The role of information in multi-period risk measurement

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Abstract

Multi-period risk functionals assign a risk value to a discrete-time stochastic process $Y = (Y_1, \ldots, Y_T)$. While convexity and monotonicity properties extend in a natural way from the single-period case and several types of translation properties may be defined, the role of information becomes crucial in the multi-period situation. In this paper, we define multi-period functionals in a generic way, such that the available information (expressed as a filtration) enters explicitly the definition of the functional. This allows to study the information monotonicity property, which comes as the counterpart of value monotonicity. We discuss several ways of constructing concrete and computable functionals out of conditional risk mappings and single-period risk functionals. Some of them appear as value functions of multi-stage stochastic programs, where the filtration appears in the non-anticipativity constraint. This approach leads in a natural way to information monotonicity. The subclass of polyhedral multi-period risk functionals becomes important for their employment in practical dynamic decision making and risk management. On the other hand, several functionals described in literature are not information-monotone, which limits their practical use.

Key words: Risk functional, acceptability functional, multi-period, conditional risk mapping, average value-at-risk, dual representation, information monotonicity

1 Introduction

Measuring and managing risk of economic decisions has become increasingly important in practically all areas of economic activity, e.g., in finance, energy, transportation telecommunication, to mention a few. This has led to a systematic study of desirable properties of (statistical) functionals that allow to quantify risk. About ten years ago, the seminal work by Artzner, Delbaen, Eber and Heath [1] devoted to an axiomatic theory of (*coherent*) risk functionals has attracted considerable attention of researchers and practitioners. It initiated a thorough discussion and (partial) extension of the axioms to *convex* risk functionals (see [12, 15]) and led to the beginnings of a theory of risk measurement (see, e.g., the monograph [13], the survey [14], the collection [40] and the original work [19, 22, 17, 26, 32, 33]). In parallel to this *static* or *single-period* setting efforts were made to extend the theory to a *dynamic* setting when time evolves and information gets available. We refer to [2, 4, 5, 7, 16, 18, 25, 28, 29, 35, 41] addressing such an extension at different levels of generality.

In this paper, we work with functionals $\mathcal{A}$ such that $\rho := -\mathcal{A}$ is a convex *risk functional.* Such functionals $\mathcal{A}$ are also known as concave monetary utility functionals in literature. We prefer the name *acceptability functional* (as, e.g., in [35]) in order to avoid any conflict with classical utility functions (cf. [13]). Given a linear space $\mathcal{Y}$ of real random variables contained in $L_1(\Omega, \mathcal{F}, \mathbb{P})$ with the partial ordering "$\leq$" defined by $Y \leq \tilde{Y}$ $\mathbb{P}$-almost surely, a functional $\mathcal{A} : \mathcal{Y} \to \overline{\mathbb{R}}$ is called acceptability functional if it is proper (i.e., $\mathcal{A}(Y) < +\infty$ for all $Y \in \mathcal{Y}$ and $\mathcal{A}(Y) > -\infty$ for some $Y \in \mathcal{Y}$) and satisfies the following axioms for all $Y,\, Y' \in \mathcal{Y}$, $c \in \mathbb{R}$, $\lambda \in [0, 1]$:

(A1) *Translation-equivariance.* $\mathcal{A}(Y + c) = \mathcal{A}(Y) + c$,

(A2) *Concavity.* $\mathcal{A}(\lambda Y + (1 - \lambda)Y') \geq \lambda\mathcal{A}(Y) + (1 - \lambda)\mathcal{A}(Y')$,

(A3) *Monotonocity.* $Y \leq Y'$ implies $\mathcal{A}(Y) \leq \mathcal{A}(Y')$.

In this paper, we deal with unbounded random variables and use $\mathcal{Y} = L_p(\Omega, \mathcal{F}, \mathbb{P}) = L_p(\mathcal{F})$ $(1 \leq p < \infty)$ by employing its Banach space and Banach lattice properties. Dual representation results of acceptability functionals $\mathcal{A}$ are immediate consequences of the Fenchel-Moreau theorem [31, Theorem 5] if $\mathcal{A}$ is upper semicontinuous. While in case $p = \infty$ a functional $\mathcal{A}$ satisfying (A1)–(A3) is always finite and continuous, the situation is different for $1 \leq p < \infty$ (cf. [27, Example 2.28]). *Version-independent* or law invariant acceptability functionals $\mathcal{A}$ (i.e. those satisfying $\mathcal{A}(Y) = \mathcal{A}(Y')$, if the probability distributions of Y and Y' coincide) are particularly important due to their specific properties and Kusuoka representation (cf. [22, 17, 20]).

A basic role will be played by the *average value-at-risk* $\mathbb{AV@R}$ at level $\alpha \in (0, 1]$, which we define as

$$\mathbb{AV@R}_\alpha(Y) := \frac{1}{\alpha}\int_0^\alpha G_Y^{-1}(u)\, du,$$

where $G_Y^{-1}(\alpha) = \inf\{u \in \mathbb{R} : G_Y(u) \geq \alpha\}$ is the quantile and $\mathbb{V@R}_\alpha(Y) = -G_Y^{-1}(\alpha)$ the value-at-risk at level α. G_Y denotes the distribution function of $Y \in L_1(\mathcal{F})$. $\mathbb{AV@R}_\alpha$ satisfies the conditions (A1)–(A3). Notice that $\mathbb{AV@R}_\alpha$ is just the negative value of the average value-at-risk as defined in [13].

When extending this single-period setting to a multi-period situation, the functional $\mathcal{A}$ should be given on a linear space $\mathcal{Y}$ of (discrete time) stochastic processes $Y = (Y_1, \ldots, Y_T)$

defined on some some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, but should also depend on the information evolving over time given by a filtration $\mathfrak{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$ and Y_t is measurable with respect to $\mathcal{F}_t$ for every $t = 1, \ldots, T$. Here Y_t is interpreted as the value of a risky cash-flow at time t, which is already discounted to present time by deterministic or stochastic discount. Typically $\mathcal{F}_t$, the information obtainable at time t, is (much) larger than the σ-field generated by $(Y_1, \ldots, Y_t)$. This becomes evident if one considers *final* processes, i.e. processes of the form $(0, \ldots, 0, Y_T)$. Here the generated σ-field is trivial up to time $T-1$, but the collected information may be much larger than that. For this reason, we define acceptability functionals $\mathcal{A}$ on pairs $(Y; \mathfrak{F})$ consisting of a process Y and a filtration $\mathfrak{F}$, such that Y is adapted to $\mathfrak{F}$.

In the literature emphasis is put on several notions of time consistency for dynamic acceptability or risk functionals (see [2, 4, 6, 9, 21, 29, 41]). For the mainly suggested notion it has been shown recently in [21] that the class of dynamic risk functionals satisfying law invariance and time consistency shrinks to the entropic one (see also Example 4.7).

In this paper, we put the emphasis on information-consistency, which compares different levels of available information, and extend the corresponding material presented in the monograph [27]. The central role is played by the notion of information-monotonicity, which is the property that $\mathfrak{F} \subseteq \mathfrak{F}'$ (meaning of course that $\mathcal{F}_t \subseteq \mathcal{F}'_t$ for all t), must imply that $\mathcal{A}(Y; \mathfrak{F}) \leq \mathcal{A}(Y; \mathfrak{F}')$. This requirement seems rather fundamental, yet it is violated in many cases. A violation leads to the paradoxical situation that the same process Y is acceptable for some information pattern, but not acceptable, if more information is available. To put it the other way round: ignoring available information could make a process acceptable.

We argue that an important setup for getting information-monotone and computable acceptability functionals is by formulating a stochastic optimization program describing some optimal risk management problem and defining the acceptability as the optimal value under the given information constraint.

This remark is not the only relation to stochastic optimization. Managing the risk of economic decisions is often based on suitable optimization models by incorporating appropriate risk functionals into the objective function or the constraints. Then the optimal solution corresponds to a decision with minimal or (properly) bounded risk. The incorporation of risk functionals into optimization models may require additional properties in addition to convexity. Such properties may be caused by complexity and computational requirements. For example, since risk functionals are nonlinear in most relevant cases, the original structure and numerical tractability of the optimization models might get lost after the incorporation of risk functionals. In this way, linear or mixed-integer linear programs might become nonlinear or mixed-integer nonlinear. For example, such an effect occurs if the entropic risk measure or alike is employed in mixed-integer linear programs. These and related optimization issues are discussed in [10, 27, 32, 37] (see also Section 4.3).

As for notation, we use the symbol $Y_1 \lhd \mathcal{F}_1$ to indicate that the random variable Y is measurable w.r.t. the σ-algebra $\mathcal{F}$. We use also the same symbol $Y \lhd \mathfrak{F}$ to indicate that the stochastic process $Y = (Y_1, \ldots, Y_T)$ is adapted to the filtration $\mathfrak{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_T)$.

Our paper is organized as follows. In Section 2 we review conditional acceptability and risk functionals, their dual representation and properties, and provide some examples. In Section 3, we discuss axioms for multi-period acceptability and risk functionals and provide dual representations. The final Section 4 is devoted to several approaches for constructing multi-period acceptability functionals, namely, scalarization, separable constructions and composition. Furthermore, we discuss the *polyhedrality* of multi-period risk functionals as important property for employing risk functionals in multi-period decision making and risk management. Roughly speaking, polyhedral multi-period risk functionals, although being nonlinear, maintain the linearity of optimization models by introducing additional variables and linear constraints. They are information-monotone and enjoy further favorable features for computations.

2 Conditional acceptability and risk mappings

Let $\mathcal{F}_1$ be any σ-field contained in $\mathcal{F}$. Let $\mathcal{Y}_1 = L_p(\mathcal{F}_1)$, $p \in [1, +\infty)$, hence $\mathcal{Y}_1 \subseteq \mathcal{Y}$. All (in)equalities between random variables in $\mathcal{Y}$ are intended to hold $\mathbb{P}$-almost surely.

A mapping $\mathcal{A}_{\mathcal{F}_1} : \mathcal{Y} \to \mathcal{Y}_1$ is called *conditional acceptability mapping* (with observable information $\mathcal{F}_1$) if the following conditions are satisfied for all Y, $\tilde{Y} \in \mathcal{Y}$, $Y_1 \in \mathcal{Y}_1$ and $\lambda \in [0, 1]$:

(CA1) **Predictable translation-equivariance.** $\mathcal{A}_{\mathcal{F}_1}(Y + Y_1) = \mathcal{A}_{\mathcal{F}_1}(Y) + Y_1$,

(CA2) **Concavity.** $\mathcal{A}_{\mathcal{F}_1}(\lambda Y + (1-\lambda)\tilde{Y}) \geq \lambda \mathcal{A}_{\mathcal{F}_1}(Y) + (1-\lambda)\mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$,

(CA3) **Monotonicity.** $Y \leq \tilde{Y}$ implies $\mathcal{A}_{\mathcal{F}_1}(Y) \leq \mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$.

We will also use the notation $\mathcal{A}(\cdot\,|\mathcal{F}_1)$ instead of $\mathcal{A}_{\mathcal{F}_1}$. The mapping $\rho = \rho_{\mathcal{F}_1} := -\mathcal{A}_{\mathcal{F}_1}$ is called *conditional risk mapping* (with observable information $\mathcal{F}_1$).
A conditional acceptability mapping $\mathcal{A}_{\mathcal{F}_1}$ is called *positively homogeneous* if $\mathcal{A}_{\mathcal{F}_1}(\lambda Y) = \lambda \mathcal{A}_{\mathcal{F}_1}(Y)$, $\forall \lambda \geq 0$. It is called *upper semicontinuous (u.s.c.)* if the mapping $Y \mapsto \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B)$ from $\mathcal{Y}$ to $\mathbb{R}$ is upper semicontinuous for every $B \in \mathcal{F}_1$.

Conditions (CA1)–(CA3) are also required in [38], but often the stronger property is imposed, namely that the concavity equation holds for all $\mathcal{F}_1$ measurable λ (e.g. in [6, 9]). We do not need this stronger condition here. However, we introduce information monotonicity as another important property. To do so, assume that conditional acceptability mappings $\mathcal{A}_{\mathcal{F}'}$ are defined for all sub σ-algebras $\mathcal{F}'$ of $\mathcal{F}$.

(CA0) **Information-monotonicity.** For all σ-fields $\mathcal{F}_1 \subseteq \mathcal{F}_1' \subseteq \mathcal{F}$

$$\mathbb{E}[\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B] \leq \mathbb{E}[\mathcal{A}_{\mathcal{F}_1'}(Y)\mathbb{1}_B] \quad \text{for all } B \in \mathcal{F}_1.$$

Condition (CA0) becomes important in Section 4. It can be reformulated equivalently due to the following Remark.

Remark 2.1 *Information monotonicity can also be rephrased as: $\mathcal{F}_1 \subseteq \mathcal{F}_1'$ implies that*

$$\mathcal{A}(Y|\mathcal{F}_1) \leq \mathbb{E}[\mathcal{A}(Y|\mathcal{F}_1')|\mathcal{F}_1] \quad a.s.$$

and this condition is also known under the name of compound convexity (see [27], Definition 2.11 and Proposition 2.12). It is also equivalent to the property that for all filtrations $\mathfrak{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$, the process $\mathcal{A}(Y|\mathcal{F}_t)$ is a submartingale w.r.t. $\mathfrak{F}$.

Next we provide characterization, representation and existence results for conditional acceptability mappings on L_p and discuss examples. We begin with a property that is established for bounded random variables in [9] under a stronger concavity condition, and also appears in [16]. The upper semicontinuity property is suitable to derive extensions for the unbounded situation.

Proposition 2.2 *Let $\mathcal{A}_{\mathcal{F}_1} : \mathcal{Y} \to \mathcal{Y}_1$ be an u.s.c. conditional acceptability mapping. Then one has*

$$\mathcal{A}_{\mathcal{F}_1}(Y\mathbb{1}_B) = \mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B \qquad \text{for all } Y \in \mathcal{Y}, B \in \mathcal{F}_1.$$

Proof. Suppose first that $|Y|$ is bounded by K. Then $-K\mathbb{1}_B \leq Y \leq K\mathbb{1}_B$ and the monotonicity (CA3) together with (CA1) implies the assertion, since

$$-K\mathbb{1}_B = \mathcal{A}_{\mathcal{F}_1}(-K\mathbb{1}_B) \leq \mathcal{A}_{\mathcal{F}_1}(Y) \leq \mathcal{A}_{\mathcal{F}_1}(K\mathbb{1}_B) = K\mathbb{1}_B\,.$$

Let now $Y_K = Y\mathbb{1}_{|Y|\leq K}$. Then $Y_K \to Y$ in L_p as $K \to \infty$. Let $C \in \mathcal{F}_1$ be such that $C \cap B = \emptyset$. By upper semicontinuity we obtain

$$0 = \limsup_{K\to\infty} \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y_K)\mathbb{1}_C) \leq \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_C) \tag{1}$$

and also

$$0 = \limsup_{K\to\infty} \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(-Y_K)\mathbb{1}_C) \leq \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(-Y)\mathbb{1}_C). \tag{2}$$

The concavity (CA2) implies that

$$0 = \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(\frac{1}{2}Y + \frac{1}{2}Y)\mathbb{1}_C) \geq \frac{1}{2}\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_C) + \frac{1}{2}\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(-Y)\mathbb{1}_C).$$

This, together with (1) and (2) implies that $\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_C) = 0$ for every $C \in \mathcal{F}_1$ with $C \cap B = \emptyset$, i.e., $\mathcal{A}_{\mathcal{F}_1}(Y)$ is zero outside B. □

Proposition 2.3 *(***Characterization***) A mapping $\mathcal{A}_{\mathcal{F}_1}$ is an u.s.c. conditional acceptability mapping if and only if the functional*

$$Y \mapsto \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) \tag{3}$$

is a finite u.s.c. acceptability functional for every $B \in \mathcal{F}_1$, which satisfies the property

$$\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y+Y_1)\mathbb{1}_B) = \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) + \mathbb{E}(Y_1\mathbb{1}_B) \qquad \text{for all } Y_1 \in L_p(\mathcal{F}_1). \tag{4}$$

Proof. The only if direction is clear due to Proposition 2.2. Assume that for all $B \in \mathcal{F}_1$ the functional (3) satisfies (A2), (A3) and the property (4). Let $B \in \mathcal{F}_1$, $Y \in \mathcal{Y}$ and $Y_1 \in \mathcal{Y}_1$. The latter property then means

$$0 = \mathbb{E}[(\mathcal{A}_{\mathcal{F}_1}(Y+Y_1)\mathbb{1}_B] - \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) - \mathbb{E}(Y_1\mathbb{1}_B) = \mathbb{E}[(\mathcal{A}_{\mathcal{F}_1}(Y+Y_1) - \mathcal{A}_{\mathcal{F}_1}(Y) - Y_1)\mathbb{1}_B].$$

Since this holds for every $B \in \mathcal{F}_1$, the element $\mathcal{A}_{\mathcal{F}_1}(Y+Y_1) - \mathcal{A}_{\mathcal{F}_1}(Y) - Y_1$ must itself be a.s. zero, since it is $\mathcal{F}_1$ measurable. Hence, (CA1) is fulfilled. Further, let $\lambda \in [0,1]$, $\tilde{Y} \in \mathcal{Y}$ and B be the set on which

$$\mathcal{A}_{\mathcal{F}_1}(\lambda Y + (1-\lambda)\tilde{Y}) < \lambda \mathcal{A}_{\mathcal{F}_1}(Y) + (1-\lambda)\mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$$

holds. Then B belongs to $\mathcal{F}_1$. If $P(B) > 0$, then taking the expectation on B of this relation leads to a contradiction. Thus, (CA2) is fulfilled. Finally, let $Y \leq \tilde{Y}$ and let B be the set where $\mathcal{A}_{\mathcal{F}_1}(Y) > \mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$. Again, taking the expectation on B leads to a contradiction and the proof is complete. □

The next representation result of u.s.c. conditional acceptability mappings is a slight extension of [27, Theorem 2.51]. To state it, we use the notion of the infimum in Banach lattices. Since the L_p-spaces ($1 \leq p \leq \infty$) are order complete Banach lattices, every nonempty collection $\mathcal{G}$ of L_p-functions, which is bounded from below has an infimum (see, e.g., [39]). Some authors call this infimum the *essential infimum*. The infimum is a measurable function $\underline{Y} = \mathbf{inf}\{Y : Y \in \mathcal{G}\}$ such that (i) $\underline{Y} \leq Y$ a.s. for all $Y \in \mathcal{Y}$ and (ii) if $Z \leq Y$ a.s. for all $Y \in \mathcal{Y}$, then $Z \leq \underline{Y}$ a.s. The infimum $\underline{Y}$ may also be characterized in a different way: Let $\underline{\mathcal{G}}$ be the collection of all minima of finitely many functions from $\mathcal{G}$. Then $\mathbf{inf}\{Y : Y \in \mathcal{G}\}$ is the uniquely determined measurable function $\underline{Y}$ such that for all measurable B

$$\mathbb{E}(\underline{Y}\mathbb{1}_B) = \inf\{\mathbb{E}(Y\mathbb{1}_B) : Y \in \underline{\mathcal{G}}\}.$$

Theorem 2.4 *(***Representation***) Let $\mathcal{A}_{\mathcal{F}_1} : \mathcal{Y} \to \mathcal{Y}_1$ be an u.s.c. conditional acceptability mapping. Then the representation*

$$\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) = \inf\{\mathbb{E}(Y\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) : Z \geq 0, \mathbb{E}(Z|\mathcal{F}_1) = 1, Z \in \mathcal{Z}(\mathcal{F}_1)\}$$

or, equivalently,

$$\mathcal{A}_{\mathcal{F}_1}(Y) = \mathbf{inf}\{\mathbb{E}(YZ|\mathcal{F}_1) - \theta_{\mathcal{F}_1}(Z) : Z \geq 0, \mathbb{E}(Z|\mathcal{F}_1) = 1, Z \in \mathcal{Z}(\mathcal{F}_1)\}$$

is valid for every pair $(Y, B) \in \mathcal{Y} \times \mathcal{F}_1$, where $\mathcal{Z}(\mathcal{F}_1)$ is a closed convex subset of $L_q(\mathcal{F})$, $\theta_{\mathcal{F}_1}$ a concave mapping from $L_q(\mathcal{F})$ to $L_q(\mathcal{F}_1)$ and $q \in (1, +\infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $B \in \mathcal{F}_1$ and the functional $\mathcal{A}_B$ on $\mathcal{Y}$ be defined by $\mathcal{A}_B(Y) := \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B)$. Since $\mathcal{A}_B$ is an u.s.c. acceptability functional according to Proposition 2.3, it satisfies (A2) and, thus the representation

$$\begin{aligned} \mathcal{A}_B(Y) &= \inf\left\{\mathbb{E}(YZ) - \mathcal{A}_B^+(Z) : Z \in L_q(\mathcal{F})\right\} \\ \mathcal{A}_B^+(Z) &= \inf\left\{\mathbb{E}(YZ) - \mathcal{A}_B(Y) : Y \in L_p(\mathcal{F})\right\} \end{aligned}$$

holds due to the Fenchel-Moreau theorem and $\mathcal{A}_B^+$ is proper, u.s.c. and concave. Since $\mathcal{A}_B$ is finite on $\mathcal{Y}$, it is continuous on $\mathcal{Y}$ (see, e.g., [3, Proposition 2.111]). Hence, the set (of supergradients of $\mathcal{A}_B$ at Y)

$$\mathcal{S}_B(Y) := \arg\min\left\{\mathbb{E}(YZ) - \mathcal{A}_B^+(Z) : Z \in L_q(\mathcal{F})\right\}$$

is nonempty, closed and convex for every $Y \in \mathcal{Y}$ (e.g., [3, Proposition 2.126]) and it holds that

$$\mathcal{A}_B(Y) = \mathbb{E}(YZ) - \mathcal{A}_B^+(Z) \qquad \text{for all } Z \in \mathcal{S}_B(Y).$$

Since $\mathcal{A}_B(Y) = \mathcal{A}_B(Y\mathbb{1}_B)$, the conjugate $\mathcal{A}_B^+$ satisfies

$$\mathcal{A}_B^+(Z) \;=\; \inf\{\mathbb{E}(ZY\mathbb{1}_B) + \mathbb{E}(ZY\mathbb{1}_{B^c}) - \mathcal{A}_B(Y\mathbb{1}_B) : Y \in L_p(\mathcal{F}_1)\}.$$

One sees that $\mathcal{A}_B^+(Z) = -\infty$, if $\mathbb{E}(|Z|\mathbb{1}_{B^c}) > 0$, where B^c is the complement of B. Thus

$$\begin{aligned} \mathcal{A}_B^+(Z) &= \inf\{\mathbb{E}(ZY\mathbb{1}_B) - \mathcal{A}_B(Y\mathbb{1}_B) : Y \in L_p(\mathcal{F}_1)\} \\ &= \mathcal{A}_B^+(Z\mathbb{1}_B). \end{aligned}$$

Let now $(B_i), 1 \le i < \infty$ be a pairwise disjoint sequence of sets in $\mathcal{F}_1$. Then

$$\begin{aligned} \mathcal{A}_{\cup_i B_i}^+(Z) &= \inf\{\mathbb{E}(ZY\mathbb{1}_{\cup_i B_i}) - \mathcal{A}_B(Y\mathbb{1}_{\cup_i B_i}) : Y \in L_p(\mathcal{F}_1)\} \\ &= \inf\{\sum_i \mathbb{E}(ZY\mathbb{1}_{B_i}) - \sum_i \mathcal{A}_{B_i}(Y\mathbb{1}_{B_i}) : Y \in L_p(\mathcal{F}_1)\} \\ &= \sum_i \inf\{\mathbb{E}(ZY\mathbb{1}_{B_i}) - \mathcal{A}_{B_i}(Y\mathbb{1}_{B_i}) : Y \in L_p(\mathcal{F}_1)\} = \sum_i \mathcal{A}_{B_i}^+(Z). \end{aligned}$$

Let $\mathcal{Z} = \mathcal{Z}(\mathcal{F}_1) = \{Z : \mathcal{A}_B^+(Z) > -\infty \text{ for all } B \text{ with } P(B) > 0\}$ and let, for fixed $Z \in \mathcal{Z}$, $\mu_Z(B) = \mathcal{A}_B^+(Z)$. μ_Z is a σ-additive signed measure on $\mathcal{F}_1$ dominated by $\mathbb{P}$. By the Radon-Nikodym Theorem there is a $\mathcal{F}_1$-measurable function denoted by $\theta_{\mathcal{F}_1}(Z)$ such that $\mathcal{A}_B^+(Z) = \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B)$ for all $B \in \mathcal{F}_1$. More precisely, $\theta_{\mathcal{F}_1}$ is a mapping from $L_q(\mathcal{F})$ to $L_q(\mathcal{F}_1)$ and its concavity follows from that of $\mathcal{A}_B^+$. The fact that for $Z \in \mathcal{Z}$, $Z \ge 0$ a.s. and $\mathbb{E}(Z|\mathcal{F}_1) = 1$ a.s. follows as in [27, Theorem 2.30]. □

Theorem 2.5 (**Existence**) *Let $\mathcal{Z}$ be a closed convex subset of $L_q(\mathcal{F})$ where $q \in (1, +\infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $Z \mapsto \theta_{\mathcal{F}_1}(Z)$ be a mapping from $L_q(\mathcal{F})$ to $L_q(\mathcal{F}_1)$. Then the equations*

$$\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) = \inf\{\mathbb{E}(Y\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) : Z \ge 0, \mathbb{E}(Z|\mathcal{F}_1) = 1, Z \in \mathcal{Z}\} \quad (B \in \mathcal{F}_1) \tag{5}$$

define an upper semicontinuous conditional acceptability mapping $\mathcal{A}_{\mathcal{F}_1} : L_p(\mathcal{F}) \to L_p(\mathcal{F}_1)$ if the infima in (5) are finite for every $Y \in \mathcal{Y}$ and $B \in \mathcal{F}_1$.

Proof. Let $B \in \mathcal{F}_1$. We show that the functional $Y \mapsto \mathcal{A}_B(Y) := \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B)$ given by (5) satisfies concavity (A2), monotonicity (A3), property (4) and upper semicontinuity on $\mathcal{Y}$. For each $Y \in \mathcal{Y}$, we have according to (5)

$$\mathcal{A}_B(Y) = \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) \le \mathbb{E}(Y\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B)$$

for every feasible $Z \in \mathcal{Z}$ such that $Z \ge 0$ and $\mathbb{E}(Z|\mathcal{F}_1) = 1$. If $Y \le \tilde{Y}$ and if (Y_n) is a sequence converging to Y in $\mathcal{Y}$, we obtain

$$\begin{aligned} \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) &\le \mathbb{E}(\tilde{Y}\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) \\ \lim_{n\to\infty} \sup \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y_n)\mathbb{1}_B) &\le \mathbb{E}(Y\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) \end{aligned}$$

for every $Z \in \mathcal{Z}$ with $Z \ge 0$. This implies

$$\begin{aligned} \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) &\le \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(\tilde{Y})\mathbb{1}_B) \\ \lim_{n\to\infty} \sup \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y_n)\mathbb{1}_B) &\le \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B). \end{aligned}$$

Hence, (A3) and upper semicontinuity are shown. Now, let $Y, \tilde{Y} \in \mathcal{Y}$, $\lambda \in [0,1]$. Then we have

$$\begin{aligned} \lambda\mathcal{A}_B(Y) + (1-\lambda)\mathcal{A}_B(\tilde{Y}) &\le \lambda\mathbb{E}(Y\ Z\ \mathbb{1}_B) + (1-\lambda)\lambda\mathbb{E}(\tilde{Y}\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) \\ &= \mathbb{E}[(\lambda Y + (1-\lambda)\tilde{Y})Z\ \mathbb{1}_B] - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) \end{aligned}$$

for every feasible $Z \in \mathcal{Z}$. This implies (A2). Finally, we obtain for $Y^{(1)} \in \mathcal{Y}_1$ and any feasible $Z \in \mathcal{Z}$

$$\begin{aligned} \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y + Y^{(1)})\mathbb{1}_B) &\le \mathbb{E}((Y + Y^{(1)})Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) \\ &= \mathbb{E}(Y\ Z\ \mathbb{1}_B) + \mathbb{E}(Y^{(1)}\mathbb{E}(Z|\mathcal{F}_1)\mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) \\ &= \mathbb{E}(Y\ Z\ \mathbb{1}_B) - \mathbb{E}(\theta_{\mathcal{F}_1}(Z)\mathbb{1}_B) + \mathbb{E}(Y^{(1)}\mathbb{1}_B) \end{aligned}$$

and, hence, property (4) by taking the infimum with respect to Z. The assertion now follows from Proposition 2.3. □

Similar as for convex functions on linear normed spaces, (cone-) convex (or concave) mappings enjoy continuity properties (see the survey [24]). In particular, the following result holds for conditional acceptability mappings (see [24, Theorem 4]).

Proposition 2.6 *(***Continuity***) A conditional acceptability mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}_1$ is continuous if it is locally bounded at some element of $\mathcal{Y}$.*

Example 2.7 *(*Conditional average value-at-risk*)*
For $Y \in L_1$, $\mathbb{A}V@R_\alpha(Y|\mathcal{F}_1)$ is given on $L_1(\mathcal{F})$ by the relation

$$\mathbb{E}(\mathbb{A}V@R_\alpha(Y|\mathcal{F}_1)\mathbb{1}_B) = \inf\left\{\mathbb{E}(Y\ Z\ \mathbb{1}_B) : 0 \le Z \le \frac{1}{\alpha}, \mathbb{E}(Z|\mathcal{F}_1) = 1\right\} \quad (B \in \mathcal{F}_1). \quad (6)$$

or, equivalently, by

$$\mathbb{A}V@R_\alpha(Y\,|\mathcal{F}_1) = \mathbf{inf}\left\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : 0 \le Z \le \frac{1}{\alpha}, \mathbb{E}(Z|\mathcal{F}_1) = 1\right\}.$$

Proposition 2.8 *The conditional average value-at-risk $\mathbb{A}V@R_\alpha(\cdot\,|\mathcal{F}_1)$ is a.s. well defined. It is a conditional acceptability mapping which is a continuous mapping from $L_1(\mathcal{F})$ to $L_1(\mathcal{F}_1)$. There is a version of this mapping such that $\alpha \mapsto \mathbb{A}V@R_\alpha(Y\,|\mathcal{F}_1)$ is monotonically increasing a.s. for $\alpha \in (0,1]$.*

Proof. By Theorem 3.3, the conditional $\mathbb{A}$V@R is a.s. well defined since the infima in (6) are finite due to the estimate

$$|\mathbb{E}(\mathbb{A}\mathrm{V@R}_\alpha(Y\,|\mathcal{F}_1)\mathbb{1}_B)| \le \frac{1}{\alpha}\mathbb{E}(|Y|\mathbb{1}_B) \le \frac{1}{\alpha}\mathbb{E}(|Y|) \tag{7}$$

for every $B \in \mathcal{F}_1$ and $Y \in \mathcal{Y}$. The estimate (7) also implies that $\mathbb{A}\mathrm{V@R}_\alpha(0\,|\mathcal{F}_1) = 0$. Since

$$|\mathbb{A}\mathrm{V@R}_\alpha(Y\,|\mathcal{F}_1)| = \mathbb{A}\mathrm{V@R}_\alpha(Y|\mathcal{F}_1)\mathbb{1}_{\mathbb{A}\mathrm{V@R}_{\alpha(Y|\mathcal{F}_1)\ge 0}} + (-\mathbb{A}\mathrm{V@R}_\alpha(Y|\mathcal{F}_1))\mathbb{1}_{\mathbb{A}\mathrm{V@R}_{\alpha(Y|\mathcal{F}_1)<0}},$$

we obtain

$$E(|\mathbb{A}\mathrm{V@R}_\alpha(Y\,|\mathcal{F}_1)|) \le \frac{2}{\alpha}\mathbb{E}(|Y|)$$

and, thus, $\mathbb{A}\mathrm{V@R}_\alpha(\cdot\,|\mathcal{F}_1)$ is locally bounded at 0. Proposition 2.6 implies its continuity. Now, let $\alpha < \beta$ and let $B = \{\omega|\mathbb{A}\mathrm{V@R}_\alpha(Y\,|\mathcal{F}_1)(\omega) > \mathbb{A}\mathrm{V@R}_\beta(Y\,|\mathcal{F}_1)(\omega)\}$. Suppose that $P(B) > 0$. Then

$$\begin{aligned} 0 &< \mathbb{E}([\mathbb{A}\mathrm{V@R}_\alpha(Y\,|\mathcal{F}_1) - \mathbb{A}\mathrm{V@R}_\beta(Y\,|\mathcal{F}_1)]\mathbb{1}_B) \\ &= \inf\{\mathbb{E}(YZ) : 0 \le Z \le \frac{1}{\alpha}\mathbb{1}_B, \mathbb{E}(Z|\mathcal{F}_1) = \mathbb{1}_B\} \\ &\quad - \inf\{\mathbb{E}(YZ) : 0 \le Z \le \frac{1}{\beta}\mathbb{1}_B, \mathbb{E}(Z|\mathcal{F}_1) = \mathbb{1}_B\} \le 0, \end{aligned}$$

which is a contradiction. By choosing a version for all rational α, which is monotonic in α a.s. and extending it by monotonicity to all real α's one may assume w.l.o.g. that almost surely

$$\alpha \mapsto \mathbb{A}\mathrm{V@R}_\alpha(Y|\mathcal{F}_1)$$

is monotonically increasing. □

By considering the trivial σ-algebra $\mathcal{F}_0 = (\emptyset, \Omega)$ one may specialize every conditional acceptability mapping to an ordinary acceptability measure. Conversely, one may lift version-independent acceptability functionals to conditional acceptability mappings. The assumption that the acceptability measure is version-independent is crucial, since the conditional mappings are based on the conditional distributions. A "lifting" method for version-independent positively homogeneous acceptability functionals is as follows:

These functionals have a Kusuoka representation

$$\mathcal{A}(Y) = \inf\left\{\int_0^1 \mathbb{AV@R}_\alpha(Y)\, dM(\alpha) : M \in \mathcal{M}_0\right\}$$

where $\mathcal{M}_0$ is a countable set of probabilities on [0,1]. Its conditional version is

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\left\{\int_0^1 \mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)\, dM(\alpha) : M \in \mathcal{M}_0\right\} \tag{8}$$

Notice that $\mathcal{A}(Y|\mathcal{F}_1)$ is $\mathcal{F}_1$ measurable, since $\mathcal{M}_0$ is countable and $\alpha \mapsto \mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)$ can be chosen to be monotonic a.s. according to Proposition 2.8.

Next we present a collection of useful conditional acceptability functionals, which may serve as the building blocks for multi-period functionals. A general reference to these examples is [27, Section 2.5]. By specialization to the trivial σ-algebra $\mathcal{F}_0$, one gets some "classical" risk and acceptability measures. Since properties (CA1)-(CA3) are well known to be satisfied, we study only property (CA0) here.

Example 2.9 *(*Conditional acceptability functionals*)*
Let h be a convex, nonnegative function satisfying $h(0) = 0$ and let $h^*(v) = \sup\{uv - h(u) : u \in \mathbb{R}\}$ be its conjugate.

(a) $\mathcal{A}(Y|\mathcal{F}_1) = \mathbb{E}(Y|\mathcal{F}_1) - \mathbb{E}[h(Y - \mathbb{E}(Y|\mathcal{F}_1))]|\mathcal{F}_1)$
This functional has a dual form defined by

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) - \mathbf{inf}\{\mathbb{E}[h^*(Z - a)|\mathcal{F}_1] : a \lhd \mathcal{F}_1\} : \mathbb{E}(Z|\mathcal{F}_1) = 1\}.$$

Since for $B \in \mathcal{F}_1$

$$\mathbb{E}[\mathcal{A}(Y|\mathcal{F}_1)\,\mathbb{1}_B] = \inf\{\mathbb{E}(Y\,Z\,\mathbb{1}_B) - \inf\{\mathbb{E}[h^*(Z - a)\,\mathbb{1}_B] : a \lhd \mathcal{F}_1\} : \mathbb{E}(Z|\mathcal{F}_1) = 1\}$$

one sees that (CA0) is fulfilled.

(b) $\mathcal{A}(Y|\mathcal{F}_1) = \mathbb{E}(Y|\mathcal{F}_1) - \mathbf{inf}\{\mathbb{E}[h(Y - a)|\mathcal{F}_1] : a \lhd \mathcal{F}_1\}$
This functional has a dual form defined by

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) + \mathbb{E}(h^*(1 - Z)|\mathcal{F}_1) : \mathbb{E}(Z|\mathcal{F}_1) = 1\}.$$

Property (CA0) follows from

$$\mathbb{E}[\mathcal{A}(Y|\mathcal{F}_1)\,\mathbb{1}_B] = \mathbb{E}(Y\,\mathbb{1}_B) - \inf\{\mathbb{E}[h(Y - a)\,\mathbb{1}_B) : a \lhd \mathcal{F}_1\}.$$

(c) $\mathcal{A}(Y|\mathcal{F}_1) = \mathbb{E}(Y|\mathcal{F}_1) - M_h(Y - \mathbb{E}(Y|\mathcal{F}_1)|\mathcal{F}_1)$
where M_h *is the conditional generalized Minkowski gauge defined by*

$$M_h(Y|\mathcal{F}_1) = \mathbf{inf}\{a \geq 0 : \mathbb{E}[h(\frac{Y}{a})|\mathcal{F}_1] \leq h(1) : a \lhd \mathcal{F}_1\}.$$

The functional has a dual form defined by

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\, Z|\mathcal{F}_1) : \mathbb{E}(Z|\mathcal{F}_1) = 1, \inf_a \{D^*_{h^*}(Z-a)|\mathcal{F}_1\} \leq 1, a \lhd \mathcal{F}_1\},$$

where

$$-D^*_{h^*}(Z|\mathcal{F}_1) = \mathbf{inf}\{-\mathbb{E}(Z\, V|\mathcal{F}_1) : \mathbb{E}[h(V)|\mathcal{F}_1] \leq h(1).$$

By

$$\mathbb{E}[\mathcal{A}(Y|\mathcal{F}_1)\, \mathbb{1}_B] = \inf\{\mathbb{E}(Y\, Z\, \mathbb{1}_B) : \mathbb{E}(Z|\mathcal{F}_1) = 1, \inf_a \{D^*_{h^*}(Z-a)|\mathcal{F}_1\} \leq 1, a \lhd \mathcal{F}_1\},$$

one sees that (CA0) is fulfilled.

(d) Let H *be a monotonic concave function (thus, a.e. differentiable) satisfying* $H(0) = 0$, $H(1) = 1$. *The pertaining conditional distortion functional is defined as*

$$\mathcal{A}(Y|\mathcal{F}_1) = -\int_\infty^0 H[P\{Y \leq u|\mathcal{F}_1\}]\, du - \int_0^\infty 1 - H[P\{Y \leq u|\mathcal{F}_1\}]\, du$$

It has a dual form defined by

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\, Z|\mathcal{F}_1) : \mathbb{E}(\phi(Z)|\mathcal{F}_1) \leq \int \phi(H'(u))\, du, \phi \text{ convex }, \phi(0) = 0\}$$

where $H' = \frac{dH}{du}$. *By*

$$\mathbb{E}(\mathcal{A}(Y|\mathcal{F}_1)\, \mathbb{1}_B) = \inf\{\mathbb{E}(Y\, Z\, \mathbb{1}_B) : \mathbb{E}(\phi(Z)|\mathcal{F}_1) \leq \int \phi(H'(u))du, \phi \text{ convex }, \phi(0) = 0\}$$

one sees that (CA0) is fulfilled. By specializing H *to* $H(u) = \min(u/\alpha, 1)$, *one sees that the conditional* $\mathbb{A}V@R$ *defined in (6) is information monotone.*

3 Information and multi-period risk measurement

Let $Y = (Y_1, \dots, Y_T)$ be a (discounted) cash-flow process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathfrak{F} = (\mathcal{F}_0, \dots, \mathcal{F}_T)$ denote a filtration modeling the available information over time, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$, and Y_t is $\mathcal{F}_t$ measurable for every $t = 1, \dots, T$. Let $\mathcal{Y} \subseteq \times_{t=1}^T L_1(\Omega, \mathcal{F}, \mathbb{P})$ be a linear normed space containing $\mathbb{R}^T$ of cash-flow processes $Y = (Y_1, \dots, Y_T)$, which are adapted to $\mathfrak{F}$.

Next we consider functionals that map the elements $Y \in \mathcal{Y}$ and filtrations $\mathfrak{F}$ (such that $Y \lhd \mathfrak{F}$) to the extended real line $\overline{\mathbb{R}}$. Furthermore, a nonempty set $\mathcal{W} = \mathcal{W}(\mathfrak{F}) \subseteq \mathcal{Y}$ is considered that contains available financial instruments and (possibly) depends on the filtration, and a functional $\pi : \mathcal{W} \to \mathbb{R}$ that determines the price of elements of $\mathcal{W}$. The latter general concept is suggested in [18].

Definition 3.1 *A multi-period functional $\mathcal{A}$ with values $\mathcal{A}(Y;\mathfrak{F})$ is called multi-period acceptability functional, if it is proper (i.e., for every filtration $\mathfrak{F}$ it holds $\mathcal{A}(Y;\mathfrak{F}) < +\infty$ for all $Y \in \mathcal{Y}$ and $\mathcal{A}(Y) > -\infty$ for some $Y \in \mathcal{Y}$) and satisfies the following properties*

(MA1) **Translation-equivariance with respect to $(\mathcal{W}, \pi)$.** *If $Y \in \mathcal{Y}$ and $W \in \mathcal{W}$, then*

$$\mathcal{A}(Y + W;\mathfrak{F}) = \mathcal{A}(Y;\mathfrak{F}) + \pi(W). \tag{9}$$

(MA2) **Concavity.** *The mapping $Y \mapsto \mathcal{A}(Y;\mathfrak{F})$ is concave on $\mathcal{Y}$ for every filtration $\mathfrak{F}$.*

(MA3) **Monotonicity.** *If $Y, \tilde{Y} \in \mathcal{Y}$ and $Y_t \leq \tilde{Y}_t$ holds a.s. for $t = 1, \ldots, T$, then*

$$\mathcal{A}(Y;\mathfrak{F}) \leq \mathcal{A}(\tilde{Y};\mathfrak{F}).$$

The functionals on $\mathcal{Y}$ given by

$$\rho(Y;\mathfrak{F}) := -\mathcal{A}(Y;\mathfrak{F}) \qquad \text{and} \qquad \mathcal{D}(Y;\mathfrak{F}) := \sum_{t=1}^{T} \mathbb{E}(Y_t) - \mathcal{A}(Y;\mathfrak{F})$$

are called multi-period risk capital and multi-period deviation risk functionals. Instead of $\mathcal{A}(Y;\mathfrak{F})$ we also use the notations $\mathcal{A}(Y_1, \ldots, Y_T; \mathcal{F}_0, \ldots, \mathcal{F}_T)$ or simply $\mathcal{A}(Y)$ if the information aspect is not in the foreground at the moment.

As in the previous section, we may define the additional property of information monotonicity.

(MA0) **Information monotonicity.** If $Y \in \mathcal{Y}$ and $\mathcal{F}_t \subseteq \mathcal{F}'_t$, for $t = 0, \ldots, T$, then

$$\mathcal{A}(Y;\mathcal{F}_0, \ldots, \mathcal{F}_T) \leq \mathcal{A}(Y;\mathcal{F}'_0, \ldots, \mathcal{F}'_T). \tag{10}$$

Condition (MA0) means that additional information (in terms of filtrations) enlarges the acceptability and reduces the risk, respectively. While the conditions (MA2) and (MA3) are straightforward extensions of the single-period axioms, and, hence, are generally accepted, the equivariance-condition (MA1) appears relatively general. Later we focus on

linear subspaces $\mathcal{W}$ of $\mathcal{Y}$ and on linear continuous price functionals π in which case (MA1) implies that $\mathcal{A}(\cdot\,;\mathfrak{F})$ is affine on $\mathcal{W}$. Often the functional $\mathcal{A}(\cdot\,;\mathfrak{F})$ is *positively homogeneous* on $\mathcal{Y}$ for all $\mathfrak{F}$ (i.e., it holds $\mathcal{A}(\lambda Y;\mathfrak{F}) = \lambda\mathcal{A}(Y;\mathfrak{F})$ for all $\lambda \geq 0$ and $Y \in \mathcal{Y}$).
Examples of $\mathcal{W}$ are $\times_{t=1}^{T} L_p(\Omega, \mathcal{F}_{t-1}, \mathbb{P})$, for some $p \in [1, +\infty)$, $\mathbb{R}^T$ and $\mathbb{R} \times \{0\}^{T-1}$. A linear continuous functional π has the general form $\pi(W) = \sum_{t=1}^{T} \mathbb{E}(Z_t^* W_t)$ with some Z^* belonging to the topological dual of $\mathcal{W}$. In many cases we use the standard choice of π, namely, $\pi(W) = \sum_{t=1}^{T} \mathbb{E}(W_t)$.

Since we are interested in unbounded discrete-time processes, we consider the linear normed space $\mathcal{Y} = \times_{t=1}^{T} L_p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, +\infty)$ and its topological dual $\mathcal{Z} = \times_{t=1}^{T} L_q(\Omega, \mathcal{F}, \mathbb{P})$ with $\frac{1}{p} + \frac{1}{q} = 1$. The corresponding dual pairing (on $\mathcal{Z} \times \mathcal{Y}$) is $\langle Z, Y\rangle = \sum_{t=1}^{T} \mathbb{E}(Z_t Y_t)$.

If $\mathcal{A} = \mathcal{A}(\cdot\,;\mathfrak{F})$ is a multi-period acceptability functional with filtration $\mathfrak{F}$ and (nonempty) domain $\mathrm{dom}(\mathcal{A}) := \{Y \in \mathcal{Y} : \mathcal{A}(Y) > -\infty\}$, its *conjugate* $\mathcal{A}^+ : \mathcal{Z} \to \overline{\mathbb{R}}$ of obtained from the Legendre-Fenchel transform is given by

$$\mathcal{A}^+(Z;\mathfrak{F}) := \inf_{Y\in\mathcal{Y}} \{\langle Z, Y\rangle - \mathcal{A}(Y;\mathfrak{F})\}. \tag{11}$$

Note that $\mathcal{A}^+ = \mathcal{A}^+(\cdot\,;\mathfrak{F})$ is again proper and concave. The Fenchel-Moreau theorem implies

$$\mathcal{A}(Y;\mathfrak{F}) = \inf_{Z\in\mathcal{Z}} \left\{\langle Z, Y\rangle - \mathcal{A}^+(Z;\mathfrak{F})\right\} \tag{12}$$

if $\mathcal{A}(\cdot\,;\mathfrak{F})$ is upper semicontinuous. If, in addition, $\mathcal{A}(\cdot\,;\mathfrak{F})$ is positively homogeneous, then $\mathcal{A}^+ = \mathcal{J}_{\mathcal{S}}$, where $\mathcal{S} = \mathcal{S}(\mathfrak{F})$ is a closed convex subset of $\mathcal{Z}$ possibly depending on $\mathfrak{F}$ and $\mathcal{J}$ is the concave indicator function

$$\mathcal{J}_{\mathcal{S}}(Z) = \begin{cases} 0 & \text{if } Z \in \mathcal{S} \\ -\infty & \text{otherwise.} \end{cases} \tag{13}$$

i.e. it holds $\mathcal{S} = \mathcal{S}(\mathfrak{F}) := \mathrm{dom}\,\mathcal{A}^+(\cdot\,;\mathfrak{F})$ and

$$\mathcal{A}(Y;\mathfrak{F}) = \inf_{Z\in\mathcal{S}(\mathfrak{F})} \langle Z, Y\rangle.$$

Remark 3.2
Notice that $\mathcal{A}(Y;\mathfrak{F})$ is information monotone (MA0), if and only if its conjugate $\mathcal{A}^+(Y;\mathfrak{F})$ is information antitone, i.e. fulfills (10) with reversed inequality sign. This can be seen from the relations (11) and (12).

Theorem 3.3 *Let $\mathcal{A}$ be an upper semicontinuous multi-period acceptability functional with linear $\mathcal{W} = \mathcal{W}(\mathfrak{F})$ and π. Then the representation*

$$\mathcal{A}(Y;\mathfrak{F}) = \inf_{Z\in\mathcal{Z}} \{\langle Z, Y\rangle - \theta(Z;\mathfrak{F}) \,|\, \pi(\cdot) = \langle Z, \cdot\,\rangle \text{ on } \mathcal{W}(\mathfrak{F}),\ Z_t \geq 0,\ t = 1, \ldots, T\} \tag{14}$$

is valid for $\theta(\cdot\,;\mathfrak{F}) = \mathcal{A}^+(\cdot\,;\mathfrak{F})$ *and every* $Y \in \mathcal{Y}$*. Moreover,* $\mathcal{A}$ *satisfies (MA0) if* $\mathcal{W}(\mathfrak{F})$ *increases with* $\mathfrak{F}$*.*
Conversely, if $\mathcal{A}$ *can be represented in the form (14) for some proper and information antitone functional* $\theta(\cdot\,;\mathfrak{F}) : \mathcal{Z} \to \overline{\mathbb{R}}$ *and for some subspace* $\mathcal{W}(\mathfrak{F})$ *increasing with* $\mathfrak{F}$*, then* $\mathcal{A}$ *is an upper semicontinuous multi-period acceptability functional satisfying (MA0).*

Proof. We show that the properties (MA1), (MA2) and (MA3) imply the representation (14) of $\mathcal{A}$ by verifying that for any Z violating some of the constraints in (14) we have $\mathcal{A}^+(Z) = -\infty$ in (12).
If for $Z \in \mathcal{Z}$ the constraint $Z_t \geq 0$ is violated for some $t \in \{1, \dots, T\}$, there exists $\bar{Y}$ such that $\bar{Y}_\tau \geq 0$ for $\tau = 1, \dots, T$ and $\langle Z, \bar{Y} \rangle = \sum_{\tau=1}^T \mathbb{E}(\bar{Y}_\tau Z_\tau) < 0$. Consider the elements $Y^s := Y + s\bar{Y}$ for $s \geq 0$. Then $Y_\tau \leq Y_\tau^s$ for $\tau = 1, \dots, T$ and, thus, $\mathcal{A}(Y^s) \geq \mathcal{A}(Y)$ for each $s \geq 0$ due to (MA3). Hence, we obtain

$$\mathcal{A}^+(Z) \leq \inf_{s \geq 0} \{\langle Z, Y^s \rangle - \mathcal{A}(Y^s)\} \leq \inf_{s \geq 0} \{\langle Z, Y \rangle - \mathcal{A}(Y) + s\langle Z, \bar{Y} \rangle\} = -\infty.$$

Now, let the first constraint in (14) be violated for some $Z \in \mathcal{Z}$, i.e., there exists $\bar{W} \in \mathcal{W}$ such that $\pi(\bar{W}) > \langle Z, \bar{W} \rangle$ holds. Then we obtain

$$\mathcal{A}^+(Z) \leq \inf_{s \geq 0} \{\langle Z, s\bar{W} \rangle - \mathcal{A}(s\bar{W})\} = \inf_{s \geq 0} \{s(\langle Z, \bar{W} \rangle - \pi(\bar{W}))\} = -\infty.$$

Conversely, suppose that the representation (14) holds for the functional $\mathcal{A} : \mathcal{Y} \to \overline{\mathbb{R}}$, where $\theta(\cdot\,;\mathfrak{F}) : \mathcal{Z} \to \overline{\mathbb{R}}$ is some proper functional. Hence, $\mathcal{A}$ is given as the infimum of continuous affine functions and, thus, is proper, concave and upper semicontinuous. It remains to verify conditions (MA0), (MA1) and (MA3).
Let $Y, \tilde{Y} \in \mathcal{Y}$ with $Y_t \leq \tilde{Y}_t$, $t = 1, \dots, T$, and $Z \in \mathcal{Z}$ be feasible (i.e., in particular, $Z_t \geq 0$ for every $t = 1, \dots, T$). Then we have

$$\langle Z, \tilde{Y} \rangle - \langle Z, Y \rangle = \sum_{t=1}^T \mathbb{E}(Z_t(\tilde{Y}_t - Y_t)) \geq 0$$

and hence (14) implies $\mathcal{A}(Y) \leq \mathcal{A}(\tilde{Y})$ and, thus, condition (MA3).
Now, let $Y \in \mathcal{Y}$, $W \in \mathcal{W}$ and $\varepsilon > 0$. We define

$$I_\varepsilon := \begin{cases} \varepsilon + \mathcal{A}(Y + W) & , \mathcal{A}(Y + W) > -\infty, \\ -\frac{1}{\varepsilon} & , \text{ otherwise.} \end{cases}$$

Then there exists $\bar{Z}$ such that $\bar{Z}_t \geq 0$, $\pi(\cdot) = \langle \bar{Z}, \cdot \rangle$ on $\mathcal{W}$ and $I_\varepsilon \geq \langle \bar{Z}, Y + W \rangle - \mathcal{A}^+(\bar{Z})$, thus,

$$I_\varepsilon \geq \langle \bar{Z}, Y \rangle + \langle \bar{Z}, W \rangle - \theta(\bar{Z}; \mathfrak{F}) = \langle \bar{Z}, Y \rangle - \theta(\bar{Z}; \mathfrak{F}) + \pi(W) \geq \pi(W) + \mathcal{A}(Y).$$

Since ε was arbitrary, we obtain $\mathcal{A}(Y + W) \geq \pi(W) + \mathcal{A}(Y)$. By changing the role of Y and $Y + W$ the converse inequality can be shown and, hence, condition (MA1) is satisfied. Condition (MA0) follows immediately from the properties of $\theta(\cdot\,;\mathfrak{F})$ and $\mathcal{W}(\mathfrak{F})$ with respect to $\mathfrak{F}$ and the proof is complete. $\square$

Note that any functional $\mathcal{A}(\cdot\,;\mathfrak{F})$ from $\mathcal{Y}$ to $\overline{\mathbb{R}}$ satisfying (MA2) and (MA3) is continuous on the interior of its domain $\operatorname{dom}\mathcal{A} = \{Y \in \mathcal{Y} \,|\, \mathcal{A}(Y) > -\infty\}$ [37, Proposition 3.1]. Hence, any multi-period acceptability functional being finite on $\mathcal{Y}$ is continuous on $\mathcal{Y}$ and, thus, admits the dual representation (14).

Notice that in case $\pi(W) = \sum_{t=1}^{T} \mathbb{E}(W_t)$ the condition $\pi(\cdot) = \langle Z, \cdot \rangle$ on $\mathcal{W} = \mathcal{W}(\mathfrak{F})$ is equivalent to

- $\mathbb{E}(Z_t|\mathcal{F}_{t-\tau}) = 1$ for every $1 \le t \le T$ if $\mathcal{W} = \times_{t=1}^{T} L_p(\Omega, \mathcal{F}_{t-\tau}, \mathbb{P})$
- $\mathbb{E}(Z_t) = 1$ for every $t = 1, \dots, \tau$ if $\mathcal{W} = \mathbb{R}^{\tau} \times \{0\}^{T-\tau}$

for some $\tau \in \{1, \dots, T\}$ and by using the convention $\mathcal{F}_{-t} = \mathcal{F}_0 = \{\emptyset, \Omega\}$ for every $t \in \{1, \dots, T\}$.

4 Construction of multi-period risk functionals

The aim of this section is to present several principles for constructing multi-period acceptability functionals including a number of examples.

4.1 Scalarized and separable constructions

The first approach consists in constructing multi-period acceptability functionals by utilizing single-period functionals and conditional acceptability mappings, respectively.

(a) Separable multi-period acceptability functionals:

$$\mathcal{A}(Y;\mathfrak{F}) := \sum_{t=1}^{T} \mathcal{A}_t(Y_t),$$

where $\mathcal{A}_t : \mathcal{Y}_t \to \overline{\mathbb{R}}$ is a single-period acceptability functional for every $t = 1, \dots, T$. Then $\mathcal{A}$ is a multi-period acceptability functional with $\mathcal{W} = \mathbb{R}^T$ and the standard choice of π. If each $\mathcal{A}_t$ is version independent, $\mathcal{A}$ does only depend on the marginal distributions $\mathbb{P}\, Y_t^{-1}$. Note that $\mathcal{A}$ satisfies (MA0) in a trivial way, since it does not depend explicitly on $\mathfrak{F}$.

(b) Scalarization:

$$\mathcal{A}(Y;\mathfrak{F}) := \mathcal{A}_0(s(Y))$$

where $\mathcal{A}_0 : L_p(\mathcal{F}) \to \overline{\mathbb{R}}$ is a single-period acceptability functional and $s : \mathcal{Y} \to L_p(\Omega, \mathcal{F}, \mathbb{P})$ a mapping satisfying concavity and monotonicity (see also [11]). Such functionals $\mathcal{A}$ have the properties (MA0), (MA2) and (MA3), but do not explicitly depend on $\mathfrak{F}$ either. Furthermore, the subspace $\mathcal{W}$ in (MA1) depends on the properties of s. For example, we have

(b1) $\mathcal{A}(Y) = \mathbb{A}\mathrm{V@R}_\alpha(\sum_{t=1}^T Y_t)$ with $s(Y) = \sum_{t=1}^T Y_t$ and $\mathcal{W} = \mathbb{R}^T$.

(b2) $\mathcal{A}(Y) = \mathbb{A}\mathrm{V@R}_\alpha(\min_{t=1,\ldots,T} \sum_{\tau=1}^t Y_\tau)$ with $s(Y) = \min_{t=1,\ldots,T} \sum_{\tau=1}^t Y_\tau$ and $\mathcal{W} = \mathbb{R} \times \{0\}^{T-1}$. This multi-period acceptability functional is suggested in [4].

(c) Separable expected conditional (SEC) multi-period acceptability functionals:

$$\mathcal{A}(Y;\mathfrak{F}) := \sum_{t=1}^T \mathbb{E}(\mathcal{A}_t(Y_t|\mathcal{F}_{t-1})) \tag{15}$$

where $\mathcal{A}_t(\cdot\,|\mathcal{F}_{t-1})$, $t = 1, \ldots, T$, are conditional acceptability mappings (see Section 2). Such multi-period functionals $\mathcal{A}$ satisfy (MA0)–(MA3).

Remark 4.1
If $\mathcal{A}_t(\cdot\,|\mathcal{F}_{t-1})$ is given by

$$\begin{aligned}\mathbb{E}[\mathcal{A}_t(Y|\mathcal{F}_{t-1})\,\mathbb{1}_B] \;=\; \inf\{&\mathbb{E}(Y\;Z\;\;\mathbb{1}_B) - \mathbb{E}[\mathcal{A}_t^+(Z|\mathcal{F}_{t-1})\,\mathbb{1}_B] : Z \geq 0, \mathbb{E}(Z|\mathcal{F}_{t-1}) = 1,\\ &Z \in \mathcal{Z}_t(\mathcal{F}_{t-1})\},\end{aligned}$$

then the SEC functional $\mathcal{A}(Y;\mathfrak{F}) := \sum_{t=1}^T \mathbb{E}(\mathcal{A}_t(Y_t|\mathcal{F}_{t-1}))$ has the dual representation

$$\begin{aligned}\mathcal{A}(Y;\mathfrak{F}) \;=\; \inf\Big\{&\sum_{t=1}^T \mathbb{E}(Y_t\,Z_t) - \sum_{t=1}^T \mathbb{E}(\mathcal{A}_t^+(Z_t|\mathcal{F}_{t-1}) : Z_t \geq 0, \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1,\\ &Z_t \in \mathcal{Z}_t(\mathcal{F}_{t-1}), t = 1, \ldots, T\Big\}\end{aligned}$$

Notice that this implies that the conjugate of $\mathcal{A}(\cdot;\mathfrak{F})$ is also SEC. SEC functionals are information-monotone, if the functionals $\mathcal{A}_t$ appearing in (15) exhibit the information-monotonicity property (CA0).

Example 4.2 *(*Multi-period average value-at-risk*)*
An example for a SEC functional is the multi-period average value-at-risk *[28]*

$$\begin{aligned}m\mathbb{A}\,V@R_\alpha(Y;\mathfrak{F}) \;:=\; &\sum_{t=1}^T \mathbb{E}(\mathbb{A}\,V@R_\alpha(Y_t|\mathcal{F}_{t-1}))\\ =\; &\inf\left\{\sum_{t=1}^T \mathbb{E}(Y_t Z_t) : 0 \leq Z_t \leq \frac{1}{\alpha}, \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1, t = 1, \ldots, T\right\}\end{aligned} \tag{16}$$

An alternate representation of $m\mathbb{A}V@R_\alpha$ is

$$\begin{aligned} & m\mathbb{A}V@R_\alpha(Y;\mathfrak{F}) \\ = & \sup\left\{\mathbb{E}\big(\sum_{t=1}^{T}[Y_t - x_{t-1}]_- + \frac{1-\alpha}{\alpha}[Y_t - x_{t-1}]_+\big) : x_t \lhd \mathcal{F}_t, t = 0,\dots,T-1\right\}. \quad (17) \end{aligned}$$

Here $[\,]_+$ resp. $[\,]_-$ denote the positive and negative part respectively. The nonanticipativity constraint $x_t \lhd \mathcal{F}_t$ reveals again that $m\mathbb{A}V@R_\alpha$ is information monotone.

4.2 Composition of conditional acceptability mappings

Let conditional acceptability mappings $\mathcal{A}_t := \mathcal{A}_t(\cdot\,|\mathcal{F}_t)$, $t = 0,\dots,T-1$, from $\mathcal{Y}_T$ to $\mathcal{Y}_t$ be given. Following [38, Section 7] we introduce a multi-period functional $\mathcal{A}$ on $\mathcal{Y}$ by a nested composition and a family $(\mathcal{A}^{(t)})_{t=1,\dots,T}$ of single-period functionals $\mathcal{A}^{(t)} : \mathcal{Y}_T \to \overline{\mathbb{R}}$ by (direct) compositions of the $\mathcal{A}_{t-1}$, $t = 1,\dots,T$, namely,

$$\begin{aligned} \mathcal{A}(Y;\mathfrak{F}) &:= \mathcal{A}_0(Y_1 + \cdots + \mathcal{A}_{T-2}(Y_{T-1} + \mathcal{A}_{T-1}(Y_T))\cdots) \quad (18) \\ \mathcal{A}^{(t)}(Y_T) &:= \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{t-1}(Y_T) \end{aligned}$$

for every $Y \in \mathcal{Y}$ and $Y_T \in \mathcal{Y}_T$. As argued in [38, Section 7], the functional $\mathcal{A}$ satisfies (MA1) with $\mathcal{W} = \mathbb{R}^T$, (MA2) and (MA3), and every $\mathcal{A}^{(t)}$ is a (single-period) acceptability functional. Moreover, it holds

$$\mathcal{A}(Y;\mathfrak{F}) = \mathcal{A}^{(T)}\left(\sum\nolimits_{t=1}^{T} Y_t\right) = \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}\left(\sum\nolimits_{t=1}^{T} Y_t\right). \quad (19)$$

The functionals $\mathcal{A}$ and $\mathcal{A}^{(t)}$, $t = 1,\dots,T$, are positively homogeneous if all $\mathcal{A}_t$ are positively homogeneous.

Example 4.3 *(*Nested average value-at-risk*)*
We consider the conditional average value-at-risk (of level $\alpha \in (0,1]$) as conditional acceptability mapping

$$\mathcal{A}_t := \mathbb{A}V@R_\alpha(\cdot\,|\mathcal{F}_t)$$

for every $t = 0,\dots,T-1$. Then the multi-period functional

$$n\mathbb{A}V@R_\alpha(Y;\mathfrak{F}) = \mathbb{A}V@R_\alpha(\cdot\,|\mathcal{F}_0) \circ \cdots \circ \mathbb{A}V@R_\alpha(\cdot\,|\mathcal{F}_{T-1})\left(\sum\nolimits_{t=1}^{T} Y_t\right)$$

satisfies (MA1) with $\mathcal{W} = \mathbb{R}^T$, (MA2), (MA3) and is positively homogeneous. It has been introduced in [6] and will be called nested average value-at-risk *because of its nested composition structure.*

Proposition 4.4 *Suppose that, for every $t = 1, \dots, T$, the conditional acceptability mapping $\mathcal{A}_{\mathcal{F}_t}$ maps $L_p(\mathcal{F})$ to $L_p(\mathcal{F}_t)$ and is given by the representation*

$$\mathbb{E}[\mathcal{A}_{\mathcal{F}_t}(Y)\mathbb{1}_B] = \inf\{\mathbb{E}(Y\ Z\ \mathbb{1}_B) - \mathbb{E}[\theta_{\mathcal{F}_t}(Z)\mathbb{1}_B] : Z \geq 0,\ \mathbb{E}(Z|\mathcal{F}_t) = 1,\ Z \in \mathcal{Z}_t(\mathcal{F}_t)\}$$

for every pair $(Y, B) \in \mathcal{Y} \times \mathcal{F}_t$ according to Theorem 2.4, where $\theta_{\mathcal{F}_t}$ is a concave mapping from $L_q(\mathcal{F})$ to $L_q(\mathcal{F}_t)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose further that there is a constant K such that for all $Z \in \mathcal{Z}_t(\mathcal{F}_t)$

$$\mathbb{E}(Z^q|\mathcal{F}_{t-1}) \leq K^q \qquad a.s. \tag{20}$$

(if $p = 1$, this has to be read that $Z \leq K$ a.s.) and

$$\mathbb{E}(|\mathcal{A}_{\mathcal{F}_t}(Z)||\mathcal{F}_{t-1}) \leq K \qquad a.s. \tag{21}$$

Then the nested acceptability functional $\mathcal{A}(Y; \mathfrak{F}) = \mathcal{A}^{(T)}(Y_1 + \cdots + Y_T)$ has the representation

$$\mathcal{A}(Y; \mathfrak{F}) = \inf\Big\{\mathbb{E}\Big[\sum_{t=1}^{T} Y_t\, M_T\Big] - \sum_{t=1}^{T} \mathbb{E}[\theta_{\mathcal{F}_t}(Z_t) M_{t-1}] : \mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t \geq 0, Z_t \in \mathcal{Z}_t(\mathcal{F}_t)\Big\} \tag{22}$$

where $M_t = \prod_{s=1}^{t} Z_s$, $M_0 = 1$ and (M_t) is a nonnegative martingale w.r.t. the filtration $\mathfrak{F}$ with $\mathbb{E}(M_t) = 1$ and $\mathbb{E}(M_t^q) < \infty$.

Proof. The proof is based on induction. It suffices to prove it for $T = 2$. Consider the two conditional mappings $\mathcal{A}_{\mathcal{F}_2}$ and $\mathcal{A}_{\mathcal{F}_1}$ defined by

$$\mathbb{E}[\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_{B_1}] = \inf\{\mathbb{E}(Y\, Z_1\, \mathbb{1}_{B_1}) - \mathbb{E}[\theta_{\mathcal{F}_1}(Z_1)\mathbb{1}_{B_1}] : Z_1 \geq 0, \mathbb{E}(Z_1|\mathcal{F}_1) = 1, Z_1 \in \mathcal{Z}_1(\mathcal{F}_1)\}$$

for every $B_1 \in \mathcal{F}_1$ and

$$\mathbb{E}[\mathcal{A}_{\mathcal{F}_2}(Y)\mathbb{1}_{B_2}] = \inf\{\mathbb{E}(Y\, Z_2\, \mathbb{1}_{B_2}) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)\mathbb{1}_{B_2}] : Z_2 \geq 0, \mathbb{E}(Z_2|\mathcal{F}_2) = 1, Z_2 \in \mathcal{Z}_2(\mathcal{F}_2)\}$$

for every $B_2 \in \mathcal{F}_2$ according to Section 2, where $\theta_{\mathcal{F}_1}$ and $\theta_{\mathcal{F}_2}$ are concave mappings. Our first goal is to show that for $X \in L_q(\mathcal{F}_2)$ it holds that

$$\mathbb{E}[\mathcal{A}_{\mathcal{F}_2}(Y)X] = \inf\{\mathbb{E}(Y\, Z_2\, X) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)X] : Z_2 \geq 0, \mathbb{E}(Z_2|\mathcal{F}_2) = 1, Z_2 \in \mathcal{Z}_2(\mathcal{F}_2)\}. \tag{23}$$

Assume first that X is of the form $X = \sum_{i=1}^{I} \alpha_i \mathbb{1}_{C_i}$ for pairwise disjoint C_i's, $C_i \in \mathcal{F}_2$. Then the minimization problem in (23) is separable and it follows that

$$\begin{aligned}
\mathbb{E}[\mathcal{A}_{\mathcal{F}_2}(Y)X] &= \sum_{i=1}^{I} \alpha_i \mathbb{E}[\mathcal{A}_{\mathcal{F}_2}(Y)\mathbb{1}_{C_i}] = \\
&= \sum_{i=1}^{I} \alpha_i \inf\left\{\mathbb{E}(Y\, Z_2\, \mathbb{1}_{C_i}) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)\mathbb{1}_{C_i}] : Z_2 \geq 0, \mathbb{E}(Z_2|\mathcal{F}_2) = 1, Z_2 \in \mathcal{Z}_2(\mathcal{F}_2)\right\} \\
&= \inf\Big\{\sum_{i=1}^{I} \alpha_i \mathbb{E}(Y\, Z_2\, \mathbb{1}_{C_i}) - \sum_{i} \alpha_i \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)\mathbb{1}_{C_i}] : Z_2 \geq 0, \mathbb{E}(Z_2|\mathcal{F}_2) = 1, Z_2 \in \mathcal{Z}_2(\mathcal{F}_2)\Big\} \\
&= \inf\{\mathbb{E}(Y\, Z_2\, X) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)X] : Z_2 \geq 0, \mathbb{E}(Z_2|\mathcal{F}_2) = 1, Z_2 \in \mathcal{Z}_2(\mathcal{F}_2)\}. \qquad (24)
\end{aligned}$$

The objective function $\mathbb{E}(Y\,Z_2\,X) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)X]$ in (24) is Lipschitz in X, since by (20) and (21)

$$\begin{aligned}
& |\mathbb{E}(Y\,Z_2\,X) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)X] - \mathbb{E}(Y\,Z_2\,X') - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)X']| \\
\leq\ & \|Y\|_p \mathbb{E}^{(1/q}[|X - X'|^q \mathbb{E}(|Z_2|^q|\mathcal{F}_1] + \mathbb{E}[|X - X'||\mathbb{E}(\mathcal{A}_{\mathcal{F}_2}||\mathcal{F}_1)] \\
\leq\ & \|Y\|_p K \|X - X'\|_q + K\|X - X'\|_q.
\end{aligned}$$

Since the functions of the form $\sum_{i=1}^{I} \alpha_i \mathbb{1}_{C_i}$ are dense in $L_q(\mathcal{F}_2)$, the validity of (24) can be extended to $X \in L_q(\mathcal{F}_2)$, i.e. (23) is proved.

Now we compose the two functionals to get for $B_1 \in \mathcal{F}_1$

$$\begin{aligned}
& \mathbb{E}[\mathcal{A}_{\mathcal{F}_1}(\mathcal{A}_{\mathcal{F}_2}(Y))\mathbb{1}_{B_1}] \\
=\ & \inf\{\mathbb{E}(\mathcal{A}_{\mathcal{F}_2}(Y)\,Z_1\,\mathbb{1}_{B_1}) - \mathbb{E}[\theta_{\mathcal{F}_1}(Z_1)\mathbb{1}_{B_1}] : Z_1 \geq 0, \mathbb{E}(Z_1|\mathcal{F}_1) = 1, Z_1 \in \mathcal{Z}_1(\mathcal{F}_1)\} \\
=\ & \inf\{\inf\{\mathbb{E}(Y\,Z_2\,Z_1\,\mathbb{1}_{B_1}) - \mathbb{E}[\theta_{\mathcal{F}_2}(Z_2)Z_1\,\mathbb{1}_{B_1}] : Z_2 \geq 0, \mathbb{E}(Z_2|\mathcal{F}_2) = 1, Z_2 \in \mathcal{Z}_2(\mathcal{F}_2)\} \\
& -\mathbb{E}[\theta_{\mathcal{F}_1}(Z_1)\mathbb{1}_{B_1}] : Z_1 \geq 0, \mathbb{E}(Z_1|\mathcal{F}_1) = 1, Z_1 \in \mathcal{Z}_1(\mathcal{F}_1)\} \\
=\ & \inf\Big\{\mathbb{E}(Y\,Z_2\,Z_1\,\mathbb{1}_{B_1}) - \sum_{t=1}^{2} \mathbb{E}[\theta_{\mathcal{F}_t}(Z_t) \prod_{s=1}^{t-1} Z_s\,\mathbb{1}_{B_1}] : Z_t \geq 0, \mathbb{E}(Z_t|\mathcal{F}_t) = 1, \\
& \qquad\qquad Z_t \in \mathcal{Z}_t(\mathcal{F}_t), t = 1, 2\Big\}
\end{aligned}$$

Here we have used (23) for $X = Z_1\mathbb{1}_{B_1}$. By setting $B_1 = \Omega$ and by induction, one gets the assertion of the proposition. Notice that the outermost functional in the composed sequence is always a nonconditional one. □

Remark 4.5
The martingale (M_t) can be seen as a sequence of probability densities $M_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}}$ and the expectations $\mathbb{E}(\cdot\, M_t)$ are expectations $\mathbb{E}_{\mathbb{Q}_t}(\cdot)$ under $\mathbb{Q}_t$.

Example 4.6 *(*Nested average value-at-risk *(continued))*
The nested average value-at-risk $n\mathbb{A}V@R$ has the following dual representation:

$$\begin{aligned}
n\mathbb{A}V@R_\alpha(Y;\mathfrak{F}) \ =\ & \inf\Big\{\mathbb{E}\Big[\sum_{t=1}^{T} Y_t M_T\Big] : 0 \leq M_t \leq \frac{1}{\alpha} M_{t-1}, \mathbb{E}(M_t|\mathcal{F}_{t-1}) = M_{t-1}, \\
& \qquad t = 1, \ldots, T, M_0 = 1\Big\}.
\end{aligned}$$

Hence, $n\mathbb{A}V@R$ is given by a linear stochastic optimization problem containing functional constraints and, thus does not belong to the class of polyhedral functionals discussed below.

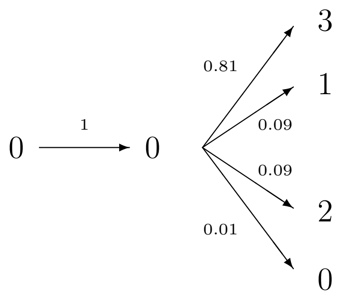

Figure 1: The final process Y and the filtration $\mathfrak{F} = (\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2)$

Example 4.7 *(*Nested entropic acceptability functional*)*
The nested entropic acceptability functional is

$$\mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}(Y)$$

with $\mathcal{A}_t(Y) = -\frac{1}{\gamma} \log \mathbb{E}[\exp(-\gamma Y)|\mathcal{F}_t]$ *for* $Y \in \mathcal{Y}$*. The entropic risk functional was introduced in [12] and further studied in [13, 9]. The representation of* $\mathcal{A}_t$ *according to Theorem 2.4 is*

$$\mathcal{A}_t(Y|\mathcal{F}_1) = \mathbf{inf} \left\{ \mathbb{E}(Y\, Z|\mathcal{F}_t) + \frac{1}{\gamma}\mathbb{E}(Z \log Z|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = 1, Z \geq 0 \right\},$$

where $0 \log 0$ *is defined as 0. By Proposition 4.4, the nested entropic acceptability functional has the representation*

$$\begin{aligned}
\mathcal{A}(Y;\mathfrak{F}) &= \inf \Big\{ \mathbb{E}\Big[\sum_{t=1}^{T} Y_t \prod_{s=1}^{T} Z_s\Big] + \frac{1}{\gamma} \sum_{t=1}^{T} \mathbb{E}[\mathbb{E}(Z_t \log Z_t|\mathcal{F}_t) \prod_{s=1}^{t-1} Z_s] : \mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t \geq 0 \Big\} \\
&= \inf \Big\{ \mathbb{E}\Big[\sum_{t=1}^{T} Y_t \prod_{s=1}^{T} Z_s\Big] + \frac{1}{\gamma}\mathbb{E}[\prod_{s=1}^{T} Z_s (\log \prod_{s=1}^{T} Z_s)] : \mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t \geq 0 \Big\} \\
&= \inf \Big\{ \mathbb{E}\Big[\sum_{t=1}^{T} Y_t M\Big] + \frac{1}{\gamma}\mathbb{E}[M \log M] : \mathbb{E}(M) = 1, M \geq 0 \Big\}
\end{aligned}$$

Here we have set $M = \prod_{s=1}^{T} Z_s$*. Notice that for any filtration* $\mathfrak{F}$*, the density process* Z_t *can be regained by* $Z_t = \mathbb{E}(M|\mathcal{F}_t)/\mathbb{E}(M|\mathcal{F}_{t-1})$*. Since the representation can be written in a form, which does not depend on the filtration, the nested entropic functional is information-monotone in a trivial way. This fact can even simpler be seen from the primal form, since it is readily seen from the definition that*

$$\mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}\Big(\sum_{t=1}^{T} Y_t\Big) = \mathcal{A}_0\Big(\sum_{t=1}^{T} Y_t\Big).$$

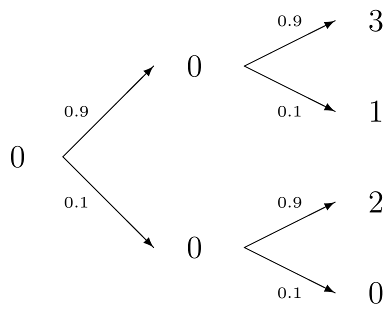

Figure 2: The same final process Y and the finer filtration $\mathfrak{F}' = (\mathcal{F}_0 \subseteq \mathcal{F}_1' \subseteq \mathcal{F}_2)$

Example 4.8 *(*Nested average value-at-risk *(continued))*
It was shown in [6] that the nested average value-at-risk is the natural time-consistent extension of the single-period $\mathbb{A}V@R$*. However, it turns out that this functional is not information-monotone. Consider Figure 1 and Figure 2. Both models describe the same final process* Y*. However, the filtration* $\mathfrak{F}'$ *in Fig. 2 is finer than the filtration* $\mathfrak{F}$ *in Fig. 1. Calculating the nested* $\mathbb{A}V@R$ *for* $\alpha = 0.1$*, we get* $\mathbb{A}V@R_{0.1}(\mathbb{A}V@R_{0.1}(Y|\mathcal{F}_1)) = 0.9$*, while* $\mathbb{A}V@R_{0.1}(\mathbb{A}V@R_{0.1}(Y|\mathcal{F}_1')) = 0$*. Thus the process in Fig. 1 seems to be more acceptable, although the information is less.*

Notice that also the composed functional $Y \mapsto \mathbb{A}V@R_{0.1}(\mathbb{E}(Y|\mathcal{F}_1))$ *is not information monotone either. One has* $\mathbb{A}V@R_{0.1}(\mathbb{E}(Y|\mathcal{F}_1)) = 2.88$*, while* $\mathbb{A}V@R_{0.1}(\mathbb{E}(Y|\mathcal{F}_1')) = 1.8$*.*

In contrast, the composed functional $Y \mapsto \mathbb{E}[\mathbb{A}V@R_{0.1}(Y|\mathcal{F}_1)]$ *is information monotone, since the conditional* $\mathbb{A}V@R$ *fulfills (CA0), which was shown earlier.*

Let us now study in detail information monotonicity of compositions. To this end, consider an unconditional acceptability functional $\mathcal{A}_1$ and a conditional one $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ and form the composition

$$\bar{\mathcal{A}}(Y; \mathcal{F}_1) = \mathcal{A}_1(\mathcal{A}_2(Y|\mathcal{F}_1)).$$

Let $\mathcal{A}_1^+$ be the conjugate of $\mathcal{A}_1$ and $\theta_{\mathcal{F}_1} = \mathcal{A}_2^+(\cdot\,|\mathcal{F}_1)$ be chosen as in the representation according to Theorem 2.4. According to Proposition 4.4 the composed functional has the representation

$$\bar{\mathcal{A}}(Y; \mathcal{F}_1) = \inf\{\mathbb{E}(Y\,Z) - \mathcal{C}_{\mathcal{F}_1}(Z)\},$$

where

$$\mathcal{C}_{\mathcal{F}_1}(Z) = \sup\{\mathcal{A}_1^+(Z_1) + \mathbb{E}[\mathcal{A}_2^+(Z_2|\mathcal{F}_1)Z_1] : Z_1 \cdot Z_2 = Z, Z_1 \lhd \mathcal{F}_1\}. \tag{25}$$

Notice that $\bar{\mathcal{A}}(Y; \mathcal{F}_1)$ is information monotone if and only if $\mathcal{C}_{\mathcal{F}_1}(Z)$ is information antitone. If the supremum in (25) is attained for $Z_1 = 1$, then the information monotonicity of $\mathcal{A}_2$ implies the information monotonicity of the composition $\bar{\mathcal{A}}$.

Example 4.9
If $\mathcal{A}_1 = \mathbb{E}$, then $\mathcal{A}_1^+ = \mathcal{J}_{\{Z=1\}}$, where $\mathcal{J}$ was introduced in (13). Therefore, trivially, the supremum in (25) is attained for $Z_1 = 1$ and the information monotonicity of $\mathcal{A}_2$ implies that of $\mathbb{E}[\mathcal{A}_2(Y|\mathcal{F}_1)]$.

Example 4.10
Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be entropic functionals with the same γ. By

$$\begin{aligned}\mathcal{A}_1^+(Z_1) &= -\frac{1}{\gamma}\mathbb{E}[Z_1 \log Z_1] + \mathcal{J}_{\{\mathbb{E}(Z_1)=1, Z_1 \geq 0\}} \\ \mathcal{A}_1^+(Z_2|\mathcal{F}_1) &= -\frac{1}{\gamma}\mathbb{E}[Z_2 \log Z_2|\mathcal{F}_1] + \mathcal{J}_{\{\mathbb{E}(Z_1|\mathcal{F}_1)=1, Z_1 > 0\}}\end{aligned}$$

one readily sees that the supremum in (25) is reached for $Z_1 = 1$ and

$$\mathcal{C}_{\mathcal{F}_1}(Z) = -\frac{1}{\gamma}\mathbb{E}(Z \log Z) + \mathcal{J}_{\{\mathbb{E}(Z)=1\}}.$$

Since $\mathcal{C}_{\mathcal{F}_1}(Z)$ is independent of $\mathcal{F}_1$ and therefore information antitone in a trivial way, we see again that the nested entropic functional is information monotone, which was already seen in Example 4.7.

The next proposition shows that these examples are in a way the only ones one can think of.

Proposition 4.11 *Let w.l.o.g. $\mathcal{A}_1^+(1) = 0$. Suppose that there is a Z in the supergradient set of the composition $\bar{\mathcal{A}}$ (i.e. there is some Y such that $\bar{\mathcal{A}}(Y) = \mathbb{E}(Y \cdot Z) - \mathcal{C}_{\mathcal{F}_1}(Z)$), for which $\mathcal{C}_{\mathcal{F}_1}(Z) > \mathbb{E}[\mathcal{A}_2^+(Z|\mathcal{F}_1)]$. Then the composition $\bar{\mathcal{A}}$ is not information monotone.*

Proof. For the trivial σ-algebra $\mathcal{F}_0$ we have that $\mathcal{C}_{\mathcal{F}_0}(Z) = \mathbb{E}[\mathcal{A}_2^+(Z|\mathcal{F}_1)]$. Therefore

$$\bar{\mathcal{A}}(Y; \mathcal{F}_1) = \mathbb{E}(Y \cdot Z) - \mathcal{C}_{\mathcal{F}_1}(Z) < \mathbb{E}(Y \cdot Z) - \mathcal{C}_{\mathcal{F}_0}(Z) = \bar{\mathcal{A}}(Y; \mathcal{F}_0).$$

This demonstrates that the composition $\bar{\mathcal{A}}$ cannot be information monotone. □

4.3 Multi-period polyhedral risk functionals

As mentioned in Section 1, the basic motivation of polyhedral risk functionals consists in maintaining linearity structures even though they are nonlinear functionals. Having this motivation in mind and recalling the representation of $\mathbb{AV@R}$, it is a natural idea to introduce acceptability functionals as optimal values of certain linear stochastic optimization problems. Extending the concept in [10] a functional $\mathcal{A}$ is called *multi-period polyhedral* if it is given by

$$\mathcal{A}(Y;\mathfrak{F}) = \sup\left\{ \mathbb{E}\left[\sum_{t=0}^{T}\langle c_t, v_t\rangle\right] \;\middle|\; \begin{array}{l} v_t \in L_p(\mathcal{F};\mathbb{R}^{k_t}),\ v_t \in V_t,\ v_t = \mathbb{E}(v_t|\mathcal{F}_t), \\ \sum\limits_{\tau=0}^{t} B_{t,\tau} v_{t-\tau} = r_t(Y_t),\ t = 0, ..., T \end{array} \right\} \tag{26}$$

for any income cash-flow process $Y = (Y_1, \ldots, Y_T)$ in $\times_{t=1}^{T} L_p(\Omega, \mathcal{F}, \mathbb{P})$ ($p \geq 1$) and filtration $\mathfrak{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$. The definition includes fixed polyhedral cones V_t in some Euclidean spaces $\mathbb{R}^{k_t}$ (e.g., $I_1 \times \ldots \times I_{k_t}$ with I_k belonging to $\{\mathbb{R}, [0,+\infty), (-\infty,0], \{0\}\}$ for every $k = 1, \ldots, k_t$) with scalar product $\langle \cdot, \cdot \rangle$, fixed matrices $B_{t,\tau}$, $\tau = 0, \ldots, t$, and vectors $c_t \in \mathbb{R}^{k_t}$, and affine mappings r_t from $\mathbb{R}$ to $\mathbb{R}^{d_t}$, $t = 0, \ldots, T$, with constant r_0. Note that functionals $\mathcal{A}$ given by (26) always satisfy information monotonicity (MA0) and concavity (MA2). The parameters may be chosen such that $\mathcal{A}$ also satisfies monotonicity (MA3) and translation-equivariance (MA1) with respect to some pair $(\mathcal{W}, \pi)$.

Note that multi-period polyhedral functionals are flexible tools. In particular, the representation (26) of the functional

$$Y \mapsto \gamma \mathcal{A}(Y;\mathfrak{F}) + \sum_{t=1}^{T} \mu_t \mathbb{E}\left[Y_t\right]$$

with a multi-period polyhedral functional $\mathcal{A}$ and real numbers γ and μ_t, $t = 1, \ldots, T$, can be fully reduced to the representation of $\mathcal{A}$ by modifying only the parameters c_t ($t = 0, \ldots, T$) of $\mathcal{A}$. Next we state a continuity and dual representation result that extends [27, Theorem 3.38] and may be proved similarly.

Proposition 4.12 *Let $\mathcal{A}$ be a multi-period polyhedral functional and assume (i) $B_{t,0}V_t = \mathbb{R}^{d_t}$, $t = 0, \ldots, T$, and (ii) there exists $\bar{u} \in \times_{t=0}^{T} \mathbb{R}^{d_t}$ such that*

$$c_t - \sum_{\tau=t}^{T} B_{\tau,\tau-t}^{\top} \bar{u}_\tau \in V_t^* \quad (t = 0, \ldots, T),$$

where the sets V_t^* are the polar cones to V_t, i.e., $V_t^* = \{v_t^* \in \mathbb{R}^{k_t} : \langle v_t^*, v_t\rangle \leq 0, \forall v_t \in V_t\}$. Then $\mathcal{A}$ is finite, concave and continuous on $\times_{t=1}^{T} L_p(\mathcal{F}_t)$ with $p \in [1, +\infty)$ and admits the representation

$$\mathcal{A}(Y;\mathfrak{F}) = \inf\left\{ \mathbb{E}\left[\sum_{t=0}^{T}((R_t^{\top} z_t)\, Y_t + \hat{r}_t^{\top} z_t)\right] \;\middle|\; \begin{array}{l} z_t \in L_q(\mathcal{F};\mathbb{R}^{d_t}),\ t = 0, \ldots, T \\ c_t - \sum_{\tau=t}^{T} B_{\tau,\tau-t}^{\top} \mathbb{E}(z_\tau|\mathcal{F}_t) \in V_t^* \end{array} \right\}, \tag{27}$$

where $q \in (1, \infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$ and r_t has the representation $r_t(Y) = R_t Y + \hat{r}_t$ with $R_t, \hat{r}_t \in \mathbb{R}^{d_t}$, $t = 0, \ldots, T$, $R_0 = 0$.

Considering $R_t^\top z_t$ as dual variable $Z_t \in L_q(\mathcal{F})$, $t = 1, \ldots, T$, the representation (27) may be compared with the general dual representation (14) to derive conditions on the parameters of (26) such that conditions (MA1) and (MA3) are satisfied and, hence, $\mathcal{A}$ represents a multi-period acceptability functional. We refer to Example 4.13 for special choices of the parameters (see also [27, Section 3.3.5] for examples with $R_t = d_t = 1$, $\hat{r}_t = 0$).

The major advantage of multi-period polyhedral acceptability functionals $\mathcal{A}$ consists in their inherent linearity structure. To see this let $\{\xi_t\}_{t=1}^T$ be a stochastic input process entering the coefficients of a linear (multi-stage) stochastic optimization model with decisions x_t (at time t) measurable with respect to the σ-field $\mathcal{F}_t$ generated by $(\xi_0, \ldots, \xi_t)$. If the stochastic revenue at time t is given by $Y_t = \langle b_t(\xi_t), x_t \rangle$, a minimum risk decision is determined by maximizing $\mathcal{A}(Y_0, Y_1, \ldots, Y_T; \mathfrak{F})$ subject to the constraints of the original optimization model. Taking into account the structure (26) of $\mathcal{A}$, the maximization problem of $\mathcal{A}(Y_1, \ldots, Y_T; \mathfrak{F})$ may be reformulated as a linear stochastic optimization model with respect to the variable (v, x) where v stems from (26).
Another advantage of multi-period polyhedral functionals consists in the possibility of using convex analysis methods for deriving continuity properties, directional derivatives and superdifferentials.

Example 4.13 *(a) Multi-period average value-at-risk* $m\mathbb{A}V@R$*: Comparing the definition (17) with the definition of polyhedral functionals, one sees that the* $m\mathbb{A}V@R$ *is of the form (26) by setting* $k_t = 3$, $d_t = 1$, $\hat{r}_t = 0$, $V_t = \mathbb{R} \times \mathbb{R}_+^2$, $t = 0, \ldots, T$, $c_0 = (1, 0, 0)$, $c_t = (1, 0, -\frac{1}{\alpha})$, $t = 1, \ldots, T-1$, $c_T = (0, 0, -\frac{1}{\alpha})$, $B_{t,0} = (0, 1, -1)$, $B_{t,1} = (1, 0, 0)$, $t = 0, \ldots, T$, $R_t = 1$, $t = 1, \ldots, T$.

(b) We consider the multi-period acceptability functional $\mathcal{A}$ *on* $\mathcal{Y} = \times_{t=1}^T L_1(\mathcal{F})$ *given by (see Section 4.1)*

$$\begin{aligned}
\mathcal{A}(Y; \mathfrak{F}) &:= \mathbb{A}V@R_\alpha\Big(\min_{t=1,\ldots,T} Y_t\Big) = \max\Big\{a - \frac{1}{\alpha}\mathbb{E}\Big(\Big[\min_{t=1,\ldots,T} Y_t - a\Big]^-\Big) : a \in \mathbb{R}\Big\} \\
&= \max\Big\{a - \frac{1}{\alpha}\mathbb{E}\Big(\max_{t=1\ldots,T}\{0, a - Y_t\}\Big) : a \in \mathbb{R}\Big\} \\
&= \max\Big\{a - \frac{1}{\alpha}\mathbb{E}\,(v_T) : v_t = \mathbb{E}(v_t|\mathcal{F}_t), a - Y_t \le v_t, v_{t-1} \le v_t, \\
&\qquad\qquad t = 1, \ldots, T,\ v_0 \in \mathbb{R}_+, a \in \mathbb{R}\Big\}
\end{aligned}$$

for any $\alpha \in (0, 1]$*. Hence,* $\mathcal{A}$ *is polyhedral.*

Some computational experience for financial and electricity portfolio management models involving the multi-period polyhedral functionals in Example 4.13 is reported in [27, Chapters 5 and 6] and [11].

Acknowledgement:
The cooperation of both authors was supported by the Wiener Wissenschafts-, Forschungs- und Technologiefonds (WWTF). In addition, the research of the second author was partially supported by the DFG Research Center MATHEON *Mathematics for Key Technologies* in Berlin (`http://www.matheon.de`).

References

[1] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, Coherent measures of risk, *Math. Finance*, 9 (1999), 203–228.

[2] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, and H. Ku, Coherent multiperiod risk adjusted values and Bellman's principle, *Annals of Operations Research* 152 (2007), 5–22.

[3] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York 2000.

[4] P. Cheridito, F. Delbaen, and M. Kupper, Coherent and convex monetary risk measures for unbounded càdlàg processes, *Finance Stochast.* 9 (2005), 369–387.

[5] P. Cheridito, F. Delbaen, and M. Kupper, Dynamic monetary risk measures for bounded discrete-time processes, *Electronic Journal of Probability* 11 (2006), 57–106.

[6] P. Cheridito and M. Kupper, Composition of time-consistent dynamic monetary risk measures in discrete time, preprint, Princeton University, 2006 and submitted (`www.princeton.edu/~dito/papers/`).

[7] P. Cheridito and T. Li, Risk measures on Orlicz hearts, *Math. Finance* (to appear).

[8] P. Cheridito and T. Li, Dual characterization of properties of risk measures on Orlicz hearts, *Mathematics and Financial Economics* (to appear).

[9] K. Detlefsen and G. Scandolo, Conditional and dynamic convex risk measures, *Finance Stochast.* 9 (2005), 539–561.

[10] A. Eichhorn and W. Römisch, Polyhedral risk measures in stochastic programming, *SIAM J. Optim.* 16 (2005), 69–95.

[11] A. Eichhorn and W. Römisch, Dynamic risk management in electricity portfolio optimization via polyhedral risk functionals, Proceedings of the 2008 PES IEEE General Meeting.

[12] H. Föllmer and A. Schied, Convex measures of risk and trading constraints, *Finance Stochast.* 6 (2002), 429–447.

[13] H. Föllmer and A. Schied, *Stochastic Finance: An Introduction in Discrete Time*, Walter de Gruyter, Berlin, 2002 (Second Edition 2004).

[14] H. Föllmer, A. Schied and S. Weber, Robust preferences and robust portfolio choice, preprint, Humboldt-University Berlin, 2007.

[15] M. Fritelli and E. Rosazza Gianin, Putting order in risk measures, *Journal of Banking & Finance*, 26 (2002), 1473–1486.

[16] M. Fritelli and E. Rosazza Gianin, Dynamic convex risk measures, in *Risk Measures for the 21st Century*, G. Szegö, ed., Wiley, Chichester, 2004, 227–248.

[17] M. Fritelli and E. Rosazza Gianin, Law invariant convex risk measures, *Advances in Mathematical Economics* 7 (2005), 33–46.

[18] M. Fritelli and G. Scandolo, Risk measures and capital requirements for processes, *Math. Finance* 16 (2006), 589–612.

[19] S. Jaschke and U. Küchler, Coherent risk measures and good-deal bounds, *Finance Stochast.* 5 (2001), 181–200.

[20] E. Jouini, W. Schachermayer, and N. Touzi, Law invariant risk measures have the Fatou property, *Advances in Mathematical Economics* 9 (2006), 49–71.

[21] M. Kupper and W. Schachermayer, Representation results for law invariant time consistent functions, preprint, TU Wien, 2008 (`www.fam.tuwien.ac.at/~wschach`).

[22] S. Kusuoka, On law invariant coherent risk measures, *Advances in Mathematical Economics* 3 (2001), 83–95.

[23] R. Kovacevic, *Conditional Acceptability Mappings: Convex Analysis in Banach Lattices*, PhD Dissertation, University of Vienna, 2008.

[24] K. Nikodem, Continuity properties of convex-type set-valued maps, *Journal of Inequalities in Pure and Applied Mathematics* 4(3) (2003), Article 52 (`jipam.vu.edu.au`).

[25] G. Ch. Pflug, The value-of-information approach to measuring risk in multiperiod economic activity, *Journal of Banking & Finance* 30 (2006), 695–715.

[26] G. Ch. Pflug, Subdifferential representations of risk measures, *Math. Programming* 108 (2006), 339–354.

[27] G.Ch. Pflug and W. Römisch, *Modeling, Measuring and Managing Risk*, World Scientific, Singapore, 2007.

[28] G. Ch. Pflug and A. Ruszczyński, Risk measures for income streams, in *Risk Measures for the 21st Century*, G. Szegö, ed., Wiley, Chichester, 2004, 249–269.

[29] F. Riedel, Dynamic coherent risk measures, *Stoch. Proc. Appl.* 112 (2004), pp. 185–200.

[30] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton 1970.

[31] R.T. Rockafellar, *Conjugate Duality and Optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics 16, SIAM, Philadelphia, 1974.

[32] R.T. Rockafellar and S. Uryasev, Conditional value-at-risk for general loss distributions, *Journal of Banking & Finance*, 26 (2002), pp. 1443–1471.

[33] R.T. Rockafellar, S. Uryasev, and M. Zabarankin, Generalized deviations in risk analysis, *Finance Stochast.* 10 (2006), 51–74.

[34] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer, Berlin 1998.

[35] B. Roorda, J.M. Schumacher, and J. Engwerda: Coherent acceptability measures in multiperiod models, *Math. Finance* 15 (2005), 589–612.

[36] A. Ruszczyński and A. Shapiro, Eds., *Stochastic Programming*, Handbooks of Operations Research and Management Science Vol. 10, Elsevier, Amsterdam, 2003.

[37] A. Ruszczyński and A. Shapiro, Optimization of convex risk functions, *Math. Oper. Research* 31 (2006), 433–452.

[38] A. Ruszczyński and A. Shapiro, Conditional risk mappings, *Math. Oper. Research* 31 (2006), 544–561.

[39] H. H. Schaefer, *Topological Vector Spaces*, Springer, New York, 1971.

[40] G. Szegö, Ed., *Risk Measures for the 21st Century*, Wiley, Chichester 2004.

[41] S. Weber, Distribution-invariant dynamic risk measures, information and dynamic consistency, *Math. Finance* 16 (2006), 419–441.