^j)\Big|,\end{aligned}$$

where $x^j = (x_1^j, \ldots, x_d^j)$, $x_i^j = \varphi_i^{-1}(\xi_i^j) \in (0,1)^d$, $i = 1, \ldots, d$, $j = 1, \ldots, n$, are the QMC points.

If the points x^j, $j = 1, \ldots, n$, are randomly shifted lattice points, n is prime and $\delta \in (0, \frac{1}{2}]$, we may continue

$$\Big(\mathbb{E}\Big|\int_{[0,1]^d} g(x)dx - n^{-1}\sum_{j=1}^{n} g(x^j)\Big|^2\Big)^{\frac{1}{2}} \leq \; C(\delta)n^{-1+\delta} + \sum_{|u|>d_S(\varepsilon)}\Big(\mathbb{E}\Big|\int_{[0,1]^{|u|}} g_u(x^u)dx^u - n^{-1}\sum_{j=1}^{n} g_u(x^j)\Big|^2\Big)^{\frac{1}{2}} \leq \; C(\delta)n^{-1+\delta} + O(\sqrt{\varepsilon})$$

if the ANOVA terms g_u, $|u| \leq d_S(\varepsilon)$, belong to $\mathbb{G}_d$ and the sequence (γ_j) is selected properly.

The condition $g_u \in \mathbb{G}_d$ is satisfied if $f_u \in \mathbb{F}_d$.

Integrands of mixed-integer two-stage stochastic programs

$$\min\left\{\langle c,x\rangle+\int_{\mathbb{R}^d}\Phi(q(\xi),h(\xi)-V(\xi)x)\rho(\xi)d\xi : x\in X\right\},$$

$$\Phi(q,h):=\inf\left\{\langle q_1,y_1\rangle+\langle q_2,y_2\rangle : W_1y_1+W_2y_2\le h, y_1\in\mathbb{R}^{m_1}, y_2\in\mathbb{Z}^{m_2}\right\}$$

for all $(q,h)\in\mathbb{R}^{m_1+m_2}\times\mathbb{R}^r$, $c\in\mathbb{R}^m$, X is a closed subset of $\mathbb{R}^m$, W_1 and W_2 are (r,m_1) and (r,m_2)-matrices, $q(\xi)\in\mathbb{R}^{m_1+m_2}$, $h(\xi)\in\mathbb{R}^r$, and the (r,m)-matrix $V(\xi)$ are affine functions of $\xi\in\mathbb{R}^d$, and ρ is a probability density on $\mathbb{R}^d$.

Assumptions:

(B1) The matrices W_1 and W_2 have only rational elements.

(B2) For each pair $(x,\xi)\in X\times\mathbb{R}^d$ it holds that $h(\xi)-V(\xi)x\in\mathcal{T}$, where

$$\mathcal{T}:=\{t\in\mathbb{R}^r : \exists(y_1,y_2)\in\mathbb{R}^{m_1}\times\mathbb{Z}^{m_2} \text{ such that } W_1y_1+W_2y_2\le t\}.$$

(B3) For each $\xi\in\mathbb{R}^d$ the recourse cost $q(\xi)$ belongs to the dual feasible set

$$\mathcal{U}:=\left\{u=(u_1,u_2)\in\mathbb{R}^{m_1+m_2} : \exists v\in\mathbb{R}^r_- \text{ such that } W_1^\top v=u_1,\, W_2^\top v=u_2\right\}.$$

Proposition:

Assume (B1)–(B3). There exist at most countably many convex polyhedra B_i, $i\in N$, covering $\mathcal{T}$ with facets parallel to suitable facets of $\mathcal{K}=W_1(\mathbb{R}^{m_1})+\mathbb{R}^r_+$. The function Φ is finite and lower semicontinuous on $\mathcal{U}\times\mathcal{T}$, Φ is bilinear on each $\mathcal{U}\times B_i$ with possible kinks or discontinuities at certain facets of $\mathcal{U}\times B_i$.

Example: (Schultz-Stougie-van der Vlerk 98)

Stochastic multi-knapsack problem:

$\min \to \max$, $m = 2$, $m_1 = 0$, $m_2 = 4$, $d = s = 2$, $X = [-5, 5]^2$, $c = (1.5, 4)$, $h(\xi) = \xi$, $q(\xi) \equiv q = (16, 19, 23, 28)$, $y_i \in \{0, 1\}$, $i = 1, 2, 3, 4$, $P \sim U(\{5, 10, 15\}^2)$ (discrete)

$$V(\xi) \equiv V = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad W = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 1 & 3 & 2 \end{pmatrix}$$

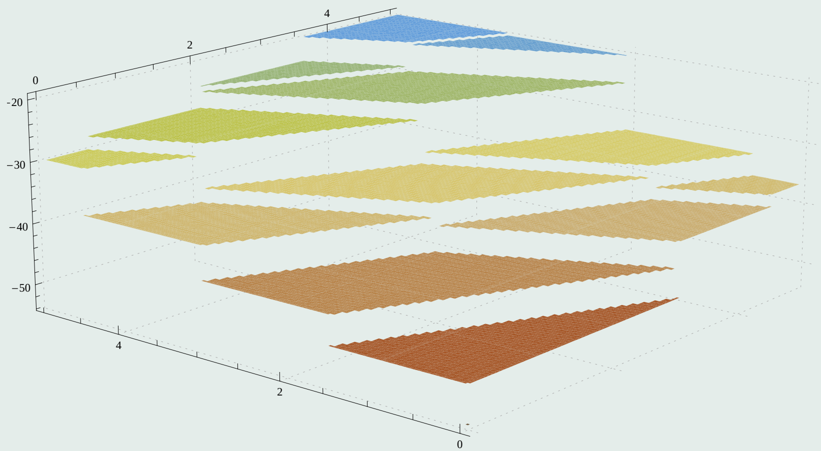


Illustration of the expected recourse function with discrete uniform probability distribution

ANOVA terms of mixed-integer two-stage integrands

Fix $x \in X$ and consider $f(\xi) = \langle c, x \rangle + \Phi(q(\xi), h(\xi) - Vx)$ which is linear-quadratic on $\Xi_i(x) = \{\xi \in \Xi : h(\xi) - Vx \in B_i\}$, $i \in N$.

Assumptions:

(B4) The density ρ has fourth order absolute moments.

(B5) $\rho(\xi) = \prod_{k=1}^{d} \rho_k(\xi_k)$ with $\rho_k \in C^d(\mathbb{R})$, $k = 1, \ldots, d$.

(B6) Let G be the finite set of vectors $g \in \mathbb{R}^d$ that generate the hyperplanes containing adjacent facets of the sets $\Xi_i(x)$, $i \in N$: For some $n \in \mathbb{N}$ and pairwise different $g_i \in G$, $i = 1, \ldots, n$, the matrix $(g_1 \cdots g_n)$ has rank $\min\{n, d\}$ (geometric condition of order n).

Theorem:

Assume (B1)–(B5) and the geometric condition of order 2 .

Then the ANOVA projections $P_u f$ of f belong to $C^{|u|-1}(\mathbb{R}^{d-|u|})$ and the ANOVA terms f_u have all mixed first derivatives in the sense of Sobolev if $1 \leq |u| \leq 2$.

Proposition: Let ρ be a multivariate normal density.

For almost every covariance matrix the geometric condition of order 2 is satisfied after principal component analysis factorization.

Numerical results

To generate RQMC samples for the load-price vector ξ with mean $m = \mathbb{E}[\xi] \in \mathbb{R}^{2T}$ and covariance matrix $\Sigma = \mathbb{E}[(\xi - m)(\xi - m)^\top]$ in our two-stage mixed-integer electricity portfolio optimization model, we first decompose Σ by a suitable matrix A such that $\Sigma = A\,A^\top$. In this way we obtain a standard normal random vector $z = (z_1, \ldots, z_{2T})^T$ such that

$$\xi = Az + m\,.$$

If ϕ denotes the standard normal distribution function, then the vector $\eta = (\eta_1, \ldots, \eta_{2T})^\top$ with $z_i = \phi^{-1}(\eta_i)$, $i = 1, \ldots, 2T$, is uniformly distributed in $[0,1]^{2T}$. We used the triangular Cholesky matrix $A = L_{\mathrm{Ch}}$ and the matrix $A = U_{\mathrm{PCA}}$ of the principal component analysis (PCA) factorization

$$U_{\mathrm{PCA}} = \left(\sqrt{\lambda_1}u_1 \cdots \sqrt{\lambda_d}u_d\right)$$

with the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and eigenvectors $u_1, \ldots, u_d$ of the covariance matrix Σ.

For our tests we used $T = 100$ and, hence, $d = 2T = 200$.

By computing the upper bounds of the effective dimension using 2^{15} randomly scrambled Sobol' points we obtained

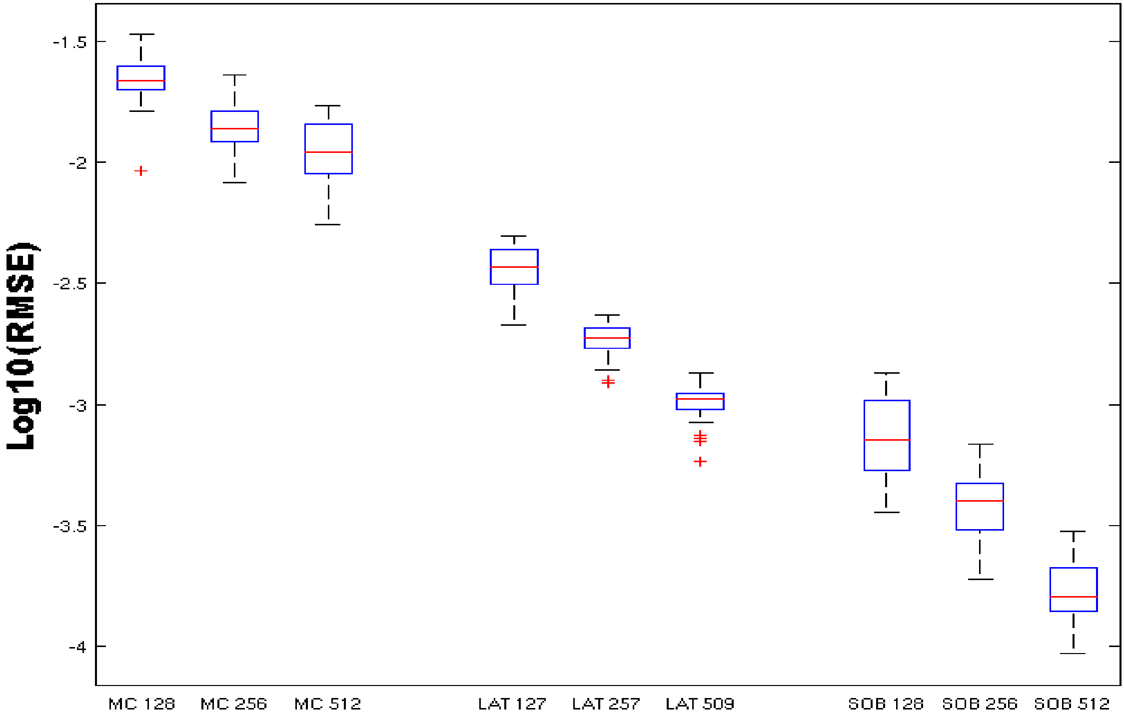
$$d_S(0.01) \leq 2 \quad \text{with PCA and} \quad 2 < d_S(0.01) \leq 200 \quad \text{with CH}\,.$$

Hence, principal component analysis leads to a strong reduction of the effective dimension.

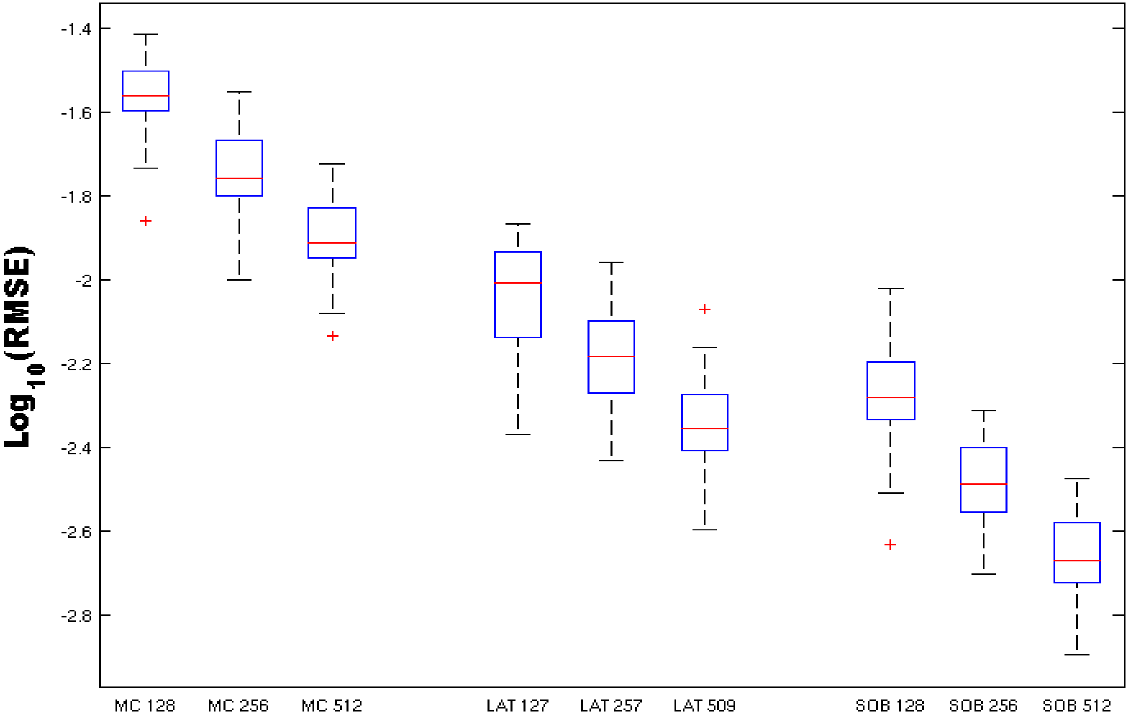
For the numerical tests n samples $\eta^j \in [0,1]^d$, $j = 1, \ldots, n$, of Mersenne Twister MC and of the two RQMC methods were generated and inserted after the transformations $z_i^j = \phi^{-1}(\eta_i^j)$, $i = 1, \ldots, 2T$, and $\xi^j = A\,z^j + m$, $j = 1, \ldots, n$, into

$$\min\Big\{\sum_{t=1}^{T}\langle c_t, x_t\rangle + \frac{1}{n}\sum_{j=1}^{n}\Phi(q(\xi^j), h(\xi^j) - Vx) : x \in X\Big\}\,.$$

For MC and randomly scrambled Sobol' points we used $n = 128, 256, 512$ and for randomly shifted lattice rules $n = 127, 257, 509$ (since prime numbers n are favorable for the latter). The Mersenne Twister was also used for the random scrambling and the random shifts.



Shown are the Log_{10} of relative RMSE for the optimal values of the two-stage model by using PCA factorization of the covariance matrix. Results for Mersenne Twister MC and randomly scrambled Sobol' QMC with $128, 256$ and 512 points and randomly shifted lattice rules QMC with $127, 257$ and 509 lattice points.



Shown are the Log_{10} of relative RMSE for the optimal values of the two-stage model by using CH factorization of the covariance matrix. Results for Mersenne Twister MC and randomly scrambled Sobol' QMC with $128, 256$ and 512 points and randomly shifted lattice rules QMC with $127, 257$ and 509 lattice points.

The relative root mean square error (RSME) of the optimal value of the mixed-integer linear two-stage model is estimated by realizing 10 runs of every experiment and repeat the process 30 times.
The lower and upper bounds of the boxes correspond to the first and third quartile and the line in between to the median. Outlier that do not belong to boxes are marked by plus signs.

The average convergence rates of three methods are -0.5 for MC, about -0.9 for randomly shifted lattice rules und -1.0 for randomly scrambled Sobol' points if PCA factorization is used.

An explanation for the much better behavior of both randomized QMC methods is the smoothing of integrands achieved by the low effective dimension due to the use of PCA.

All three methods showed only average convergence rate $-0,5$ if CH factorization is used. However, it is also visible that also under CH both randomized QMC methods lead to smaller errors than MC.

Conclusions

- Randomized Quasi-Monte Carlo methods are advantageous compared to MC methods also for integrands having kinks or even discontinuities at least in case of normal distributions and if the effective dimension of the integrand is low.
- Instead of 10^4 MC samples one only needs about 10^2 samples for randomly scrambled Sobol' point sets and randomly shifted lattice rules. The advantages consist in the improved accuracy for given sample size or in smaller running times for identical sample sizes.
- The presented results extend our earlier work, for example, in (Leövey-Römisch 15).

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