Multi-Period Risk Functionals

G. Ch. Pflug and W. Römisch

University of Vienna
Institute of Statistics and Decision Support Systems
and
Humboldt-University Berlin
Institute of Mathematics

http://www.math.hu-berlin.de/~romisch

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Multistage stochastic programs

Let $\{\xi_t\}_{t=1}^T$ be a discrete-time stochastic data process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with ξ_1 deterministic. The stochastic decision x_t at period t is assumed to be measurable with respect to $\mathcal{F}_t := \sigma(\xi_1, \ldots, \xi_t)$ (nonanticipativity).

Multistage stochastic optimization model:

$$\max \left\{ \mathbb{E}\left[\sum_{t=1}^{T} \langle b_t(\xi_t), x_t \rangle\right] \middle| \begin{array}{l} x_t \in X_t, x_t \text{ is } \mathcal{F}_t\text{-measurable}, t = 1, \ldots, T \\ A_{t,0}x_t + A_{t,1}x_{t-1} = h_t(\xi_t), t = 2, \ldots, T \end{array} \right\}$$

where the sets X_t, $t = 1, \ldots, T$, are closed and their convex hulls polyhedral, the vectors $b_t(\cdot)$ and $h_t(\cdot)$ are affine functions of ξ_t.

Typical applications: Power production and trading planning, revenue and portfolio management models.

Question: How to incorporate risk into multi-period models ?

Axiomatic characterization of single-period risk

Let $\mathcal{Y} = L_p(\Omega, \mathcal{F}, \mathbb{P}) = L_p(\mathcal{F})$, $1 \le p \le +\infty$. A mapping $\mathcal{A} : \mathcal{Y} \to \overline{\mathbb{R}}$ is called acceptability functional if it satisfies the following conditions for all $Y, \tilde{Y} \in \mathcal{Y}$, $r \in \mathbb{R}$, $\lambda \in [0,1]$:

(A1) $\mathcal{A}(Y + r) = \mathcal{A}(Y) + r$ (translation-equivariance),

(A2) $\mathcal{A}(\lambda Y + (1-\lambda)\tilde{Y}) \ge \lambda \mathcal{A}(Y) + (1-\lambda)\mathcal{A}(\tilde{Y})$ (concavity),

(A3) $Y \le \tilde{Y}$ implies $\mathcal{A}(Y) \le \mathcal{A}(\tilde{Y})$ (monotonicity).

An acceptability functional $\mathcal{A}$ is called
positively homogeneous if $\mathcal{A}(\lambda Y) = \lambda \mathcal{A}(Y)$, $\forall \lambda \ge 0$, $Y \in \mathcal{Y}$.
strict if $\mathcal{A}(Y) \le \mathbb{E}(Y)$, $\forall Y \in \mathcal{Y}$.
version-independent if $\mathcal{A}(Y)$ depends only on the distribution $\mathbb{P}\,Y^{-1}$.
Given an acceptability functional $\mathcal{A}$, the mappings

$$\rho := -\mathcal{A} \quad \text{and} \quad \mathcal{D} := \mathbb{E} - \mathcal{A}$$

are called capital risk and deviation risk functional, respectively.

References: Artzner-Delbaen-Eber-Heath 99, Föllmer-Schied 02, Pflug-Römisch 07

Examples:

(a) Lower semi standard deviation corrected expectation:

$$\mathcal{A}(Y) := \mathbb{E}(Y) - \left(\mathbb{E}([Y - \mathbb{E}(Y)]^-)^2\right)^{\frac{1}{2}}$$

(Markowitz' mean-(lower)variance model)

(b) Average value-at-risk

The Average value-at-risk of Y at level $\alpha \in (0,1]$ is defined as

$$\mathbb{AV@R}_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha G^{-1}(u)du = \max\left\{x - \frac{1}{\alpha}\mathbb{E}([Y - x]^-) : x \in \mathbb{R}\right\},$$

where G is the distribution function of Y.

Conditional risk mappings

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{F}_1$ be a σ-field contained in $\mathcal{F}$. Let $\mathcal{Y} = L_p(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{Y}_1 = L_p(\Omega, \mathcal{F}_1, \mathbb{P})$ for some $p \in [1, +\infty)$, hence $\mathcal{Y}_1 \subseteq \mathcal{Y}$. All (in)equalities between random variables in $\mathcal{Y}$ are intended to hold $\mathbb{P}$-almost surely.

A mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}_1$ is called conditional acceptability mapping (with observable information $\mathcal{F}_1$) if the following conditions are satisfied for all $Y, \tilde{Y} \in \mathcal{Y}$, $Y^{(1)} \in \mathcal{Y}_1$, $\lambda \in [0, 1]$:

(CA1) $\mathcal{A}(Y + Y^1) = \mathcal{A}(Y) + Y^{(1)}$ (predictable translation-equivariance),

(CA2) $\mathcal{A}(\lambda Y + (1-\lambda)\tilde{Y}) \geq \lambda \mathcal{A}(Y) + (1-\lambda)\mathcal{A}(\tilde{Y})$ (concavity),

(CA3) $Y \leq \tilde{Y}$ implies $\mathcal{A}(Y) \leq \mathcal{A}(\tilde{Y})$ (monotonicity).

The conditional acceptability mapping $\mathcal{A}$ is called
positively homogeneous if $\mathcal{A}(\lambda Y) = \lambda \mathcal{A}(Y)$, $\forall \lambda > 0$.
upper semicontinuous if $\mathbb{E}(\mathcal{A}(\cdot)\mathbb{1}_B) : \mathcal{Y} \to \overline{\mathbb{R}}$ is upper semicontinuous $\forall B \in \mathcal{F}_1$.

For a conditional acceptability mapping with observable information $\mathcal{F}_1$ we will use the notations $\mathcal{A}(\cdot|\mathcal{F}_1)$ or $\mathcal{A}_{\mathcal{F}_1}$. The mapping $\rho = \rho_{\mathcal{F}_1} := -\mathcal{A}_{\mathcal{F}_1}$ is called conditional risk mapping (with observable information $\mathcal{F}_1$).

Theorem 1: (representation theorem)
Let $\mathcal{A} = \mathcal{A}_{\mathcal{F}_1} : \mathcal{Y} \to \mathcal{Y}_1$ be an upper semicontinuous and positively homogeneous conditional acceptability mapping. Then the representation

$$\mathbb{E}(\mathcal{A}(Y)\mathbb{1}_B) = \inf_{Z \in \mathcal{S}_B} \{\mathbb{E}(YZ) : Z \geq 0, \mathbb{E}(Z|\mathcal{F}_1) = \mathbb{1}_B\}$$

is valid for every $Y \in \mathcal{Y}$ and $B \in \mathcal{F}_1$ with a closed convex set $\mathcal{S}_B = \{Z \in L_q(\mathcal{F}) : A_B(Z) \geq 0\}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$A_B(Z) := \inf\{\mathbb{E}(YZ) - \mathbb{E}(\mathcal{A}(Y)\mathbb{1}_B) : Y \in \mathcal{Y}\}$$

for every $Z \in L_q(\mathcal{F})$ and $B \in \mathcal{F}_1$.

A partial converse of Theorem 1 on L_1:

Theorem 2: (existence theorem)
Let $\mathcal{S}$ be a closed convex subset of $L_\infty(\mathcal{F})$ such that $\mathbb{1} \in \mathcal{S}$ and $Z\mathbb{1}_B \in \mathcal{S}$ for every $B \in \mathcal{F}_1$ and $Z \in S$. Then the equations

$$\mathbb{E}(\mathcal{A}(Y)\mathbb{1}_B) = \inf_{Z\in\mathcal{S}} \{\mathbb{E}(YZ) : Z \geq 0, \mathbb{E}(Z|\mathcal{F}_1) = \mathbb{1}_B\}, \ \forall B \in \mathcal{F}_1,$$

define an upper semicontinuous and positively homogeneous conditional acceptability mapping $\mathcal{A} : L_1(\mathcal{F}) \to L_1(\mathcal{F}_1)$.

Proof: using the Radon-Nikodym theorem for σ-additive signed measures which are absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_1$.

Proposition: (continuity)
A conditional acceptability mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}_1$ is continuous if it is locally bounded at some element of $\mathcal{Y}$.

Proof: follows from a more general continuity result for cone-convex mappings, see a survey of Nikodem 03.

Examples:

(a) Conditional expectation: The defining equation for the conditional expectation $\mathbb{E}(\cdot\,|\mathcal{F}_1)$, namely,

$$\mathbb{E}(\mathbb{E}(Y|\mathcal{F}_1)\,\mathbb{1}_B) = \mathbb{E}(Y\mathbb{1}_B) \quad (\forall B \in \mathcal{F}_1)$$

can be recovered from Theorem 2 by

$$\begin{aligned}\mathbb{E}(\mathbb{E}(Y|\mathcal{F}_1)\mathbb{1}_B) &= \inf\{\mathbb{E}(YZ) : 0 \le Z \le 1, \mathbb{E}(Z|\mathcal{F}_1) = \mathbb{1}_B\} \\ &= \mathbb{E}(Y\mathbb{1}_B).\end{aligned}$$

It is a mapping from $L_p(\mathcal{F})$ onto $L_p(\mathcal{F}_1)$ for $p \in [1, \infty)$.

(b) Conditional average value-at-risk: $\mathbb{AV@R}_\alpha(Y\,|\mathcal{F}_1)$ is defined on $L_1(\mathcal{F})$ by the relation

$$\begin{aligned}\mathbb{E}(\mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)\mathbb{1}_B) &= \inf\{\mathbb{E}(YZ) : 0 \le Z \le \frac{1}{\alpha}\mathbb{1}_B, \\ &\qquad \mathbb{E}(Z|\mathcal{F}_1) = \mathbb{1}_B\}.\end{aligned}$$

for every $B \in \mathcal{F}_1$. Due to Theorem 2 and the Proposition the mapping $Y \mapsto \mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)$ is positively homogeneous, continuous and satisfies conditions (CA1)–(CA3).

Multi-period acceptability functionals

Let a filtration of σ-fields $\mathscr{F} = (\mathcal{F}_t)_{t=0}^T$ with $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (information flow for income processes) be given. A functional $\mathcal{A} = \mathcal{A}(\cdot\,;\mathscr{F}) : \mathcal{Y} := \times_{t=1}^T L_p(\mathcal{F}_t) \to \overline{\mathbb{R}}$ is called multi-period acceptability functional if it satisfies the following conditions for all $Y, \tilde{Y} \in \times_{t=1}^T L_p(\mathcal{F}_t)$:

(MA0) $\mathcal{F}_t \subseteq \mathcal{F}'_t$, $\forall t$, implies $\mathcal{A}(Y_1, \ldots, Y_T; \mathscr{F}) \le \mathcal{A}(Y_1, \ldots, Y_T; \mathscr{F}')$ (information monotonicity),

(MA1) $\tilde{Y}_t \in L_p(\mathcal{F}_{t-1})$ implies $\mathcal{A}(Y_1, \ldots, Y_t + \tilde{Y}_t, \ldots, Y_T) = \mathbb{E}(\tilde{Y}_t) + \mathcal{A}(Y_1, \ldots, Y_T)$ ((predictable) translation-equivariance),

(MA2) $\mathcal{A}$ is concave on $\mathcal{Y}$ (concavity),

(MA3) $Y_t \le \tilde{Y}_t$, $\forall t$, implies $\mathcal{A}(Y_1, \ldots, Y_T) \le \mathcal{A}(\tilde{Y}_1, \ldots, \tilde{Y}_T)$ (monotonicity).

Notation: $\mathcal{A}(Y; \mathscr{F})$ or $\mathcal{A}(Y_1, \ldots, Y_T; \mathcal{F}_1, \ldots, \mathcal{F}_T)$.

The functionals $\rho := -\mathcal{A}$ and $\mathcal{D}(Y) := \sum_{t=1}^T \mathbb{E}(Y_t) - \mathcal{A}(Y)$ are called a multi-period capital risk and deviation risk functionals.

Weaker translation-equivariance conditions:

(MA1)' $\mathcal{A}(Y_1,\ldots,Y_t+c_t,\ldots,Y_T;\boldsymbol{\mathcal{F}}) = c_t + \mathcal{A}(Y_1,\ldots,Y_T;\boldsymbol{\mathcal{F}})$ for all $c_t \in \mathbb{R}$, $t = 1,\ldots,T$ (weak translation-equivariance).

(MA1)" $\mathcal{A}(Y_1+c_1,Y_2,\ldots,Y_T;\boldsymbol{\mathcal{F}}) = c_1 + \mathcal{A}(Y_1,Y_2,\ldots,Y_T;\boldsymbol{\mathcal{F}})$ for all $c_1 \in \mathbb{R}$ (first-period translation-equivariance).

General translation-equivariance condition: (Frittelli-Scandolo, Math.Fin. 06)

(MA1)* $\mathcal{A}(Y+W;\boldsymbol{\mathcal{F}}) = \mathcal{A}(Y;\boldsymbol{\mathcal{F}}) + \pi(W)$ for all $W \in \mathcal{W}$, where $\mathcal{W}$ is a closed linear subspace of $\mathcal{Y}$ and $\pi : \mathcal{W} \longrightarrow \mathbb{R}$ is linear and continuous ($(\pi,\mathcal{W})$-translation-equivariance).

Special cases:

$$\left.\begin{array}{lcl} \text{(MA1)} & \Leftrightarrow & \text{(MA1)}^* : \ \mathcal{W} := \times_{t=0}^{T-1} L_p(\mathcal{F}_t) \\ \text{(MA1)'} & \Leftrightarrow & \text{(MA1)}^* : \ \mathcal{W} := \mathbb{R}^T \\ \text{(MA1)"} & \Leftrightarrow & \text{(MA1)}^* : \ \mathcal{W} := \mathbb{R} \times \{0\}^{T-1} \end{array}\right\} \pi(W) := \sum_{t=1}^{T} \mathbb{E}(W_t).$$

Dual representations and properties

Let $\mathcal{Z}$ denote the topological dual of $\mathcal{Y}$ for $p \in [1, +\infty)$, i.e., $\mathcal{Z} := \times_{t=1}^{T} L_q(\mathcal{F}_t)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $\langle Z, Y \rangle = \sum_{t=1}^{T} \mathbb{E}(Z_t Y_t)$ be the dual pairing between $\mathcal{Z}$ and $\mathcal{Y}$.

A multi-period acceptability functional $\mathcal{A} = \mathcal{A}(\cdot\,; \mathscr{F})$ is called proper if $\mathcal{A}(Y) < +\infty$ for all $Y \in \mathcal{Y}$ and its domain $\mathrm{dom}(\mathcal{A}) := \{Y \in \mathcal{Y} : \mathcal{A}(Y) > -\infty\}$ is nonempty.

The conjugate $\mathcal{A}^+ : \mathcal{Z} \to \overline{\mathbb{R}}$ of $\mathcal{A}$ is given by

$$\mathcal{A}^+(Z) := \inf_{Y \in \mathcal{Y}} \{\langle Z, Y \rangle - \mathcal{A}(Y)\}.$$

The Fenchel-Moreau-Rockafellar theorem implies

$$\mathcal{A}(Y) = \inf_{Z \in \mathcal{Z}} \{\langle Z, Y \rangle - \mathcal{A}^+(Z)\}$$

if $\mathcal{A}$ is a proper and upper semicontinuous multi-period acceptability functional. If, in addition, $\mathcal{A}$ is positively homogeneous, then

$$\mathcal{A}(Y) = \inf_{Z \in \mathcal{S}} \langle Z, Y \rangle,$$

where $\mathcal{S}$ is the closed convex set $\mathcal{S} := \mathrm{dom}(\mathcal{A}^+)$.

(Ruszczyński-Shapiro, MOR 06)

Theorem 3:

Let $\mathcal{A} = \mathcal{A}(\cdot\,;\mathscr{F}) : \mathcal{Y} \to \overline{\mathbb{R}}$ be a proper, positively homogeneous and upper semicontinuous multi-period acceptability functional satisfying (MA1)*. Then the representation

$$\mathcal{A}(Y) = \inf_{Z\in\mathcal{S}} \left\{ \sum_{t=1}^{T} \mathbb{E}(Z_t Y_t) : \pi(\cdot) = \langle Z, \cdot\rangle,\ Z_t \geq 0,\ t = 1,\ldots,T \right\}$$

is valid for every $Y \in \mathcal{Y}$, where $\mathcal{S} = \mathrm{dom}(\mathcal{A}^{+}) \subseteq \mathcal{Z}$. Notice that
(MA1) $\quad \pi(\cdot) = \langle Z, \cdot\rangle \;\Leftrightarrow\; \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1,\ t = 1,\ldots,T,$
(MA1)' $\quad \pi(\cdot) = \langle Z, \cdot\rangle \;\Leftrightarrow\; \mathbb{E}(Z_t) = 1,\ t = 1,\ldots,T.$
Conversely, if $\mathcal{A}$ can be represented as above with a nonempty, closed and convex set $\mathcal{S} \subseteq \mathcal{Z}$, then $\mathcal{A}$ is a proper, positively homogeneous and upper semicontinuous multi-period acceptability functional satisfying (MA1)*.

Moreover, $\mathcal{A}$ is locally Lipschitz continuous, superdifferentiable and Hadamard directionally differentiable on $\mathrm{int}\,\mathrm{dom}(\mathcal{A})$

(Ruszczynski-Shapiro, MOR 06).

Examples: (Separable constructions)

(a) Separable multi-period acceptability functionals:

$$\mathcal{A}(Y;\mathscr{F}) := \sum_{t=1}^{T} \mathcal{A}_t(Y_t),$$

where $\mathcal{A}_t$ are single-period acceptability functionals, satisfy (MA1)', (MA2) and (MA3), but do not depend on $\mathscr{F}$.

(b) SEC multi-period acceptability functionals:

$$\mathcal{A}(Y;\mathscr{F}) := \sum_{t=1}^{T} \mathbb{E}(\mathcal{A}_t(Y_t|\mathcal{F}_{t-1}))$$

where $\mathcal{A}_t(\cdot\,|\mathcal{F}_{t-1})$, $t=1,\ldots,T$, are conditional (single-period) acceptability functionals, satisfy (MA0)–(MA3).

Example: (Multi-period average value-at-risk, Pflug-Ruszczyński 04)

$$m\mathbb{AV@R}_\alpha(Y;\mathscr{F}) := \sum_{t=1}^{T} \mathbb{E}(\mathbb{AV@R}_\alpha(Y_t|\mathcal{F}_{t-1}))$$

$$= \inf\Big\{\sum_{t=1}^{T} \mathbb{E}(Y_t Z_t) : Z_t \in [0,\frac{1}{\alpha}], \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1, \forall t\Big\}$$

Multi-period polyhedral acceptability functionals

It is a natural idea to introduce acceptability and risk functionals as optimal values of certain stochastic programs.

Definition: (Eichhorn-Römisch, SIAM J. Opt. 05)
A multi-period functional $\mathcal{A}$ on $\times_{t=1}^{T} L_p(\mathcal{F}_t)$ is called polyhedral if there are $k_t \in \mathbb{N}$, $c_t \in \mathbb{R}^{k_t}$, $t = 1, \ldots, T$, $w_{t\tau} \in \mathbb{R}^{k_{t-\tau}}$, $t = 1, \ldots, T$, $\tau = 0, \ldots, t-1$, (convex) polyhedral sets $V_t \subset \mathbb{R}^{k_t}$, $t = 1, \ldots, T$, such that

$$\mathcal{A}(Y) = \sup \left\{ \mathbb{E}\left[\sum_{t=1}^{T} \langle c_t, v_t \rangle\right] \middle| \begin{array}{l} v_t \in L_p(\mathcal{F}_t; \mathbb{R}^{k_t}),\ v_t \in V_t, \\ \sum_{\tau=0}^{t-1} \langle w_{t,\tau}, v_{t-\tau} \rangle = Y_t,\ t = 1, \ldots, T \end{array} \right\}.$$

Result: There exist multi-period polyhedral acceptability functionals satisfying (MA0), (MA1) ((MA1)',(MA1)''), (MA2), (MA3) (strictness, positive homogeneity).

Multi-period polyhedral acceptability functionals preserve linearity, decomposition structures and stability properties of multi-stage stochastic programming models. When replacing $\mathbb{E}$ by $\mathcal{A}$ we obtain a linear multi-stage stochastic program of the form

$$\max\left\{\mathbb{E}\left[\sum_{t=1}^{T}\langle c_t, v_t\rangle\right] \middle| \begin{array}{l} v_t \text{ and } x_t\ \mathcal{F}_t\text{-measurable, } v_t \in V_t,\ x_t \in X_t, \\ \sum_{\tau=0}^{t-1}\langle w_{t,\tau}, v_{t-\tau}\rangle = \langle b_t(\xi_t), x_t\rangle, t = 1, \ldots, T, \\ A_{t,0}x_t + A_{t,1}x_{t-1} = h_t(\xi_t), t = 2, ..., T. \end{array}\right\}$$

by introducing the additional variables v_t, $t = 1, \ldots, T$.

Examples:

(a) Multi-period average value-at-risk $m\mathbb{AV}@\mathsf{R}$.

(b) $\mathcal{A}_2(Y) := \mathbb{AV}@\mathsf{R}_\alpha(\sum_{\tau=1}^{t(\cdot)} Y_\tau)$, where $t(\cdot)$ is uniformly distributed on $\{1, \ldots, T\}$ and independent of $(Y_\tau)_{\tau=1}^{T}$, is polyhedral (Eichhorn 07).

(c) $\mathcal{A}_6(Y) := \mathbb{AV}@\mathsf{R}_\alpha(\min\{Y_1, \ldots, \sum_{\tau=1}^{t} Y_\tau, \ldots, \sum_{\tau=1}^{T} Y_\tau\})$ is polyhedral (Eichhorn 07; Artzner-Delbaen-Eber-Heath-Ku 07).

Both acceptability mappings satisfy (MA0), (MA1)", (MA2), (MA3) and positive homogeneity.

Composition of conditional acceptability mappings

Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathfrak{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$ of σ-fields $\mathcal{F}_t$, $t = 0, ..., T$, with $\mathcal{F}_T = \mathcal{F}$ be given. We consider the Banach spaces $\mathcal{Y}_t := L_p(\mathcal{F}_t)$ of $\mathcal{F}_t$-measurable (real) random variables for $t = 1, \ldots, T$ and some $p \in [1, +\infty)$.

Let, for each $t = 1, \ldots, T$, conditional acceptability mappings $\mathcal{A}_{t-1} := \mathcal{A}(\cdot\,|\mathcal{F}_{t-1})$ from $\mathcal{Y}_T$ to $\mathcal{Y}_{t-1}$ be given satisfying the following conditions for all Y_T and $\tilde{Y}_T$ in $\mathcal{Y}_T$. We introduce a multi-period probability functional $\mathcal{A}$ on $\mathcal{Y} := \times_{t=1}^{T} \mathcal{Y}_t$ and a family $(\mathcal{A}^{(t)})_{t=1}^{T}$ of single-period probability functionals $\mathcal{A}^{(t)}$ by compositions of the conditional acceptability mappings $\mathcal{A}_{t-1}$, $t = 1, \ldots, T$, namely,

$$\begin{aligned} \mathcal{A}(Y; \mathfrak{F}) &:= \mathcal{A}_0[Y_1 + \cdots + \mathcal{A}_{T-2}[Y_{T-1} + \mathcal{A}_{T-1}(Y_T)]\cdot] \\ \mathcal{A}^{(t)}(Y_T) &:= \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{t-1}(Y_T) \end{aligned}$$

for every $Y \in \mathcal{Y}$ and $Y_T \in \mathcal{Y}_T$.

(Ruszczynski-Shapiro, Math. OR 06)

Proposition: (Ruszczynski-Shapiro, Math. OR 06)
The multi-period functional $\mathcal{A}(\cdot;\mathscr{F}) : \mathcal{Y} \to \overline{\mathbb{R}}$ satisfies the conditions (MA1'), (MA2) and (MA3). Every $\mathcal{A}^{(t)} : \mathcal{Y}_T \to \mathbb{R}$ is a (single-period) acceptability functional. Moreover, it holds

$$\mathcal{A}(Y;\mathscr{F}) = \mathcal{A}^{(T)}(Y_1 + \cdots + Y_T).$$

The functionals $\mathcal{A}$ and $\mathcal{A}^{(t)}$, $t = 1, \ldots, T$, are positively homogeneous if all $\mathcal{A}_t$ are positively homogeneous.

Example:
We consider the conditional average value-at-risk (of level $\alpha \in (0, 1]$) as conditional acceptability mapping

$$\mathcal{A}_{t-1}(Y_t) := \mathbb{AV@R}_\alpha(\cdot\,|\mathcal{F}_{t-1})$$

for every $t = 1, \ldots, T$. Then the multi-period probability functional

$$n\mathbb{AV@R}_\alpha(Y;\mathscr{F}) = \mathbb{AV@R}_\alpha(\cdot\,|\mathcal{F}_0) \circ \cdots \circ \mathbb{AV@R}_\alpha(\cdot\,|\mathcal{F}_{T-1})\Big(\sum_{t=1}^{T} Y_t\Big)$$

satisfies (MA0), (MA1'), (MA2), (MA3) according to the Proposition. It is called the nested average value-at-risk.

Proposition:

The nested $n\mathbb{AV}@\mathsf{R}$ has the following dual representation:

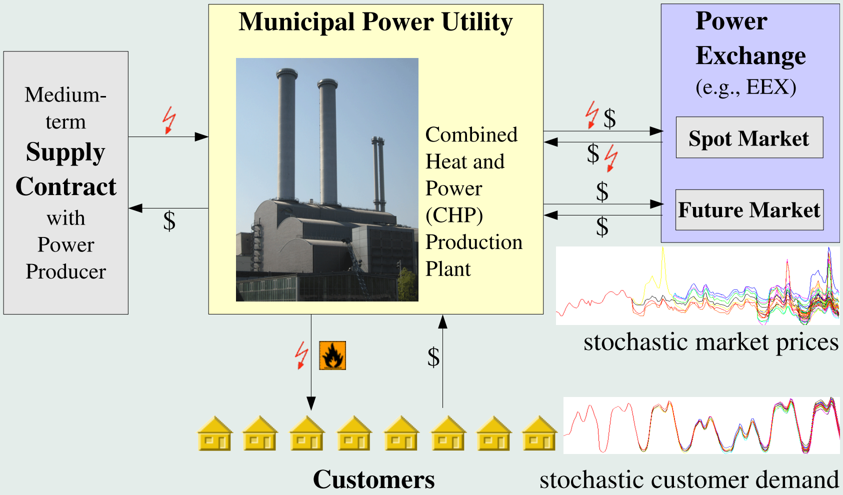
$$n\mathbb{AV}@\mathsf{R}_\alpha(Y;\mathscr{F}) = \inf\{\mathbb{E}[(Y_1+\ldots+Y_T)Z_T] : 0 \le Z_t \le \frac{1}{\alpha}Z_{t-1}, \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = Z_{t-1}, Z_0 = 1, t = 1,\ldots,T\}.$$

Notice that the (dual) process (Z_t) is a martingale and that $n\mathbb{AV}@\mathsf{R}$ isn't polyhedral, but given by a linear stochastic program (with operator constraints).

Electricity Portfolio Management

We consider the electricity portfolio management of a municipal electric utility. Its portfolio consists of the following positions:

- power and heat production (by company-owned thermal units),
- (physical) (day-ahead) spot market trading (e.g., EEX) and
- (financial) trading of derivatives (here, futures).



Schematic diagram for the optimization model components

(Eichhorn-Römisch-Wegner 05, Eichhorn-Heitsch-Römisch 07)

The yearly time horizon is discretized into hourly intervals.

Objective: Maximizing the expected revenue and/or the acceptability of its production and trading decisions.

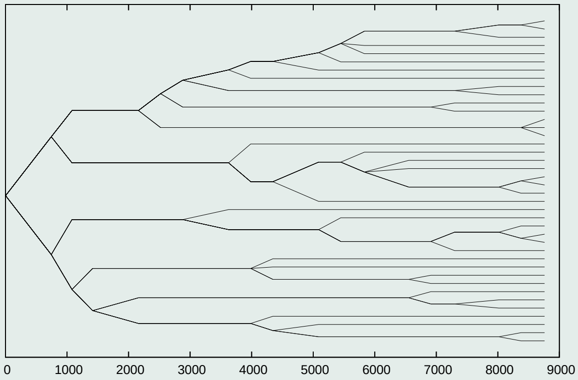
For the stochastic input data of the optimization model, here (yearly electricity and heat demand, and electricity spot prices), a statistical model is employed. It is adapted to historical data as follows:

- cluster classification for the intra-day (demand and price) profiles

- 3-dimensional time series model for the daily average values (deterministic trend functions, a trivariate ARMA model for the (stationary) residual time series)

- simulation of an arbitrary number of three dimensional sample paths (scenarios) by sampling the white noise processes for the ARMA model and by adding on the trend functions and matched intra-day profiles from the clusters afterwards.

- generation of scenario trees as in Heitsch-Römisch 05.



Scenario tree with 40 scenarios for electricity and heat demand, and spot prices

Test runs were performed on real-life data of the utility DREWAG Stadtwerke Dresden GmbH leading to a linear program containing $T = 365 \cdot 24 = 8760$ time steps and about 150.000 nodes. The objective function is of the form

$$\text{Maximize} \quad \gamma \mathcal{A}(Y) + (1-\gamma)\mathbb{E}\left(\sum_{t=1}^{T} Y_t\right)$$

with a (multi-period) acceptability functional $\mathcal{A}$ and coefficient $\gamma \in [0, 1]$ ($\gamma = 0$ corresponds to no risk). $\mathbb{E}(\sum_{t=1}^{T} Y_t)$ denotes the overall expected revenue.

The model is implemented and solved with ILOG CPLEX 9.1 on a 2 GHz Linux PC with 1 GB memory.

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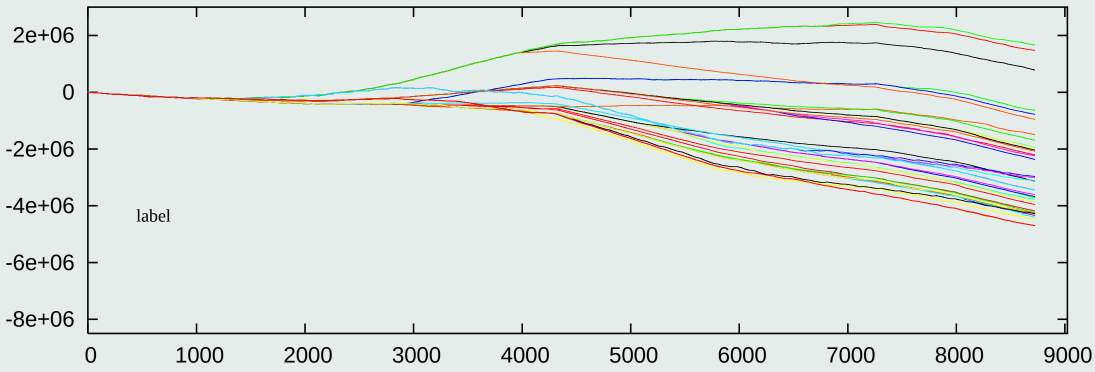
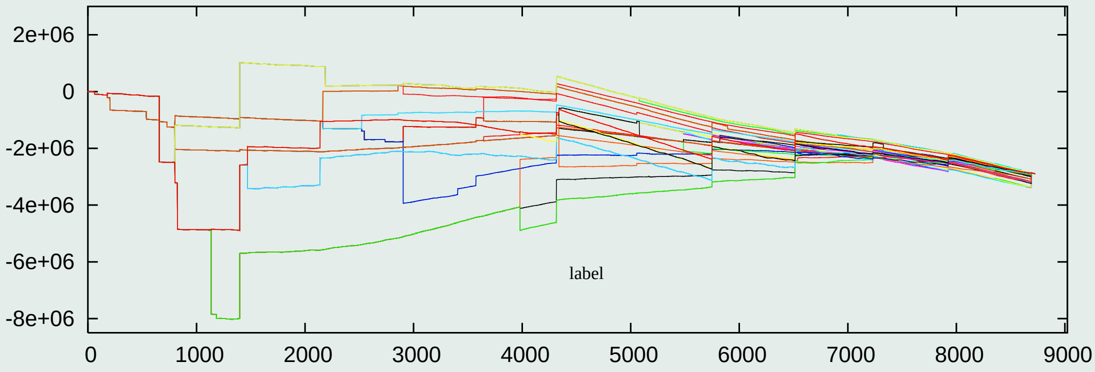
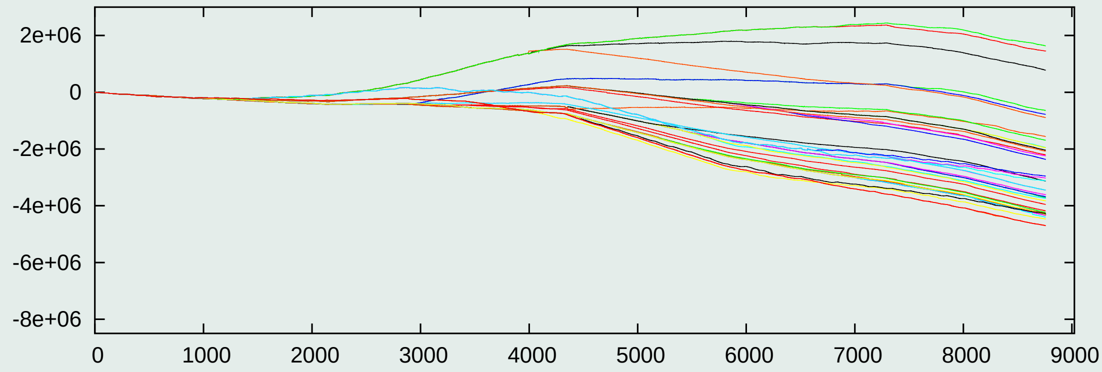


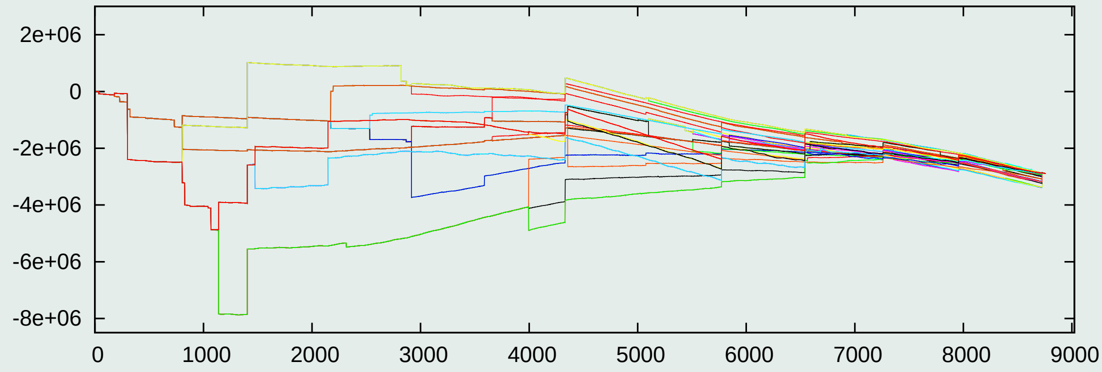
Total revenue and $\gamma = 0$



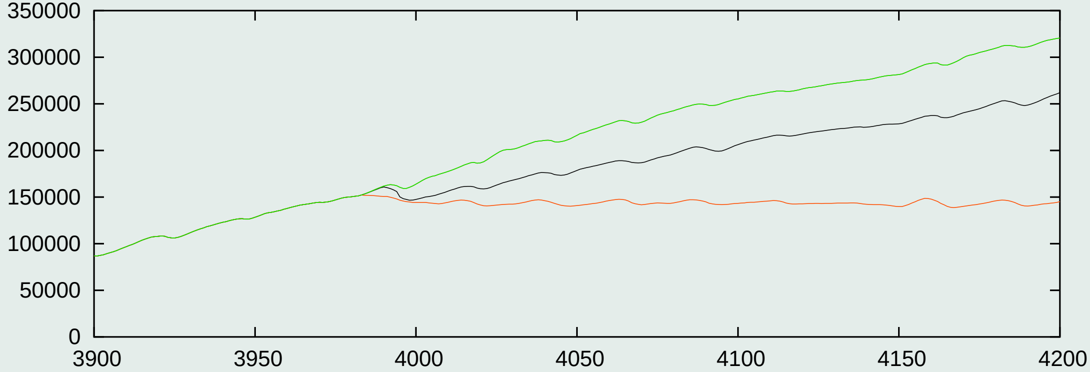
Total revenue with $\mathcal{A}(Y) = \mathbb{AV}@\mathrm{R}_{0.05}(\sum_{t=1}^{T} Y_t)$ and $\gamma = 0.9$



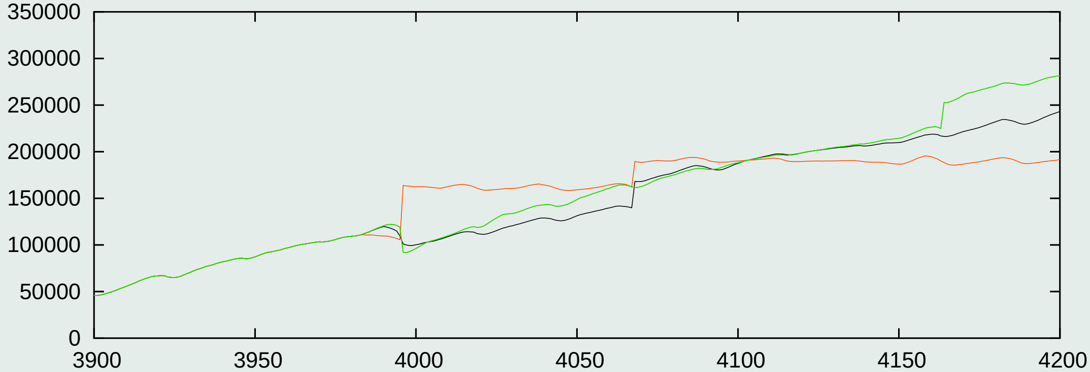
Total revenue with $\mathcal{A} = m\mathbb{AV}@\mathsf{R}_{0.05}$ and $\gamma = 0.9$



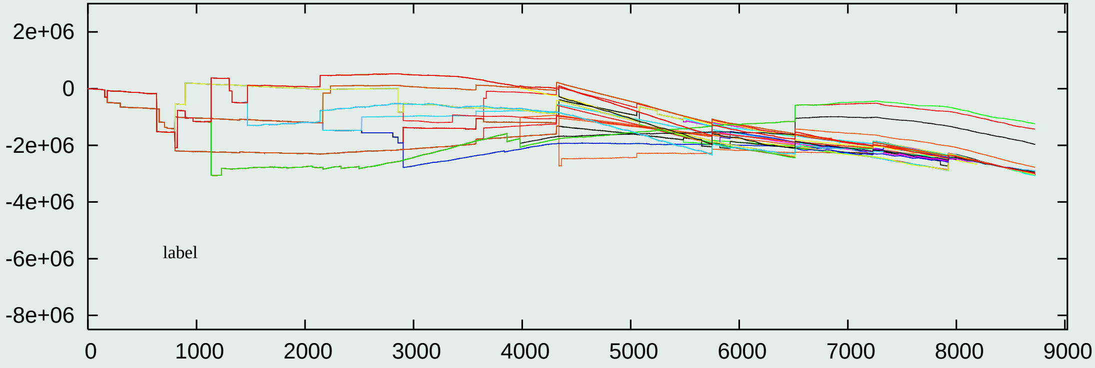
Total revenue with $\mathcal{A} = n\mathbb{AV}@\mathsf{R}_{0.05}$ and $\gamma = 0.9$



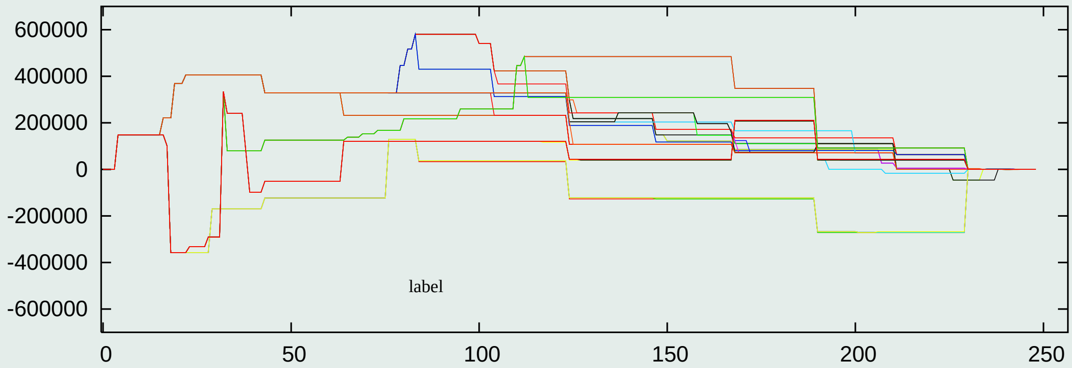
Zoom of total revenue and $\gamma = 0$



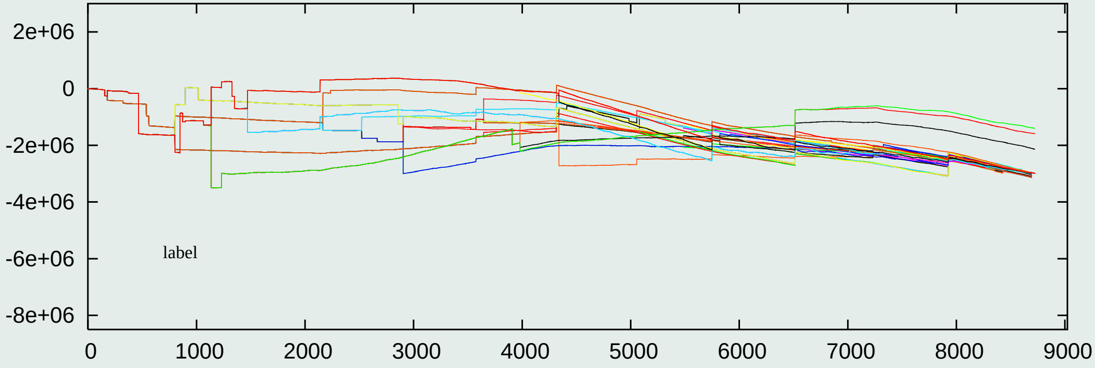
Zoom of total revenue with $\mathcal{A} = m\mathbb{AV}@\mathrm{R}_{0.05}$ and $\gamma = 0.9$



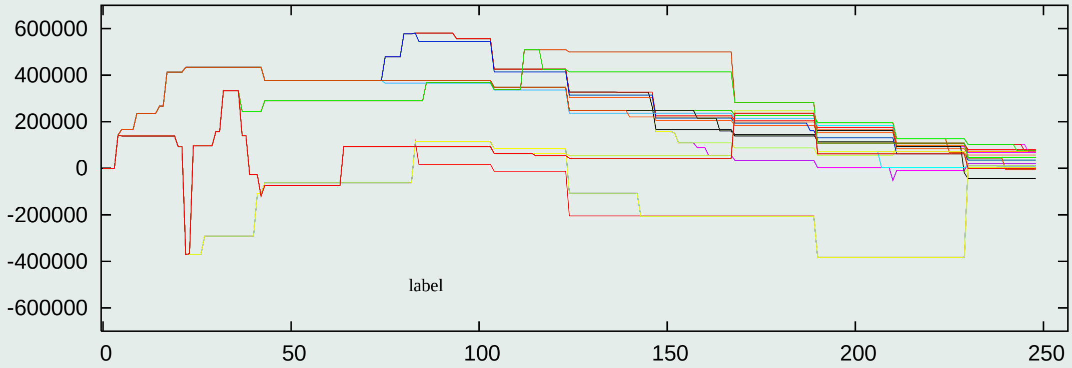
Total revenue with $\mathcal{A} = \mathcal{A}_2$ and $\gamma = 0.9$



Future trading for $\mathcal{A} = \mathcal{A}_2$ and $\gamma = 0.9$



Total revenue with $\mathcal{A} = \mathcal{A}_6$ and $\gamma = 0.9$



Future trading for $\mathcal{A} = \mathcal{A}_6$ and $\gamma = 0.9$

The risk aversion strategies of $\mathcal{A}_2$ and $\mathcal{A}_6$ by trading at derivative markets require less than additional 1% of the optimal expected revenue.

Conclusions

- Concepts for multi-period acceptability and risk functionals and their dual representations were presented,
- several approaches for deriving multi-period acceptability functionals and specific examples were proposed,
- an application to risk management in electricity production and trading was discussed.

Reference:

G. Ch. Pflug and W. Römisch: Modeling, Measuring and Managing Risk, World Scientific, Singapore, 2007.