Abelianization and Floer homology of Lagrangians in clean intersection

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Preface

Summary

This thesis is split up into two parts each revolving around Floer homology and quantum cohomology of closed monotone symplectic manifolds. In the first part we consider symplectic manifolds obtained by symplectic reduction. Our main result is that a quantum version of an abelianization formula of Martin [50] holds, which relates the quantum cohomologies of symplectic quotients by a group and by its maximal torus. Also we show a quantum version of the Leray-Hirsch theorem for Floer homology of Lagrangian intersections in the quotient.

The second part is devoted to Floer homology of a pair of monotone Lagrangian submanifolds in clean intersection. Under these assumptions the symplectic action functional is degenerated. Nevertheless Frauenfelder [33] defines a version of Floer homology, which is in a certain sense an infinite dimensional analogon of Morse-Bott homology. Via natural filtrations on the chain level we were able to define two spectral sequences which serve as a tool to compute Floer homology. We show how these are used to obtain new intersection results for simply connected Lagrangians in the product of two complex projective spaces.

The link between both parts is that in the background the same technical methods are applied; namely the theory of holomorphic strips with boundary on Lagrangians in clean intersection. Since all our constructions rely heavily on these methods we also give a detailed account of this theory although in principle many results are not new or require only straight forward generalizations.

Zusammenfassung

Diese Dissertation ist in zwei Abschnitte gegliedert, die sich beide mit Floer Homologie und Quantenkohomologie von geschlossenen monotonen symplektischen Mannigfaltigkeiten beschäftigen. Im ersten Abschnitt betrachten wir symplektische Mannigfaltigkeiten die durch symplektische Reduktion hervorgehen. Unser Hauptresultat ist, dass eine Abelisierungsformel die von Martin [50] für gewöhnliche Kohomologie beschrieben wurde unter bestimmten Vorraussetzungen auch für Quantenkohomologie gilt. Genauer stellt diese Formel eine Beziehung zwischen der Quantenkohomolgie von symplektischen Quotienten bezüglich einer Gruppe und der des maximalen Toruses her. Des weiteren zeigen wir eine Verallgemeinerung des Leray-Hirsch Theorems für Floer Homologie von Lagrangeschen Untermannigfaltigkeiten im Quotienten.

Im zweiten Abschnitt widmen wir uns der Floer Homologie eines Paars Lagrangescher Untermannigfaltigkeiten mit sauberem Schnitt. In diesem Fall ist das symplektische Wirkungsfunktional degeneriert. Frauenfelder [33] beschreibt dafür eine Version von Floer Homologie welche in einem gewissen Sinne ein unendliches Analogon von Morse-Bott Homologie ist. Mithilfe von natürlicher Filtrierungen des Kettenkomplexes sind wir in der Lage Spektralsequenzen zu definieren, welche als Werkzeug zur Berrechnung der Floer Homologie dienen. Wir zeigen anhand eines Beispiels wie diese verwendet werden um neue Resultate über das Schnittverhalten von einfach zusammenhängenden Lagrangeschen Untermannigfaltikeiten in einem Produkt von zwei komplexen projektiven Räumen zu erhalten.

Das Bindeglied zwischen beiden Abschnitten ist, dass wir im Hintergrund dieselben technischen Methoden verwenden; nämlich das Studium holomorpher Streifen mit Rand auf sich sauber schneidenden Lagrangeschen Untermannigfaltigkeiten. Da all unsere Konstruktionen stark auf diesen Methoden beruhen, geben wir auch eine detailierte Darstellung dieser Theorie. Dieser Teil der Arbeit ist grösstenteils eine Zusammenfassung bereits bekannter Resultate.

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1. Introduction

This thesis is devoted to the computation of symplectic invariants for monotone closed symplectic manifolds, namely quantum cohomology and Floer homology. In contrast to ordinary homology theories the mentioned invariants lack obvious functoriality properties which would facilitate computations. Surprisingly despite of this many formulas which hold for ordinary homology theories have a corresponding formula for quantum cohomology or Floer homology. In the first part of the thesis we demonstrate this phenomenon in two examples.

We consider closed symplectic manifolds obtained by symplectic reduction. Let G be a compact Lie group and $T \subset G$ a maximal torus. Assume that G acts on the symplectic manifold M via Hamiltonian diffeomorphisms. We denote the Hamiltonian quotients by $M/\!\!/G$ and $M/\!\!/T$ respectively and assume in the following that they are regular and monotone. Relations between the usual cohomology rings of $M/\!\!/G$ and $M/\!\!/T$ were studied by Ellingsrud-Stømme [23] and Martin [50] and the authors proved a specific isomorphism. The main result of the first part of the thesis is that under some topological assumtions there is a corresponding isomorphism for quantum cohomology rings with rational coefficients; namely

$$QH^*(M/\!\!/G) \cong QH^*(M/\!\!/T)^W / Q\mathrm{ann}\,(\mathrm{D})\,.$$

On the right-hand side $QH^*(M/\!\!/T)^W$ denotes the ring elements which are invariant under the natural action of the Weyl group $W = N(T)/T$ on $QH(M/\!\!/T)$ and $Q\mathrm{ann}\,(\mathrm{D})$ denotes the ideal of invariant elements such that the quantum cup product with the canonical invariant class D vanishes. We conclude that under the above mentioned topological assumptions the quantum cohomology ring of a symplectic quotient $M/\!\!/G$ is completely determined by the quantum cohomology of the so called *abelian quotient* $M/\!\!/T$ and the action of the finite group W. Furthermore if the quantum cohomology of $M/\!\!/T$ is known, we are able to compute the quantum cohomology of $M/\!\!/G$ by means of standard algebraic operations. For instance this is the case if $M \cong \mathbb{C}^n$ is a complex vector space equipped with the standard symplectic form and G acts via linear maps. Then the abelian quotient $M/\!\!/T$ is a toric manifold with quantum ring given by Batyrev's formula (cf. Corollary 2.1.3).

The quantum cohomology ring and more generally the Gromov-Witten invariants of symplectic quotients were previously studied by many authors. On one side there is an approach by Cieliebak-Gaio-Mundet-Salamon [18], Ziltener [81] and others via the *symplectic vortex equations*. Roughly speaking their results relate the Gromov-Witten invariants of the symplectic quotient $M/\!\!/G$ to invariants based on the symplectic vortex equations in M. In particular this yields under suitable topological assumptions a surjective ring homomorphism from the equivariant cohomology of M to the quantum cohomology of $M/\!\!/G$ (cf. [39]). On the other side there is an algebro-geometric approach by Givental [40], Iritani [45], Bertram-Ciocan-Kim [9] and others. The authors express the Gromov-Witten invariants of the quotient $M/\!\!/G$ in terms of *equivariant Gromov-Witten invariants* or *twisted Gromov-Witten invariants* in M. For example in [9, 20] it is conjectured that the Gromov-Witten invariants of $M/\!\!/G$ are determined by the twisted Gromov-Witten invariants of $M/\!\!/T$. The conjecture is proven in the case where $M/\!\!/G$ is a flag manifold. Finally there is an approach by Gonzales-Woodward [41, 42], Nguyen-Woodward-Ziltener [56] and Woodward [80, 79, 78] which relies on both of the above mentioned approaches. The authors were able to construct a *quantum Kirwan map* which intertwines the Gromov-Witten invariants of the quotient with the *gauged Gromov-Witten invariants* in M. Moreover in [42] an identification between the gauged Gromov-Witten invariants of $M/\!\!/G$ and $M/\!\!/T$ is deduced. The idea of comparing Gromov-Witten invariants in the respective quotients was first mentioned by the physicists Hori-Vafa [44, Appendix A].

Our approach is different. We do not express the Gromov-Witten invariants of the quotient $M/\!\!/G$ in terms of some new invariants in M or $M/\!\!/T$. Instead our statement about the ring homomorphism is in fact a relation for usual Gromov-Witten invariants in the respective quotients. The morphism is constructed via a count of J-holomorphic disks in $M/\!\!/G \times M/\!\!/T$ with boundary on the *abelian/non-abelian correspondence* $V := \mu^{-1}(0)/T$ where $\mu : M \to \mathfrak{g}^\vee$ denotes the moment map. In a certain sense these disks are degenerated symplectic vortices. The non-trivial step in the proof of the aforementioned isomorphism is to show that the chain homomorphism defined by the particular count of these disks descends to a ring homomorphism for the quantum cohomologies.

The advantage of our more geometric approach is that the techniques easily generalize to other situations. Indeed we were able to show a generalization of the Leray-Hirsch theorem to Floer homology of Lagrangian intersections. Let $V \subset Y$ be a coisotropic submanifold and assume that the projection π to the set of equivalence classes $X = V/\!\sim$ with respect to the isotropic leaf relation is a locally trivial fibre bundle. For example if G acts on M as above, then $Y = M/\!\!/T$, $V = \mu^{-1}(0)/T$ and $X = M/\!\!/G$. The quotient X is canonically a symplectic manifold and given a Lagrangian submanifold $L \subset X$ the space $L^V := \pi^{-1}(L)$ is a Lagrangian submanifold of Y. At the same time L^V is a fibre bundle over L restricting π. If the fibre bundle $L^V \to L$ satisfies the assumption of the Leray-Hirsch theorem and under some index assumption we show

$$HF_*(L^V, L^V) \cong HF_*(L, L) \otimes H_*(F), \qquad F = \pi^{-1}(\mathrm{pt})\,.$$

With the same index assumption and the additional assumption that F is homeomorphic to a sphere, Perutz [60] obtained a Gysin sequence for Floer homology groups, which implies our result. However his proof is different to ours and uses perturbations. We use the isomorphism to obtain new rigidity results for Lagrangian embeddings into symplectic quotients of linear group actions.

The second part of the thesis is devoted to the study of Floer homology of Lagrangian intersection, which is a module $HF_*(L_0, L_1)$ associated to two Lagrangians $L_0, L_1 \subset M$ of a symplectic manifold (M, ω). Let $\mathcal{P}(L_0, L_1)$ be the space of paths $x : [0,1] \to M$ such that $x(0) \in L_0$ and $x(1) \in L_1$. Consider the symplectic action one-form on $\mathcal{P}(L_0, L_1)$ given by

$$\alpha(x)\xi = \int_0^1 \omega(\dot{x}, \xi)\,\mathrm{d}t\,, \qquad \xi \in T_x\mathcal{P}(L_0, L_1)\,.$$

The form is always closed and exact under suitable topological assumptions on L_0 and L_1. In that case Floer [25] constructed his homology as a sort of Morse homology for a primitive of α, which is the *symplectic action functional*. Later it was noticed that his constructions extend to the non-exact case in the sense of Novikov (cf. [43] and [58]). If the Lagrangians L_0, L_1 intersect cleanly the symplectic action functional is degenerated. This situation was first studied by Pozniak in [62], where he carefully choose perturbations by Hamiltonian diffeomorphisms to move L_0 and L_1 into transverse position and then identified certain holomorphic strips which appear in the definition of the boundary operator of the Floer homology complex with Morse trajectories on the intersection manifold $L_0 \cap L_1$.

Instead we leave the Lagrangians as they are and treat the action functional for the degenerate situation as a Morse-Bott function using cascades in the sense of [33]. We obtain a complex which we call the *pearl chain complex for* (L_0, L_1) that also computes Floer homology. The part is mostly of expository nature because this complex was previously studied by Fukaya-Ohta-Ono-Oh [35, 36, 37] and Frauenfelder [33]. However we include some details which have not been treated (eg. surjectivity of gluing) and also give a slightly different approach to orientations which is more adapted to the interpretation of the Floer complex as a Morse complex. Based on ideas of Oh [59], Biran-Cornea [12] and Seidel [69] we construct two spectral sequences $E^{\mathrm{loc},*}_{**}$ and E^*_{**}. The spectral sequence $E^{\mathrm{loc},*}_{**}$ has E^1-term given by

$$E^{\mathrm{loc},1}_{ij} = \begin{cases} H_{i+j-\mu(C_i)}(C_i, \mathbb{Z}_2) & \text{if } 1 \le i \le k\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $C_1, C_2, \ldots, C_k$ are the connected components of $L_0 \cap L_1$ ordered in a way determined by the symplectic action functional and $\mu(C_i) \in \mathbb{Z}$ is a Viterbo type index. The sequence $E^{\mathrm{loc},*}_{**}$ collapses and gives the E^1-term of the second sequence as follows

$$E^1_{pq} = \begin{cases} \bigoplus_{i+j=q} E^{\mathrm{loc},\infty}_{ij} & \text{if } p \in N\mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We show that the sequence E^*_{**} collapses and converges to $HF_*(L_0, L_1)$. As both sequences are homological spectral sequences their r-th boundary operator has degree $(-r, r+1)$. We use these spectral sequences to obtain a new result about the intersection of simply connected Lagrangians in $\mathbb{CP}^n \times \mathbb{CP}^n$.

The thesis is structured as follows. In Chapter 2 we give an overview over the main results, which are announced here with precise statements. In Chapter 3 is devoted to recall the necessary background material. The proofs of the main results are deferred to the end in Chapters 11 and 10. In Chapter 10 we give additionally a construction of the pearl homology complex associated to two Lagrangians in clean intersection. All proofs require a treatment of the moduli space of holomorphic strips with boundary on cleanly intersecting Lagrangians along the standard programm; viz. compactness, transversality, gluing and orientations. These steps are well-known and treated in various sources for the case of holomorphic strips with boundary on transversely intersecting Lagrangians. For holomorphic strips with boundary on cleanly intersecting Lagrangians the steps are also more or less done or require only small generalizations of existing theory. However since the proofs are spread out over the literature we felt it worthwhile to give a self-contained treatment. This is worked out in Chapters 4–9. Estimates which are used frequently in these chapters are collected in Appendices A and B. In Appendix C we provide a small generalization of the Viterbo index; in particular an index for holomorphic strips with boundary on Lagrangians in clean intersection. In Appendix D we give a short introduction to all required material about principle bundles and Lie group actions. In Chapter 12 we give two examples how the main theorems are applied.

2. Overview of the main results

2.1. Abelianization

Let G be a compact connected Lie group which acts on the symplectic manifold (M, ω) via Hamiltonian diffeomorphisms with moment map $\mu_G : M \to \mathfrak{g}^\vee$. The *symplectic reduction* is defined as the quotient space

$$M/\!\!/G := \mu_G^{-1}(0)/G\,.$$

We call $M/\!\!/G$ *regular* if $0 \in \mathfrak{g}^\vee$ is a regular value of μ_G and $\mu_G^{-1}(0)$ is a closed manifold on which G acts freely. If this happens, the space $M/\!\!/G$ is naturally a closed symplectic manifold. Any subgroup $T \subset G$ acts on M by Hamiltonian diffeomorphisms with moment map $\mu_T = i^\vee \circ \mu_G : M \to \mathfrak{t}^\vee$, where $\mathfrak{t}$ denotes the Lie algebra of T and $i^\vee : \mathfrak{g}^\vee \to \mathfrak{t}^\vee$ is the canonical projection induced by the inclusion $i : \mathfrak{t} \to \mathfrak{g}$. From now on, we assume that $T \subset G$ is a maximal torus. If so, the *Weyl group* $W = N(T)/T$ acts naturally on $M/\!\!/T$ via symplectomorphisms and there exists a particular cohomology class, the *canonical anti-invariant class* $\mathrm{D} \in H^*(M/\!\!/T)$ (cf. §11.1). We denote by $QH^*(M/\!\!/G; \Lambda)$ the quantum cohomology ring over $\Lambda := \mathbb{Q}[\lambda, \lambda^{-1}]$ (cf. [53, Ex. 11.1.4.(i)]) and by $QH^*(M/\!\!/T; \Lambda)^W \subset QH^*(M/\!\!/T; \Lambda)$ the subring of invariant elements.

Theorem 2.1.1. *Assume that $M/\!\!/G$ is regular, simply connected and that $M/\!\!/T$ is regular, monotone and with minimal Chern number $c_{M/\!\!/T}$ satisfying the bound*

$$2c_{M/\!\!/T} \geq \dim G/T + 2\,. \tag{2.1.1}$$

Then there exists a ring isomorphism

$$QH^*(M/\!\!/G; \Lambda) \cong QH^*(M/\!\!/T; \Lambda)^W / Q\mathrm{ann}\,(\mathrm{D})\,, \tag{2.1.2}$$

where $Q\mathrm{ann}\,(\mathrm{D})$ is the ideal of invariant elements such that the quantum cup product with D vanishes.

In [50] Martin gave a similar isomorphism for rational cohomology rings. More precisely he proved that there exists a ring isomorphism

$$H^*(M/\!\!/G; \mathbb{Q}) \cong H^*(M/\!\!/T; \mathbb{Q})^W / \,\mathrm{ann}(\mathrm{D})\,, \tag{2.1.3}$$

where $\mathrm{ann}(\mathrm{D})$ is the ideal such that the ordinary cup product with D vanishes. We obtain the desired isomorphism (2.1.2) on the level of modules already from (2.1.3). The nontrivial content of the theorem is that there exists a ring isomorphism for the quantum product. Of course this is generally not the Λ-extension of the isomorphism (2.1.3).

Remark 2.1.2. The isomorphism (2.1.2) holds for more general situations. We did not state the theorem in the utmost generality for the sake of a cleaner exposition.

1. All the arguments go through with $\mathbb{Q}$ replaced by a different (commutative and unital) ring A and $\Lambda := A[\lambda, \lambda^{-1}]$, provided that the pull-back $i^* : H^*(M/\!\!/T, A)^W \to H^*(V, A)^W$ is surjective. By Kirwan surjectivity this always holds if $A = \mathbb{Q}$.

2. If $M/\!\!/G$ is not simply connected, then (2.1.2) holds as long as $V \subset M/\!\!/G \times M/\!\!/T^-$ is monotone and the number $2c_{M/\!\!/T}$ in the bound (2.1.1) is replaced by the minimal Maslov number of V.

We already obtain non-trivial results in the case when M is a complex vector space equipped with the standard symplectic form on which which G acts via linear unitary maps. In other words we are given an unitary representation of G on M. By definition the reduction with respect to the subgroup $T \subset G$ is a toric variety which is canonically a symplectic manifold, if it is regular. Moreover it is well-known that there is an value $w \in \mathfrak{t}^\vee$ (unique up to scaling) such that the symplectic quotient $M/\!\!/_w T := \mu_T^{-1}(w)/T$ is a monotone symplectic manifold, if it is regular. The quantum cohomology ring is given by

$$\Lambda[x_1, \ldots, x_k]/Q\mathrm{SR}\,,$$

where $Q\mathrm{SR} \subset \Lambda[x_1, \ldots, x_k]$ is the *quantum Stanley-Reisner ideal.* The Weyl group W acts naturally on $\Lambda[x_1, \ldots, x_k]$ and there exists a class which we also denote by D and which divides every anti-invariant class. In [23, §4] it is deduced that the map

$$p : \Lambda[x_1, \ldots, x_k] \to \Lambda[x_1, \ldots, x_k]^W, \qquad r \mapsto \mathrm{D}^{-1} \sum_{w \in W} \mathrm{sign}(w) w.r\,, \tag{2.1.4}$$

is well-defined and induces an isomorphism when restricted to the anti-invariant subspace. We conclude the following corollary. For more details see Section 3.1.2.

Corollary 2.1.3. *Let $w \in \mathfrak{t}^\vee$ be such that $Y := M/\!\!/_w T = \mu_T^{-1}(w)/T$ is regular, monotone and assume that $X := M/\!\!/_w G = \mu_G^{-1}(w)/G$ is regular. Suppose that $2c_Y \geq \dim G/T + 2$ then the quantum cohomology ring of X is given by*

$$\Lambda[x_1, \ldots, x_k]^W/p(Q\mathrm{SR})\,.$$

Remark 2.1.4. The previous example gives a formula to compute the quantum cohomology ring of many monotone symplectic manifolds; like Grassmannians, partial flag manifolds and more generally quiver varieties, as they all arise as symplectic quotients of linear group actions by compact groups. In Lemma 3.1.6 we show how to compute the minimal Chern number c_Y in terms of the weight vectors of the induced representation of T on M. Unfortunately the condition $2c_Y \geq \dim G/T + 2$ seems to be very restrictive. We do not know of any example where the above formula fails whenever $2c_Y < \dim G/T + 2$.

2.2. Quantum Leray-Hirsch theorem

Given a symplectic manfold (Y, ω_Y) and $V \subset Y$ a *regular coisotropic submanifold*, i.e. a coisotropic submanifold such that the quotient by the isotropic-leaf relation defines a locally trivial fibration. Let $X := V/\sim$ be the quotient and $\pi : V \to X$ the canonical projection. The space X is canonically equipped with a symplectic form ω_X uniquely determined by the requirement that $\pi^*\omega_X = i^*\omega_Y$ where $i : V \to Y$ denotes the inclusion. We conclude that via $i \times \pi$ the space V is a Lagrangian submanifold of $Y \times X^-$ where X^- denotes the space X equipped with the symplectic form $-\omega_X$. It is an easy observation that given a Lagrangian submanifold $L \subset X$ we obtain a Lagrangian submanifold $L^V := \pi^{-1}(L)$ embedded into Y via i. At the same time L^V is also a fibration over L via π.

Theorem 2.2.1. *Given a regular coisotropic submanifold $V \subset Y$ with quotient X such that it is embedded via $i \times \pi$ into $Y \times X^-$ as a monotone Lagrangian submanifold. Let $L \subset X$ be a monotone Lagrangian submanifold such that*

- *the space $L^V := \pi^{-1}(L)$ is a monotone Lagrangian submanifold of Y,*
- *the pull-back $H^*(L^V; \mathbb{Z}_2) \to H^*(F; \mathbb{Z}_2)$ of $F := \pi^{-1}(\mathrm{pt}) \subset L^V$ is surjective,*
- *the minimal Maslov number N of the pair $(L \times L^V, V)$ satisfies the bound*

$$N \geq \dim F + 2 \,. \tag{2.2.1}$$

Then there exists an isomorphism

$$HF_*(L^V, L^V; \mathbb{Z}_2) \cong HF_*(L, L; \mathbb{Z}_2) \otimes H_*(F; \mathbb{Z}_2) \,. \tag{2.2.2}$$

For the definition of the minimal Maslov number of a pair of Lagrangian submanifolds see Section 3.1. If L, F and X are simply connected then the number N in the bound (2.2.1) is given by twice the minimal Chern number of Y (cf. [62, Remark 3.3.2]). Similarly if V is simply connected then N is given by minimal Maslov number of the Lagrangian L. The proof of the following corollary is given in Section 12.2.

Corollary 2.2.2. *With the same assumptions as Corollary 2.1.3. Suppose additionally that there exists a closed Lagrangian submanifold $L \subset X$ with $H^*(L, \mathbb{Z}_2) \cong H^*(S^n, \mathbb{Z}_2)$. Then one of the following holds*

- $2c_Y$ *divides* $n+1$,
- $\dim G/T \leq 2$ *and* $n \leq 4$.

Remark 2.2.3. The second condition is sharp in the sense that for $G = U(2)$ acting on $\mathbb{C}^{4\times 2}$ from the right the symplectic quotient is the complex Grassmannian $\mathrm{Gr}(4, 2)$ which is a quadric and contains a Lagrangian sphere as the fixed point set of the anti-symplectic involution. We do not know if there are examples of Hamiltonian quotients containing Lagrangian spheres L and $2c_Y$ divides $\dim L + 1$ or $2c_Y < \dim G/T + 2$.

2.3. Floer homology of Lagrangians in clean intersection

Let (M, ω) be a symplectic manifold, $L_0, L_1 \subset M$ be two Lagrangian submanifolds and A be a unital commutative ring. We now state a topological condition under which the Floer homology of the pair (L_0, L_1) is well-defined. All terms are explained with much detail in Sections 3.1 and 9.2.

Assumption 2.3.1. *We assume that*

- *the pair (L_0, L_1) is monotone,*
- *the minimal Maslov number N is greater or equal to 3,*
- *the Lagrangians L_0 and L_1 intersect cleanly,*
- *if $2A \neq 0$ then we fix a relative spin structure for (L_0, L_1).*

Let $\mathcal{P}(L_0, L_1)$ be the space of paths $x : [0,1] \to M$ such that $x(0) \in L_0$ and $x(1) \in L_1$. We decompose $L_0 \cap L_1$ into connected components $C_1, \dots, C_k$. Fix an element $x_* \in C_1$. For every $j = 1, \dots, k$ we choose a path $u_j : [-1,1] \times [0,1] \to M$ such that $u_j(s, \cdot) \in \mathcal{P}(L_0, L_1)$, $u_j(-1) = x_*$ and $u_j(1) \in C_j$. Define

$$\mathcal{A}(C_j) := -\int u_j^* \omega, \qquad \mu(C_j) := -\mu_{\mathrm{Vit}}(u_j) - \frac{1}{2}\dim C_j + \frac{1}{2}\dim C_1\,,$$

in which $\mu_{\mathrm{Vit}}(u_j)$ denotes the Viterbo index of u_j. Let τ be the monotonicity constant of (L_0, L_1). Without loss of generality we assume that the maps u_j are chosen such that for all $j = 1, \dots, k$ we have

$$0 \leq \mathcal{A}(C_j) < \tau N\,.$$

Define the action values $0 \leq a_1 < a_2 < \dots < a_\kappa < \tau N$ as the values attained by $\mathcal{A}(C_j)$ for $j = 1, \dots, k$. If $2A \neq 0$, let $\mathcal{L} = \mathcal{O} \times_{\mathbb{Z}_2} A$ be the local system associated to the relative spin structure (cf. Definition 9.3.4 and Lemma 3.3.2). If $2A = 0$ set $\mathcal{L} = A$.

Theorem 2.3.2. *With Assumption 2.3.1. There exists two spectral sequences E^*_{**} and $E^{\mathrm{loc},*}_{**}$ such that*

(i) $E^{\mathrm{loc},1}_{ij} \cong \bigoplus_{\{\ell | \mathcal{A}(C_\ell) = a_i\}} H_{i+j-\mu(C_\ell)}(C_\ell; \mathcal{L})$ *for all $1 \leq i \leq \kappa$, $j \in \mathbb{Z}$ and 0 otherwise,*

(ii) $E^1_{pq} \cong \bigoplus_{i+j=q} E^{\mathrm{loc},\infty}_{ij}$ *for all $p \in N\mathbb{Z}$, $q \in \mathbb{Z}$ and 0 otherwise,*

(iii) $\bigoplus_{p+q=*} E^\infty_{pq} \cong HF_*(L_0, L_1)$.

For both spectral sequences the r-th boundary operator has degree $(-r, r-1)$.

We immediately obtain some displacement results. We say that L_0 is *displaceable from* L_1 if there exists a Hamiltonian diffeomorphism such that $\varphi(L_0) \cap L_1 = \emptyset$.

Corollary 2.3.3. *With Assumption 2.3.1. Suppose that L_0 is displaceable from L_1 and $C := L_0 \cap L_1$ consists of only one connected component, then we have*

- $N \leq \dim C + 1$,
- *if moreover* $2N > \dim C + 1$ *then for all* $k \in \mathbb{N}$

$$H_k(C;\mathbb{Z}_2) \cong \begin{cases} H_{k+N-1}(C;\mathbb{Z}_2) & \text{if } 0 \leq k \leq \dim C - N + 1 \\ 0 & \text{if } \dim C - N + 2 \leq k \leq N - 2 \\ H_{k-N+1}(C;\mathbb{Z}_2) & \text{if } N - 1 \leq k \leq \dim C\,. \end{cases}$$

Proof. Since there is only one connected component the sequence $E^{\mathrm{loc},*}_{**}$ collapses at the first page and thus

$$E^1_{pq} \cong \begin{cases} H_q(C) & \text{if } p \in N\mathbb{Z}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the spectral sequence E^*_{**}. If of page r the boundary operator ∂^r is non-trivial then necessarily $r \in N\mathbb{N}$. If $r = N$ we have for all $q, \bar{p} \in \mathbb{Z}$

$$H_q(C,\mathbb{Z}_2) \cong E^N_{\bar{p}N,q} \xrightarrow{\partial^N} E^N_{(\bar{p}-1)N,q+N-1} \cong H_{q+N-1}(C,\mathbb{Z}_2)\,. \tag{2.3.1}$$

Suppose by contradiction that $N > \dim C + 1$. Then we conclude from (2.3.1) that ∂^N is trivial. Inductively we show that $\partial^{\bar{r}N}$ is trivial for all $\bar{r} \in \mathbb{N}$. Hence $HF_*(L_0, L_1) \cong H_*(C,\mathbb{Z}_2) \otimes \Lambda$. But if L_0 is displaceable from L_1 the module $HF_*(L_0, L_1)$ vanishes. This shows the first claim.

Suppose now that $2N > \dim C + 1$. In a similary manner we show that the only possibly non-trivial boundary operator is on page $r = N$. By assumption $HF_*(L_0, L_1) \cong 0$ and thus $E^\infty_{pq} = \ker \partial^N / \operatorname{im} \partial^N \cong 0$. Using (2.3.1) we conclude the second statement. □

The case $L_0 = L_1 = L$ is a special case of a clean intersection and the previous corollary implies the well-known result about closed monotone Lagrangians submanifolds.

Theorem 2.3.4 (Polterovich, Oh)**.** *If a monotone Lagrangian submanifold L is displaceable, then the minimal Maslov number N_L satisfies*

$$N_L \leq \dim L + 1\,.$$

As an illustration, we apply the spectral sequences to obtain a new intersection result of simply connected Lagrangians in $\mathbb{C}\mathbf{P}^n \times \mathbb{C}\mathbf{P}^n$, which generalize results of Fortune [32] about fixed points of symplectomorphisms of $\mathbb{C}\mathbf{P}^n$. Let ω_{FS} denote the Fubini-Study symplectic form on $\mathbb{C}\mathbf{P}^n$.

Proposition 2.3.5. *Let $\mathbb{C}\mathbf{P}^n \oplus \mathbb{C}\mathbf{P}^n$ be equipped with the symplectic form $\omega_{\mathrm{FS}} \oplus -\omega_{\mathrm{FS}}$. Give two simply connected Lagrangians $L_0, L_1 \subset \mathbb{C}\mathbf{P}^n \times \mathbb{C}\mathbf{P}^n$ intersecting cleanly with $L_0 \neq L_1$. Then $L_0 \cap L_1$ has at least two connected components. Moreover assume that the intersection $L_0 \cap L_1$ consists of two disjoint connected components one of which is a point, then we have*

$$H_*(C,\mathbb{Z}_2) \cong H_*(\mathbb{C}\mathbf{P}^{n-1},\mathbb{Z}_2)\,,$$

where C denotes the other connected component.

Proof. Suppose $A = \mathbb{Z}_2$. The Lagrangians have to intersect by a result of Albers [6]. Assume that $L_0 \cap L_1 = C$ has only one connected component. The minimal Chern number of $\mathbb{C}\mathbf{P}^n \times \mathbb{C}\mathbf{P}^n$ equals $n+1$. Since L_0 and L_1 are simply connected, the pair (L_0, L_1) is monotone with minimal Maslov number $N = 2(n+1)$. We conclude that $HF_*(L_0, L_1) \cong H_*(C;\mathbb{Z}_2) \otimes \Lambda$ with $\Lambda = \mathbb{Z}_2[\lambda, \lambda^{-1}]$ and $\deg \lambda = -2n-2$ (see the proof of Corollary 2.3.3). By the quantum action Floer homology $HF_*(L_0, L_1)$ is a module over the quantum cohomology ring of $\mathbb{C}\mathbf{P}^n \times \mathbb{C}\mathbf{P}^n$ which contains an invertible element of degree two. Hence $HF_k(L_0, L_1) \cong HF_{k+2}(L_0, L_1)$ for all $k \in \mathbb{Z}$. But this leads to the contradiction

$$\mathbb{Z}_2 \cong HF_0(L_0, L_1) \cong HF_2(L_0, L_1) \cong \ldots \cong HF_{2n}(L_0, L_1) \cong 0\,. \tag{2.3.2}$$

Now assume that $L_0 \cap L_1 = \{\mathrm{pt}\} \cup C$. Without loss of generality we assume that the base point lies on the component C and let $d = \mu(\mathrm{pt}) \in \mathbb{Z}$ denote the index of the intersection point which does not lie on C. Then the local spectral sequence collapses at the second page and we have $E_*^{\mathrm{loc},\infty} \cong \ker \partial^{\mathrm{loc},1} \oplus \operatorname{coker} \partial^{\mathrm{loc},1}$, where $\partial^{\mathrm{loc},1} : H_*(C;\mathbb{Z}_2) \to \mathbb{Z}_2[d]$ (here $\mathbb{Z}_2[d]$ denotes the group $\mathbb{Z}_2$ in degree d).

Assume that $\partial^{\mathrm{loc},1} \neq 0$. Then $\operatorname{coker} \partial^{\mathrm{loc},1} = 0$ and $E_*^{\mathrm{loc},\infty}$ is only supported in degrees $0, 1, \ldots, \dim C < 2n$. Similarly as above we conclude that $HF_*(L_0, L_1) \cong E_*^{\mathrm{loc},\infty} \otimes \Lambda$ which by degree reasons leads to a contradiction as in (2.3.2).

Assume that $\partial^{\mathrm{loc},1} = 0$. Then $E_*^{\mathrm{loc},\infty} \cong H_*(C;\mathbb{Z}_2) \oplus \mathbb{Z}_2[d]$. If $0 \leq d \leq 2n-1$, we obtain as above a contradiction via (2.3.2). If $d > 2n-1$ we can not conclude by degree reasons that E_{**}^* collapses at the first page, since there might possibly exist $r \in \mathbb{N}$ and $0 \leq j \leq \dim C$ such that $j + rN - 1 = d$ and $\partial^r \neq 0$. Yet, if this were the case then the spectral sequence collapses at the $r+1$-page and we have $HF_*(L_0, L_1) \cong H_* \otimes \Lambda$ in which $H_* \subset H_*(C;\mathbb{Z}_2)$ is a subspace of codimension one. Unless $H_* = 0$ and $n = 1$ we again obtain a contradiction via (2.3.2). But if $H_* = 0$ and $n = 1$ then $H_*(C;\mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H_*(\mathbb{C}\mathbf{P}^0;\mathbb{Z}_2)$ as claimed. Finally if $\partial^r = 0$ for all $r \geq 1$, then $HF_*(L_0, L_1) \cong \big(H_*(C;\mathbb{Z}_2) \oplus \mathbb{Z}_2[d]\big) \otimes \Lambda$. The only possibility which does not lead to a contradiction via (2.3.2) is if $d = 2n \mod 2n+2$ and

$$H_k(C;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, 2, \ldots, 2n-n, \\ 0 & \text{otherwise}\,. \end{cases}$$

But then C has exactly the same homology as $\mathbb{C}\mathbf{P}^{n-1}$. $\square$

3. Background

3.1. Symplectic geometry

3.1.1. Symplectic manifolds and Lagrangians

A *symplectic manifold* (M, ω) is a $2n$-dimensional manifold M equipped with a *symplectic form* ω, which is a 2-form that is closed (i.e. $\mathrm{d}\omega = 0$) and non-degenerated (i.e. $\omega^{\wedge n} \neq 0$). An *almost complex structure on* M is a complex structure on the tangent bundle TM given by an endomorphism $J : TM \to TM$ such that $J^2 = -\mathbb{1}$. The almost complex structure is called ω*-compatible*, if

$$g_J = \omega(\cdot, J\cdot)$$

defines a Riemannian metric on M. We denote by $\mathrm{End}(TM, \omega)$ the space of all almost complex structures on M which are compatible to a fixed ω. A complex structure on TM induces a first Chern class (see [16, Section 20]). Since the space of compatible almost complex structures is contractible the Chern class does not depent on the choice $J \in \mathrm{End}(TM, \omega)$ and is denoted $c_1(\omega) \in H^2(M, \mathbb{Z})$.

Let $H_2^S(M)$ be the image of the Hurewicz morphism $\pi_2(M) \to H_2(M, \mathbb{Z})$. Evaluation of $c_1(\omega)$ and $[\omega]$ on elements in $H_2^S(M)$ defines two homomorphisms

$$I_c : H_2^S(M) \to \mathbb{Z}, \qquad\qquad I_\omega : H_2^S(M) \to \mathbb{R}\ .$$

We define the *minimal Chern number of* M as the smallest positive value of I_c, i.e. $c_M := \min\{I_c(A) \mid A \in H_2^S(M) \text{ with } I_c(A) > 0\}$. A symplectic manifold (M, ω) is

- *symplectically aspherical*, if for all classes $a \in H_2^S(M)$ we have $I_\omega(a) = I_c(a) = 0$,
- *monotone* or more precisely τ*-monotone*, if there exists a constant $\tau > 0$, such that for all classes $a \in H_2^S(M)$ we have $I_\omega(a) = 2\tau I_c(a)$.

These assumptions where introduced by Floer and lead to a simplification of the analysis. Unless otherwise noted all symplectic manifolds in this work are either symplectically aspherical or monotone.

Lagrangians A submanifold $L \subset M$ is *isotropic* if ω vanishes on all pairs of vectors tangent to L. A *Lagrangian submanifold* is an isotropic submanifold $L \subset M$ such that $\dim L = n$. By the non-degeneracy of ω, any isotropic submanifold has a dimension of at most n. From that viewpoint Lagrangian submanifolds are sometimes called *maximally* *isotropic*. Correspondingly we have similar homological requirements, which where introduced by [58] and again lead to simplification of the analysis. Let $H_2^S(M,L)$ be the image of the relative Hurewicz morphism $\pi_2(M,L) \to H_2(M,L)$. Evaluation of $[\omega]$ and the Maslov index (cf. [53, Section C.3]) defines two homomorphisms

$$I_\mu : H_2^S(M,L) \to \mathbb{Z}, \qquad\qquad I_\omega : H_2^S(M,L) \to \mathbb{R}\ .$$

Similarly as above we define the *minimal Maslov number of* L as the smallest positive value of I_μ, i.e. $N_L := \min\{I_\mu(A) \mid A \in H_2^S(M,L) \text{ with } I_\mu(A) > 0\}$. A Lagrangian submanifold $L \subset M$ is

- *symplectically aspherical* if for all classes $a \in H_2^S(M,L)$ we have $I_\omega(a) = I_\mu(a) = 0$,
- *τ-monotone* if there exists a constant $\tau > 0$ such that for all classes $a \in H_2^S(M,L)$ we have $I_\omega(a) = \tau I_\mu(a)$.

Remark 3.1.1. If $L \subset M$ is symplectically aspherical then M is necessarily symplectically aspherical as well and if L is τ-monotone then M is τ-monotone or symplectically aspherical. For that reason we purposely included the factor 2 in the definition of the monotonicity constant of a monotone symplectic manifold. Another basic observation is that the minimal Maslov number of a Lagrangian $L \subset M$ always divides $2c_M$.

Lemma 3.1.2. *Let (M,ω) be a monotone symplectic manifold and $L \subset M$ be a Lagrangian submanifold such that the fundamental group $\pi_1(L)$ is finite, then L is monotone. Suppose that $\pi_1(L)$ is trivial, then the minimal Maslov number of L equals $2c_M$, where c_M is the minimal Chern number of M.*

Proof. Let $u : (D,\partial D) \to (M,L)$ be a disc with boundary on L. After a suitable cover $\varphi : D \to D$ of some degree $k \in \mathbb{N}$ the boundary of the composition $\tilde{u} = u \circ \varphi$ is contractible within L, i.e. there exists $v : D \to L$ such that $v|_{\partial D} = \tilde{u}|_{\partial D}$. Let $w = u \sqcup v/\sim$ be the map defined on $S^2 \cong D \sqcup D/\sim$ with boundary points identified. Hence

$$I_\omega([w]) = \int w^*\omega = \int \tilde{u}^*\omega + \int v^*\omega = k\int u^*\omega = kI_\omega(u)\,,$$

and by [53, Thm. C.3.10]

$$2I_c([w]) = 2\langle c_1(TM),[w]\rangle = \mu_{\mathrm{Mas}}(\tilde{u}) + \mu_{\mathrm{Mas}}(v) = k\mu_{\mathrm{Mas}}(u) = kI_\mu(u)\ .$$

According to the assumption there exists a $\tau > 0$ such that $kI_\omega([u]) = I_\omega([w]) = 2\tau I_c([w]) = \tau k I_\mu([u])$. This shows that L is monotone. If $\pi_1(L)$ is trivial then $k = 1$ and $I_\mu([u]) = 2I_c([w]) \in 2c_M\mathbb{Z}$ for all u. This shows that $2c_M$ divides the minimal Maslov number of L, denoted N_L. But since N_L always divides $2c_M$ we have $N_L = 2c_M$. □

Lemma 3.1.3. *The diagonal $\Delta = \{(p,p) \mid p \in M\}$ is a Lagrangian submanifold of $(M \times M, \omega \oplus -\omega)$ with minimal Maslov number given by twice the minimal Chern number of M. Moreover M is monotone if and only if Δ is monotone.*

Proof. Identify a disk $u = (u_0, u_1) : (D, \partial D) \to (M \times M, \Delta)$ uniquely with a sphere $v : \mathbb{P}^1 \to M$ via $v(z) := u_0(z)$ for $|z| \leq 1$ and $v(z) := u_1(1/\bar{z})$ for $|z| \geq 1$. Conversely given a sphere v we obtain a disk $u : (D, \partial D) \to (M \times M, \Delta)$ by the same identification. Choose trivializations $\Phi_0 : u_0^* TM \to D \times \mathbb{C}^n$ and $\Phi_1 : u_1^* TM \to D \times \mathbb{C}^n$. Denote by $\Psi : S^1 \to U(n)$, $\theta \mapsto \Phi_1(\theta)\Phi_0(\theta)^{-1}$ the trivialization change along $\partial D = S^1$. For every $\theta \in S^1$ define the linear Lagrangian subspace $F(\theta) := (\Phi_0(\theta) \oplus \Phi_1(\theta)) T_{(u_0,u_1)}\Delta = \operatorname{graph} \Psi(\theta) \subset \mathbb{C}^n \times \mathbb{C}^n$. By definition of the Maslov index (see [53, Theorem C.3.6]) we have

$$I_\mu([u]) = \mu_{\mathrm{Mas}}(F) = \deg\det \Psi^2 = 2 \deg\det \Psi = 2\langle c_1, [v] \rangle = 2 I_c([v]) \; .$$

This shows the claim by choosing u such that $I_\mu([u])$ equals the minimal Maslov number. The supplement follows directly since $I_\omega([u]) = I_\omega([v])$. □

Lagrangian pairs Given two Lagrangian submanifolds $L_0, L_1 \subset M$. We denote the *path space*

$$\mathcal{P}(L_0, L_1) := \{x \in C^\infty([0,1], M) \mid x(0) \in L_0,\ x(1) \in L_1\} \,. \tag{3.1.1}$$

Fix an element $x_* \in \mathcal{P}(L_0, L_1)$. Given a smooth map $u : [-1,1] \times [0,1] \to M$ such that

$$u(-1, \cdot) = u(1, \cdot) = x_*, \qquad u(\cdot, 0) \subset L_0, \qquad u(\cdot, 1) \subset L_1 \,,$$

the map $s \mapsto u(s, \cdot)$ defines a loop in $\mathcal{P}(L_0, L_1)$. Every loop in $\mathcal{P}(L_0, L_1)$ based in x_* is homotop to a loop of this type. Integrating the symplectic form over u or by evaluating the Maslov index on u we obtain two ring homomorphisms

$$I_\omega : \pi_1(\mathcal{P}(L_0, L_1), x_*) \to \mathbb{R}, \qquad\qquad I_\mu : \pi_1(\mathcal{P}(L_0, L_1), x_*) \to \mathbb{Z} \; .$$

We define the *minimal Maslov number of* (L_0, L_1) *with respect to* x_* as smallest positive value of I_μ. We have corresponding homological requirements. The pair (L_0, L_1) is called

- *symplectically aspherical with respect to* x_* if for all $a \in \pi_1(\mathcal{P}(L_0, L_1); x_*)$ we have $I_\omega(a) = I_\mu(a) = 0$,
- τ*-monotone with respect to* x_* if there exists a constant $\tau > 0$ such that for all $a \in \pi_1(\mathcal{P}(L_0, L_1), x_*)$ we have $I_\omega(a) = \tau I_\mu(a)$.

For simplicity we will write that (L_0, L_1) is *monotone* if the choice of base point x_* is self-understood. If (L_0, L_1) is monotone with minimal Maslov number N, then again M, L_0 and L_1 is monotone or symplectically aspherical. Moreover N divides $2c_M$ and the minimal Maslov number of each L_0 and L_1 (cf. [62, Remark 3.3.2]).

Coisotropics A submanifold $V \subset M$ is *coisotropic* if at each point $p \in V$ we have

$$T_p V^\omega := \{\xi \in T_p M \mid \omega_p(\xi, \xi') = 0 \ \forall\ \xi' \in T_p V\} \subset T_p V \; .$$

Consequently the bundle $TV^\omega = \bigcup_{p \in V} T_p V^\omega \subset TV$ has constant rank and defines a foliation of V by isotropic leaves (see [52, Lemma 5.33]). The foliation induces an equivalence relation on V where two points are equivalent if they lie on the same leaf. The space V is *fibered coisotropic submanifold* if the projection $\pi : V \to X$ onto the quotient $X := V/\sim$ is a local trivial fibration. When this happens, X has an induced smooth structure, π becomes a smooth submersion and the leaves through two different points are diffeomorphic. Via symplectic reduction the quotient X also carries an induced symplectic form uniquely determined by

$$\pi^*\omega_X = i^*\omega_M \ ,$$

where $i : V \to M$ denotes the embedding. Equivalently ω_X is uniquely determined by requiring that V embeds via $i \times \pi$ into $(M \times X, \omega_M \oplus -\omega_X)$ as a Lagrangian submanifold. By abuse of language we write that V is *τ-monotone* (resp. *symplectically aspherical*) if it defines a τ-monotone (resp. symplectically aspherical) Lagrangian submanifold in that way.

Proposition 3.1.4. *Suppose that (M, ω_M) is a τ-monotone symplectic manifold and $V \subset M$ a fibred coisotropic submanifold with simply connected leaves, then the quotient (X, ω_X) is τ-monotone. Moreover the minimal Chern number of M divides the minimal Chern number of X.*

Proof. As in the proof of [50, Proposition 1.2] we construct compatible almost complex structures J_M and J_X on M and X respectively such that there exists a splitting of complex vector bundles

$$i^*TM \cong \pi^*TX \oplus \left(\ker \mathrm{d}\pi \otimes \mathbb{C}\right) \ ,$$

where $\ker \mathrm{d}\pi \otimes \mathbb{C}$ denotes the complexification of the real vector bundle $\ker \mathrm{d}\pi \subset TV$. It is a classical fact that $2c_1(F \otimes \mathbb{C}) = 0$ for any real vector bundle F (cf. [54, p.174]). We conclude with the above splitting over $\mathbb{R}$

$$i^*c_1(M) = \pi^*c_1(X) \ .$$

Fix a point $p \in X$ and let $D = \{(s,t) \in \mathbb{R}^2 \mid s^2 + t^2 \leq 1\}$ denote the unit disc with boundary ∂D. Every spherical homology class $a \in H_2^S(X)$ is represented by a map $u : (D, \partial D) \to (X, p)$. There exists a lift $\tilde{u} : (D, \partial D) \to (V, F)$ such that $\pi \circ \tilde{u} = u$ where $F = \pi^{-1}(p)$. Since F is simply connected there exists a map $v : D \to F$ such that $v|_{\partial D} = \tilde{u}|_{\partial D}$. The connected sum $w := \tilde{u} \# v : S^2 \to V$ defines a spherical class $\tilde{a} \in H_2^S(V)$ such that by construction $\pi_* \tilde{a} = a$. Since $F \subset M$ is isotropic we have $\int v^*\omega_M = 0$. We conclude

$$\begin{aligned}\int u^*\omega_X = \int \tilde{u}^*\pi^*\omega_X = \int \tilde{u}^*i^*\omega_M = \int \tilde{u}^*i^*\omega_M + \int v^*i^*\omega_M = \int w^*i^*\omega_M \\ = \tau\langle i^*c_1(M), \tilde{a}\rangle = \tau\langle \pi^*c_1(X), \tilde{a}\rangle = \tau\langle c_1(X), \pi_*\tilde{a}\rangle = \tau\langle c_1(X), a\rangle \ ,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. By the same token we see that if $a \in H_2^S(X)$ is such that $c_X = \langle c_1(X), a\rangle$, there exists $\tilde{a} \in H_2^S(V)$ with $c_X = \langle c_1(X), a\rangle = \langle c_1(M), i_*\tilde{a}\rangle \in c_M\mathbb{Z}$. $\square$

A fibered coisotropic submanifold V is an example of a Lagrangian correspondence and is used to transfer Lagrangians in X to Lagrangians in M. More precisely given a Lagrangian submanifold $L \subset X$ it is easy to see that $L^V := \pi^{-1}(L)$ is a Lagrangian submanifold of M embedded via $i|_{L^V}$ and that $\pi|_{L^V} : L^V \to L$ is a fibre bundle. The next proposition gives a sufficient condition when this transfer preserves monotonicity.

Proposition 3.1.5. *Assume that V is τ-monotone and that X is simply connected. Given a τ-monotone Lagrangian submanifold $L \subset X$, then $L^V := \pi^{-1}(L) \subset M$ is τ-monotone Lagrangian submanifold. Moreover if N_L, N_V denotes the minimal Maslov number of L and V respectively then* $\gcd(N_L, N_V)$ *divides the minimal Maslov number of L^V.*

Proof. Lets first show that $L^V \subset M$ is indeed Lagrangian. The submanifold L^V is isotropic because given any point $p \in L^V$ and two vectors $\xi, \xi' \in T_pL^V$, we have

$$(\omega_M)_p(\xi, \xi') = (\omega_X)_{\pi(p)}(\mathrm{d}_p\pi\xi, \mathrm{d}_p\pi\xi') = 0\,.$$

It remains to show that L^V has the correct dimension. Let $F \subset V$ denote a leaf of V. We have the identities $2\dim V = \dim X + \dim M$ (V is Lagrangian), $\dim V = \dim X + \dim F$ (V is a fibre bundle), $\dim L^V = \dim L + \dim F$ (L^V is a fibre bundle) and $2\dim L = \dim X$ (L is Lagrangian). Combined we have

$$\begin{aligned} 2\dim L^V = 2\dim L + 2\dim F = 2\dim V - \dim X = \dim X + \dim M - \dim X = \\ = \dim M\,. \end{aligned}$$

We show that L^V is τ-monotone. Given a map $u : (D, \partial D) \to (M, L^V)$. The loop $\pi \circ u|_{\partial D}$ is contractible in X and there exists a map $v : (D, \partial D) \to (X, L)$ such that $v|_{\partial D} = \pi \circ u|_{\partial D}$. The pair $w := (u, v)$ satisfies $w : (D, \partial D) \to (M \times X, V)$. By monotonicity of V and L we have

$$\mu_{\mathrm{Mas}}(w) = \tau^{-1}\int u^*\omega_M - \tau^{-1}\int v^*\omega_X = \tau^{-1}\int u^*\omega_M - \mu_{\mathrm{Mas}}(v)\,.$$

Hence $\int u^*\omega_M = \tau\mu_{\mathrm{Mas}}(w) + \tau\mu_{\mathrm{Mas}}(v)$. The bundle pair splits

$$\left(w^*(TM \oplus TX), w|_{\partial D}^*(TL^V \oplus TL)\right) = \left(u^*TM \oplus v^*TX, u|_{\partial D}^*TL^V \oplus v|_{\partial D}^*TL\right)\,.$$

That implies $\mu_{\mathrm{Mas}}(w) = \mu_{\mathrm{Mas}}(u) - \mu_{\mathrm{Mas}}(v)$ and with the above $\int u^*\omega_M = \tau\mu_{\mathrm{Mas}}(u)$. From $\mu_{\mathrm{Mas}}(w) = \mu_{\mathrm{Mas}}(u) - \mu_{\mathrm{Mas}}(v)$ we also conclude that $\gcd(N_L, N_V)$ divides the minimal Maslov number of L^V. □

3.1.2. Hamiltonian group actions

We give a short introduction of Hamiltonian group actions. For a reference see for example the book [8]. Let G be a compact Lie group and M be a manifold. A *(left) action* of G on M is a smooth map $\varphi : G \times M \to M$, $\varphi_g = \varphi(g, \cdot)$ with the properties that for all g_0, $g_1 \in G$ and $x \in M$ we have $\varphi_{g_0}(\varphi_{g_1}(x)) = \varphi_{g_0 g_1}(x)$ and $\varphi_e(x) = x$ where $e \in G$ is the neutral element. If (M, ω) is a symplectic manifold, then the group action is *symplectic* if for all $g \in G$ we have $\varphi_g^* \omega = \omega$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^\vee$ its dual. Consider the exponential map $\exp : \mathfrak{g} \to G$. For every $\xi \in \mathfrak{g}$ we have a one-parameter family of diffeomorphisms $t \mapsto \varphi_{\exp t\xi}$ generated by the vector field $\underline{\xi} = \partial_t \varphi_{\exp(t\xi)}|_{t=0} \in \mathrm{Vect}(M)$. The vector field $\underline{\xi}$ is called *fundamental vector field to* ξ. The action is *Hamiltonian* if there exists a map $\mu : M \to \mathfrak{g}^\vee$ such that

- $\omega(\underline{\xi}, \cdot) = \mathrm{d}H_\xi$ with $H_\xi = \langle \mu, \xi \rangle$ for all $\xi \in \mathfrak{g}$,
- $\omega(\underline{\xi}, \underline{\eta}) = \langle \mu, [\xi, \eta] \rangle$ for all $\xi, \eta \in \mathfrak{g}$ in which $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g}$.

Note that if there is a map which only satisfies the first property, the second is accheived by adding a constant (cf. [8, Rmk. III.1.2]). The map μ is called *moment map*. Every Hamiltonian action is symplectic. Conversely every symplectic action is locally a Hamiltonian action and if the group is semi-simple, i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, every symplectic action is also Hamiltonian (cf. [8, p. 75]). A prominent example is action of the rotation group $SO(3)$ on $\mathbb{R}^3 \oplus \mathbb{R}^3$ via $g(q,p) = (gq, gp)$. The action is Hamiltonian if $\mathbb{R}^3 \oplus \mathbb{R}^3$ is equipped with the standard symplectic structure. In that case the dual of the Lie algebra of $SO(3)$ is canonically isomorphic to $\mathbb{R}^3$ and the moment map is the cross product.

Linear group actions

Let G act on a complex vector space M via linear maps. In other words we are given a complex representation of G on M. Since G is compact we assume without loss of generality that the representation is unitary and in particular symplectic for the standard symplectic form on M. In the following we also assume that the representation is faithful. After choosing a basis we have $M \cong \mathbb{C}^n$ and we identify G with a subgroup $G \subset U(n)$ of the uniary group. The linear action of the unitary group $U(n)$ on $\mathbb{C}^n$ is Hamiltonian with moment map

$$\mu_{U(n)} : \mathbb{C}^n \to \mathfrak{h}(n), \qquad (z_1, \dots, z_n) \mapsto (\bar{z}_j z_i)_{1 \le i,j \le n} \,,$$

where we have identified the dual $\mathfrak{u}(n)^\vee$ with the space of Hermitian matrices $\mathfrak{h}(n)$ using the standard inner product on $\mathfrak{u}(n)$ given by $\langle \xi, \eta \rangle = -\operatorname{Tr}(\xi\eta)$ and multiplication with the imaginary unit $\sqrt{-1}$. Because G is subgroup of a group with Hamiltonian action the action of G is Hamiltonian as well.

Let $T \subset G$ be a maximal torus. The embedding $T \subset G \subset U(n)$ induces an embedding of Lie algebras $\mathfrak{t} \subset \mathfrak{g} \subset \mathfrak{h}(n)$. Let

$$\mathfrak{t}_{\mathbb{Z}} = \{\xi \in \mathfrak{t} \mid \exp(2\pi\sqrt{-1}\xi) = e\} \subset \mathfrak{t} \,,$$

be the unit lattice of $\mathfrak{t}$. With the identification every $\xi \in \mathfrak{t}_{\mathbb{Z}}$ is an Hermitian matrix thus has real eigenvalues. Moreover if $\lambda \in \mathbb{R}$ is an eigenvalue to the eigenvector $v \in \mathbb{C}^n$ we have $\exp(2\pi\sqrt{-1}\xi)v = \exp(2\pi\sqrt{-1}\lambda)v = v$. We conclude that $\lambda \in \mathbb{Z}$. It is easy to check that $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{t}$ is a free $\mathbb{Z}$-module of rank $\dim \mathfrak{t}$. Let $\xi_1, \dots, \xi_k$ be an integer basis of $\mathfrak{t}_{\mathbb{Z}}$. With the identification above $\xi_1, \ldots, \xi_k$ are Hermitian matrices which pair-wise commute. Thus there exists a basis of mutual eigenvectors $v_1, \ldots, v_n \in \mathbb{C}^n$ such that $\langle v_i, v_j \rangle = \delta_{ij}$ where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian product on $\mathbb{C}^n$. The *weights* are the tuples of eigenvalues $w_1, \ldots, w_n \in \mathbb{Z}^k$, i.e. $w_j = (\xi_1 v_j, \ldots, \xi_k v_j)$ for all $j = 1, \ldots, n$. In the basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ and $\xi_1, \ldots, \xi_k$ of $\mathfrak{t}$ the moment map of the T-action is

$$\mathbb{C}^n \mapsto \mathbb{R}^k, \qquad (z_1, \ldots, z_n) \mapsto \sum_{j=1}^n w_j |z_j|^2 \,.$$

Denote the subgroup $\mathbb{T}^n \subset U(n)$ which consists of diagonal matrices. Without loss of generality $T \subset \mathbb{T}^n$. The quotient $Y_w := \mu_T^{-1}(w)/T$ at any $w \in \mathbb{R}^k$ inherits an action of the torus $\mathbb{T}^n/T$. We conclude that Y_w is a toric manifold, if the reduction is regular. In [53, §11.3.1]) it is deduced that the quotient Y_w is monotone if

$$w = w_1 + w_2 + \cdots + w_n \in \mathbb{Z}^k \,.$$

In the following we always assume that w is of that form and the reduction is regular. In [19, §1] it is shown that the minimal Chern number c_Y of Y_w is the greatest common divisor of the k-tuple w of integers. We wish to warn the reader that this only holds if the action of the torus T on M is faithfull as we have assumed in the section. Together with Proposition 3.1.4 we conclude the following

Lemma 3.1.6. *If the symplectic quotients $X := \mu_G^{-1}(w)/G$ and $Y = \mu_T^{-1}(w)/T$ at $w = w_1 + \cdots + w_n \in \mathbb{Z}^k$ are regular, then X is a monotone symplectic manifold and we have $c_Y = \gcd w$.*

We describe the cohomology and the quantum cohomology rings. All material is taken from [53, §11.3.1]. To see that the statements remain true over the integers see for example [38, §5.2]. Abbreviate the index set $I_0 = \{1, \ldots, n\}$. For any subset $I \subset I_0$ we consider its *cone* given by

$$\operatorname{cone}(I) = \{\sum_{i \in I} a_i w_i \in \mathbb{R}^k \mid a_i \geq 0\} \,.$$

A subset $I \subset I_0$ is *primitive* if $w \notin \operatorname{cone}(I_0 \setminus I)$ but $w \in \operatorname{cone}(I_0 \setminus J)$ for any $J \subsetneq I$. We identify $\mathbb{R}^n$ with the Lie algebra of $\mathbb{T}^n$ and $\mathbb{Z}^n \subset \mathbb{R}^n$ with its unit lattice. The inclusion $T \subset \mathbb{T}^n$ induces and an injection of $\mathbb{Z}$-modules $\mathfrak{t}_{\mathbb{Z}} \subset \mathbb{Z}^n$. In the basis $\xi_1, \ldots, \xi_k$ of $\mathfrak{t}_{\mathbb{Z}}$ and the standard basis of $\mathbb{Z}^n$ the matrix of the inclusion has row vectors $w_1, \ldots, w_n$. Using the dual of the standard basis of $\mathbb{Z}^n$ and dual of the basis of $\xi_1, \ldots, \xi_k$ of $\mathfrak{t}_Z$ we obtain identifications of symmetric algebras $\mathbb{Z}[y_1, \ldots, y_n] \cong \operatorname{Sym}(\mathbb{Z}^n)^\vee$ and $\mathbb{Z}[x_1, \ldots, x_k] \cong \operatorname{Sym} \mathfrak{t}_{\mathbb{Z}}^\vee$ respectively. Via the injection $\mathfrak{t}_{\mathbb{Z}} \subset \mathbb{Z}^n$ together with the identifications we obtain a canonical ring morphism

$$\mathbb{Z}[y_1, \ldots, y_n] \to \mathbb{Z}[x_1, \ldots, x_k] \,, \tag{3.1.2}$$

which is equally obtained by quotienting the relations of the form $\sum_i a_i y_i = 0$ for all $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $\sum_i a_i w_i = 0$. Let $\mathrm{SR}' \subset \mathbb{Z}[y_1, \ldots, y_n]$ be the ideal generated by $\prod_{i \in I} y_i$ for all primitive index sets $I \subset I_0$ and $\mathrm{SR} \subset \mathbb{Z}[x_1, \ldots, x_k]$ the image of SR' under the map (3.1.2). The singular cohomology ring of Y is given by

$$H^*(Y, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_k]/\mathrm{SR}\,.$$

For the quantum cohomology ring with coefficient ring $\Lambda = \mathbb{Q}[\lambda, \lambda^{-1}]$ where $\deg \lambda = 2c_Y$ we tensor (3.1.2) with Λ and obtain

$$\Lambda[y_1, \ldots, y_n] \to \Lambda[x_1, \ldots, x_k]\,. \tag{3.1.3}$$

Let $Q\mathrm{SR}' \subset \Lambda[y_1, \ldots, y_n]$ be the ideal generated by

$$\prod_{i \in I} y_i - \lambda^{|d|/c_Y} \prod_{i \notin I} y_i^{-d_i}\,,$$

over all primitive index sets I where $|d| = d_1 + d_2 + \cdots + d_n$ and $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ is the unique vector in the image of the injection $\mathfrak{t}_{\mathbb{Z}} \subset \mathbb{Z}^n$ such that $d_i = 1$ for all $i \in I$ and $d_i \leq 0$ for all $i \notin I$. By definition the product over the empty set is just 1. The ideal $Q\mathrm{SR} \subset \Lambda[x_1, \ldots, x_k]$ given by the image of $Q\mathrm{SR}'$ under (3.1.3) is called the *quantum Stanley-Reisner ideal.* The quantum cohomology of Y is the quotient ring

$$QH^*(Y, \Lambda) \cong \Lambda[x_1, \ldots, x_k]/Q\mathrm{SR}\,.$$

Because $\Lambda[x_1, \ldots, x_k]$ is identified with the symmetric algebra over $\mathfrak{t}_{\mathbb{Z}}^\vee$ tensored with Λ, we obtain a canonical action of the Weyl group on $\Lambda[x_1, \ldots, x_k]$. By the same reasoning any root is a linear element in $\Lambda[x_1, \ldots, x_k]$ and the canonical anti-invariant class D is given by the product of all positive roots. Now Theorem 2.1.1 adapted to our setting states that the quantum cohomology of $X := \mu_G^{-1}(w)/G$ is given by

$$QH^*(X, \Lambda) \cong \Lambda[x_1, \ldots, x_k]^W / (Q\mathrm{SR} : \mathrm{D} \cap \Lambda[x_1, \ldots, x_k]^W)\,,$$

where $Q\mathrm{SR} : \mathrm{D}$ denotes the ideal quotient of $Q\mathrm{SR}$ by D. To show Corollary 2.1.3 it remains to prove the following lemma.

Lemma 3.1.7. *We have* $Q\mathrm{SR} : \mathrm{D} = p(Q\mathrm{SR})$ *with* p *given in* (2.1.4) .

Proof. Abbreviate $S := \Lambda[x_1, \ldots, x_k]$ and by S^W the ring of invariants. Given $f \in (Q\mathrm{SR} : \mathrm{D}) \cap S^W$. In other words $f \in S^W$ and $f\mathrm{D} \in Q\mathrm{SR}$. Set $g := |W|^{-1} f\mathrm{D} \in Q\mathrm{SR}$ then we have

$$p(g) = |W|^{-1}\, \mathrm{D}^{-1} \sum_{w \in W} \operatorname{sign} w\, f w.\mathrm{D} = |W|^{-1}\, \mathrm{D}^{-1} f\mathrm{D}\, |W| = f\,.$$

This shows that $f \in p(Q\mathrm{SR})$. Conversely given $f \in p(Q\mathrm{SR})$. Hence there must exist $g \in Q\mathrm{SR}$ such that $p(g) = f$. By definition of p and since $Q\mathrm{SR}$ is invariant as a set under the action of W

$$f\mathrm{D} = \sum_{w \in W} \operatorname{sign} w\, w.g \in Q\mathrm{SR}\,.$$

We conclude that $f \in S^W$ and $f \in Q\mathrm{SR} : \mathrm{D}$. □

Remark 3.1.8. The subring of invariants $\Lambda[x_1, \ldots, x_k]^W \subset \Lambda[x_1, \ldots, x_k]$ is generated by k homogeneous polynomials of degrees $2m_1 + 2, \ldots, 2m_k + 2$ (with $\deg x_i = 2$) where $(2m_1 + 1, \ldots, 2m_k + 1)$ is the rational type of G (cf. Section D.1).

3.2. Hamiltonian action functional

Fix a symplectic manifold (M, ω) and two Lagrangians submanifolds $L_0, L_1 \subset M$. We replace the path space $\mathcal{P} = \mathcal{P}(L_0, L_1)$ with its Sobolev extension

$$\mathcal{P} := \left\{x \in H^{1,2}([0,1], M) \mid (x(0), x(1)) \in L_0 \times L_1\right\} \ .$$

Fix an element $x_* \in \mathcal{P}$ and a Hamiltonian function $H \in C^\infty([0,1] \times M)$. Central to our study is the *Hamiltonian action functional* $\mathcal{A}_H$,

$$\mathcal{A}_H(u_x, x) = -\int u_x^*\omega - \int_0^1 H(t, x)\mathrm{d}t \ , \tag{3.2.1}$$

where $x \in \mathcal{P}$ lies in the same connected component of x_* and $u_x : [-1,1] \times [0,1] \to M$ satisfies

$$u_x(1, \cdot) = x, \qquad u_x(-1, \cdot) = x_*, \qquad (u_x(\cdot, 0), u_x(\cdot, 1)) \subset L_0 \times L_1 \, . \tag{3.2.2}$$

In other words $s \mapsto u_x(s, \cdot)$ is a path in $\mathcal{P}$ from x_* to x. We call u_x a *cap of* x. Unfortunately $\mathcal{A}_H$ is not well-defined on the path-space $\mathcal{P}$ since it depends on the choice of the cap u_x. To take that into account we define a certain cover of $\mathcal{P}$ on which $\mathcal{A}_H$ becomes well-defined.

Domain of $\mathcal{A}_H$ Two caps u_x and u_y are *equivalent* if $x = y$ and $[u_x \# u_y^\vee] \in \ker I_\omega \cap \ker I_\mu$, in which $u_x \# u_y^\vee : [-1,1] \times [0,1] \to M$ denotes the connected sum of u_x with the reversed map $u_y^\vee$, that is

$$(u_x \# u_y^\vee)(s,t) := \begin{cases} u_x(2s+1, t) & \text{if } -1 \le s \le 0 \\ u_y(1-2s, t) & \text{if } 0 \le s \le 1 \, . \end{cases} \tag{3.2.3}$$

Let $\widetilde{\mathcal{P}}$ denote the set of equivalence classes.

Remark 3.2.1. We have incorporated a finer equivalence relation than actually necessary at this point. Instead of $\ker I_\omega \cap \ker I_\mu$, we could have just used $\ker I_\omega$. The finer equivalence relation will become useful when we define the index later on (cf. Section 3.4). Note that in the case of a monotone pair these two subgroups are the same.

Since $\mathcal{P}$ is locally path-connected $\widetilde{\mathcal{P}}$ carries an induced topology and is in fact a covering space over the connected component of $\mathcal{P}$ containing x_*, denoted by $\mathcal{P}_{[x_*]}$. The covering map is given by

$$\widetilde{\mathcal{P}} \to \mathcal{P}_{[x_*]}, \qquad [u_x, x] \mapsto x \ .$$

The group of Deck transformations of this cover is $\Gamma := \pi_1(\mathcal{P}; x_*)/(\ker I_\omega \cap \ker I_\mu)$ which is acting transitively and effectively via

$$\Gamma \times \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}}, \qquad ([u], [u_x, x]) \mapsto [u \# u_x, x] \ .$$

We obtain a well-defined map with domain a covering space of $\mathcal{P}_{[x_*]}$

$$\mathcal{A}_H : \widetilde{\mathcal{P}} \to \mathbb{R}, \qquad [u_x, x] \to \mathcal{A}_H(u_x, x) \, . \tag{3.2.4}$$

Critical points of $\mathcal{A}_H$ Critical points of $\mathcal{A}_H$ correspond to solutions of the Hamiltonian equation. We choose the following convention in order to define the *Hamiltonian vector field* X_H,

$$\omega(X_H, \cdot) = \mathrm{d}H \ . \tag{3.2.5}$$

Define the *perturbed intersection points*

$$\mathcal{I}_H(L_0, L_1) := \{x : [0,1] \to M \mid \dot{x} = X_H(x), \ (x(0), x(1)) \in L_0 \times L_1\} \ . \tag{3.2.6}$$

Note that the set $\mathcal{I}_H(L_0, L_1)$ is in bijection with $\varphi_H(L_0) \cap L_1$, where φ_H denotes the Hamiltonian diffeomorphism associated to H, i.e. the time-one map of the flow associated to the Hamiltonian vector field X_H.

Lemma 3.2.2. *Critical points of $\mathcal{A}_H$ are exactly the points $[u_x, x] \in \widetilde{\mathcal{P}}$ with $x \in \mathcal{I}_H(L_0, L_1)$.*

Proof. To compute the directional derivative of $\mathcal{A}_H$ we fix $\varepsilon > 0$ and let $u : (-\varepsilon, \varepsilon) \times [-1,1] \times [0,1] \to M$ be a smooth map such that the maps $u_\tau = u(\tau, \cdot)$ satisfy $u_\tau\big|_{t=0,1} \subset L_{0,1}$ and $u_\tau(-1, \cdot) = x_*$ for all $\tau \in (-\varepsilon, \varepsilon)$. We write $x_\tau = u_\tau(1, \cdot)$ and $\xi = \partial_\tau|_{\tau=0}\, u_\tau(1, \cdot)$.

$$\begin{aligned}
\frac{d}{d\tau}\bigg|_{\tau=0} & \mathcal{A}_H\left(u_\tau, x_\tau\right) = \\
&= -\int_{[-1,1]\times[0,1]} \partial_\tau \omega\left(\partial_s u_\tau, \partial_t u_\tau\right) \mathrm{d}s\mathrm{d}t\bigg|_{\tau=0} - \int_0^1 \partial_\tau H(t, x_\tau(t))\mathrm{d}t\bigg|_{\tau=0} \\
&= -\int_0^1 \omega\left(\partial_\tau u_\tau, \partial_t u_\tau\right)\, \mathrm{d}t\bigg|_{\tau=0, s=1} - \int_0^1 dH(t, \cdot)\xi \, \mathrm{d}t \\
&= -\int_0^1 \omega\left(\xi, \dot{x}\right)\, \mathrm{d}t - \int_0^1 \omega(X_H, \xi)\, \mathrm{d}t \\
&= \int_0^1 \omega\left(\dot{x} - X_H, \xi\right)\, \mathrm{d}t \ .
\end{aligned}$$

For the second line we use

$$0 = d\omega\left(\partial_\tau u, \partial_s u, \partial_t u\right) = \partial_\tau \omega\left(\partial_s u, \partial_t u\right) - \partial_s \omega\left(\partial_\tau u, \partial_t u\right) + \partial_t \omega\left(\partial_\tau u, \partial_s u\right) \ ,$$

and that integration over the ∂_t-part vanishes by the Lagrangian boundary conditions. By non-degeneracy of the symplectic form ω we conclude that $[u_x, x]$ is a critical point of $\mathcal{A}_H$ if and only if $\dot{x} = X_H(x)$. □

Gradient of $\mathcal{A}_H$ A key observation of Floer was that the gradient of $\mathcal{A}_H$ with respect to a certain L^2-metric on $\mathcal{P}$ establishes ties between Morse theory and J-holomorphic curve theory. More precisely fix a path $J : [0,1] \to \mathrm{End}(TM, \omega)$, we define an L^2-metric on the path space $\mathcal{P}$ via

$$\langle \xi, \eta \rangle_J = \int_0^1 \langle \xi(t), \eta(t) \rangle_{J_t} \mathrm{d}t = \int_0^1 \omega_{x(t)}\big(\xi(t), J_t(x(t))\eta(t)\big)\mathrm{d}t \ , \tag{3.2.7}$$

for all sections $\xi, \eta \in \Gamma(x^*TM)$ and $x \in \mathcal{P}$. The metric canonically lifts to $\widetilde{\mathcal{P}}$. As one sees at the formula (3.2.8) of the next lemma, the gradient of $\mathcal{A}_H$ is independent of the choice of the base point and descends to a vector field on the path space $\mathcal{P}$.

Lemma 3.2.3. *The gradient of the functional $\mathcal{A}_H$ with respect to the metric* (3.2.7) *is given by*

$$\operatorname{grad}_J \mathcal{A}_H(u_x, x) = J\left(\partial_t x - X_H(x)\right) . \tag{3.2.8}$$

Proof. Given $\xi \in C^\infty(x^*TM)$, continuing the computation in the proof of Lemma 3.2.2 we see that

$$\begin{aligned} \mathrm{d}\mathcal{A}_H(u_x, x)[\xi] &= \int_0^1 \omega_{x(t)}\left(\dot{x}(t) - X_H(t, x(t)), \xi(t)\right) \mathrm{d}t \\ &= \int_0^1 \omega_{x(t)}\big(\xi(t), J_t^2(x(t))\left(\dot{x}(t) - X_H(t, x(t))\right)\big) \mathrm{d}t \\ &= \int_0^1 \langle \xi(t), J_t(x(t))\left(\dot{x}(t) - X_H(t, x(t))\right)\rangle_J \,\mathrm{d}t\,. \\ &= \langle \xi, J\left(\partial_t x - X_H(x)\right)\rangle_J\,. \end{aligned}$$

This shows the claim. □

Another crucial idea of Floer was that despite the fact that the negative gradient flow of $\mathcal{A}_H$ is not well-defined, finite energy negative-gradient flow-lines between any two critical points are. A gradient flow line between the critical points $[u_-, x_-]$ and $[u_+, x_+]$ is given by a map $u : \mathbb{R} \times [0, 1] \to M$ such that

$$\begin{gathered} \partial_s u + J(u)\left(\partial_t u - X_H(u)\right) = 0\,, \\ u\big|_{t=0} \subset L_0, \qquad u\big|_{t=1} \subset L_1\,, \\ \int_{\mathbb{R}\times[0,1]} |\partial_s u|_J^2 \,\mathrm{d}s\mathrm{d}t < \infty\,, \\ \lim_{s\to-\infty} u(s, \cdot) = x_-, \qquad \lim_{s\to\infty} u(s, \cdot) = x_+\,, \end{gathered} \tag{3.2.9}$$

where the limits in the last line are in uniform topology. We call u a *finite-energy* (J, H)*-holomorphic strip with boundary in* (L_0, L_1) *connecting* x_- *to* x_+. These "generalized" flow-lines satisfy the same properties of negative gradient flow lines in Morse theory. For example if $[u_+, y] = [u_-\#u, y]$ with $u_-\#u$ defined via (3.2.3) we have the *action-energy relation*

$$E(u) = \int_{\mathbb{R}\times[0,1]} |\partial_s u|_J^2 \,\mathrm{d}s\mathrm{d}t = \mathcal{A}_H(u_-, x_-) - \mathcal{A}_H(u_+, x_+)\,. \tag{3.2.10}$$

There is a standard trick to transform a solution of (3.2.9) into a solution with $H = 0$ but changing L_1 and J. We will use it at several places in the paper.

Lemma 3.2.4. *Given a Hamiltonian function $H \in C^\infty([0,1] \times M)$ and an almost complex structure $J \in C^\infty([0,1], \mathrm{End}(TM, \omega))$. Let $u : \mathbb{R} \times [0,1] \to M$ be solution of*

$$\partial_s u + J(u)\big(\partial_t u - X_H(u)\big) = 0\,,$$

which satisfies the boundary condition

$$u(\cdot, 0) \subset L_0, \qquad u(\cdot, 1) \subset L_1\,.$$

Then the map $v : \mathbb{R} \times [0,1] \to M$ defined by $v(s,t) = \varphi_H^t(u(s,t))$ is a solution of

$$\partial_s v + J'(v)\partial_t v = 0, \qquad J'_t := (\mathrm{d}\varphi_H^t)^{-1} \circ J_t \circ \mathrm{d}\varphi_H^t\,,$$

which satisfies the boundary condition

$$v(\cdot, 0) \subset L_0, \qquad v(\cdot, 1) \subset \varphi_H^{-1}(L_1)\,.$$

Moreover we have $E(u) = E(v)$.

Proof. Obviously the curve v satisfies the boundary condition by construction. We check the differential equation. We have $\partial_s u = \mathrm{d}\varphi_H \partial_s v$ and $\partial_t u = X_H(u) + \mathrm{d}\varphi_H \partial_t v$ and thus

$$\mathrm{d}\varphi_H \left(\partial_s v + J'(v)\partial_t v\right) = \partial_s u + J(u)\left(\partial_t u - X_H(u)\right) = 0\,.$$

This shows that v is J'-holomorphic. Then

$$|\partial_s u|^2 = \omega(\partial_s u, J\partial_s u) = \omega(\mathrm{d}\varphi_H \partial_s v, J\mathrm{d}\varphi_H \partial_s v) = \omega(\partial_s v, J'\partial_s v) = |\partial_s v|^2\,.$$

This shows $E(u) = E(v)$. $\square$

Hessian of $\mathcal{A}_H$ Let ∇^t denote the Levi-Civita connection with respect to the metric $\omega(\cdot, J_t \cdot)$ for each $t \in [0,1]$. Given $x \in \mathcal{P}$, we define the *Hessian of the Hamiltonian action functional* as the operator

$$A_x : T_x\mathcal{P}(L_0, L_1) \to L^2(x^*TM), \qquad \xi \mapsto J(x)\left(\nabla_t \xi - \nabla_\xi X_H\right)\,, \tag{3.2.11}$$

with domain $T_x\mathcal{P}(L_0, L_1) \subset L^2(x^*TM)$ given by

$$T_x\mathcal{P}(L_0, L_1) = \{\xi \in H^{1,2}(x^*TM) \mid \xi(0) \in T_{x(0)}L_0,\ \xi(1) \in T_{x(1)}L_1\}\,.$$

Remark 3.2.5. One can show that the operator (3.2.11) is the Hessian of the Hamiltonian action functional with respect to the Levi-Civita connection of $\mathcal{P}$ induced from the metric (3.2.7) and whenever $x \in \mathcal{I}_H(L_0, L_1)$ the operator is independent of the choice of the connection.

The eigenvalues and eigenfunctions of A_x play an important role for the study of the asymptotic behavior of solutions of (3.2.9). We have the following result due to Frauenfelder.

Proposition 3.2.6 ([33, Theorem 4.1])**.** *For any $x \in \mathcal{P}(L_0, L_1)$ the operator A_x is self-adjoint with respect to the inner product* (3.2.7) *and has a closed range. The spectrum $\sigma(A_x) \subset \mathbb{R}$ is discrete and consists purely of eigenvalues.*

We prove in Chapter 4 that the gap in the spectrum around zero of the Hessian A_x controls the decay rate of finite energy (J, H)-holomorphic strips. Given $x \in \mathcal{I}_H(L_0, L_1)$ we define

$$\iota_x(J, H) := \inf \{|\alpha| \mid 0 \neq \alpha \in \sigma(A_x)\} \ , \tag{3.2.12}$$

and moreover for any subset $C \subset \mathcal{I}_H(L_0, L_1)$ we define

$$\iota(J, H) := \inf_{x \in \mathcal{I}_H(L_0, L_1)} \iota_x(J, H), \qquad \iota(C; J, H) := \inf_{x \in C} \iota_x(J, H) \, . \tag{3.2.13}$$

If $H \equiv 0$, then we abbreviate $\iota(x; J, 0)$, $\iota(C; J, 0)$ and $\iota(J, 0)$ by $\iota(x; J)$, $\iota(C; J)$ and $\iota(J)$ respectively.

Remark 3.2.7. Whenever $H \equiv 0$, $J_t = J_0$ for all $t \in [0, 1]$ and $\dim M = 2$ there is a geometric interpretation of the spectrum of A_x as angle at the intersection point $x = p \in L_0 \cap L_1$. For example if $M = \mathbb{C}$, $\omega = \omega_{\mathrm{std}}$, $J_t = J_{\mathrm{std}}$ for all $t \in [0, 1]$, $L_0 = \mathbb{R}$ and $L_1 = e^{i\alpha}\mathbb{R}$, then the spectrum is given by

$$\sigma(0; J_{\mathrm{std}}, 0) = \alpha + \pi\mathbb{Z} \ ,$$

and $\iota := \iota(0; J_{\mathrm{std}})$ is the unique constant such that $\iota \in (0, \pi/2]$ and $\iota = |\alpha + \pi k|$ with $k \in \mathbb{Z}$. Geometrically it corresponds to the acute angle of the intersection L_0 with L_1.

Lemma 3.2.8. *For all $x \in \mathcal{I}_H(L_0, L_1)$ we have $\iota_x(J, H) = \iota_p(\varphi_H^* J)$ with $p = x(0) \in L_0 \cap \varphi_H^{-1}(L_1)$ and $(\varphi_H^* J)_t = \mathrm{d}\varphi_H^t \circ J_t \circ \left(\mathrm{d}\varphi_H^t\right)^{-1}$.*

Proof. Abbreviate $J_t' = (\varphi_H^* J)_t$ and $L_1' = \varphi_H^{-1} L_1$. Consider the operator

$$A_p : T_p\mathcal{P}(L_0, L_1') \to L^2([0, 1], T_pM), \qquad \xi \mapsto J'\partial_t\xi \, .$$

It suffices to show that the operators A_p and A_x are conjugated by isomorphisms

$$T_p\mathcal{P}(L_0, L_1') \to T_x\mathcal{P}(L_0, L_1), \qquad L^2([0, 1], T_pM) \to L^2(x^*TM) \, ,$$

both given by $\xi \mapsto (t \mapsto \mathrm{d}\varphi_H^t \xi(t))$. For that it suffices to show that for all smooth $\xi : [0, 1] \to T_pM$ we have

$$J\left(\nabla_t \mathrm{d}\varphi_H \xi - \nabla_{\mathrm{d}\varphi_H \xi} X_H\right) = \mathrm{d}\varphi_H J' \partial_t \xi \ .$$

Suppose that $\partial_t \xi = 0$ for a moment, then since ∇ is torsion free and $\partial_t x = X_H$ the equation holds after $\nabla_t \mathrm{d}\varphi_H \xi = \nabla_{X_H} \mathrm{d}\varphi_H \xi = \nabla_{\mathrm{d}\varphi_H \xi} X_H$. In general any other ξ is given as $\xi = \sum_j f_j \xi_j$ with ξ_j constant and $f_j \in H^{1,2}([0, 1], \mathbb{R})$. We compute

$$\begin{aligned} J\left(\nabla_t \mathrm{d}\varphi_H \xi - \nabla_{\mathrm{d}\varphi_H \xi} X_H\right) &= \sum_j J\big(\nabla_t f_j \mathrm{d}\varphi_H \xi_j - f_j \nabla_{\mathrm{d}\varphi_H \xi_j} X_H\big) \\ &= \sum_j J\mathrm{d}\varphi_H(\partial_t f_j)\xi_j + f_j\left(\nabla_t \mathrm{d}\varphi_H \xi_j - \nabla_{\mathrm{d}\varphi_H \xi_j} X_H\right) \\ &= \sum_j \mathrm{d}\varphi_H J' \partial_t(f_j \xi_j) = \mathrm{d}\varphi_H J' \partial_t \xi \, . \end{aligned}$$

Thus the last equation holds for all ξ. We conclude that A_x and A_p are conjugated. $\square$

Clean intersections Two submanifolds $L_0, L_1 \subset M$ *intersect cleanly along a submanifold* $C \subset M$, if $C \subset L_0 \cap L_1$ and for all p in C we have

$$T_pC = T_pL_0 \cap T_pL_1 \ .$$

Moreover L_0, L_1 are *in clean intersection* if they intersect cleanly along $L_0 \cap L_1$. Every transverse intersection is also clean but certainly the converse is not true. Pozniak [62] gave a normal form for Lagrangian submanifolds in clean intersection. Let $C \subset L$ be a submanifold of a manifold L. The *conormal bundle* $TC^\omega \subset T^*L$ *of* C is defined by

$$TC^\omega = \{(q,p) \in T^*L \mid q \in C,\ p(v) = 0 \quad \forall\ v \in T_qC\} \ .$$

Note that $TC^\omega \subset (T^*L, \omega_{\text{can}})$ is an exact Lagrangian submanifold, which intersects the zero section cleanly along C.

Proposition 3.2.9 ([62, Proposition 3.4.1])**.** *Let (M,ω) be a symplectic manifold and $L_0, L_1 \subset M$ be two Lagrangian submanifolds intersecting cleanly along a compact submanifold $C \subset M$, then there exists a vector bundle $E \to C$, open sets $V \subset T^*E$, $U \subset M$ and a diffeomorphism $\varphi : U \to V$ such that such that $C \subset U$, $\varphi^*\omega_{\text{std}} = \omega$ and*

$$\varphi\left(L_0 \cap U_{\text{Poz}}\right) = E \cap V, \qquad \varphi\left(L_1 \cap U_{\text{Poz}}\right) = TC^\omega \cap V \,,$$

in which E and C are identified with their image under the zero section in the bundles $T^*E \to E$ and $E \to C$ respectively.

Lemma 3.2.10. *With the same assumption as Proposition 3.2.9. For all $p \in C$ there exists open sets $U \subset M$, $V \subset \mathbb{R}^{2n}$ and a diffeomorphism $\varphi : U \to V$ such that $p \in U$, $\varphi(p) = 0$, $\varphi^*\omega_{\text{std}} = \omega$ and*

$$\varphi\big(L_0 \cap U\big) = \Lambda_0 \cap V, \qquad \varphi\big(L_1 \cap U\big) = \Lambda_1 \cap V \,,$$

in which $\Lambda_0, \Lambda_1 \subset \mathbb{R}^{2n}$ are linear subspaces which are Lagrangian with respect to the standard symplectic form ω_{std}.

Proof. According to Proposition 3.2.9 we assume without loss of generality that $M = T^*L_0$, $\omega = \omega_{\text{can}}$ and $L_1 = TC^\omega$ for some submanifold $C \subset L_0$ of dimension k. There exists local coordinates $\psi : V \xrightarrow{\cong} W$ with $W \subset L_0$ and $V \subset \mathbb{R}^n$ is an open ball such that $\psi(V \cap \mathbb{R}^k) = W \cap C$. We define $U := T^*W$ and $\varphi = \psi^* : U \to T^*V$. $\square$

Lemma 3.2.11. *With the same assumption as Proposition 3.2.9. Let $J : [0,1] \to \operatorname{End}(TM,\omega)$ be a path of compatible almost complex structures. For all $p \in L_0 \cap L_1$ there exists an open neighborhood $U \subset M$ and a local trivialization*

$$\Phi : [0,1] \times U \times \mathbb{R}^{2n} \to TM_{|U}, \quad (t,q,v) \mapsto \Phi_t(q)v \in T_qM \ ,$$

such that we have

- *$J_t(q)\Phi_t(q) = \Phi_t(q)J_{\mathrm{std}}$ for all $t \in [0,1]$ and $q \in U$,*
- *$\omega_q(\Phi_t(q)\xi, \Phi_t(q)\xi') = \omega_{\mathrm{std}}(\xi, \xi')$ for all $t \in [0,1]$, $q \in U$ and $\xi, \xi' \in T_qM$*
- *$T_qL_k = \Phi_k(q)(\mathbb{R}^n \oplus \{0\})$ for all $q \in L_k \cap U$ and $k = 0, 1$.*

where $J_{\mathrm{std}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the standard complex structure with matrix representation given by

$$J_{\mathrm{std}} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{3.2.14}$$

Proof. Choose local coordinates (cf. Lemma 3.2.10) and assume U is an open subset of $\mathbb{R}^{2n}$, $\omega = \omega_{\mathrm{std}}$, $L_0, L_1 = \Lambda_0, \Lambda_1$ are linear Lagrangian subspaces and the almost complex structure is given by a matrix valued function $J : [0,1] \times U \to \mathbb{R}^{2n\times 2n}$. Choose a smooth path of linear Lagrangian subspaces $F : [0,1] \to \mathcal{L}(n)$ such that $F(0) = \Lambda_0$ and $F(1) = \Lambda_1$. Choose functions $e_1, \dots, e_n : [0,1]\times U \to \mathbb{R}^{2n}$ such that $(e_1(t,q), \dots, e_n(t,q))$ is a frame of F_t and after Gram-Schmidt satisfies $\omega(e_i(t,q), J_t(q)e_j(t,q)) = \delta_{ij}$ for all $i, j = 1, \dots, n$, $t \in [0,1]$ and $q \in U$. Then the linear map $\Phi_t(q)$ given as matrix with column vectors $(e_1, \dots, e_n, Je_1, \dots, Je_n)$ satisfies all required properties. □

Clean and transverse Hamiltonians Given a Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$ we denote by $\varphi_H : M \to M$ the corresponding Hamiltonian diffeomorphism, i.e. the time-one map of the Hamiltonian flow.

Definition 3.2.12. Given two Lagrangian submanifolds $L_0, L_1 \subset M$. An Hamiltonian H is

(i) *clean for* (L_0, L_1), if L_0 and $\varphi_H^{-1}(L_1)$ are in clean intersection,

(ii) *transverse for* (L_0, L_1), if L_0 and $\varphi_H^{-1}(L_1)$ are in transverse intersection.

If there is no risk of confusion we just write H is *clean* or *transverse.*

Usually only transverse Hamiltonians are considered for the definition of Floer homology of Lagrangian intersections and in that case the action function is Morse, i.e. critical points are non-degenerated. The next lemma shows that, if the Hamiltonian H is clean, then the action functional $\mathcal{A}_H$ is Morse-Bott.

Lemma 3.2.13. *Suppose that the Hamiltonian H is clean, then every connected component of $\mathcal{I}_H(L_0, L_1)$ is a manifold and for all $x \in \mathcal{I}_H(L_0, L_1)$ we have $\ker A_x = T_x\mathcal{I}_H(L_0, L_1)$ as subspaces of $T_x\mathcal{P}$.*

Proof. Via $x \mapsto x(0)$ the space $\mathcal{I}_H(L_0, L_1)$ is isomorphic to $L_0 \cap \varphi_H^{-1}(L_1)$ which is component-wise a manifold and provides the chart maps. Set $p = x(0)$. Given $\xi_0 \in T_pL_0$, consider the vector field $\xi(t) := \mathrm{d}\varphi_H^t\xi_0$, which is a vector field along x. Since ∇ is torsion free and $\partial_t x = X_H$ we have $\nabla_t\xi = \nabla_{X_H}\xi = \nabla_\xi X_H$. We conclude that every element in the kernel of A_x is of the form $t \mapsto \xi(t) = \mathrm{d}\varphi_H^t\xi_0$ with $\xi_0 \in T_pL_0 \cap T_p\varphi_H^{-1}(L_1)$. If the Hamiltonian is clean then $T_pL_0 \cap T_p\varphi_H^{-1}(L_1) = T_p(L_0 \cap \varphi_H^{-1}(L_1))$, which under the identification of $\mathcal{I}_H(L_0, L_1)$ with $L_0 \cap \varphi_H^{-1}(L_1)$ is the tangent space of $\mathcal{I}_H(L_0, L_1)$ at x. □

Proposition 3.2.14. *Suppose that the Hamiltonian H is clean, then for any compact subset $C \subset \mathcal{I}_H(L_0, L_1)$ we have $\inf_{p\in C} \iota_p(J,H) > 0$.*

Proof. With loss of generality we assume that $H = 0$ and L_0, L_1 are in clean intersection (cf. Lemmas 3.2.8 and 3.2.10). Suppose by contradiction that there exists a sequence of points $(p_\nu) \subset C$ such that $\lim_{\nu\to\infty} \iota_{p_\nu}(J) = 0$. Since C is compact, we assume that (p_ν) converges to $p \in C$. Using the trivialization Φ from Lemma 3.2.11 we define matrix valued functions $\sigma_\nu, \sigma_\infty : [0,1] \to \mathbb{R}^{2n\times 2n}$ by the requirement

$$J(p)\partial_t\Phi(p)\xi = \Phi(p)\left(J_{\mathrm{std}}\partial_t\xi + \sigma_\infty\xi\right), \quad J(p_\nu)\partial_t\Phi(p_\nu)\xi = \Phi(p_\nu)\left(J_{\mathrm{std}}\partial_t\xi + \sigma_\nu\xi\right) ,$$

for all smooth $\xi : [0,1] \to \mathbb{R}^{2n}$. Because J and Φ are smooth there exists a uniform constant c_1 such that for all $t \in [0,1]$ and $\nu \geq 1$

$$|\sigma_\infty(t) - \sigma_\nu(t)| \leq c_1 \mathrm{dist}\,(p_\nu, p) \ .$$

We define the unbounded operators in the Hilbert space $L^2([0,1], \mathbb{R}^{2n})$ via

$$\left(A_\infty\xi\right)(t) = J_{\mathrm{std}}\partial_t\xi(t) + \sigma_\nu(t)\xi(t), \qquad \left(A_\nu\xi\right)(t) = J_{\mathrm{std}}\partial_t\xi(t) + \sigma_\infty(t)\xi(t) \ ,$$

with dense domain $\{\xi \in H^{1,2}([0,1], \mathbb{R}^{2n}) \mid \xi(0), \xi(1) \in \mathbb{R}^n \times \{0\}\}$. Being conjugated to the Hessians A_p, A_{p_ν} the operators A_∞, A_ν are self-adjoint and have a closed range (cf. Proposition 3.2.6). The difference $A_\infty - A_\nu$ extends to a bounded operator which converges to zero as ν tends to infinity. By Lemma 3.2.13 the kernels of A_∞, A_ν have the same dimension. Then, by Lemma B.1.3 there exists ν_0 such that for all $\nu \geq \nu_0$ we have $\iota_{p_\nu}(J) = \iota(A_\nu) \geq 1/2\iota(A) > 0$ in contradiction to $\iota_{p_\nu}(J) \to 0$. $\square$

3.3. Morse homology

Let C be a closed manifold. A *Morse function* $f : C \to \mathbb{R}$ is a smooth function such that the Hessian at any critical point $p \in \mathrm{crit}\, f$ is non-degenerate. Necessarily the set of critical points is isolated. We choose a Riemannian metric g on C and assume that the negative gradient flow $\psi : \mathbb{R} \times C \to C$, $\psi^s = \psi(s, \cdot)$ exists for all times. Define the *unstable* (resp. *stable*) *manifold* of a critical point $p \in \mathrm{crit}\, f$ by

$$W^u(p; f) := \{u \in C \mid \psi^s(u) \to p \text{ for } s \to -\infty\}$$
$$(\text{resp. } W^s(p; f) := \{u \in C \mid \psi^s(u) \to p \text{ for } s \to \infty\}) \ .$$

Without risk of confusion we write $W^u(p)$ and $W^s(q)$ to denote $W^u(p; f)$ and $W^s(q; f)$ respectively. We call the pair (f, g) *Morse-Smale*, if for any two critical points $p, q \in \mathrm{crit}\, f$ the unstable manifold $W^u(p; f)$ intersects the stable manifold $W^s(q; f)$ transversely.

If (f, g) is Morse-Smale, then Morse homology is well-defined. We define the *space of parametrized Morse trajectories*

$$\widetilde{\mathcal{M}}_0(p, q) = W^u(p) \cap W^s(q) \ .$$

The negative gradient flow preserves $\widetilde{\mathcal{M}}_0(p, q)$ and induces an action of $\mathbb{R}$. Whenever $p \neq q$ the action is free and we denote the quotient by $\mathcal{M}_0(p, q)$.

Orientation We choose an orientation of the unstable manifolds $W^u(p)$ for each critical point $p \in \operatorname{crit} f$, which is always possible because $W^u(p)$ is contractible. Once a choice is made, the stable manifolds $W^s(q)$ are automatically cooriented for all $q \in \operatorname{crit} f$ and we obtain an orientation of $\widetilde{\mathcal{M}}_0(p,q)$ for all pairs of critical points $p,q \in \operatorname{crit} f$ via a canonical construction (cf. equation (9.1.6)). With standard orientation of $\mathbb{R}$, we also obtain orientations of the quotient $\mathcal{M}_0(p,q)$, which at elements $[u]$ in the zero dimensional component is just a number in $\{\pm 1\}$, denoted $\operatorname{sign} u$.

Morse complex Let A be any commutative ring with unit. We define the *Morse chain complex* $C_*(f,A)$ as the free A-module generated by the critical points $\operatorname{crit} f$, graded by $|p| = \mu_{\mathrm{Mor}}(p) = \dim W^u(p)$ and equipped with the boundary operator

$$\partial : C_*(f;A) \to C_{*-1}(f;A), \qquad p \mapsto \sum_{\mu(q)=\mu(p)-1} \sum_{[u]\in\mathcal{M}_0(p,q)} \operatorname{sign} u \cdot q \ .$$

Note that if $|p|-|q| = 1$ then the sum $\sum_{[u]\in\mathcal{M}_0(p,q)} \operatorname{sign} u$ equals the intersection number of $W^u(p)$ with $W^s(q)$. The next theorem is a classical result. A modern proof is found in [1] or [67]

Theorem 3.3.1. *We have $\partial \circ \partial = 0$. The associated homology group $H_*(f;A) := \ker\partial/\operatorname{im}\partial$ is independent of the function f, the metric and the choices of orientations up to isomorphism and we have the natural isomorphism*

$$H_*(f;A) \cong H_*(C;A) \ . \tag{3.3.1}$$

Functoriality Let $\varphi : C \to C'$ be a smooth map between the manifolds C and C' which are equipped with Morse-Smale pairs (f,g) and (f',g') respectively. Given two critical points $p \in \operatorname{crit} f$ and $p' \in \operatorname{crit} f'$ define the space

$$\mathcal{M}^\varphi(p,p') := W^u(p) \cap \varphi^{-1}\big(W^s(p')\big) \ . \tag{3.3.2}$$

Generically the intersection is transverse and hence $\mathcal{M}^\varphi(p,p')$ is a manifold of dimension $\mu(p') - \mu(p)$. If $W^u(p)$ is oriented and $W^s(p')$ is cooriented, then $\mathcal{M}^\varphi(p,p')$ carries an induced orientation. We define the morphism

$$C\varphi_* : C_*(f;A) \to C_*(f';A), \qquad p \mapsto \sum\nolimits_{\mu(p')=\mu(p)} \sum\nolimits_{u\in\mathcal{M}^\varphi(p,p')} \operatorname{sign} u \cdot p' \ .$$

The homomorphism $C\varphi_*$ is a chain map. We denote the induced map on homology by $\varphi_* : H_*(f;A) \to H_*(f';A)$. In [3, §2.2] it is proven that φ_* is the push-forward in homology under the identification (3.3.1).

Cohomology By definition the cohomology complex $C^*(f;A)$ is given by the module $\operatorname{Hom}(C_*(f;A),A)$ equipped with the boundary operator $d : C^*(f;A) \to C^{*+1}(f;A)$, $\varphi \mapsto (p \mapsto \varphi(\partial p))$. One shows that $d \circ d = 0$ and the associated cohomology is isomorphic to $H^*(C;A)$. For any critical point $p \in \operatorname{crit} f$ let $\delta_p \in C^*(f;A)$ be the homomorphism that is 1 on p and 0 otherwise. Since any element in $C^*(f;A)$ is a linear combination of these, we see that Morse cohomology is alternatively defined by the free module generated by critical points $p \in \operatorname{crit} f$, graded by the Morse index and equipped with differential

$$d : C^*(f;A) \to C^{*+1}(f;A), \qquad p \mapsto \sum_{\mu(q)=\mu(p)+1} \sum_{[u]\in\mathcal{M}_0(q,p)} \operatorname{sign} u \cdot q\,. \tag{3.3.3}$$

For more details see [67].

Local coefficients Let $\mathcal{C}$ be the the category of points in C with morphisms given by homotopy classes of paths and composition law by concatenation of paths. A *local system* $\mathcal{L}$ is a functor from $\mathcal{C}$ into the category of A-modules. An isomorphism class of a local system is given by a representation of $\pi_1(C)$ on an A-module. Every local system arises from the following general construction:

Lemma 3.3.2. *Let Γ be the group of Deck transformations of a covering $\tilde{C} \to C$. Assume that Γ acts on an A-module M by A-module morphisms. We obtain a corresponding local system, which associates to a point $p \in C$ the fibre in the associated covering $\tilde{C} \times_\Gamma M \to C$ and to a homotopy class of paths the parallel transport. If $\mathcal{L}$ arises in that way we denote the local system by $\mathcal{L} = \tilde{C} \times_\Gamma M$.*

Morse homology with coefficients in $\mathcal{L}$ is the A-module

$$C_*(f;\mathcal{L}) = \bigoplus_{p\in\operatorname{crit} f} \mathcal{L}(p)\,,$$

graded by the Morse index and equipped with the boundary operator

$$\partial : C_*(f;\mathcal{L}) \to C_{*-1}(f;\mathcal{L}), \qquad \mathcal{L}(p) \ni a \mapsto \sum_{\mu(q)=\mu(p)-1} \sum_{[u]\in\mathcal{M}_0(p,q)} \operatorname{sign} u \cdot \mathcal{L}(u)a\,.$$

Alternatively we choose an isomorphism $\mathcal{L}(p) \cong M$ for any $p \in \operatorname{crit} f$. Then $C_*(f;\mathcal{L}) = C_*(f) \otimes M$ and the boundary operator is given by the same formula where now $\mathcal{L}(u)$ is an automorphism of the module M. In [57, §7.2] it is shown (with the minor difference that the argument there is for cohomology) that $\partial \circ \partial = 0$ and the associated homology is isomorphic to $H_*(A;\mathcal{L})$, which is the homology of C with values in the local system $\mathcal{L}$. For more details see also [4, Appendix A].

3.4. Floer homology

Fix a symplectic manifold (M,ω), Lagrangians $L_0, L_1 \subset M$ and a coefficient ring A which satisfy Assumption 2.3.1. We give a short introduction to Floer homology of the pair (L_0, L_1) with coefficients in the Novikov ring $\Lambda = A[\lambda, \lambda^{-1}]$.

Floer trajectories Choose a Hamiltonian $H \in C^\infty([0,1] \times M)$ and a path of almost complex structures $J : [0,1] \to \mathrm{End}(TM, \omega)$, $J_t = J(t, \cdot)$. For two Hamiltonian arcs $x_-, x_+ \in \mathcal{I}_H(L_0, L_1)$ we define the space of *parametrized finite energy Floer trajectories*

$$\widetilde{\mathcal{M}}(x_-, x_+; J, H) = \{u \in C^\infty(\mathbb{R} \times [0,1], M) \mid (3.2.9)\}.$$

The Hamiltonian function H is *transverse* if $\varphi_H(L_0)$ intersects L_1 transversely, where φ_H denotes the Hamiltonian diffeomorphism associated to H. In [31] it is shown that being transverse is a generic condition, i.e. can always be fulfilled after an arbitrary small perturbation of H. Moreover for an transverse Hamiltonian function H it is shown in [31], that for a generic almost complex structure J each connected component of the space $\widetilde{\mathcal{M}}(x_-, x_+; J, H)$ is a manifold and the dimension of a component containing u is given by the Viterbo index $\mu(u)$. Let us fix generic data J and H. We abbreviate $\widetilde{\mathcal{M}}(x, y) := \widetilde{\mathcal{M}}(x, y; J, H)$ for any two arcs $x, y \in \mathcal{I}_H(L_0, L_1)$. There exists an $\mathbb{R}$-action on the space $\widetilde{\mathcal{M}}(x, y)$ by translation on the domain, i.e. $(a.u)(s,t) = u(s-a, t)$. If $x \neq y$ the action is free and we denote the quotient by $\mathcal{M}(x, y)$.

Grading Let $N \in \mathbb{N}$ denote the minimal Maslov number of the pair (L_0, L_1) and let $\mathcal{P}$ denote the space of paths $\gamma : [0,1] \to M$ with $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$. For every $x \in \mathcal{I}_H(L_0, L_1)$ which is in the same connected component of $\mathcal{P}$ as x_* we choose $u_x : [-1,1] \times [0,1] \to M$ such that $u(s) \in \mathcal{P}$ for all $s \in [-1,1]$, $u_x(-1) = x_*$ and $u_x(1) = x$. Then define the grading as the Viterbo index of u_x, i.e.

$$|x| := -\mu(u_x).$$

Orientation If the characteristic of A is not two, we need to orient the spaces $\mathcal{M}(x, y)$. This is done as follows. Let D_{u_x} be the linearized Cauchy-Riemann-Floer operator of the cap u_x extended constantly which by Theorem 6.1.10 is a Fredholm operator. There is a natural notion of an orientation of a Fredholm operator and we denote by $|D_{u_x}|$ the space of orientations of D_{u_x}. Fix an orientation $o_x \in |D_{u_x}|$ for all perturbed intersection points $x \in \mathcal{I}_H(L_0, L_1)$ which are connected to x_* within $\mathcal{P}$. Given $u \in \widetilde{\mathcal{M}}(x, y)$ there exists an orientation gluing operation which lifts the linear gluing map $|D_{u_x}| \otimes |D_u| \cong |D_{u_x \# u}|$ (cf. Lemma 9.3.3). Provided that the pair (L_0, L_1) is equipped with a relative spin structure $|D_{u_x \# u}|$ and $|D_{u_y}|$ are naturally isomorphic. Hence by the orientation gluing map and our choices, we obtain an orientation of D_u which induces an orientation o_u of $\widetilde{\mathcal{M}}(x, y)$ (cf. Theorem 9.3.6). By associativity of the orientation gluing operation (cf. Lemma 9.4.2), the constructed orientations satisfy $o_u \# o_v = o_{u \# v}$ for all $(u, v) \in \widetilde{M}(x, z) \times \widetilde{\mathcal{M}}(z, y)$. i.e. are coherent. With standard orientation on $\mathbb{R}$ we also obtain an orientation of the quotient space $\mathcal{M}(x, y)$ for all $x, y \in \mathcal{I}_H(L_0, L_1)_{[x_*]}$ (cf. equation (9.1.8)). Let $\mathcal{M}(x, y)_{[0]}$ be the union of all zero-dimensional components. An orientation of an element $[u] \in \mathcal{M}(x, y)_{[0]}$ is a number in $\{\pm 1\}$, which we denote by $\operatorname{sign} u$.

Floer complex Let $N \in \mathbb{N}$ be the minimal Maslov number of the pair (L_0, L_1). Denote by $\Lambda := A[\lambda, \lambda^{-1}]$ the ring of Laurent polynomials in the variable λ of degree given by $-N$. Let $\mathcal{I}_H(L_0, L_1)_{[x_*]} \subset \mathcal{I}_H(L_0, L_1)$ denoted the subset of elements which are connected to x_* within $\mathcal{P}$. The *Floer complex* is given by the free Λ-module generated by $\mathcal{I}_H(L_0, L_1)_{[x_*]}$, graded by $\left|x \otimes \lambda^k\right| = -\mu(u_x) - kN$ and equipped with the Λ-linear operator

$$\partial : CF_*(L_0, L_1) \to CF_{*-1}(L_0, L_1),$$
$$x \mapsto \sum_{y \in \mathcal{I}_H(L_0,L_1)_{[x_*]}} \sum_{[u] \in \mathcal{M}(x,y)_{[0]}} \operatorname{sign} u \cdot y \otimes \lambda^{(|y|-|x|+1)/N} .$$

That ∂ is a boundary operator is a highly non-trivial fact and the central result of Floer's papers [25, 27, 29] with details of the monotone case worked out by Oh [58]. The supplement with orientations is in the books of Fukaya et al. [35, 36] and their paper [37].

Theorem 3.4.1 (Floer)**.** *We have $\partial \circ \partial = 0$. The associated homology group*

$$HF_*(L_0, L_1) = \ker \partial / \operatorname{im} \partial\,, \tag{3.4.1}$$

is independent of choices of J, H and orientations up to isomorphism.

4. Asymptotic analysis

We study the asymptotic behavior holomorphic strips with boundary and finite energy on Lagrangian submanifolds. More precisely we show that if the Lagrangians intersect cleanly such strips decay exponentially and approach an eigenfunction of the asymptotic operator up to an error of higher exponential decay. If the Lagrangians intersect transversely this was proven by Robbin and Salamon in [65]. The generalization to holomorphic strips with boundary on cleanly intersecting Lagrangians was mainly done by Frauenfelder in [33]. The only part which we have not found in the literature is the fact that the decay parameter has an upper bound by the spectral gap of the asymptotic operator and the above mentioned convergence to the eigenfunction. These improvements however are necessary to embed the space of holomorphic curves in a suitable Banach manifold.

4.1. Main statement

Given a compact symplectic manifold (M,ω), two Lagrangian submanifolds $L_0, L_1 \subset M$. Fix an almost complex structure $J \in C^\infty([0,1], \mathrm{End}(TM,\omega))$. In this chapter we study the asymptotic behavior of smooth maps $u : [0,\infty) \times [0,1] \to M$ satisfying the Cauchy-Riemann equation

$$\partial_s u(s,t) + J_t(u)\partial_t u(s,t) = 0\ , \tag{CR}$$

and the boundary condition

$$u|_{t=0} \subset L_0, \qquad\qquad u|_{t=1} \subset L_1\ . \tag{BC}$$

For each point $p \in L_0 \cap L_1$ we consider the linear differential operator (cf. equation (3.2.11))

$$A_p : T_p\mathcal{P}(L_0, L_1) \to L^2([0,1], T_pM), \qquad \xi \mapsto J_t(p)\partial_t \xi\ .$$

In [33, Theorem 4.1] it is shown that A_p is an operator with discrete spectrum consisting only of eigenvalues. We define the *spectral gap at p*

$$\iota_p := \min\{|\alpha| \mid \alpha \in \sigma(A_p) \setminus \{0\}\}\ . \tag{4.1.1}$$

Up to the quantitative estimate on the decay parameter the next theorem is proven in [33, Thm. 3.16].

Theorem 4.1.1 (exponential decay)**.** *Assume that L_0 and L_1 intersect cleanly. Given a map $u : [0,\infty) \times [0,1] \to M$ which satisfies* (CR) *and* (BC)*. Then the following three statements are equivalent.*

(i) *We have that*

$$E(u) = \int_0^\infty \int_0^1 |\partial_s u|^2 \mathrm{d}t\mathrm{d}s < \infty \,. \tag{E}$$

(ii) *There exists a point* $p \in L_0 \cap L_1$ *such that*

$$\lim_{s\to\infty} u(s,t) = p, \qquad \lim_{s\to\infty} |\partial_s u(s,t)| = 0 \,, \tag{4.1.2}$$

where the limits exist uniformly for all $t \in [0,1]$.

(iii) *For any positive constant* $\mu < \iota_p$ *with* ι_p *given in* (4.1.1) *and integer* $k \in \mathbb{N}_0$*, there exists a constant* $c_k = c_k(\mu)$ *such that*

$$\|\partial_s u\|_{C^k([s,\infty)\times[0,1])} \leq c_k e^{-\mu s} \,, \tag{4.1.3}$$

for all $s \geq 0$.

Let u be a finite energy J-holomorphic strip which approaches the intersection point $p = \lim_{s\to\infty} u(s,t) \in L_0 \cap L_1$. The next theorem states that in a chart which is centered at p we have the approximation $u(s,t) \approx e^{-\alpha s}\zeta(t)$ up to an error of higher exponential decay for some eigenfunction ζ of the asymptotic operator A_p and $\alpha > 0$ the corresponding eigenvalue.

Theorem 4.1.2 (Convergence to eigenfunction)**.** *Assume* L_0 *and* L_1 *intersect cleanly. Given a non-constant map* $u : [0,\infty) \times [0,1] \to M$ *satisfying* (CR), (BC) *and* (4.1.2). *Then there exists an non-zero eigenvalue* α *of* A_p *with corresponding eigenfunction* $\zeta \in \ker(A_p - \alpha)$ *and a constant* s_0 *such that the function* $w : [s_0,\infty) \times [0,1] \to T_pM$, $(s,t) \mapsto w(s,t)$ *defined by*

$$u(s,t) = \exp_p\left(e^{-\alpha s}\zeta(t) + w(s,t)\right) \,,$$

satisfies the following: For any $\mu < \iota_p$ *and number* $k \in \mathbb{N}$ *there exists a constant* c_k *such that for all* $s \geq s_0$ *we have*

$$\|w\|_{C^k([s,\infty)\times[0,1])} \leq c_k e^{-(\mu+\alpha)s} \,.$$

Corollary 4.1.3. *Assume that* u *is non-constant and satisfies* (CR), (BC) *and* (E). *There exists a point* $p \in L_0 \cap L_1$*, a non-zero eigenvalue* $\alpha \in \sigma(A_p)$ *and constants* c*,* s_0 *such that we have*

$$c^{-1}e^{-\alpha s} \leq |\partial_s u(s,t)| \leq c e^{-\alpha s} \,,$$

for all $s \geq s_0$ *and* $t \in [0,1]$*. In particular* $\partial_s u(s,t)$ *is not zero for all* $s \geq s_0$ *and* $t \in [0,1]$.

Proof. Using a uniform bound on the derivative of the exponential map (cf. equation (A.1.5)) and Theorem 4.1.2 we have constants c_1 and c_2 such that

$$|\partial_s u(s,t)| \leq c_1 \left|e^{-\alpha s}\zeta(t) + \partial_s w(s,t)\right| \leq c_2 e^{-\alpha s} + c_2 e^{-(\alpha+\mu)s} \leq 2c_2 e^{-\alpha s} \,.$$

To show the second inequality observe that since ζ solves a linear first order ordinary differential equation we have $c_3 := \inf_{t\in[0,1]} |\zeta(t)| > 0$. Hence with uniform bounds on the derivative of the exponential map (cf. equation (A.1.6)) and Theorem 4.1.2, there exists a constant c_4 such that

$$\begin{aligned} c_3 e^{-\alpha s} \leq \left|e^{-\alpha s}\zeta(t)\right| &\leq \left|\partial_s(-\alpha e^{-\alpha s}\zeta(t) + w(s,t))\right| + |\partial_s w(s,t)| \\ &\leq c_4 \left|\partial_s u(s,t)\right| + c_4 e^{-\mu s} e^{-\alpha s}\,. \end{aligned}$$

Since $\mu > 0$ we have for s_0 sufficiently large that $c_4 e^{-\mu s} \leq c_3/2$ for all $s \geq s_0$. This shows the second estimate by subtracting $c_3 e^{-\alpha s}/2$ and dividing by c_4 in the last inequality. □

We prepare the necessary material for the proofs. The proofs themselves are deferred to the end of the chapter. The proof of the Theorem 4.1.2 will closely follow [65, Thm. B] once provided with the adaptation of certain lemmas; in particular [65, Lmm. 3.6]. For the proof of Theorem 4.1.1 we differ from the proof given in [65] and make use of the isoperimetric inequality for arcs between cleanly intersecting Lagrangians (see Proposition 4.3.1). The idea stems from [33] and uses the special nature of the symplectic action functional. It is a short-cut of the argument. We just want to state that it is not necessary and one can prove Theorem 4.1.1 without the isoperimetric inequality, sticking with the methods of [65].

4.2. Mean-value inequality

The mean-value inequality states that for J-holomorphic curves with sufficiently small energy the norm of the gradient is controlled by the energy. The fact is well-known for almost complex structures which do not explicitly depend on the domain (cf. [53, Sec. 4]). The generalization for almost complex structures which do depend on the domain was done in [33] with the minor restriction that the argument was for Lagrangians which are the fixed point set of anti-symplectic involutions. However if we slightly change the assumptions the proof is easily adapted for the general case. In the following we identify the half-strip $[0,\infty)\times[0,1]$ with $\Sigma_+ := \{z = s + it \in \mathbb{C} \mid s \geq 0, t \in [0,1]\}$.

Proposition 4.2.1 (Mean value inequality)**.** *There exists constants $\hbar$ and c such that for any $r < 1/2$, $z_0 := (s_0, t_0) \in \Sigma_+$ and map $u : \Sigma_+ \to M$ which satisfies* (CR) *and* (BC) *we have*

$$\int_{B_r(z_0)} |\partial_s u(s,t)|^2 \mathrm{d}s\mathrm{d}t < \hbar \quad \Longrightarrow \quad |\mathrm{d}u(s_0,t_0)|^2 \leq \frac{c}{r^2}\int_{B_r(z_0)} |\partial_s u(s,t)|^2 \mathrm{d}s\mathrm{d}t\,,$$

in which $B_r(z_0) := \{z \in \Sigma_+ \mid |z - z_0| < r\}$ denotes the open ball of radius r centered at z_0.

Proof. See [33, Lemma 3.13]. The present situation is slightly different. We bound the radius r by $1/2$ since we need to assure that $B_r(s_0,t_0)$ touches at most one of the faces of $[0,\infty)\times[0,1]$. This is necessary because unlike in [33] we do not assume symmetries for the almost complex structure which was necessary to extend the solutions. The rest goes through directly. We give the whole argument for completeness. Except for some minor changes the computation is the same as in the proof of [53, Lemma 4.3.1].

Since M is compact any two metrics are equivalent. We assume without loss of generality that the metric is given as in [53, Lemma 4.3.3] with respect to J_0 and L_0 if $s_0 < 1/2$ (resp. J_1 and L_1 if $s_0 \geq 1/2$). Let ∇ denote the Levi-Civita connection of that metric. Abbreviate $\xi = \partial_s u$ and $\eta = \partial_t u$ and define the function

$$w : B_r(s_0, t_0) \to \mathbb{R}, \qquad (s,t) \mapsto \frac{1}{2}\big|\xi(s,t)\big|^2 .$$

Let $\Delta = \partial_s^2 + \partial_t^2$ denote the Laplace operator. We want to show that w satisfies the inequality

$$\Delta w \geq -c_1(w + w^2)\,, \tag{4.2.1}$$

for some positive constant $c_1 > 0$. We compute

$$\Delta w = |\nabla_s \xi|^2 + |\nabla_t \xi|^2 + \langle \nabla_s \nabla_s \xi + \nabla_t \nabla_t \xi, \xi \rangle\,.$$

We abbreviate by $\partial_t J$ the derivative of the path of endomorphisms $t \mapsto J_t$ and $\nabla_\eta J$ the covariant derivative of J_t for a fixed t with respect to the vector field η along the curve u. We compute

$$\begin{aligned}
\nabla_s \nabla_s \xi + \nabla_t \nabla_t \xi &= \nabla_s(\nabla_s \xi + \nabla_t \eta) + \nabla_t \nabla_s \eta - \nabla_s \nabla_t \eta \\
&= \nabla_s\left(\nabla_s(-J\eta) + \nabla_t(J\xi)\right) - R(\xi,\eta)\eta \\
&= \nabla_s\left((\partial_t J)\xi + (\nabla_\eta J)\xi - (\nabla_\xi J)\eta\right) - R(\xi,\eta)\eta\,,
\end{aligned}$$

in which we denote by $R(\xi,\eta)\eta := (\nabla_s \nabla_t - \nabla_t \nabla_s)\,\eta$ and R the curvature tensor. The last two equalities combined give

$$\Delta w = |\nabla_s \xi|^2 + |\nabla_t \xi|^2 - \langle R(\xi,\eta)\eta, \xi\rangle + \langle \nabla_s(\partial_t J)\xi + \nabla_s(\nabla_\eta J)\xi - \nabla_s(\nabla_\xi J)\eta, \xi\rangle\,.$$

Let κ denote the last term on the right-hand side. There exists a constant $c_2 > 1$ depending only on the norm of the derivatives of J up to order two such that

$$\begin{aligned}
\kappa &\geq -c_2\left(|\xi|^3 + |\xi|\,|\nabla_s \xi| + |\xi|^2\left(|\xi|^2 + 2|\nabla_t \xi| + 2|\nabla_s \xi|\right)\right) \\
&\geq -c_2^2\,|\xi|^2 - \frac{1}{4}\,|\xi|^4 - c_2^2\,|\xi|^2 - \frac{1}{4}\,|\nabla_s \xi|^2 - c_2\,|\xi|^4 - 8c_2^2\,|\xi|^4 - \frac{1}{4}(|\nabla_t \xi|^2 + |\nabla_s \xi|^2) \\
&\geq -\frac{1}{2}(|\nabla_s \xi|^2 + |\nabla_t \xi|^2) - 10c_2^2(|\xi|^2 + |\xi|^4)\,,
\end{aligned}$$

in which for the second estimate we have used the inequality $-ab \geq -a^2 - \frac{1}{4}b^2$ for all $a, b \in \mathbb{R}$. Since M is assumed to be compact there exists $c_3 > 0$ depending only on the curvature of the metric and the norm of J such that

$$\langle R(\xi,\eta)\eta, \xi\rangle \geq -c_3\,|\xi|^4\ .$$

Combining the last three estimates we obtain the constant $c_1 > 0$ such that inequality (4.2.1) holds. Then after [33, Lemma D.1] this proves the assertion in the case when $B_r(s_0, t_0)$ does not intersect the boundary of $[0, \infty) \times [0, 1]$. If it does we extend w via $w(s, -t) = w(s, t)$ for $t > 0$ if $s_0 < 1/2$ (resp. via $w(s, 1 + t) = w(s, 1 - t)$ for $t > 1$ if $s_0 \geq 1/2$) and conclude by the same argument as on [53, Page 84]. $\square$

A corollary of the mean-value inequality and bounded gradient compactness (cf. Lemma 5.2.1) is that $\partial_s u(s, t)$ converges uniformly to zero with all derivatives as s tends to ∞.

Corollary 4.2.2. *Assume that u satisfies* (CR), (BC) *and* (E). *Then for any $k \in \mathbb{N}$ we have*

$$\lim_{s\to\infty} \|\partial_s u\|_{C^k([s,\infty)\times[0,1])} = 0 \, .$$

Proof. Suppose by contradiction that we find constants $\varepsilon > 0$, $k \in \mathbb{N}$ and a sequence $s_\nu \to \infty$ such that for all $\nu \in N$

$$\|\partial_s u\|_{C^k([s_\nu-1,s_\nu+1]\times[0,1])} > \varepsilon \, . \tag{4.2.2}$$

We define $u_\nu : [-2, 2] \times [0, 1] \to M$ via $u_\nu(s, t) := u(s + s_\nu, t)$. By the mean-value inequality we have that

$$\sup_{\nu\in\mathbb{N},(s,t)\in[-1,1]\times[0,1]} |\mathrm{d}u_\nu(s, t)| < \infty \, .$$

By bounded gradient compactness (cf. Lemma 5.2.1) we conclude that after possibly passing to a subsequence there exists a map $v : [-1, 1] \times [0, 1] \to M$ such that (u_ν) converges to v uniformly with all derivatives. In particular $E(u_\nu; [-1, 1] \times [0, 1]) \to E(v; [-1, 1] \times [0, 1]) = 0$, hence v is constant. We conclude that

$$\|\partial_s u\|_{C^k([s_\nu-1,s_\nu+1]\times[0,1])} = \|\partial_s u_\nu\|_{C^k([-1,1]\times[0,1])} \to 0 \, ,$$

which contradicts (4.2.2). $\square$

Remark 4.2.3. In the previous corollary we have not used that L_0 and L_1 intersect cleanly.

4.3. Isoperimetric inequality

For paths $\gamma : [0, 1] \to M$ with endpoints $\gamma(k) \in L_k$ for $k = 0, 1$ and with image sufficiently close to the intersection $L_0 \cap L_1$ we define the *local action*

$$\mathcal{A}_{\mathrm{loc}}(\gamma) := \int \gamma^*\lambda \, ,$$

in which λ is any primitive of the symplectic form restricted to a neighborhood of $L_0 \cap L_1$ such that $\lambda|_{L_k} = 0$ (see Proposition 3.2.9 to show that such λ exist). That the inequality which we are about to show is true for *some* constant μ is well-known and previously proven in [33, Lmm. 3.17] or [62, Lmm. 3.4.5].

Proposition 4.3.1 (Isoperimetric inequality)**.** *Assume that L_0 and L_1 are in clean intersection. For every point $p \in L_0 \cap L_1$ and constant $\mu < \iota_p$ with ι_p defined in (4.1.1) there exists a constant $\rho > 0$ with the following significance: For any smooth curve $\gamma : [0,1] \to M$ satisfying $\gamma(0) \in L_0$, $\gamma(1) \in L_1$ and $\operatorname{dist}(\gamma(t), p) < \rho$ for all $t \in [0,1]$ we have*

$$2\mu \, |\mathcal{A}_{\mathrm{loc}}(\gamma)| \le \int_0^1 |\partial_t \gamma|_J^2 \mathrm{d}t \, .$$

If moreover $\mu < \inf\{\iota_p \mid p \in L_0 \cap L_1\}$ then there exists $\ell_0 > 0$ such that for all $\gamma : [0,1] \to M$ with $\gamma(0) \in L_0$, $\gamma(1) \in L_1$ and $\ell(\gamma) := \int_0^1 |\partial_t \gamma| \, \mathrm{d}t < \ell_0$ the same conclusion holds.

Proof. By Lemma 3.2.10 we assume that $\gamma : [0,1] \to \mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ equipped with standard symplectic form and the Lagrangians L_0, L_1 are linear subspaces Λ_0, Λ_1 respectively. Let Φ be the trivialization constructed in Lemma 3.2.10, which we think of as an matrix valued function and abbreviate $\Phi_\gamma(t) := \Phi_t(\gamma(t))$ and $J_\gamma(t) := J_t(\gamma(t))$ for all $t \in [0,1]$. The matrix Φ_γ is symplectic and satisfies $\Phi_\gamma J_{\mathrm{std}} = J_\gamma \Phi_\gamma$. Consider the Hilbert space $H = L^2([0,1], \mathbb{R}^{2n})$ equipped with standard inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We conclude

$$\begin{aligned}\int_0^1 \omega(\partial_t \gamma, J_\gamma \partial_t \gamma) \mathrm{d}t = \langle J_{\mathrm{std}} \partial_t \gamma, J_\gamma \partial_t \gamma \rangle = \langle J_{\mathrm{std}} \partial_t \gamma, \Phi_\gamma \Phi_\gamma^{-1} J_\gamma \partial_t \gamma \rangle = \\ = \left\| \Phi_\gamma^{-1} J_\gamma \partial_t \gamma \right\|^2 \, . \end{aligned} \tag{4.3.1}$$

We extend $\gamma : [0,1] \to \mathbb{R}^{2n}$ to a map $u : [0,1]^2 \to \mathbb{R}^{2n}$ via $u(s,t) = s\gamma(t)$ and compute

$$\mathcal{A}_{\mathrm{loc}}(\gamma) = \int_{[0,1]} \gamma^* \lambda = \int_{[0,1]^2} u^* \omega = \frac{1}{2} \int_0^1 \omega(\partial_t \gamma, \gamma) \mathrm{d}t = \frac{1}{2} \langle J_{\mathrm{std}} \partial_t \gamma, \gamma \rangle \, ,$$

in which for the second equality we have used Stokes and the fact that by construction $u|_{t=k} \subset \Lambda_k$ for $k = 0, 1$. Abbreviate $\Phi_\infty(t) := \Phi_t(0)$ and $J_\infty(t) := J_t(0)$ for all $t \in [0,1]$. Define the unbounded operator A_∞ via (4.4.4). The function $\xi : [0,1] \to \mathbb{R}^{2n}$, $t \mapsto \xi(t) = \Phi_\infty(t)^{-1} \gamma(t)$ lies in the domain of A_∞. Continue the computation

$$2\mathcal{A}_{\mathrm{loc}}(\gamma) = \langle J_{\mathrm{std}} \partial_t \gamma, \gamma \rangle = \langle J_{\mathrm{std}} \partial_t \gamma, \Phi_\infty \xi \rangle = \langle \Phi_\infty^{-1} J_\infty \partial_t \gamma, \xi \rangle = \langle A_\infty \xi, \xi \rangle \, .$$

By construction the operator A_∞ is conjugated to A_p. In particular these two operators have the same spectral gap. By Corollary B.1.2 we have

$$2 \, |\mathcal{A}_{\mathrm{loc}}(\gamma)| = |\langle A_\infty \xi, \xi \rangle| \le \frac{1}{\iota_p} \left\| A_\infty \xi \right\|^2 = \frac{1}{\iota_p} \left\| \Phi_\infty^{-1} J_\infty \partial_t \gamma \right\|^2 \, . \tag{4.3.2}$$

The matrix $G_t(q) := \Phi_t(q)^{-1} J_t(q)$ is invertible for all $(t, q) \in [0,1] \times U$, $\xi \in \mathbb{R}^{2n}$. Moreover we have

$$\|G_t(0)\xi\| \le \|G_t(q)\xi\| + \|G_t(0) - G_t(q)\| \left\| G_t(q)^{-1} \right\| \|G_t(q)\xi\| \, .$$

Thus there exists a constant $c > 0$ such that for all $\rho < 1$ and curves γ with distance to p bounded by ρ we have

$$\left\| \Phi_\infty^{-1} J_\infty \partial_t \gamma \right\|^2 = \|G(0) \partial_t \gamma\|^2 \le (1 + c\rho) \, \|G(\gamma) \partial_t \gamma\|^2 = (1 + c\rho) \left\| \Phi_\gamma^{-1} J_\gamma \partial_t \gamma \right\|^2 \, . \tag{4.3.3}$$

Together with (4.3.1) and (4.3.2) we conclude

$$2\iota_p \, |\mathcal{A}_{\mathrm{loc}}(\gamma)| \leq (1 + c\rho) \int_0^1 \omega(\partial_t \gamma, J_\gamma \partial_t \gamma) \mathrm{d}t \, .$$

Then the first claim follows if we choose $\rho < (\iota_p - \mu)/(c\mu)$.

We show the second statement. Repeat the argument above with each point $p \in L_0 \cap L_1$ and let c_p denote the corresponding constant from (4.3.3). Since $p \mapsto c_p$ is upper semi-continuous and $\iota_p - \mu$ bounded away from zero the constant $\rho := \inf\{\frac{\iota_p - \mu}{c_p \mu} \mid p \in L_0 \cap L_1\}$ is positive. Abbreviate by $\mathcal{P}$ the space of paths $\gamma : [0, 1] \to M$ with $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$. We denote by $B_\rho(p) \subset M$ the open ball about p with radius ρ. We claim that there exists ℓ_0 such that for any $\gamma \in \mathcal{P}$ we have

$$\ell(\gamma) := \int_0^1 |\dot{\gamma}(t)| \mathrm{d}t < \ell_0 \quad \Rightarrow \quad \gamma(0) \in V := \bigcup_{p \in L_0 \cap L_1} B_{\rho/2}(p) \, . \tag{4.3.4}$$

If not, there exists sequences $(\gamma_\nu) \subset \mathcal{P}$ such that for all $\nu \in \mathbb{N}$ we have $\ell(\gamma_\nu) < 1/\nu$ and $\gamma_\nu(0)$ lies in the complement of V. By compactness of L_0, there exists a sub-sequence, still denoted (γ_ν), such that $\gamma_\nu(0)$ converges to a point $p \in L_0$. Moreover since $\ell(\gamma_\nu) \to 0$ and L_1 is closed we have that $\gamma_\nu(1) \in L_1$ converges to p and thus $p \in L_0 \cap L_1$, contradicting the fact that $\gamma_\nu(0) \notin V$ for all $\nu \in \mathbb{N}$ since V is an open neighborhood of p. To show the lemma we assume without loss of generality that $\ell_0 < \rho/2$. Indeed given any $\gamma \in \mathcal{P}$ with $\ell(\gamma) < \ell_0$, by (4.3.4) there exists $p \in L_0 \cap L_1$ such that $\gamma(0) \in B_{\rho/2}(p)$ and hence $\gamma \subset B_\rho(p)$. □

The next lemma is a direct consequence of the isoperimetric inequality. It is the generalization of a version for J-holomorphic cylinders as given in [53, Lemma 4.7.3]. The assertion is that the energy of a J-holomorphic half-strip with boundary in (L_0, L_1) decays exponentially and the energy of a J-holomorphic strip of finite length with boundary in (L_0, L_1) can not spread out uniformly but must be concentrated at the ends, provided that the energy is sufficiently small.

Lemma 4.3.2 (Energy decay). *Assume that L_0 and L_1 are in clean intersection. For any constant $\mu < \inf\{\iota_p \mid p \in L_0 \cap L_1\}$ there exists constants ε_0 and c with the following significance:*

(i) For any map $u : [0, \infty) \times [0, 1] \to M$ satisfying (CR), (BC) *and $E(u) < \varepsilon_0$, then for all $s \geq 1$ we have*

$$E(u; [s, \infty) \times [0, 1]) \leq E(u) e^{-2\mu s} \, . \tag{4.3.5}$$

Moreover there exists a point $p \in L_0 \cap L_1$ such that for all $s \geq 1$ and $t \in [0, 1]$ we have

$$\operatorname{dist}(u(s, t), p) + |\mathrm{d}u(s, t)| \leq c e^{-\mu s} \, . \tag{4.3.6}$$

(ii) *For all* $s_0 < s_1$ *and any map* $u : [s_0, s_1] \times [0,1] \to M$ *satisfying* (CR), (BC) *and* $E(u) < \varepsilon_0$ *we have*

$$E(u; [a+s, b-s] \times [0,1]) \le E(u)e^{-2\mu s}\,. \tag{4.3.7}$$

for all $1 \le s \le (s_1 - s_0)/2$. *Moreover for all* $\sigma, \sigma' \in [s_0 + s, s_1 - s]$ *and* $t, t' \in [0,1]$ *we have*

$$|\mathrm{d}u(\sigma, t)| + \mathrm{dist}\,\big(u(\sigma,t), u(\sigma',t')\big) \le ce^{-\mu s}\,. \tag{4.3.8}$$

If instead $p \in L_0 \cap L_1$ *is a point and* $\mu < \iota_p$, *then there exists constants* ε, c *and* ρ *satisfying the statements above after replacing the manifold* M *with the open ball* $B_\rho(p)$.

Proof. We show the first statement. Assume that ε_0 is smaller than the constant $\hbar$ from the mean-value inequality (cf. Prop. 4.2.1). Let $u : [0,\infty)\times[0,1] \to M$ satisfy (CR), (BC) and $E(u) < \varepsilon_0$. That assured the mean-value inequality provides a constant c_1 independent of u, such that for any $s > 1/2$ we have

$$|\mathrm{d}u(s,t)|^2 \le c_1 E(u; [s-1/2, s+1/2] \times [0,1]) \le c_1\varepsilon_0\,. \tag{4.3.9}$$

Abbreviate $\gamma_s(t) = u(s,t)$. Let ℓ_0 denote the constant from the isoperimetric inequality (cf. Prop. 4.3.1). By possibly decreasing ε_0 we assume that for all $s > 1/2$ we have

$$\ell(\gamma_s) = \int_0^1 |\partial_t \gamma_s(t)|\,\mathrm{d}t \le \sqrt{c_1\varepsilon_0} < \ell_0\,. \tag{4.3.10}$$

By the choice of ℓ_0 the point $u(s,t)$ lies inside the Pozniak neighborhood where the symplectic form is exact for all $s \ge 1$ and $t \in [0,1]$. Hence $\omega = \mathrm{d}\lambda$ for some one form λ and by the isoperimetric inequality we get

$$\begin{aligned} f(s) := E(u; [s,\infty) \times [0,1]) &= \int_0^1 \gamma_s^*\lambda - \lim_{b\to\infty} \int_0^1 \gamma_b^*\lambda \\ &\le \frac{1}{2\mu}\int_0^1 |\dot\gamma_s(t)|^2\,\mathrm{d}t + \frac{1}{2\mu}\lim_{b\to\infty}\int_0^1 |\dot\gamma_b(t)|^2\,\mathrm{d}t \\ &\le -\frac{\partial_s f(s)}{2\mu} + \frac{1}{2\mu}\lim_{b\to\infty} E(u; [b-1, b+1] \times [0,1]) \\ &= -\frac{\partial_s f(s)}{2\mu}\,. \end{aligned}$$

Hence $2\mu f(s) + \partial_s f(s) \le 0$ which gives (4.3.5).

We show (4.3.6). Since L_0 is compact there exists a sequence $s_\nu \to \infty$ and a point $p \in L_0$ such that $p_\nu := u(s_\nu, 0) \to p$. Given any s we find ν_0 such that $s_\nu > s$ for all $\nu \ge \nu_0$ and by the exponential decay of the energy and the mean-value inequality we have

$$\begin{aligned}
\operatorname{dist}(u(s,t),p) &\le \operatorname{dist}(u(s,t),u(s_\nu,0)) + \operatorname{dist}(p_\nu,p) \\
&\le \int_s^{s_\nu} |\partial_\sigma u(\sigma,t)|\,\mathrm{d}\sigma + \int_0^t |\partial_\tau u(s_\nu,\tau)|\,\mathrm{d}\tau + \operatorname{dist}(p_\nu,p) \\
&\le \sqrt{c_1\varepsilon_0}\int_s^\infty e^{-\mu\sigma}\mathrm{d}\sigma + \sqrt{c_1\varepsilon_0}e^{-\mu s_\nu} + \operatorname{dist}(p_\nu,p) \\
&\le \sqrt{c_1\varepsilon_0}e^{-\mu s} + \operatorname{dist}(p_\nu,p) \to \sqrt{c_1\varepsilon_0}e^{-\mu s}\,.
\end{aligned}$$

To see that $p \in L_0 \cap L_1$, we consider $p'_\nu := u(s_\nu,1) \in L_1$ for all $\nu \in \mathbb{N}$. By the previous estimate we have $p'_\nu \to p$. Since L_1 is closed we conclude $p \in L_0 \cap L_1$. The last estimate together with (4.3.9) and (4.3.5) shows (4.3.6).

We show (4.3.7). Let $u : [s_0,s_1]\times[0,1] \to M$ be a map that satisfies (CR), (BC) and $E(u) < \varepsilon_0$. The equations (4.3.9) and (4.3.10) still hold. In particular $u(s,t)$ lies inside the Pozniak neighborhood for all $(s,t) \in [s_0+1/2, s_1-1/2]\times[0,1]$ and we have

$$\begin{aligned}
f(s) := E(u;[s_0+s,s_1-s]\times[0,1]) &= \int_0^1 \gamma^*_{s_0+s}\lambda - \int_0^1 \gamma^*_{s_1-s}\lambda \\
&\le \frac{1}{2\mu}\int_0^1 |\dot\gamma_{s_0+s}|^2\,\mathrm{d}t + \frac{1}{2\mu}\int_0^1 |\dot\gamma_{s_1-s}|^2\,\mathrm{d}t = -\frac{1}{2\mu}\partial_s f(s)\,,
\end{aligned}$$

for all $1/2 \le s \le (s_1-s_0)/2$. Hence $\partial_s f(s) + 2\mu f(s) \le 0$ which implies (4.3.7).

To show (4.3.8) we assume without loss of generality that $s_0 = -s_1$, after possibly replacing u with the shifted map $\tilde u$ given by $\tilde u(s,t) = u(s-(s_1+s_0)/2,t)$. By the mean value inequality and the energy decay we have for $0 \le \sigma \le s_1 - 1$

$$\begin{aligned}
|\mathrm{d}u(\sigma,t)|^2 &\le c_1 E(u;[\sigma-1/2,\sigma+1/2]\times[0,1]) \\
&\le c_1 E(u;[-\sigma-1/2,\sigma+1/2]\times[0,1]) \\
&\le c_1 e^{2\mu}\varepsilon_0 e^{-2\mu(s_1-\sigma)}\,,
\end{aligned}$$

where in the last estimate we used (4.3.7) with $s = s_1 - \sigma - 1/2$. Note that because $\sigma \le s_1 - 1$ we have $s \ge 1/2$ as required. Fix some $s \in [1,s_1]$, $\sigma_0 \in [0,s_1-s]$ and $t_0 \in [0,1]$. We compute with $c_2 = \sqrt{c_1\varepsilon_0}e^\mu$

$$\begin{aligned}
\operatorname{dist}(u(\sigma_0,t_0),u(0,0)) &\le \int_0^{\sigma_0} |\partial_s u(\sigma,0)|\,\mathrm{d}\sigma + \int_0^{t_0} |\partial_t u(\sigma_0,t)|\,\mathrm{d}t \\
&\le c_2\int_0^{\sigma_0} e^{-\mu(s_1-\sigma)}\mathrm{d}\sigma + c_2\int_0^{t_0} e^{-\mu(s_1-\sigma_0)}\mathrm{d}t \\
&\le c_2(\mu^{-1}+1)e^{-\mu(s_1-\sigma_0)} \le c_2(\mu^{-1}+1)e^{-\mu s}\,.
\end{aligned}$$

We conclude the same estimate for every $\sigma_1 \in [-s_0+s,0]$ and $t_1 \in [0,1]$. Hence

$$\operatorname{dist}(u(\sigma_0,t_0),u(\sigma_1,t_1)) \le 2c_2(1/\mu+1)e^{-\mu s}\,.$$

This shows (4.3.8). □

4.4. Linear theory

This section is mainly an exposition of the results from [65]. We have included it to introduce the necessary notations. Following the ideas of [65] we reformulate the linearization of (CR) and (BC) as an operator of the form $\partial_s + A(s) + B(s)$ where $s \mapsto A(s)$ is a path of unbounded operators converging to a self-adjoined operator as s tends to ∞ and $s \mapsto B(s)$ is a path of anti-symmetric bounded operators converging to zero as s tends to ∞.

Fix a point $p \in L_0 \cap L_1$ and neighborhood $U \subset M$ from Lemma 3.2.10. Given a map $u : [0, \infty) \times [0, 1] \to M$ which satisfies (CR) and (BC). Assume that the image of u is completely contained in U. We consider the linearized Cauchy-Riemann operator

$$D_u : \Gamma(u^*TM) \to \Gamma(u^*TM), \qquad \xi \mapsto \nabla_s \xi + J(u)\nabla_t \xi + \nabla_\xi J(u)\partial_t u\,. \tag{4.4.1}$$

Let Φ be the trivialization from Lemma 3.2.11 and abbreviate $\Phi_u(s,t) := \Phi_t(u(s,t))$ for all $(s,t) \in [0,\infty)\times[0,1]$. We define the matrix valued function $S : [0,\infty)\times[0,1] \to \mathbb{R}^{2n\times 2n}$ by

$$\Phi_u \left(\partial_s \xi + J_{\mathrm{std}}\partial_t \xi + S\xi\right) = D_u \Phi_u \xi\,, \tag{4.4.2}$$

for all smooth $\xi : [0,\infty) \times [0,1] \to \mathbb{R}^{2n}$. Abbreviate $\Phi_\infty(t) := \Phi_t(0)$ and $J_\infty(t) := J_t(0)$ for all $t \in [0,1]$. Similarly we define $S_\infty : [0,1] \to \mathbb{R}^{2n\times 2n}$ via

$$\Phi_\infty \left(J_{\mathrm{std}}\partial_t \xi + S_\infty \xi\right) = J_\infty \partial_t \Phi_\infty \xi\,, \tag{4.4.3}$$

for all smooth $\xi : [0,1] \to \mathbb{R}^{2n}$. The next lemma relates the asymptotic behavior of S to the asymptotic behavior of u. Since the proof does not use the fact that L_0 and L_1 intersect transversely, we quote directly from [65].

Lemma 4.4.1. *The matrix $S_\infty(t)$ symmetric for all $t \in [0,1]$. There exists constants s_0 and $c > 0$ such that*

$$\left|S(s,t) - S_\infty(t)\right| \le c\Big(\left|\partial_s u(s,t)\right| + \operatorname{dist}\left(u(s,t), p\right)\Big)\,,$$

for all $s \ge s_0$ and $t \in [0,1]$. Moreover if u satisfies an uniform C^k-bound for some $k \ge 0$, then there exist a constant $c_k > 0$ such that

$$\|S - S_\infty\|_{C^k([s,\infty)\times[0,1])} \le c_k\Big(\,\|\partial_s u\|_{C^k([s,\infty)\times[0,1])} + \sup_{s\le\sigma, 0\le t\le 1} \operatorname{dist}\left(u(\sigma,t), p\right)\Big)\,.$$

Proof. See [65, Lemma 2.2]. □

Consider the Hilbert space $H = L^2([0,1], \mathbb{R}^{2n})$ equipped with standard inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Consider the dense subspace $V \subset H$ given by

$$V = \left\{\xi \in H^{1,2}([0,1], \mathbb{R}^{2n}) \mid \xi(0), \xi(1) \in \mathbb{R}^n \times \{0\}\right\}\,.$$

Given $s \in [0,\infty)$ we define the linear operators $A(s) : V \to H$, $\xi \mapsto A(s)\xi$ where

$$(A(s)\xi)(t) = J_{\mathrm{std}}\partial_t \xi(t) + \frac{1}{2}\left(S(s,t) + S(s,t)^T\right)\xi(t)\,,$$

and the operator $A_\infty : V \to H$, $\xi \mapsto A_\infty \xi$ where

$$(A_\infty \xi)(t) = J_{\mathrm{std}}\partial_t \xi(t) + S_\infty(t)\xi(t)\,. \tag{4.4.4}$$

Moreover define the linear operator $B(s) : H \to H$, $\eta \mapsto B(s)\eta$ given by

$$(B(s)\eta)(t) = \frac{1}{2}\left(S(s,t) - S(s,t)^T\right)\eta(t)\,.$$

We quote the next lemma directly from [65]. It states that the paths $s \mapsto A(s)$ and $s \mapsto B(s)$ are continuously differentiable. We denote the derivatives by $\dot{A}(s)$ and $\dot{B}(s)$ respectively.

Lemma 4.4.2. *The operators $A(s) - A_\infty$, $\dot{A}(s)$, $B(s)$ and $\dot{B}(s)$ have extensions to bounded linear operators on H. Moreover there exists a constant $c > 0$ such that for ever $s \geq 0$,*

$$\|A(s) - A_\infty\| + \|B(s)\| \leq c \sup\nolimits_{t\in[0,1]}\left(\left|\partial_s u(s,t)\right| + \mathrm{dist}\,(u(s,t),p)\right)\,,$$

$$\left\|\dot{A}(s)\right\| + \left\|\dot{B}(s)\right\| \leq c \sup\nolimits_{t\in[0,1]}\left(\left|\nabla_s \partial_s u(s,t)\right| + \left|\partial_s u(s,t)\right| + \mathrm{dist}\,(u(s,t),p)\right)\,.$$

In which $\|\cdot\|$ denotes the operator norm on bounded linear operators.

Proof. [65, Lemma 2.3] □

Define the function $\xi_u : [0,\infty) \times [0,1] \to \mathbb{R}^{2n}$, $(s,t) \mapsto \xi_u(s,t)$

$$\xi_u(s,t) = \Phi_u(s,t)^{-1}\partial_s u(s,t)\,. \tag{4.4.5}$$

Since u solves the Cauchy-Riemann equation (CR) and J is s-independent, the vector field $\partial_s u$ lies in the kernel of D_u and with the above definition we have

$$\partial_s \xi_u(s,t) + J_{\mathrm{std}}\partial_t \xi_u(s,t) + S(s,t)\xi_u(s,t) = 0\,. \tag{4.4.6}$$

By construction we have $\xi_u(0,\cdot), \xi_u(1,\cdot) \subset \mathbb{R}^n \times 0$ for $k = 0, 1$, in particular $\xi_u(s,\cdot) \in V$ for all $s \geq 0$. Abusing notation we denote the path $[0,\infty) \to V$, $s \mapsto \xi_u(s,\cdot)$ also by ξ_u. According to (4.4.6) we have for all $s \geq 0$

$$\partial_s \xi_u(s) + A(s)\xi_u(s) + B(s)\xi_u(s) = 0\,. \tag{4.4.7}$$

In contrast to the setting of [65], the asymptotic operator A_∞ is no longer injective in our situation. To be able to conclude we need that the component of ξ_u is the kernel is controlled by ξ_u as provided in the next lemma. Let $\ker A_\infty \subset H$ denote the kernel of A_∞ considered as a closed subspace of H and $P : H \to \ker A_\infty$ the orthogonal projection to the kernel of A_∞.

Lemma 4.4.3. *There exists a uniform constant c such that for all $s \geq 0$ we have*

$$\|P\xi_u(s)\| \leq c\ \sup_{t\in[0,1]} \operatorname{dist}(u(s,t),p)\, \|\xi_u(s)\| \ .$$

Proof. Via Lemma 3.2.10 we assume without loss of generality that $U \subset \mathbb{R}^{2n}$ equipped with the standard symplectic structure and L_0, L_1 are fixed linear Lagrangian subspaces. Moreover we think of the almost complex structure J and Φ as matrix valued functions. Fix $s \geq 0$ and abbreviate $\gamma_s(t) := u(s,t)$ for all $t \in [0,1]$. The path $t \mapsto \Phi_\infty^{-1}(t)\gamma_s(t)$ is an element of the domain of A_∞. Let $e \in \ker A_\infty$ be an element with $\|e\| = 1$. Since A_∞ is symmetric we compute using the definition of A_∞ (cf. equations (4.4.4) and (4.4.3))

$$\langle \Phi_\infty^{-1} J_\infty \partial_t \gamma_s, e\rangle = \langle \Phi_\infty^{-1} J_\infty \partial_t \Phi_\infty \Phi_\infty^{-1}\gamma_s, e\rangle = \langle A_\infty \Phi_\infty^{-1}\gamma_s, e\rangle = 0\,.$$

Abbreviate $G_\infty(t) := \Phi_\infty(t)^{-1}J_\infty(t)$ and $G_u(t) := \Phi_u(t)^{-1}J_t(u(s,t))$ for all $t \in [0,1]$. By definition of ξ_u (cf. equation (4.4.5)) and since u solves (CR) we have $\xi_u = \Phi_u^{-1}\partial_s u = -\Phi_u^{-1}J(u)\partial_t u = -G_u\partial_t\gamma_s$. Thus

$$\begin{aligned}\langle \xi_u(s), e\rangle = -\langle G_u \partial_t\gamma_s, e\rangle = \langle (G_\infty - G_u)\partial_t\gamma_s, e\rangle \leq \\ \leq \|G_\infty - G_u\|_{C^0} \left\|G_u^{-1}\right\|_{C^0} \|\xi(s)\| \ .\end{aligned}$$

The matrix $G_t(q) := \Phi_t(q)^{-1}J_t(q)$ is invertible for all $(t,q) \in [0,1]\times U$ and satisfies an uniform C^1-bound. In particular there exists a uniform constant c such that for all $s \geq 0$ we have

$$\langle \xi_u(s), e\rangle \leq c \sup_{t\in[0,1]} \operatorname{dist}(u(s,t),p)\, \|\xi_u(s)\| \ .$$

The claim follows after taking the supremum over all $e \in \ker A_\infty$ with $\|e\| = 1$ of the last estimate. □

4.5. Proofs

Proof of Theorem 4.1.1. Given u which satisfies (CR) and (BC). Assume additionally that u satisfies (4.1.3) then (E) clearly follows. Also if u satisfies (E), then (4.1.2) follows by the estimate (4.3.6). In order to prove the theorem it suffices to show that if u satisfies (4.1.2) then (4.1.3) follows. Provided with the exponential decay of the energy this follows from elliptic bootstrapping as explained on [65, Page 594]. We quickly repeat the argument.

Given u such that (CR), (BC) and (4.1.2) holds. Let $\rho = \rho(\mu, p)$ denote the constant from the isoperimetric inequality (cf. Prop. 4.3.1). By (4.1.2) we assume without loss of generality that $u(s,t) \in B_\rho(p)$ for all $s \geq 0$ and $t \in [0,1]$. Moreover we assume that the image of u lies in a suitable symplectic chart as considered in Section 4.4. The map $\xi := \xi_u$ defined in (4.4.5) solves (4.4.6), i.e. for all $(s,t) \in [0,\infty)\times[0,1]$ we have

$$\partial_s\xi(s,t) + J_{\mathrm{std}}\partial_t\xi(s,t) + S(s,t)\xi(s,t) = 0\ . \tag{4.5.1}$$

Fix $k \in \mathbb{N}_0$ and $s \geq k$. For all $\nu \in \mathbb{N}_0$ we define the shifted maps

$$\xi_\nu(\sigma, t) := \xi(\sigma + s + \nu, t), \qquad S_\nu(\sigma, t) := S(\sigma + s + \nu, t) \, .$$

For any $a < b$ with possibly $b = \infty$ we abbreviate $\Sigma_a^b = [a, b] \times [0, 1]$ and $\Sigma_a^\infty = [a, \infty) \times [0, 1]$ if $b = \infty$. For any $\ell \in \mathbb{N}_0$ let $\|\cdot\|_{\ell,2;\Sigma_a^b}$ denote the standard Sobolev norm of $H^{\ell,2}(\Sigma_a^b, \mathbb{R}^{2n})$. Using elliptic bootstrapping (cf. [65, Lemma C.1]) and since ξ solves (4.5.1) we have a constant $c_1 = c_1(\ell)$ which depends on ℓ but is independent of ξ, S and ν such that

$$\|\xi\|^2_{\ell,2;\Sigma_s^\infty} = \sum_{\nu=0}^{\infty} \|\xi_\nu\|^2_{\ell,2;\Sigma_0^1} \leq c_1 \sum_{\nu=0}^{\infty} \left(\|S_\nu \xi_\nu\|^2_{\ell-1,2;\Sigma_{-1}^2} + \|\xi_\nu\|^2_{\ell-1,2;\Sigma_{-1}^2} \right) \, .$$

According to Lemmas 4.4.1 and Corollary 4.2.2 the smooth maps S_ν satisfy an uniform C^ℓ-bound, hence there exists a uniform constant $c_2 = c_2(\ell)$ such that

$$\|\xi\|^2_{\ell,2;\Sigma_s^\infty} \leq c_2 \sum_{\nu=0}^{\infty} \|\xi_\nu\|^2_{\ell-1,2;\Sigma_{-1}^2} = 3c_2 \, \|\xi\|^2_{\ell-1,2;\Sigma_{s-1}^\infty} \, .$$

Repeating the previous k times we conclude that for each $k \in \mathbb{N}_0$ we have constant $c_3 = c_3(k)$ depending on k such that

$$\|\xi\|^2_{k,2;\Sigma_s^\infty} \leq c_3 \, \|\xi\|^2_{0,2;\Sigma_{s-k}^\infty} = c_3 E(u; \Sigma_{s-k}^\infty) \, .$$

The C^k-norm of Φ is bounded and after Corollary 4.2.2 so is the C^k-norm of the map $(s, t) \mapsto \Phi_u(s, t) = \Phi_t(u(s, t))$, hence there exists a constant c_4 such that

$$\|\partial_s u\|_{C^k(\Sigma_s^\infty)} \leq c_4 \, \|\xi\|_{C^k(\Sigma_s^\infty)} \, .$$

By Sobolev embedding, the last two estimates and Lemma 4.3.2 we have constants c_5 and c_6 such that

$$\|\partial_s u\|_{C^k(\Sigma_s^\infty)} \leq c_5 \, \|\xi\|_{k+2,2;\Sigma_s^\infty} \leq c_3 c_5 \, \|\xi\|_{L^2(\Sigma_{s-k-2}^\infty)} \leq c_6 e^{-\mu(s-k-2)} \, .$$

This shows (4.1.3) and hence the theorem. □

Proof of Theorem 4.1.2. We follow closely the line of arguments from the proof of [65, theorem B]. By Theorem 4.1.1 we assume without loss of generality that the image of u lies in a suitable symplectic chart. With notations from Section 4.4 we see that $\xi := \xi_u : [0, \infty) \to V$ satisfies (4.4.7), i.e. for all $s \geq 0$ we have

$$\partial_s \xi(s) + A(s)\xi(s) + B(s)\xi(s) = 0 \, .$$

In [33, theorem 4.1] it is proven that A_∞ is Fredholm and self-adjoined considered as an unbounded operator in H. Using Lemmas 4.4.2 and 4.4.3 together with the exponential decay of u given in equation (4.1.3) all the requirements for Lemma B.2.5 are fulfilled.

Hence there exists an eigenvalue α of A_∞, an eigenvector $\zeta \in \ker(A_\infty - \alpha)$ and a constant c such that for all $s \geq 0$ we have

$$\int_0^1 \left|e^{\alpha s}\xi(s,t) - \zeta(t)\right|^2 \mathrm{d}t \leq ce^{-2\mu s}\,.$$

Abbreviate $\Sigma_s := [s,\infty) \times [0,1]$. We prove by induction that for each $k \in \mathbb{N}_0$ there exists a constant c_k such that for all $s \geq 0$

$$\|e^{\alpha s}\xi - \zeta\|_{H^{k,2}(\Sigma_s)} \leq c_k e^{-\mu s}\,. \tag{4.5.2}$$

For $k = 0$ this follows by the last estimate. Now assume that (4.5.2) has been established for some $k \geq 0$. Abbreviate $\theta(s,t) := e^{\alpha s}\xi(s,t) - \zeta(t)$ for all $(s,t) \in [0,\infty) \times [0,1]$. The map $\theta : [0,\infty) \times [0,1] \to \mathbb{R}^{2n}$ satisfies

$$\partial_s\theta(s,t) + J_{\mathrm{std}}\partial_t\theta(s,t) = \eta(s,t), \qquad \theta(s,0),\ \theta(s,1) \subset \mathbb{R}^n \times \{0\}\ ,$$

for all $(s,t) \in [0,\infty) \times [0,1]$, in which

$$\eta(s,t) = (\alpha - S_\infty(t))\,\theta(s,t) + (S_\infty(t) - S(s,t))\,(\theta(s,t) - \zeta(t))\ .$$

By the C^k-bounds of S (cf. Lemma 4.4.1) and the exponential decay for $\partial_s u$ (cf. equation (4.1.3)) we have a constant $c = c(k)$ such that for all $s \geq 0$

$$\|S - S_\infty\|_{C^k(\Sigma_s)} \leq ce^{-\mu s}\,.$$

By this estimate and the induction hypotheses there exists another constant $c = c(k)$ such that for all $s \geq 0$

$$\|\partial_s\theta + J_{\mathrm{std}}\partial_t\theta\|_{H^{k,2}(\Sigma_s)} \leq ce^{-\mu s}\,.$$

Then after elliptic bootstrapping (cf. [65, Lemma C.1]) we conclude that (4.5.2) holds with k replaced with $k+1$. This shows (4.5.2) for all $k \in \mathbb{N}$. By the Sobolev embedding we also conclude for all $k \in \mathbb{N}$ we have a possibly larger constant c_k such that

$$\|e^{\alpha s}\xi - \zeta\|_{C^k(\Sigma_s)} \leq c_k e^{-\mu s}\,. \tag{4.5.3}$$

By construction the Hessian A_p and the operator A_∞ are conjugated via Φ_∞. In particular the path $[0,1] \to T_pM$, $t \mapsto \Phi_\infty(t)\zeta(t)$ is an eigenvector of A_p with eigenvalue α. Define the map $w : [s_0,\infty) \times [0,1] \to T_pM$ by

$$u(s,t) = \exp_p(-\alpha^{-1}e^{-\alpha s}\Phi_\infty(t)\zeta(t) + w(s,t))\,.$$

We derive the equation by ∂_s and obtain

$$\partial_s u = E(\tilde{u})e^{-\alpha s}\Phi_\infty\zeta + E(\tilde{u})\partial_s w\,,$$

in which $E(\tilde{u})$ denotes the derivative of the exponential map at

$$\tilde{u} := -\alpha^{-1}e^{-\alpha s}\Phi_\infty\zeta + w\,.$$

Rewriting the last equation gives

$$\begin{aligned}\partial_s w &= E(\tilde{u})^{-1}\partial_s u - e^{-\alpha s}\Phi_\infty \zeta \\ &= E(\tilde{u})^{-1}\Phi_u(\xi - e^{-\alpha s}\zeta) + e^{-\alpha s}(E(\tilde{u})^{-1}\Phi_u - \Phi_\infty)\zeta\,.\end{aligned}$$

By the exponential decay of u, since the C^k-norms of E and Φ are uniformly bounded and $E(0)$ is the identity we conclude that there exists a possibly larger constant c_k such that

$$\left\|E(\tilde{u})^{-1}\Phi_u - \Phi_\infty\right\|_{C^k(\Sigma_s)} \leq \left\|(E(\tilde{u})^{-1} - \mathbb{1})\Phi_u\right\|_{C^k(\Sigma_s)} + \|\Phi_u - \Phi_\infty\|_{C^k(\Sigma_s)}\,,$$

is bounded by $c_k e^{-\mu s}$. Hence with together with estimate (4.5.3) we obtain a possibly larger constant c_k such that for all $s \geq 0$

$$\|\partial_s w\|_{C^k(\Sigma_s)} \leq c_k e^{-(\mu+\alpha)s}\,.$$

By construction we see that $\lim_{s\to\infty} w(s,t) = 0$ for each fixed $t \in [0,1]$ and thus

$$w(s,t) = -\int_s^\infty \partial_\sigma w(\sigma,t)\mathrm{d}\sigma\,.$$

Using the previous estimate on $\partial_s w$ we conclude that w also satisfies an exponential decay. This proves the theorem. $\square$

5. Compactness

We study sequences of (perturbed) holomorphic strips with boundary on two Lagrangians. We show that if the energy of the sequence is uniformly bounded, then a subsequence converges in a certain sense to a broken strip. The convergence is a very crude version of Gromov compactness, which forgets the so called "bubbles" and just remembers their energies. If the Lagrangians are monotone, this will prove to be sufficient for our purposes. Convergence of holomorphic strips has originally been studied by Floer in [25] in which he a priory excluded the bubbles and later by Oh in [58] for the monotone case. Both of the results are formulated under the assumption that the Lagrangians intersect transversely. Here we give a refinement which allows cleanly intersecting Lagrangians. In the special case where both Lagrangians are the same and the almost complex structure does not depend on the domain a sequence of holomorphic strips is nothing but a sequence of holomorphic disks and Gromov compactness of these is fully described in [34]. Most proofs are straight forward generalizations of this special case. An alternative approach is developed Ivashkovich-Shevchishin in [46].

5.1. Cauchy-Riemann-Floer equation

Let (M,ω) be a symplectic manifold and $L_0, L_1 \subset M$ be two Lagrangian submanifolds not necessarily in clean intersection. We abbreviate the strip $\Sigma := \mathbb{R} \times [0,1]$. Further denote by $X \in C^\infty(\Sigma, \mathrm{Vect}(X))$ and $J \in C^\infty(\Sigma, \mathrm{End}(TM,\omega))$ a vector field and an almost complex structure respectively. A *non-trivial finite-energy (J,X)-holomorphic strip u with boundary in* (L_0, L_1) is a map $u : \mathbb{R} \times [0,1] \to M$ which satisfies

$$\begin{aligned} \partial_s u + J(u)\,(\partial_t u - X(u)) = 0\,, \\ u|_{t=0} \subset L_0, \qquad u|_{t=1} \subset L_1\,, \\ 0 < \int |\partial_s u|_J^2 \,\mathrm{d}s\mathrm{d}t < \infty\,. \end{aligned} \tag{5.1.1}$$

By convenience we often just write that u is a (J,X)-*holomorphic strip*. For an open subset $\Omega \subset \mathbb{R} \times [0,1]$, we define the *energy of u on* Ω by

$$E(u) := \int |\partial_s u|_J^2 \,\mathrm{d}s\mathrm{d}t \qquad E(u;\Omega) := \int_\Omega |\partial_s u|_J^2 \;\mathrm{d}s\mathrm{d}t\,.$$

For technical reasons we need to assume that J is asymptotically constant and X is asymptotically constant to a Hamiltonian vector field of a clean Hamiltonian (cf. Definition 3.2.12).

Definition 5.1.1. Given $J \in C^\infty(\Sigma, \mathrm{End}(TM, \omega))$ and $X \in C^\infty(\Sigma, \mathrm{Vect}(X))$,

- we call J *admissible* if there exists s_0 and paths J_- and J_+ such that $J(-s, \cdot) = J_-$ and $J(s, \cdot) = J_+$ for all $s \geq s_0$ and
- we call X *admissible* if there exists s_0 and clean Hamiltonians H_- and H_+ such that $X(-s, \cdot) = X_{H_-}$ and $X(s, \cdot) = X_{H_+}$ for all $s \geq s_0$.

We call J (resp. X) $\mathbb{R}$-*invariant* if the same holds for $s_0 = 0$. Necessarily for $\mathbb{R}$-invariant structures we have $J_- = J_+$ (resp. $H_- = H_+$).

Lemma 5.1.2. *Given admissible J and X. For any (J, X)-holomorphic strip u the limits $u(-\infty) := \lim_{s \to -\infty} u(s, \cdot)$ and $u(\infty) := \lim_{s \to \infty} u(s, \cdot)$ exists and with the notation above we have for all s large enough we have $u(-\infty) \in \mathcal{I}_{H_-}(L_0, L_1)$ and $u(\infty) \in \mathcal{I}_{H_+}(L_0, L_1)$*

Proof. Use Theorem 4.1.1 and Lemma 3.2.4 □

Definition 5.1.3. Given a sequence of admissible almost complex structures $(J_\nu)_{\nu \in \mathbb{N}}$ and a sequence of admissible vector fields $(X_\nu)_{\nu \in \mathbb{N}}$ converging to J and X respectively. A sequence $(u_\nu)_{\nu \in \mathbb{N}}$ of (J_ν, X_ν)-holomorphic strips *Floer-Gromov converges modulo bubbling* to a tuple $v = (v_1, \ldots, v_k)$ if there exists sequences $(a_1^\nu), \ldots, (a_k^\nu) \subset \mathbb{R}$ and empty or finite sets $Z_1, \ldots, Z_k \subset \Sigma$ such that for all $j = 1, \ldots, k$ we have

(i) the sequence $u_\nu \circ \tau_{a_j^\nu}$ converges to v_j in $C^\infty_{\mathrm{loc}}(\Sigma \setminus Z_j)$

(ii) for all $z \in Z_j$ the limit $m_{j,z} := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E(u_\nu \circ \tau_{a_j^\nu}, B_\varepsilon(z))$ exists and is strictly positive,

(iii) if v_j is constant then Z_j is not empty,

(iv) $\lim_{\nu \to \infty} u_\nu(-\infty) = v_1(-\infty)$, $\lim_{\nu \to \infty} u_\nu(\infty) = v_k(\infty)$ and if $j \neq k$ then $v_j(\infty) = v_{j+1}(-\infty)$.

Moreover we have

$$\lim_{\nu \to \infty} E(u_\nu) = \sum_{j=1}^{k} E(v_j) + m, \qquad m := \sum_{j=1}^{k} \sum_{z \in Z_j} m_{j,z}\,.$$

If the sets $Z_1, \ldots, Z_k$ are all empty we say that (u_ν) *Floer-Gromov converges.*

Theorem 5.1.4. *Given a sequence of admissible almost complex structures $(J_\nu)_{\nu \in \mathbb{N}}$ and a sequence of admissible vector fields $(X_\nu)_{\nu \in \mathbb{N}}$ converging to J and X respectively. Any sequence $(u_\nu)_{\nu \in \mathbb{N}}$ of (J_ν, X_ν)-holomorphic with uniformly bounded energies has a subsequence which Floer-Gromov converges modulo bubbling.*

Proof. Iteratively apply Lemma 5.2.2 and Lemma 5.3.2 given below. □

5.2. Local convergence

In this section we provide local convergence results. We use a well-known trick and transform the statement of perturbed holomorphic curves into a statement for holomorphic curves at the cost of turning the target space into a non-compact space. Then the results follows from standard theory on holomorphic curves. Given an open subspace $\Omega \subset \Sigma$, we say that $u : \Omega \to M$ is a (J, X)-holomorphic map if u satisfies (5.1.1) wherever it is defined.

Lemma 5.2.1 (bounded gradient compactness)**.** *Given*

- *a sequence $\Omega_1, \Omega_2, \dots \subset \Sigma$ of open subsets which exhaust $\Omega \subset \Sigma$,*
- *a sequence $J_1, J_2, \dots$ such that $J_\nu : \Omega_\nu \to \mathrm{End}(TM, \omega)$ are almost complex structures converging to $J : \Omega \to \mathrm{End}(TM, \omega)$ in C^∞_{loc},*
- *a sequence $X_1, X_2, \dots$ such that $X_\nu : \Omega_\nu \to \mathrm{Vect}(X)$ are vector fields converging to $X : \Omega \to \mathrm{Vect}(M)$ in C^∞_{loc},*

then for any sequence $u_1, u_2, \dots$ such that $u_\nu : \Omega_\nu \to M$ is a (J_ν, X_ν)-holomorphic map and assume that

$$\sup_{\nu \in \mathbb{N}} \|\partial_s u_\nu\|_{C^0} < \infty \,,$$

there exists subsequence which converges to a map $u : \Omega \to M$ in C^∞_{loc}. Moreover the map u is (J, X)-holomorphic.

Proof. Define the manifold $\widetilde{M} := \Omega \times M$ with submanifolds

$$\widetilde{L}_0 = (\mathbb{R} \times \{0\} \cap \Omega) \times L_0, \qquad \widetilde{L}_1 = (\mathbb{R} \times \{1\} \cap \Omega) \times L_1 \,.$$

Define almost complex structures $\widetilde{J}_\nu, \widetilde{J} \in \mathrm{End}(T\widetilde{M})$ via

$$\widetilde{J}_\nu(s,t,p) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ X_\nu(s,t,p) & -J_\nu(s,t,p) X_\nu(s,t,p) & J_\nu(s,t,p) \end{pmatrix} \,,$$

and similarly $\widetilde{J}$. One checks directly that the manifolds $\widetilde{L}_0$ and $\widetilde{L}_1$ are totally real with respect to $\widetilde{J}$ and that the curves $\widetilde{u}_\nu(s,t) = (s, t, u_\nu(s,t))$ solve

$$\bar{\partial}_{\widetilde{J}_\nu} \widetilde{u}_\nu = \partial_s \widetilde{u}_\nu + \widetilde{J}_\nu(\widetilde{u}_\nu) \partial_t \widetilde{u}_\nu = 0 \,,$$

with boundary conditions

$$\widetilde{u}_\nu(\cdot, 0) \subset \widetilde{L}_0, \qquad \widetilde{u}_\nu(\cdot, 1) \subset \widetilde{L}_1 \,. \tag{5.2.1}$$

We equip $\widetilde{M}$ with the product symplectic structure, then $\widetilde{J}_\nu$ is compatible and for the associated metric we have

$$|\partial_s \widetilde{u}_\nu|^2 = |\partial_t \widetilde{u}_\nu|^2 = 1 + \omega(\partial_s u_\nu, \partial_t u_\nu) = 1 + |\partial_s u_\nu|^2 + \omega(\partial_s u_\nu, X_\nu) \,.$$

We see that the gradient of $\widetilde{u}_\nu$ is uniformly bounded. By the Theorem of Arzelà-Ascoli there exists $\widetilde{u} : \Omega \to \widetilde{M}$ such that $\widetilde{u}_\nu$ converges to $\widetilde{u}$ in C^0_{loc}. It is easy to see that $\widetilde{u}$ satisfies the boundary condition (5.2.1) and $\widetilde{u}(s,t) = (s,t,u(s,t))$ for all $(s,t) \in \Omega$ with some map $u : \Omega \to M$. To improve the convergence and show that $\widetilde{u}$ is $\widetilde{J}$-holomorphic (thus u is (J,X)-holomorphic) we proceed as in proof of [53, Theorem B.4.2]. □

Lemma 5.2.2 (Convergence modulo bubbling)**.** *Assume that $\Omega, J, X, \Omega_\nu, J_\nu, X_\nu$ satisfy the hypotheses of Lemma 5.2.1. Let $u_1, u_2, \dots$ be a sequence of maps such that $u_\nu : \Omega_\nu \to M$ is a (J_ν, X_ν)-holomorphic map and assume that*

$$\sup_{\nu \in \mathbb{N}} E(u_\nu; \Omega_\nu) < \infty \ ,$$

then there exists a subsequence, still denoted by (u_ν), a (J,X)-holomorphic map $u : \Omega \to M$ and an empty or finite set of points $Z = \{z_1, \dots, z_\ell\} \subset \Omega$ such that the following holds

(i) u_ν converges to u in $C^\infty_{\mathrm{loc}}(\Omega \setminus Z)$

(ii) for every $i = 1, \dots, \ell$ and every $\varepsilon > 0$ such that $B_\varepsilon(z_i) \cap Z = \{z_i\}$, the limit

$$m_\varepsilon(z_i) := \lim_{\nu \to \infty} E(u_\nu; B_\varepsilon(z_i) \cap \Omega_\nu) \ ,$$

exists. Moreover

$$m_i := m(z_i) := \lim_{\varepsilon \to 0} m_\varepsilon(z_i),$$

is the energy of a non-constant holomorphic sphere of disk.

(iii) For every compact subset $K \subset \Omega$ with $Z \subset \mathrm{int}(K)$,

$$\lim_{\nu \to \infty} E(u_\nu; K) = E(u; K) + \sum_{j=1}^{\ell} m_i \, .$$

Proof. See [53, Theorem 4.6.1] provided with Lemma 5.2.1. □

5.3. Convergence on the ends

In this section we consider convergence of (J,X)-holomorphic curves restricted to the half-strip $\Sigma^+ := [0,\infty) \times [0,1]$. We assume without loss of generality that $X(s,\cdot) = X_H$ for all $s \geq 0$ and some clean Hamiltonian function H.

Lemma 5.3.1 (C^0-convergence on ends)**.** *Fix a clean Hamiltonian H and an almost complex structure $J \in C^\infty([0,1], \mathrm{End}(TM, \omega)$. Given a sequence $u_1, u_2, u_3, \dots : \Sigma^+ \to M$ of (J,H)-holomorphic half-strips and assume that (u_ν) converges to a half-strip u in $C^\infty_{\mathrm{loc}}(\Sigma^+, M)$. If we have*

$$\lim_{\nu \to \infty} E(u_\nu) = E(u) < \infty \, ,$$

then (u_ν) converges to u in the topology of $C^0(\Sigma^+, M)$. Moreover $u_\nu(\infty)$ converges to $u(\infty)$ as ν tends to ∞.

Proof. See [68, proposition 4.3.10] and [68, proposition 4.3.11] for the proof for case of (J,H)-holomorphic cylinders asymptotic to non-degenerate Hamiltonian orbits. The proof here is a little different and uses the isoperimetric inequality. According to Lemma 3.2.4 we assume without loss of generality that $H = 0$ and L_0, L_1 intersect cleanly. Let U_{Poz} denote the neighborhood of $L_0 \cap L_1$ given by Proposition 3.2.9. We decompose

$$L_0 \cap L_1 = C_1 \cup C_2 \cup \cdots \cup C_m \ ,$$

into connected components and by possibly making U_{Poz} smaller we obtain a respective decomposition

$$U_{\mathrm{Poz}} = U_1 \cup U_2 \cup \cdots \cup U_m \ ,$$

such that $C_i \subset U_i$ for all $i = 1, \ldots, m$ and $U_i \cap U_j = \emptyset$ whenever $i \neq j$. In view of Theorem 4.1.1 we assume without loss of generality that $u(s,t) \in U_1$ for all $s \geq 0$ and $t \in [0,1]$.

Step 1. There exists an s_0 and ν_0 such that $u_\nu(s,t) \in U_1$ for all $s \geq s_0$, $t \in [0,1]$ and $\nu \geq \nu_0$.

By contradiction assume that there exists a sequence (s_ν, t_ν) with $s_\nu \to \infty$ such that

$$u_\nu(s_\nu, t_\nu) \in M \setminus U_1 \ , \tag{5.3.1}$$

for all $\nu \geq 0$. For $0 \leq a < b$ we abbreviate

$$E_\nu(a,b) = E(u_\nu; [a,b] \times [0,1]), \qquad E(a,b) = E(u; [a,b] \times [0,1]) \ ,$$

and similarly $E_\nu(a,\infty)$ and $E(a,\infty)$. Since $E(u_\nu) \to E(u)$ we have

$$0 \leq \lim_{\nu\to\infty} E_\nu(s_\nu - a, \infty) \leq \lim_{\nu\to\infty} E_\nu(b,\infty) = E(b,\infty) \ ,$$

for any $0 \leq a < b$. This shows that

$$\lim_{\nu\to\infty} E_\nu(s_\nu - a, \infty) = 0 \ ,$$

for all $a > 0$. In particular $E_\nu(s_\nu - a, s_\nu + a) \to 0$ and thus $u_\nu(s_\nu, t_\nu) \to x_2 \in L_0 \cap L_1$ in C^∞_{loc}. Because of (5.3.1) we must have $x_2 \notin U_1$. Lets assume without loss of generality that $x_2 \in U_2$. But since $u_\nu \to u$ in C^∞_{loc} and the image of u lies completely in U_1 we find another sequence (s^2_ν, t^2_ν) such that

$$u_\nu(s^2_\nu, t^2_\nu) \in M \setminus (U_1 \cup U_2) \ ,$$

for all $\nu \geq 1$. Repeating the same argument we see that $u_\nu(s^2_\nu, t^2_\nu) \to x_3 \in L_0 \cap L_1$. With $x_3 \notin U_1 \cup U_2$. Lets say $x_3 \in U_3$. Yet again we find a sequence (s^3_ν, t^3_ν) with $u(s^3_\nu, t^3_\nu) \notin U_1 \cup U_2 \cup U_3$ for all $\nu \geq 1$ and eventually a sequence (s^m_ν, t^m_ν) with $s^m_\nu \to \infty$ as ν tends to ∞ such that

$$u_\nu(s^m_\nu, t^m_\nu) \notin U_1 \cup U_2 \cup \cdots \cup U_m = U_{\mathrm{Poz}} \ , \tag{5.3.2}$$

for all $\nu \geq 1$. On the other hand, as before, we also have $u_\nu(s^m_\nu, t^m_\nu) \to x_{m+1} \in L_0 \cap L_1$. This contradicts (5.3.2) and consequently shows the claim.

Step 2. The map u_ν convergences to u in $C^0([0,\infty)\times[0,1])$.
Assume by contradiction that there exists a sequence $(s_\nu,t_\nu)\in\Sigma_0^\infty$ and a constant $\varepsilon>0$ such that

$$\operatorname{dist}\left(u_\nu(s_\nu,t_\nu),u(s_\nu,t_\nu)\right)\geq\varepsilon\ , \tag{5.3.3}$$

for all $\nu\geq 1$. Since $u_\nu\to u$ in C^∞_{loc} we must necessarily have $s_\nu\to\infty$. By the last step, Lemma 4.3.2 and Proposition 4.2.1 there exists constants $c_1,c_2,\delta>0$ such that

$$|\partial_s u_\nu(s,t)|^2\leq c_1E_\nu(s-1,\infty)\leq c_1E_\nu(s_0,\infty)e^{-2\delta(s-s_0-1)}\leq c_2e^{-2\delta s}\ , \tag{5.3.4}$$

for all $s\geq s_0$ and $\nu\geq\nu_0$. Thus

$$\operatorname{dist}\left(u_\nu(a,t),u_\nu(s_\nu,t)\right)\leq\int_a^{s_\nu}|\partial_s u_\nu|\,\mathrm{d}s\leq\frac{c_2}{\delta}\left(e^{-\delta a}-e^{-\delta s_\nu}\right)\leq\frac{c_2}{\delta}e^{-\delta a}\ , \tag{5.3.5}$$

for any $a>s_0$. By the same reasoning for u and possibly making c_2 larger we also have

$$|\partial_s u(s,t)|\leq c_2e^{-\delta s},\qquad\operatorname{dist}\left(u(s,t),u(s_\nu,t)\right)\leq\frac{c_2}{\delta}e^{-\delta s}\ , \tag{5.3.6}$$

for all $s>s_0$ and $t\in[0,1]$. Choose a large enough such that $c_2/\delta e^{-\delta a}\leq\varepsilon/4$ and ν_1 large enough such that the distance from $u_\nu(a,t_\nu)$ to $u(a,t_\nu)$ is smaller than $\varepsilon/4$ for all $\nu\geq\nu_1$ then we finally have

$$\begin{aligned}\operatorname{dist}\left(u_\nu(s_\nu,t_\nu),u(s_\nu,t_\nu)\right)&\leq\operatorname{dist}\left(u_\nu(s_\nu,t_\nu),u_\nu(a,t_\nu)\right)\\&\quad+\operatorname{dist}\left(u_\nu(a,t_\nu),u(a,t_\nu)\right)+\operatorname{dist}\left(u(a,t_\nu),u(s_\nu,t_\nu)\right)\leq\frac{3}{4}\varepsilon\ .\end{aligned}$$

This is a contradiction to (5.3.3) hence proves the claim. The last inequality also shows that $u_\nu(\infty)\to u(\infty)$ as ν tends to ∞. □

Given a clean Hamiltonian $H\in C^\infty([0,1]\times M)$. We say that a map $u:\Sigma^+\to M$ is an (J,H)-*holomorphic* half-strip, if it satisfies (5.1.1) with $X=X_H$.

Lemma 5.3.2 (soft rescaling on ends)**.** *Given a clean Hamiltonian $H:[0,1]\times M\to\mathbb{R}$ and a path of almost complex structure $J:[0,1]\to\operatorname{End}(TM,\omega)$. Suppose that a sequence (u_ν) of (J,H)-holomorphic half-strips converges to $u:\Sigma^+\to M$ in C^∞_{loc} such that the limit*

$$m=\lim_{s\to\infty}\lim_{\nu\to\infty}E(u_\nu;[s,\infty)\times[0,1])>0\ ,$$

exists and is positive. Then there exists a subsequence of (u_ν), still denoted (u_ν), a sequence $(b_\nu)\subset\mathbb{R}$ with $b_\nu\to\infty$, a (J,H)-holomorphic strip $v:\Sigma\to M$ with boundary in (L_0,L_1) and a finite set $Z=\{z_1,\dots,z_\ell\}\subset\Sigma$ such that

(*i*) *The rescaled sequence $v_\nu:=u_\nu\circ\tau_{b_\nu}$ converges to v in $C^\infty_{\mathrm{loc}}(\Sigma\setminus Z)$,*

(*ii*) *the limit*

$$m_j:=m(z_j):=\lim_{\varepsilon\to 0}\lim_{k\to\infty}E(v_k;B_\varepsilon(z_j)\cap\Sigma)\ ,$$

exists and is the energy of a non-constant holomorphic sphere or disk, (*iii*) *if* v *is constant then* $Z \neq \emptyset$,

(*iv*) *the limits*

$$m_0 := m(-\infty) := \lim_{s\to\infty} \lim_{\nu\to\infty} E(v_\nu; (-b_\nu, -s) \times [0,1])$$
$$m_{\ell+1} := m(\infty) := \lim_{s\to\infty} \lim_{\nu\to\infty} E(v_\nu; (s,\infty) \times [0,1]) \ ,$$

exists and we have

$$\lim_{\nu\to\infty} E(u_\nu) = E(v) + \sum_{j=0}^{\ell+1} m_j \ , \qquad m = E(v) + \sum_{j=1}^{\ell+1} m_j \ .$$

(*v*) $u(\infty) = v(-\infty)$.

Proof. This is the adaption of [34, Theorem 3.5] and [34, Lemma 3.6] to the setting of strips. Essentially all arguments work analogous provided with the energy decay (cf. Lemma 4.3.2).

Step 1. We claim that $m \geq \hbar$, where $\hbar$ is smaller than the constant from Proposition 5.4.1 and the constant from Proposition 5.4.2 for $\mathcal{J} = \{J_{s,t} \mid (s,t) \in [-s_0, s_0] \times [0,1]\}$.

After transforming u_ν we assume that $H = 0$ (see Lemma 3.2.4) and L_0, L_1 are in clean and compact intersection. Let $U_{\mathrm{Poz}} \subset M$ denote the neighborhood of $L_0 \cap L_1$ given by Proposition 3.2.9. We claim there there exists a sequence $(s_\nu, t_\nu) \in \Sigma$ such that

$$\lim_{\nu\to\infty} s_\nu = \infty, \qquad \forall\ \nu \geq 1 \ : \ u_\nu(s_\nu, t_\nu) \in M \setminus U_{\mathrm{Poz}} \ . \tag{5.3.7}$$

Otherwise we find $s_1 \geq 0$ such that $u_\nu(s,t) \in U_{\mathrm{Poz}}$ for all $s \geq s_1$, $t \in [0,1]$ and $\nu \geq 1$. Then by Lemma 4.3.2 there exists a constant $\delta > 0$ such that

$$E(u_\nu; (s,\infty)) \leq E(u_\nu; (s_1,\infty)) e^{-2\delta(s-s_1)} \ ,$$

for all $s \geq s_1$ and $\nu \geq 1$, which implies

$$m = \lim_{s\to\infty} \lim_{\nu\to\infty} E(u_\nu; (s,\infty)) \leq \sup_\nu E(u_\nu) \lim_{s\to\infty} e^{-2\delta(s-s_1)} = 0 \ .$$

This contradicts the fact that $m > 0$ and shows the existence of the sequence s_ν satisfying (5.3.7). Now we claim that

$$\liminf_{\nu\to\infty} \sup_{t\in[0,1]} |\mathrm{d}u_\nu(s_\nu, t)| > 0 \ . \tag{5.3.8}$$

If not, then we find a subsequence ν_k such that $u_{\nu_k}(s_{\nu_k}, \cdot)$ converges to a constant arc in $L_0 \cap L_1$ and hence $u_{\nu_k}(s_{\nu_k}, t_{\nu_k}) \in U_{\mathrm{Poz}}$ for k large enough, contradicting (5.3.7). This shows (5.3.8). Define the sequences $c_\nu \in \mathbb{R}$ and $t'_\nu \in [0,1]$ by

$$c_\nu = \left|\mathrm{d}u_\nu(s_\nu, t'_\nu)\right| = \sup\nolimits_t |\mathrm{d}u_\nu(s_\nu, t)| \ .$$

We distinguish two cases. If c_ν is unbounded, then after passing to a subsequence (still denoted ν) we assume that $c_\nu \to \infty$ and $t'_\nu \to t'_\infty \in [0,1]$. By [53, Lemma 4.6.5] we have that for every $\varepsilon > 0$ sufficiently small

$$\hbar \leq \liminf_{\nu\to\infty} E(u_\nu; B_\varepsilon(s_\nu, t'_\infty) \cap \Sigma) \leq \lim_{\nu\to\infty} E(u_\nu; (s,\infty)) \ ,$$

for all $s \geq 0$ and taking the limit

$$\hbar \leq \lim_{s\to\infty} \lim_{\nu\to\infty} E(u_\nu; (s,\infty)) = m \ .$$

This shows the claim in the case when c_ν is unbounded. Now assume that c_ν is bounded. By Lemma 5.2.2 the rescaled sequence $u_\nu \circ \tau_{s_\nu}$ converges to a J-holomorphic strip $v' : \Sigma \to M$ in $C^\infty_{\mathrm{loc}}(\Sigma \setminus Z')$ for some finite set $Z' \subset \Sigma$ and we have

$$E(v'; (-s,\infty)) \leq \lim_{\nu\to\infty} E(u_\nu; (s_\nu - s, \infty)) \leq \lim_{\nu\to\infty} E(u_\nu; (s,\infty)) \ , \tag{5.3.9}$$

for all $s \geq 0$. Since c_ν is bounded we must have $Z' \cap \{0\} \times [0,1] = \emptyset$ and thus C^∞-convergence of $u_\nu \circ \tau_{s_\nu} \to v'$ on $\{0\} \times [0,1]$. We assume without loss of generality that $t'_\nu \to t'_\infty$. By (5.3.8)

$$\left|\mathrm{d}v(0, t'_\infty)\right| = \lim_{\nu\to\infty} \left|\mathrm{d}u_\nu(s_\nu, t'_\nu)\right| > 0 \ .$$

Hence v' is non-constant. Proposition 5.4.1 and equation (5.3.9) imply

$$\hbar \leq \lim_{s\to\infty} E(v'; (-s,\infty)) \leq \lim_{s\to\infty} \lim_{\nu\to\infty} E(u_\nu; (s,\infty)) = m \ .$$

This shows the claim.

Step 2. There exists a sequence $a_\nu \to \infty$ such that

$$\lim_{\nu\to\infty} E(u_\nu, (a_\nu - s, \infty) \times [0,1]) = m \ ,$$

for all $s \geq 0$.

Given $a < b$ we abbreviate

$$E_\nu(a) = E(u_\nu; (a,\infty) \times [0,1]), \qquad E_\nu(a,b) = E(u_\nu; (a,b) \times [0,1]) \ .$$

For $\ell \in \mathbb{N}$ we find $a_\ell > \ell$ and ν_ℓ such that $|E_\nu(a_\ell) - m| \leq 1/\ell$ for all $\nu \geq \nu_\ell$. Without loss of generality we assume that $\nu_\ell < \nu_{\ell+1}$ and define $a_\nu = a_\ell$ if $\nu_\ell \leq \nu < \nu_{\ell+1}$. This shows that

$$\lim_{\nu\to\infty} E(u_\nu; (a_\nu, \infty) \times [0,1]) = m, \qquad \lim_{\nu\to\infty} a_\nu = \lim_{\ell\to\infty} a_\ell = \infty \ . \tag{5.3.10}$$

Let $\varepsilon > 0$ there exists s_0, ν_0 such that $E_\nu(s_0) \leq m + \varepsilon$ for all $\nu \geq \nu_0$, by definition of m. Secondly given any $s \geq 0$, we find $\nu_1 \geq \nu_0$ such that $a_\nu - s > s_0$ for all $\nu \geq \nu_1$ and hence

$$E_\nu(a_\nu) \leq E_\nu(a_\nu - s) \leq E_\nu(s_0) \leq m + \varepsilon \ .$$

for any $\nu \geq \nu_1$. Taking the limit of that inequality as ν tends to ∞ and then as $\varepsilon \to 0$ we have with (5.3.10)

$$m = \lim_{\nu\to\infty} E_\nu(a_\nu) \leq \lim_{\nu\to\infty} E_\nu(a_\nu - s) \leq m \; .$$

This shows the claim.

Step 3. There exists ν_0 and a sequence $b_\nu \to \infty$ such that

$$E(u_\nu; (b_\nu, \infty) \times [0,1]) = m - \hbar/2 \; , \tag{5.3.11}$$

for all $\nu \geq \nu_0$ and we have

$$\lim_{s\to\infty} \lim_{\nu\to\infty} E(u_\nu; (b_\nu - s, \infty) \times [0,1]) = m \; . \tag{5.3.12}$$

By definition of m and since $\sup_\nu E(u_\nu; \Sigma_0^\infty)$ is finite there exists ν_0 such that

$$E_\nu(0) \geq m, \qquad\qquad \lim_{s\to\infty} E_\nu(s) = 0 \; ,$$

for all $\nu \geq \nu_0$. Due to the first step $m \geq \hbar$. By the intermediate value theorem there exists $b_\nu \geq 0$ such that

$$E_\nu(b_\nu) = m - \hbar/2 \; ,$$

for all $\nu \geq \nu_0$. Since for every bounded sequence $\sup_\nu s_\nu \leq c$ it holds

$$\lim_{\nu\to\infty} E_\nu(s_\nu) \geq \lim_{\nu\to\infty} E_\nu(c) \geq \lim_{s\to\infty} \lim_{\nu\to\infty} E_\nu(s) = m \; ,$$

we necessarily have $b_\nu \to \infty$. This shows (5.3.11). A similar argument as in step 2 shows that for all $s \geq 0$ we have

$$m - \hbar/2 = \lim_{\nu\to\infty} E_\nu(b_\nu) \leq \lim_{\nu\to\infty} E_\nu(b_\nu - s) \leq m \; .$$

By contradiction, assume that (5.3.12) is false. Then we find $0 < \rho \leq \hbar/2$ such that

$$\lim_{\nu\to\infty} E_\nu(b_\nu - s) \leq m - \rho \; , \tag{5.3.13}$$

for every $s \geq 0$. We claim that this implies

$$\lim_{\nu\to\infty} b_\nu - a_\nu = \infty \; . \tag{5.3.14}$$

Arguing indirectly, we assume that there exists $s_0 \geq 0$ such that $b_\nu - a_\nu \leq s_0$ for all $\nu \geq 1$. This leads to the following contradiction

$$m = \lim_{\nu\to\infty} E_\nu(a_\nu) \leq \lim_{\nu\to\infty} E_\nu(b_\nu - s_0) \leq m - \rho < m \; .$$

So (5.3.14) is true and thus for any $s \geq 0$ we have

$$\begin{aligned} \lim_{\nu\to\infty} E_\nu(a_\nu - s, a_\nu + s) &\leq \lim_{\nu\to\infty} E_\nu(a_\nu - s, b_\nu) \\ &= \lim_{\nu\to\infty} E_\nu(a_\nu - s) - \lim_{\nu\to\infty} E_\nu(b_\nu) = m - m + \hbar/2 = \hbar/2 \; . \end{aligned}$$

By Lemma 5.2.2 the rescaled strip $w_\nu = u_\nu \circ \tau_{a_\nu}$ converges modulo bubbling to a (J, H)-holomorphic strip w and we have

$$\hbar/2 \geq \lim_{\nu\to\infty} E(w_\nu; (-s, s) \times [0, 1]) = E(w; (-s, s) \times [0, 1]) + m' \ .$$

But if w were non-constant or $m' > 0$ then right-hand side is larger than $\hbar$. This shows that w must be constant and $w_\nu \to w$ in $C^\infty_{\mathrm{loc}}(\Sigma)$. In particular

$$\lim_{\nu\to\infty} E_\nu(a_\nu - s, a_\nu + s) = 0 \ , \tag{5.3.15}$$

for all $s \geq 0$. We also have

$$\lim_{\nu\to\infty} E_\nu(a_\nu, b_\nu) = \lim_{\nu\to\infty} E_\nu(a_\nu) - \lim_{\nu\to\infty} E_\nu(b_\nu) = \hbar/2 \ .$$

By possibly making $\hbar/2$ smaller, we assume that $E_\nu(a_\nu, b_\nu) < \varepsilon_0$ for ν large enough, where ε_0 is given by Lemma 4.3.2. Hence there exists constants $\nu_0, \delta > 0$ such that

$$E_\nu(a_\nu + s, b_\nu - s) \leq E_\nu(a_\nu, b_\nu)e^{-\delta s} \leq \hbar/2e^{-\delta s} \ ,$$

for all $s \geq 1$ and $\nu \geq \nu_0$. This shows

$$\lim_{s\to\infty} \lim_{\nu\to\infty} E_\nu(a_\nu + s, b_\nu - s) \leq \hbar/2 \lim_{s\to\infty} e^{-\delta s} = 0 \ . \tag{5.3.16}$$

Now for $s \geq 0$ we have

$$E_\nu(b_\nu - s) = E_\nu(a_\nu - s) - E_\nu(a_\nu - s, a_\nu + s) - E_\nu(a_\nu + s, b_\nu - s) \ .$$

Combining (5.3.10), (5.3.16) and (5.3.15) this shows that

$$\lim_{s\to\infty} \lim_{\nu\to\infty} E_\nu(b_\nu - s) = m \ ,$$

contradicting (5.3.13) and proving (5.3.12).

Step 4. We show points (*i*), (*ii*) and (*iii*) of the theorem.

With b_ν given in (5.3.11) from last step we define $v_\nu := u_\nu \circ \tau_{b_\nu}$. The existence of the strip v, the set Z and the limits m_j is provided by Lemma 5.2.2. This shows (*i*) and (*ii*). We show (*iii*). With the following argument we even locate the bubbling point. By the definition of b_ν there exists a constant ν_0 such that

$$E_\nu(b_\nu - s_1, b_\nu - s_0) \leq E_\nu(b_\nu - s_1) - E_\nu(b_\nu) \leq m - (m - \hbar/2) = \hbar/2 \ ,$$

for all $\nu \geq \nu_0$ and $0 < s_0 < s_1$. The same argument leading to (5.3.15) shows that no bubbling can occur on $(b_\nu - s_1, b_\nu - s_0) \times [0, 1]$ and provided that v is constant we get

$$\lim_{\nu\to\infty} E(v_\nu; (-s_1, -s_0) \times [0, 1]) = E(v; (-s_1, -s_0) \times [0, 1]) = 0 \ .$$

This shows that

$$\lim_{\nu\to\infty} E(v_\nu;(-s_1,\infty)\times[0,1]) = \lim_{\nu\to\infty} E(v_\nu;(-s_0,\infty)\times[0,1]) + \lim_{\nu\to\infty} E(v_\nu;(-s_1,-s_0)\times[0,1])$$

is independent of s_1. With (5.3.12) we have

$$\lim_{\nu\to\infty} E(v_\nu;(-s_1,\infty)\times[0,1]) = \lim_{s\to\infty}\lim_{\nu\to\infty} E_\nu(b_\nu - s) = m\ ,$$

for all $s_1 > 0$. But on the other hand

$$\lim_{\nu\to\infty} E(v_\nu;(0,\infty)\times[0,1]) = \lim_{\nu\to\infty} E_\nu(b_\nu) = m - \hbar/2\ .$$

This implies that there must be a bubbling point on $\{0\}\times[0,1]$ for v_ν.

Step 5. We show (*iv*).

Let s_0 be so large that $Z \subset \Sigma_{-s_0}^{s_0}$. By possible passing to a further subsequence still denoted by (v_ν), we assume that

$$\rho(s_0) := \lim_{\nu\to\infty} E(v_\nu;(s_0,\infty)\times[0,1])\ ,$$

is well-defined. Then for any $s \geq s_0$, we have C^∞-convergence of v_ν to v on $\Sigma_{s_0}^{s}$ and thus

$$\rho(s) := \rho(s_0) - \lim_{\nu\to\infty} E(v_\nu;(s_0,s)\times[0,1]) = \rho(s_0) - E(v;(s_0,s)\times[0,1])\ .$$

This shows that $\rho(s)$ is well-defined and monotone decreasing. Hence the limit

$$m_{\ell+1} = \lim_{s\to\infty}\lim_{\nu\to\infty} E(v_\nu;(s,\infty)\times[0,1]) = \lim_{s\to\infty}\rho(s)\ ,$$

exists and moreover

$$\rho(s_0) = m_{\ell+1} + E(v;(s_0,\infty)\times[0,1])\ . \tag{5.3.17}$$

Secondly by definition of v_ν and after assumption the limit

$$\begin{aligned} m_0 &= \lim_{s\to\infty}\lim_{\nu\to\infty} E(v_\nu;(-b_\nu,-s)\times[0,1]) = \lim_{s\to\infty}\lim_{\nu\to\infty} E(u_\nu;(0,b_\nu - s)\times[0,1]) \\ &= \lim_{s\to\infty} E(u;(0,s)\times[0,1]) = E(u)\ , \end{aligned}$$

exists. Now by (5.3.12) and Lemma 5.2.2 we have

$$\begin{aligned} m &= \lim_{s\to\infty}\lim_{\nu\to\infty} E(u_\nu;(b_\nu - s,\infty)\times[0,1]) \\ &= \lim_{s\to\infty}\lim_{\nu\to\infty} E(v_\nu;(-s,\infty)\times[0,1]) \\ &= \lim_{s\to\infty}\lim_{\nu\to\infty} E(v_\nu;(-s,-s_0)\times[0,1]) + \lim_{\nu\to\infty} E(v_\nu;(-s_0,s_0)\times[0,1]) + \rho(s_0) \\ &= E(v) + \sum_{j=1}^{\ell+1} m_j\ , \end{aligned}$$

where in the last step we used (5.3.17). Finally using the last two equations

$$\begin{aligned}\lim_{\nu\to\infty} E(u_\nu) &= \lim_{s\to\infty}\lim_{\nu\to\infty} E(u_\nu;[0,s)\times[0,1]) + \lim_{s\to\infty}\lim_{\nu\to\infty} E(u_\nu;(s,\infty)\times[0,1]) \\ &= E(u) + m = E(v) + \sum_{j=0}^{\ell+1} m_j \ .\end{aligned}$$

Step 6. We show (v).
Let $s \geq 0$ be large enough. We have by assumption

$$\lim_{\nu\to\infty} E(u_\nu;(s,\infty)\times[0,1]) = m + E(u;(s,\infty)\times[0,1]) \ ,$$

and after (iv)

$$\begin{aligned}\lim_{\nu\to\infty} E(u_\nu;(b_\nu - s,\infty)\times[0,1]) &= \lim_{\nu\to\infty} E(v_\nu;(-s,\infty)\times[0,1]) \\ &= E(v;(-s,\infty)\times[0,1]) + \sum_{j=1}^{\ell+1} m_j = m - E(v;(-\infty,-s)\times[0,1]) \ .\end{aligned}$$

Subtracting these two identities gives

$$\lim_{\nu\to\infty} E(u_\nu;(s,b_\nu - s)\times[0,1]) = E(u;(s,\infty)\times[0,1]) + E(v;(-\infty,-s)\times[0,1]) \ .$$

If s tends to ∞ the right-hand side approaches zero. Hence there exists constants s_0 and ν_0 such that

$$E(u_\nu;(s,b_\nu - s)\times[0,1]) \leq \varepsilon_0 \ ,$$

for all $s \geq s_0$ and $\nu \geq \nu_0$, where ε_0 is the constant from Lemma 4.3.2. Using this proposition with $a = s_0$, $b = b_\nu - s_0$, $\sigma = b_\nu - s$ and $\sigma' = s$ we see that there exists a constant c_1 such that

$$\mathrm{dist}\left(u_\nu(s,0), u_\nu(b_\nu - s,0)\right) \leq c_1 e^{-\delta(s-s_0)} \ ,$$

for all $s \geq s_0 + 1$. Now estimate using the triangle inequality

$$\begin{aligned}\mathrm{dist}\left(u(\infty), v(-\infty)\right) \leq \mathrm{dist}\left(u(\infty), u(s,0)\right) + \mathrm{dist}\left(u(s,0), v(-s,0)\right) \\ + \mathrm{dist}\left(v(-s,0), v(-\infty)\right) \ ,\end{aligned}$$

and

$$\begin{aligned}\mathrm{dist}\left(u(s,0), v(-s,0)\right) \leq \mathrm{dist}\left(u(s,0), u_\nu(s,0)\right) + \mathrm{dist}\left(u_\nu(s,0), u_\nu(b_\nu - s,0)\right) \\ + \mathrm{dist}\left(v_\nu(-s,0), v(-s,0)\right) \ .\end{aligned}$$

Using theorem 4.1.1 there exists a constant c_2 such that

$$\mathrm{dist}\left(u(\infty), u(s,0)\right) + \mathrm{dist}\left(v(-s,0), v(-\infty)\right) \leq c_2 e^{-\delta s} \ ,$$

for all $s \geq s_0$. Given any $\varepsilon > 0$ choose $s \geq s_0 + 1$ such that

$$c_2 e^{-\delta s} + c_1 e^{-\delta(s-s_0)} \leq \varepsilon/2 ,$$

then choose ν such that

$$\operatorname{dist}(u(s,0), u_\nu(s,0)) + \operatorname{dist}(v_\nu(-s,0), v(-s,0)) \leq \varepsilon/2 .$$

That is possible because for s sufficiently large and fixed, $u_\nu(s,0)$ converges to $u(s,0)$ and $v_\nu(-s,0)$ converges to $v(-s,0)$ as ν tends to ∞. Combining the last six estimates shows that the distance from $u(\infty)$ to $v(-\infty)$ is lesser than ε and hence the claim. □

5.4. Minimal energy

We establish lower bounds on the energy. We denote by $\mathcal{J}_{\mathrm{adm}}$ the space of admissible almost complex structures and $\mathcal{X}_{\mathrm{adm}}$ the space of admissible vector fields.

Proposition 5.4.1. *Given path of almost complex structures $J : [0,1] \to \operatorname{End}(TM, \omega)$ and a Hamiltonian $H \in C^\infty([0,1] \times M)$ such that $\varphi_H(L_0)$ and L_1 are in clean intersection. There exists a positive constant $\hbar > 0$ such that for every non-constant (J,H)-holomorphic strip $u : \Sigma \to M$ with boundary in (L_0, L_1) we have $E(u) \geq \hbar$.*

Proof. See [53, Prop. 4.1.4.] for the analogous proposition for holomorphic spheres or disks. Note that we can not apply the proof technique from there directly because there is no mean-value inequality of large radius. We have to argue indirectly. After a transformation we assume without loss of generality that $H = 0$ (see Lemma 3.2.4). Assume by contradiction that there exists a sequence u_ν of non-constant J-holomorphic strips such that

$$0 < E(u_\nu), \qquad \lim_{\nu \to \infty} E(u_\nu) = 0 . \tag{5.4.1}$$

Let U_{Poz} denote the Poźniak neighborhood of $L_0 \cap L_1$ given by Proposition 3.2.9. We claim that there exists ν_0 such that $u_\nu(s,t) \in U_{\mathrm{Poz}}$ for all $(s,t) \in \Sigma$ and $\nu \geq \nu_0$. To show that we assume by contradiction that there exists a sequence $(s_\nu, t_\nu) \in \Sigma$ such that

$$u_\nu(s_\nu, t_\nu) \in M \setminus U_{\mathrm{Poz}} , \tag{5.4.2}$$

for all $\nu \geq 1$. But since $E(u_\nu) \to 0$ there exists a subsequence such that $u_{\nu_k}(s_{\nu_k}, t_{\nu_k}) \to x$ converges to a point $x \in L_0 \cap L_1$ as k tends to ∞. This contradicts (5.4.2) and we have proven that $u_\nu(s,t) \in U_{\mathrm{Poz}}$ for all $(s,t) \in \Sigma$ and $\nu \geq \nu_0$. Now inside U_{Poz} the symplectic form $\omega = \mathrm{d}\lambda$ is exact with $\lambda|_{TL_k} = 0$ for $k = 0,1$. We have

$$E(u_\nu) = \int_\Sigma |\mathrm{d}u_\nu|^2 = \int_\Sigma u_\nu^* \omega = \int_{\partial\Sigma} u_\nu^* \lambda = 0 .$$

This shows that $E(u_\nu) = 0$ for all $\nu \geq \nu_0$, which contradicts (5.4.1). □

Proposition 5.4.2. *Let $\mathcal{J} \subset \mathrm{End}(TM, \omega)$ be a compact subset of almost complex structures. There exists a positive constant $\hbar > 0$ such that $\int u^*\omega \geq \hbar$ for any non-constant J-holomorphic sphere $u : S^2 \to M$ or non-constant J-holomorphic disk $u : (D^2, \partial D^2) \to (M, L_k)$ with $k = 0, 1$ and $J \in \mathcal{J}$.*

Proof. For every $J \in \mathcal{J}$ let $\hbar(J)$ be the minimal energy of a non-constant J-holomorphic sphere. For $k = 0, 1$ let $\hbar_k(J)$ be the minimal energy of a non-constant J-holomorphic disk $u : (D, \partial D^2) \to (M, L_k)$. In [53, Prop 4.1.4] we see that the maps $J \mapsto \hbar(J)$ and $J \mapsto \hbar_k(J)$ are lower semi-continuous and everywhere positive. Let $\hbar$ be smaller than their minimum which is positive since $\mathcal{J}$ is compact. □

5.5. Action, energy and index estimates

We denote by $\mathcal{J}_{\mathrm{adm}}$ and $\mathcal{X}_{\mathrm{adm}}$ the space of admissible almost complex structures and admissible vector fields respectively (cf. Definition 5.1.1).

Lemma 5.5.1 (action-energy estimate)**.** *Given $J \in \mathcal{J}_{\mathrm{adm}}$ and $X \in \mathcal{X}_{\mathrm{adm}}$, there exists a constant $c > 0$ such that for any finite energy (J, X)-holomorphic strip u with boundary in (L_0, L_1) we have*

$$\frac{1}{2}E(u) - c \leq \int u^*\omega \leq \frac{3}{2}E(u) + c\,.$$

Proof. Fix a (J, X)-holomorphic strip u with finite energy. We denote the asymptotic points $u(-\infty) = x_-$ and $u(\infty) = x_+$ and estimate

$$|\omega(\partial_s u, X)| = |\langle \partial_s u, JX\rangle_J| \leq \frac{1}{2}\left|\partial_s u\right|_J^2 + \frac{1}{2}\left|X\right|_J^2\,.$$

By definition of an admissible vector field we have $X(\pm s, \cdot) = X_{H_\pm}$ for all $s \geq s_0$. This shows

$$\begin{aligned}
&\int_\Sigma \omega(X, \partial_s u)\mathrm{d}s\mathrm{d}t = \\
&= \int_{\Sigma_{-\infty}^{-s_0}} \partial_s H_-(u)\mathrm{d}s\mathrm{d}t + \int_{\Sigma_{-s_0}^{s_0}} \omega(X, \partial_s u)\mathrm{d}s\mathrm{d}t + \int_{\Sigma_{s_0}^{\infty}} \partial_s H_+(u)\mathrm{d}s\mathrm{d}t \\
&= \int_0^1 H_-(u(-s_0, t)) - H_-(x_-(t))\mathrm{d}t + \int_{\Sigma_{-s_0}^{s_0}} \omega(X, \partial_s u)\mathrm{d}s\mathrm{d}t + \\
&\qquad + \int_0^1 H_+(x_+(t)) - H_+(u(s_0, t))\mathrm{d}t\,.
\end{aligned}$$

With the last estimate we see

$$\begin{aligned}
\frac{1}{2}\int_\Sigma \left|\partial_s u\right|_J^2 + \sup H_- - \inf H_- + \sup H_+ - \inf H_- + s_0\left\|X\right\|_\infty^2 &\geq \int_\Sigma \omega(X, \partial_s u) \\
&\geq -\frac{1}{2}\int_\Sigma \left|\partial_s u\right|_J^2 - \sup H_- + \inf H_- - \sup H_+ + \inf H_+ - s_0\left\|X\right\|_\infty^2\,.
\end{aligned}$$

This shows the claim using $\left|\partial_s u\right|_J^2 = \omega(\partial_s u, \partial_t u) + \omega(X, \partial_s u)$. □

Given an admissible vector field $X \in \mathcal{X}_{\text{adm}}$ such that $X(\pm s, \cdot) = X_{H_\pm}$ for all $s \geq s_0$ and an almost complex structure $J \in \mathcal{J}_{\text{adm}}$. For every (J, X)-holomorphic strip u asymptotic to $x := \lim_{s \to -\infty} u(s, \cdot)$ and $y := \lim_{s \to \infty} u(s, \cdot)$ we define the *action-energy defect*

$$\Delta(u) := E(u) - \int u^*\omega - \int_0^1 H_+(t, y(t))\mathrm{d}t + \int_0^1 H_-(t, x(t))\mathrm{d}t\,. \tag{5.5.1}$$

The quantity is called action-energy defect, because if X is $\mathbb{R}$-invariant then $\Delta(u)$ vanishes and the equation above is the action-energy relation (cf. equation (3.2.10)). The next lemma states that the defect is continuous under Gromov converge.

Lemma 5.5.2. *Given sequences $(J^\nu) \subset \mathcal{J}_{\text{adm}}$ and $(X^\nu) \subset X_{\text{adm}}$ converging in C^∞-topology to J and X respectively. Let (u^ν) be a sequence of (J^ν, X^ν)-holomorphic strips with boundary in (L_0, L_1). Fix a finite subset $Z \subset \Sigma$ and assume that (u^ν) converges in $C^\infty_{\text{loc}}(\Sigma \setminus Z)$ to the (J, X)-holomorphic map u, then we have $\lim_{\nu \to \infty} \Delta(u^\nu) = \Delta(u)$.*

Proof. Let $B_\varepsilon(z) \subset \Sigma$ denote the open ball with radius $\varepsilon > 0$. Fix some $\varepsilon > 0$ and denote the thickened set

$$Z_\varepsilon = \bigcup_{z \in Z} B_\varepsilon(z)\,.$$

Then after convergence of $u^\nu \to u$ and $(J^\nu, X^\nu) \to (J, X)$ in $C^\infty(\Sigma_{-s_0}^{s_0} \setminus Z_\varepsilon)$ there exists a ν_0 such that for all $\nu \geq \nu_0$

$$\left| \int_{\Sigma_{-s_0}^{s_0} \setminus Z_\varepsilon} \langle \partial_s u^\nu, J^\nu(u^\nu) X^\nu(u^\nu) \rangle - \langle \partial_s u, J(u) X(u) \rangle \mathrm{d}s\mathrm{d}t \right| \leq \varepsilon\,,$$

and

$$\int_{\{-s_0\} \times [0,1] \setminus Z_\varepsilon} |H_-(u^\nu) - H_-(u)|\, \mathrm{d}t + \int_{\{s_0\} \times [0,1] \setminus Z_\varepsilon} |H_+(u^\nu) - H_+(u)|\, \mathrm{d}t \leq \varepsilon\,.$$

Moreover using the Cauchy-Schwarz inequality we obtain for any $z \in Z$

$$\begin{aligned} &\left| \int_{B_\varepsilon(z)} \langle \partial_s u^\nu, J^\nu(u^\nu) X^\nu(u^\nu) \rangle \mathrm{d}s\mathrm{d}t \right| \\ &\qquad \leq \left(\int_{B_\varepsilon(z)} |X^\nu|^2\, \mathrm{d}s\mathrm{d}t \int_{B_\varepsilon(z)} |\partial_s u^\nu|^2\, \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{2}} \leq \sqrt{\pi}\varepsilon\, \|X^\nu\|_{C^0} \sup_\nu \sqrt{E(u^\nu)}\,. \end{aligned}$$

We have similar estimates for u^ν, X^ν and J^ν replaced by u, X and J respectively. A plain C^0-estimate gives for every $z \in Z$

$$\begin{aligned} \int_{\{s_0\} \times [0,1] \cap B_\varepsilon(z)} |H_+(u^\nu) - H_+(u)|\, \mathrm{d}t + \int_{\{-s_0\} \times [0,1] \cap B_\varepsilon(z)} |H_-(u^\nu) - H_-(u)|\, \mathrm{d}t& \\ \leq 4\varepsilon \left(\|H_-\|_{C^0} + \|H_+\|_{C^0} \right)&\,. \end{aligned}$$

Putting all together we obtain a constant c independent of ε such that

$$|\Delta(u^\nu) - \Delta(u)| \le c\varepsilon\,,$$

for all $\nu \in \mathbb{N}$ larger than ν_0. This shows the claim. $\square$

Lemma 5.5.3 (action-index relation)**.** *Assume that the pair (L_0, L_1) is τ-monotone. Given Hamiltonians H_-, H_+. Fix connected components $C_- \subset \mathcal{I}_{H_-}(L_0, L_1)$ and $C_+ \subset \mathcal{I}_{H_+}(L_0, L_1)$. Given two maps $u, v : [-1,1] \times [0,1] \to M$ such that $u(\cdot, k), v(\cdot, k) \subset L_k$ for $k = 0, 1$, $x := u(-1, \cdot), x' := v(-1, \cdot) \in C_-$ and $y := u(1, \cdot), y' := v(1, \cdot) \in C_+$ then we have*

$$\begin{aligned}\tau(\mu(u) - \mu(v)) = \int u^*\omega + \int H_+(y)\,\mathrm{d}t - \int H_-(x)\,\mathrm{d}t \\ - \int v^*\omega - \int H_+(y')\,\mathrm{d}t + \int H_-(x')\,\mathrm{d}t\,.\end{aligned}$$

Proof. Let $u_- : [-1,1] \times [0,1] \to M$ and $u_+ : [-1,1] \times [0,1] \to M$ be such that $u_\pm(s, \cdot) \in C_\pm$ for all $s \in [-1, 1]$ and $u_-(-1, \cdot) = x'$, $u_-(1, \cdot) = x$ as well as $u_+(-1, \cdot) = y'$, $u_+(1, \cdot) = y$. The connected sum $u_-\#u\#u_+^\vee\#v^\vee$ defines a map $w : [-1,1] \times [0,1] \to M$ with $w(\cdot, k) \subset L_k$ for $k = 0, 1$ and $w(-1, \cdot) = w(1, \cdot)$. By monotonicity we have $\int w^*\omega = \tau\mu_{\mathrm{Mas}}(w)$. The additivity of the Viterbo index shows

$$\int u_-^*\omega + \int u^*\omega - \int u_+^*\omega - \int v^*\omega = \tau\big(\mu(u_-) + \mu(u) - \mu(u_+) - \mu(v)\big)\,.$$

By the zero axiom the index $\mu(u_-) = \mu(u_+) = 0$. We compute

$$\begin{aligned}\int u_-^*\omega = \int \omega(\partial_s u_-, \partial_t u_-) = \int \omega(\partial_s u_-, X_{H_-}(u_-)) = -\int \partial_s H_-(u_-)\,\mathrm{d}s\mathrm{d}t = \\ = \int H_-(x')\,\mathrm{d}t - \int H_-(x)\,\mathrm{d}t\,.\end{aligned}$$

Similarly for $\int u_+^*\omega = \int H_+(y') - \int H_+(y)$. We conclude by plugging these two equations into the last one. $\square$

Lemma 5.5.4. *With the same assumptions as Theorem 5.1.4. Assume additionally that the pair (L_0, L_1) is monotone. Suppose that $(u_\nu)_{\nu\in\mathbb{N}}$ Floer-Gromov converges modulo bubbling to $(v_1, \ldots, v_k)$ then either all $Z_1, \ldots, Z_k$ are empty or we have for all $\nu \in \mathbb{N}$ large enough*

$$\mu(u_\nu) \ge \sum_{j=1}^{k} \mu(v_j) + N\,, \tag{5.5.2}$$

where N is the minimum of the minimal Maslov numbers of L_0 and L_1.

Proof. Assume that for some $j = 1, \dots, k$ the set Z_j is non-empty. After rescaling and removal of singularities we see that $m_{j,z}$ is the energy of a non-constant holomorphic sphere or disk with boundary on L_0 or L_1. Hence by monotonicity $m_{j,z}/\tau$ is a positive multiple of the minimal Malsov number of L_0 or L_1 and thus $m/\tau \geq N$. Abbreviate $x := v_1(-\infty)$, $y := v_k(\infty)$, $x_\nu := u_\nu(-\infty)$ and $y_\nu := u_\nu(\infty)$. Further abbreviate the connected sum $v = v_1 \# v_2 \# \dots \# v_k$. We have $\mu(v) := \sum_{j=1}^k \mu(v_j)$ and $E(v) := \sum_{j=1}^k E(v_j)$. Let $j_0 = 1, \dots, k$ be the unique index such that $a_{j_0}^\nu = 0$ for all $\nu \in \mathbb{N}$ and hence (u_ν) converges to v_{j_0} in $C^\infty_{\mathrm{loc}}(\Sigma \setminus Z_{j_0})$. By the action-index relation (cf. Lemma 5.5.3) we have

$$
\begin{aligned}
&\tau(\mu(u_\nu) - \mu(v)) \\
&= \int u_\nu^* \omega + \int H_+(y_\nu)\,\mathrm{d}t - \int H_-(x_\nu)\,\mathrm{d}t - \int v^* \omega - \int H_+(y)\,\mathrm{d}t + \int H_-(x)\,\mathrm{d}t \\
&= E(u_\nu) - \Delta(u_\nu) - E(v) + \Delta(v_{j_0}) \to m\,.
\end{aligned}
$$

where we have used Lemma 5.5.2. $\square$

6. Fredholm Theory

We define Banach manifolds and Banach bundles such that the moduli problem of the perturbed Cauchy-Riemann equation becomes the zero set of a Fredholm section. This step is part of the standard program in order to put a smooth structure on the moduli spaces and was pioneered by Floer in [25] under the assumption that the Lagrangians intersect transversely. Frauenfelder constructed the Banach manifolds for the degenerate case of clean intersecting Lagrangians in [33]. Besides recalling these well-known constructions we also give a formula for the index in the degenerate case, which was not done before.

6.1. Banach manifold

Given a compact symplectic manifold (M,ω), two Lagrangian submanifolds $L_0, L_1 \subset M$ and two clean Hamiltonians H_-, H_+ with perturbed intersection points $\mathcal{I}_-$, $\mathcal{I}_+$ respectively (cf. 3.2.12 and (3.2.6)). To construct the Banach manifold we need some auxillary choices. Choose a Riemannian metric on M and denote by $\varepsilon > 0$ its injectivity radius. For two points $p, q \in M$ which are close enough, we denote by $\Pi_p^q : T_pM \to T_qM$ the parallel transport along the unique shortest geodesic joining p to q. More generally if $p, q \in M$ are arbitrary, we define the linear map

$$\widehat{\Pi}_p^q : T_pM \to T_qM, \qquad \widehat{\Pi}_p^q = \beta(\varepsilon^{-1}\mathrm{dist}\,(p,q))\Pi_p^q\,, \tag{6.1.1}$$

in which β is a smooth cut-off function supported in $[0,1]$ and $\beta \equiv 1$ on $[0,1/2]$. For maps $u, v : \Sigma \to M$ we denote $\widehat{\Pi}_u^v : u^*TM \to v^*TM$, $(\widehat{\Pi}_u^v)(z) = \widehat{\Pi}_{u(z)}^{v(z)}$.

Definition 6.1.1. Fix numbers $p > 2$ and $\delta > 0$ we define

$$\mathcal{B}^{1,p;\delta} \subset C^0(\mathbb{R} \times [0,1], M)\,,$$

to be the space of maps u such that

(i) u is of local regularity $H^{1,p}$,

(ii) u satisfies the boundary condition $(u(s,0), u(s,1)) \in L_0 \times L_1$ for all $s \in \mathbb{R}$,

(iii) there exists $x_- := u(-\infty) \in \mathcal{I}_-$ and $x_+ := u(\infty) \in \mathcal{I}_+$ such that

$$\int_{\Sigma_\pm} \Big(\mathrm{dist}\,(u, x_\pm)^p + |\partial_s u|^p + \big|\partial_t u - \widehat{\Pi}_x^u \partial_t x_\pm\big|^p\Big)\, e^{\delta p|s|}\mathrm{d}s\mathrm{d}t < \infty\,, \tag{6.1.2}$$

with $\Sigma_- = (-\infty, 0] \times [0,1]$ and $\Sigma_+ = [0,\infty) \times [0,1]$.

For two subspaces $C_- \subset \mathcal{I}_-$ and $C_+ \subset \mathcal{I}_+$ we denote by $\mathcal{B}^{1,p;\delta}(C_-, C_+) \subset \mathcal{B}^{1,p;\delta}$ the subspace of all u such that $u(-\infty) \in C_-$ and $u(\infty) \in C_+$.

Remark 6.1.2. Since any two metrics on the compact manifold M are equivalent, the space $\mathcal{B}^{1,p;\delta}$ does not depend on the specific choice of the metric. For the construction of the charts we employ a domain dependent metric, which is explained below but for the mere definition of the space it suffices to consider a simple metric on M.

6.1.1. Tangent space

Let ∇ denote the Levi-Civita connection associated to the axillary metric. We show in Lemma 3.2.13 that the tangent spaces of $\mathcal{I}_-$ and $\mathcal{I}_+$ are computed.

Definition 6.1.3. For any $u \in \mathcal{B}^{1,p;\delta}$ we define $T_u\mathcal{B}^{1,p;\delta}$ to be the space of sections ξ of u^*TM such that

(i) ξ is of local regularity $H^{1,p}$,

(ii) ξ satisfies the linearized boundary condition

$$\xi(s,0) \in T_{u(s,0)}L_0, \qquad \xi(s,1) \in T_{u(s,1)}L_1\ ,$$

(iii) there exists vector fields $\xi_- \in T_{x_-}\mathcal{I}_-$ and $\xi_+ \in T_{x_+}\mathcal{I}_+$ such that the following norm is finite

$$\begin{aligned}\|\xi\|_{1,p;\delta} := \Big(&\|\xi_-\|^p_{L^\infty} + \|\xi_+\|^p_{L^\infty} + \\ &+ \int_{\Sigma_-} \Big(\big|\xi - \widehat{\Pi}^u_{x_-}\xi_-\big|^p + \big|\nabla\big(\xi - \widehat{\Pi}^u_{x_-}\xi_-\big)\big|^p\Big)\, e^{\delta p|s|}\mathrm{d}s\mathrm{d}t \\ &+ \int_{\Sigma_+} \Big(\big|\xi - \widehat{\Pi}^u_{x_+}\xi_+\big|^p + \big|\nabla\big(\xi - \widehat{\Pi}^u_{x_+}\xi_+\big)\big|^p\Big)\, e^{\delta p|s|}\mathrm{d}s\mathrm{d}t\Big)^{1/p}.\end{aligned}$$

Furthermore, we define $\mathcal{E}^{p;\delta}_u$ to be the space of all sections $\eta \in \Gamma(u^*TM)$ of local regularity L^p which are bounded in the norm

$$\|\eta\|_{p;\delta} := \left(\int_\Sigma |\eta(s,t)|^p\, e^{\delta p|s|}\mathrm{d}s\mathrm{d}t\right)^{1/p}.$$

If u is smooth we define the spaces $T_u\mathcal{B}^{1,p;\delta}$ and $\mathcal{E}^{p;\delta}_u$ for any constants $p > 1$ and $\delta \in \mathbb{R}$.

Remark 6.1.4. Since M is compact the norms $\|\cdot\|_{1,p;\delta}$ and $\|\cdot\|_{p;\delta}$ with respect to two different connections and metrics are equivalent. Hence $T_u\mathcal{B}^{1,p;\delta}$ is well-defined independently of the choice of the connection. Similarly $\mathcal{E}^{p;\delta}_u$ is well-defined independent of the metric and connection.

Lemma 6.1.5. *If the Hamiltonians H_- and H_+ are clean (cf. Definition 3.2.12) each path-connected component of $\mathcal{B}^{1,p;\delta}$ is a Banach manifold and the vector bundles*

$$TB^{1,p;\delta} := \bigsqcup_{u\in B^{1,p;\delta}} T_u\mathcal{B}^{1,p;\delta}, \qquad \mathcal{E}^{p;\delta} := \bigsqcup_{u\in\mathcal{B}^{1,p;\delta}} \mathcal{E}_u^{p;\delta} ,$$

carry the structure of Banach bundles. Moreover $T\mathcal{B}^{1,p;\delta}$ is the tangent bundle of $\mathcal{B}^{1,p;\delta}$.

Proof. We give a sketch since the proof is basically already given in [33, Section 4.2]. We also refer the reader to [68, Theorem 2.1.7] and [68, Theorem 2.2.1].

In order to construct local charts we choose a metric which depends on the domain, denoted $(g_{s,t})_{(s,t)\in\Sigma}$. For every $(s,t)\in\Sigma$ we obtain a Levi-Civita connection, norm, distance function, exponential map and parallel transport associated to $g_{s,t}$, denoted by $\nabla^{s,t}$, $|\cdot|_{s,t}$, $\mathrm{dist}_{s,t}$, $\exp^{s,t}$ and ${}^{s,t}\Pi$ respectively. Fix metrics g_- (resp. g_+) such that L_0 and $L_1^- := \varphi_{H_-}^{-1}(L_1)$ (resp. L_0 and $L_1^+ := \varphi_{H_+}^{-1}(L_1)$) are totally geodesic (cf. Lemma 6.1.6 to see that such metrics exists). We assume that the family $(g_{s,t})$ satisfies

(a) L_k is totally geodesic with respect to $g_{s,k}$ for $k=0,1$ and all $s\in\bar{\mathbb{R}}$,

(b) $g_{s,t} = (\varphi^t_{H_+})_* g_+$ and $g_{-s,t} = (\varphi^t_{H_-})_* g_-$ for all $s\geq 1$ and $t\in[0,1]$.

Because the family only varies over a compact domain the minimal injectivity radius of $g_{s,t}$ is uniformly bounded from below by a constant $\varepsilon>0$. Given a map $u\in C^0(\Sigma,M)$ and a continuous vector field $\xi\in\Gamma(u^*TM)$ such that $\sup_{s,t}|\xi(s,t)|_{s,t}<\varepsilon$ we define the map $u_\xi\in C^0(\mathbb{R}\times[0,1],M)$ via

$$u_\xi(s,t) := \exp^{s,t}_{u(s,t)}\xi(s,t) \tag{6.1.3}$$

and the *parallel transport map*

$$\Pi_u^{u_\xi} : \Gamma(u^*TM)\to\Gamma(u_\xi^*TM), \qquad \xi'\mapsto \big(\Pi_u^{u_\xi}\xi'\big)(s,t) := {}^{s,t}\Pi_{u(s,t)}^{u_\xi(s,t)}\xi'(s,t)\,. \tag{6.1.4}$$

Let $u\in\mathcal{B}$ be a strip which is smooth and asymptotically constant, i.e. there exists $s_0>0$ such that $u(\pm s,\cdot)=u(\pm\infty)$ for all $s\geq s_0$. Define the subset

$$\mathcal{V}_u := \{\xi\in T_u\mathcal{B}\mid \sup_{s,t}|\xi(s,t)|_{s,t}<\varepsilon\}\subset T_u\mathcal{B}\,.$$

Since $p>2$ and $\delta>0$, it follows by the Sobolev embedding that $\mathcal{V}_u$ is an open subset. We define the chart map by $\exp_u:\mathcal{V}_u\to\mathcal{B}$, $\xi\mapsto u_\xi$.

We explain why $u_\xi\in\mathcal{B}$. By the property (a), the map u_ξ satisfies the boundary condition. By Corollary A.1.2 we have $|\mathrm{d}u_\xi|\leq c(|\mathrm{d}u|+|\nabla\xi|)$ for some uniform constant c, which readily shows that u_ξ is of regularity $H^{1,p}_{\mathrm{loc}}$. To show that the integral (6.1.2) is finite we need a sharper estimate. By symmetry it suffices to consider only the positive end. Abbreviate $v(s,t) := (\varphi^t_{H_+})^{-1}(u(s,t))$ and $\zeta(s,t) := (\mathrm{d}\varphi^t_{H_+})^{-1}(\xi(s,t))$. Let exp denotes the exponential map of the fixed metric g^+ and abbreviate $v_\zeta(s,t) := \exp_{v(s,t)}\zeta(s,t)$. By property (b) we have

$$\exp^{s,t}\circ\mathrm{d}\varphi^t_{H_+} = \varphi^t_{H_+}\circ\exp\,,$$

and thus $u_\xi(s,t) = \varphi^t_{H_+}(v_\zeta(s,t))$ for all $s \geq 1$ and $t \in [0,1]$. Let s_0 be a constant such that $u(s,\cdot) = x_+$ for all $s \geq s_0$. We have $\nabla^{s,t}_s \xi(s,t) = \partial_s \xi(s,t) = \mathrm{d}\varphi^t_{H_+} \partial_s \zeta$. Denote by $|\cdot|_+$ the norm induced by g_+. Estimate for all $s \geq s_0$ and $t \in [0,1]$ using Corollary A.1.2

$$|\partial_s u_\xi(s,t)|_t = \left|\mathrm{d}\varphi^t_{H_+} \partial_s v_\zeta\right|_t = |\partial_s v_\zeta|_+ \leq c\,|\partial_s \zeta|_+ = c\,|\partial_s \xi|_t = c\,|\partial_s(\xi - \xi_+)|_t \,.$$

Moreover we have $\partial_t u_\xi = X_{H_+}(u_\xi) + \mathrm{d}\varphi^t\,\partial_t v_\zeta$ and $\nabla^{s,t}_t \xi = \nabla^{s,t}_\xi X_{H_+} + \mathrm{d}\varphi^t \partial_t \zeta$. Thus

$$\begin{aligned}\left|\partial_t u_\xi - X_{H_+}(u_\xi)\right|_t &= |\partial_t v_\zeta|_+ \\ &\leq c\,|\partial_t \zeta|_+ = c\,|\nabla_t \xi - \nabla_\xi X_H|_t = c\left|\nabla_t \xi - \nabla_{(\xi-\xi_+)} X_H - \nabla_t \xi_+\right| \\ &\leq c(1 + \|X_H\|)(|\nabla_t(\xi - \xi_+)|_t + |\xi - \xi_+|_t)\,.\end{aligned}$$

Now use these two stronger estimates to show that the integral is bounded for u_ξ. Now one shows that the collection of all $(\mathcal{V}_u, \exp_u)$ indexed over all smooth and asymptotically constant curves u gives an atlas of $\mathcal{B}$. This completes the proof of the first statement.

We construct local trivializations of the bundles over the images of our chart maps given by $\mathcal{U}_u := \{u' \in \mathcal{B} \mid \sup_{s,t\in\Sigma} \mathrm{dist}_{s,t}(u(s,t), u'(s,t)) < \varepsilon\}$ where again u is smooth an asymptotically constant. The trivializations are defined using (6.1.4)

$$\begin{aligned}&\mathcal{U}_u \times T_u\mathcal{B} \to T\mathcal{B}\big|_{\mathcal{U}_u}, \qquad (u_\xi, \xi') \mapsto \Pi^{u_\xi}_u \xi' \in T_{u_\xi}\mathcal{B}\,,\\ &\mathcal{U}_u \times \mathcal{E}^{p,\delta}_u \to \mathcal{E}^{p;\delta}\big|_{\mathcal{U}_u}, \qquad (u_\xi, \eta) \mapsto \Pi^{u_\xi}_u \eta \in \mathcal{E}^{p;\delta}_{u_\xi}\,,\end{aligned}$$

where for the second map we actually use the unique continuous extension of the densely defined operator $\Pi^{u_\xi}_u : \mathcal{E}^{p,\delta}_u \to \mathcal{E}^{p;\delta}_{u_\xi}$. It is again straight-forward to check that the trivialization change is smooth using the estimates from Section A.1. □

Lemma 6.1.6. *Let M be a manifold and $L_0, L_1 \subset M$ be two submanifolds in clean intersection. There exists a metric on M such that L_0 and L_1 are totally geodesic. Moreover given a submanifold $W \subset L_0 \cap L_1$, then there exists a metric such that W, L_0 and L_1 are totally geodesic.*

Proof. We construct the metric in suitable charts and patch it together at the end. For any point $p \in L_0 \cap L_1$ we find a chart identifying a neighborhood of p with a ball in $\mathbb{R}^{2n}$ such that p is identified with zero, L_0 is identified with the vector space $V_0 \subset \mathbb{R}^{2n}$ and L_1 is a graph over the vector space $V_1 \subset \mathbb{R}^{2n}$ of a function with vanishing differential at 0. Since L_0 and L_1 intersect cleanly the intersection $L_0 \cap L_1$ is a graph over $K := V_0 \cap V_1$. Decompose $\mathbb{R}^{2n} = K \oplus V_0' \oplus V_1' \oplus R$ such that $K \oplus V_0' = V_0$ and $K \oplus V_1' = V_1$. In the decomposition a point in L_1 has coordinates $(x, \varphi(x,y), y, \psi(x,y))$ for functions $\varphi : V_1 \to V_0'$ and $\psi : V_1 \to R$ with vanishing differentials at 0 and the property that $\psi(x,0) = 0$ for all $x \in K$. Consider the map

$$\begin{aligned}\Phi : K \oplus V_0' \oplus V_1' \oplus R &\to K \oplus V_0' \oplus V_1' \oplus R\\ (x, x', y, y') &\mapsto (x, x' - \varphi(x,y), y, y' - \psi(x,y))\,.\end{aligned}$$

The differential of Φ at 0 is the identity, hence by possibly making the ball smaller we assume that Φ is a diffeomorphism. By construction $\Phi(L_0) = \Phi(V_0) = V_0$ and $\Phi(L_1) = V_1$. Hence composing the chart map with Φ we have found a chart such that L_0 and L_1 are identified with the vector spaces V_0 and V_1 respectively. Now take any metric on $V_0 \cap V_1$ such that W is totally geodesic and extend it over the chart such that V_0 and V_1 are totally geodesic. □

6.1.2. Cauchy-Riemann-Floer operator

We fix a vector field $X \in C^\infty(\Sigma, \mathrm{Vect}(M))$ and an almost complex structure $J \in C^\infty(\Sigma, \mathrm{End}(TM, \omega))$ which are admissible in the sense of Definition 5.1.1. We define the *non-linear Cauchy-Riemann-Floer operator*

$$\overline{\partial}_{J,X} : C^\infty(\Sigma, M) \to C^\infty(\Sigma, TM), \qquad u \mapsto \partial_s u + J(u)\left(\partial_t u - X(u)\right) .$$

Let $T^\varepsilon M$ denote the disk-bundle of vectors ξ with norm bounded by ε. For $\varepsilon > 0$ small enough we define the local representative of $\overline{\partial}_{J,X}$ at a given $u : \Sigma \to M$ by $\mathcal{F}_u : C^\infty(\Sigma, T^\varepsilon M) \to C^\infty(\Sigma, TM)$ with

$$\mathcal{F}_u(\xi) = \Pi^u_{u_\xi} \left(\partial_s u_\xi + J(u_\xi)\big(\partial_t u_\xi - X(u_\xi)\big)\right) . \tag{6.1.5}$$

Here $u_\xi = \exp_u \xi$ and $\Pi^u_{u_\xi}$ are given by (6.1.3) and (6.1.4) respectively. The *linearized Cauchy-Riemann-Floer operator* as the differential of the map (6.1.5) at zero, which is given by (cf. [53, Prop. 3.1.1])

$$D_u\xi = \nabla_s\xi + J(u)\big(\nabla_t\xi - \nabla_\xi X(u)\big) + \big(\nabla_\xi J(u)\big)\left(\partial_t u - X(u)\right) . \tag{6.1.6}$$

Remark 6.1.7. If u is satisfies $\overline{\partial}_{J,X} u = 0$, the operator D_u is defined independently of the choice of ∇ and is the vertical differential of $\overline{\partial}_{J,X}$ at u.

Definition 6.1.8. Given a constant $\mu > 0$. A smooth map $u : \Sigma \to M$ has *μ-decay* if there exists constants c, s_0 and a smooth paths $x_-, x_+ : [0,1] \to M$ such that for all $s \geq s_0$ and $t \in [0,1]$ we have

$$\mathrm{dist}\left(u(s,t), x_+(t)\right) + \|\partial_s u\|_{C^1(\Sigma_s^\infty)} + \left\|\partial_t u - \Pi^u_{x_+}\partial_t x_+\right\|_{C^1(\Sigma_s^\infty)} \leq c\, e^{-\mu s} , \tag{6.1.7}$$

with $\Sigma_s^\infty := [s,\infty) \times [0,1]$ and a similar estimate for the negative end. We write $C^{\infty;\mu}(\Sigma, M)$ for the space of all maps with μ-decay.

The next proposition states that the space $\mathcal{B}^{1,p;\delta}$ contains all finite energy (J,X)-holomorphic strips (cf. equation (5.1.1)), provided that δ is sufficiently small. The constants $\iota(J_-, H_-)$ and $\iota(J_+, H_+)$ are defined in equation (3.2.13).

Proposition 6.1.9. *Set $\iota := \min\{\iota(J_-, H_-), \iota(J_+, H_+)\}$ with $J_\pm := J(\pm s, \cdot)$ and $H_\pm$ given by $X_{H_\pm} = X(\pm s, \cdot)$ for some s large enough. For any $\mu < \iota$ we have that all (J,X)-holomorphic strips have μ-decay. In particular $u \in \mathcal{B}^{1,p;\delta}$ for any $\delta < \iota$.*

Proof. Define the map $v : \Sigma_0^\infty \to M$, $(s,t) \mapsto (\varphi^t_{H_+})^{-1}(u(s,t))$. The map has boundary in (L_0, L_1') with $L_1' = \varphi^{-1}_{H_+}(L_1)$. Since by Lemma 3.2.4 the map v is J'-holomorphic with $J'_t := (\varphi^t_{H_+})^*(J_+)_t$ for $t \in [0,1]$, and L_0 and L_1' are in clean intersection by assumption, we conclude that $v(s,\cdot)$ converges to an intersection point $p \in L_0 \cap L_1'$ as $s \to \infty$ (cf. Theorem 4.1.1). By Lemma 3.2.8 we have $\iota(J_+, H_+) = \iota(J')$. Hence $\mu < \iota_p(J')$ and we conclude that $\|\mathrm{d}v\|_{C^1(\Sigma_s^\infty)} \leq O(e^{-\mu s})$. Moreover the path $[s,\infty) \to M$, $\sigma \to v(\sigma,t)$ extends to a continuous path from $v(s,t)$ to p, hence

$$\operatorname{dist}(v(s,t),p) \leq \int_s^\infty |\partial_\sigma v(\sigma,t)| \mathrm{d}\sigma \leq O(1) \int_s^\infty e^{-\mu\sigma} \mathrm{d}\sigma \leq O(e^{-\mu s}) \,.$$

By construction $\partial_s u = \mathrm{d}\varphi^t_H \, \partial_s v$. Set $x : [0,1] \to M$, $t \mapsto \varphi^t_H(p)$. Since $\partial_t x = X_H(x)$ we also have

$$\partial_t u - \Pi^u_x \partial_t x = \mathrm{d}\varphi^t_H \partial_t v + X_H(u) - \Pi^u_x X_H(x) \,.$$

Using these identities and the estimates for v we conclude that u has μ-decay for the positive end (cf. estimate (6.1.7)). We proceed similarly for the negative end. Now since u has μ-decay on both ends we conclude that the integral (6.1.2) in the definition of $\mathcal{B}^{1,p;\delta}$ is finite. □

Theorem 6.1.10. *With $\iota > 0$ as in Proposition 6.1.9. Choose constants δ and μ such that $\delta < \mu < \iota$. For any smooth map $u \in \mathcal{B}^{1,p;\delta}$ with μ-decay the linearized Cauchy-Riemann operator D_u defined in* (6.1.6) *extends to a bounded Fredholm operator $D_u : T_u\mathcal{B}^{1,p;\delta} \to \mathcal{E}^{1,p;\delta}_u$ of index*

$$\operatorname{ind} D_u = \mu_{\mathrm{Vit}}(u) + \frac{1}{2}\left(\dim C_- + \dim C_+\right) \,, \tag{6.1.8}$$

in which $C_- \subset \mathcal{I}_-$ and $C_+ \subset \mathcal{I}_+$ are connected components such that $u(-\infty) \in C_-$ and $u(\infty) \in C_+$.

Proof. We describe how to conjugate the operator D_u to an operator $D = \partial_s + J_{\mathrm{std}}\partial_t + S$ as considered in section 6.2. The statements then follow from the fact that D is Fredholm proven in Lemma 6.2.4 and the index formula.

In a first step we construct trivializations of u^*TM. Let s_0 be such that $X(s,t) = X_{H_+}(t)$ for all $s \geq s_0$. The map $v : [s_0,\infty) \times [0,1] \to M$, $v(s,t) = (\varphi^t_{H_+})^{-1}(u(s,t))$ is a J'-holomorphic half-strip where $J' = (\varphi_{H_+})^*J$ and with boundary condition $v(s,0) \in L_0$ and $v(s,1) \in L_1' := (\varphi_{H_+})^{-1}(L_1)$ for all $s \geq s_0$ (cf. Lemma 3.2.4). By asymptotic analysis the point $v(s,t)$ lies inside a suitable neighborhood of $p_+ = v(\infty) \in L_0 \cap L_1'$ for all $t \in [0,1]$ and s sufficiently large (cf. Theorem 4.1.1). Let Φ_+ be the trivialization constructed in Lemma 3.2.11 with respect to J', L_0 and L_1. Then define $\Phi_u(s,t) = \mathrm{d}\varphi^t_{H_+} \circ \Phi_+(t, v(s,t)) : \mathbb{R}^{2n} \to T_{u(s,t)}M$. We end up with a trivialization Φ_u of $u^*TM|_{[s_0,\infty)\times[0,1]}$ which is

- symplectic, i.e. $\omega_{u(s,t)}(\Phi_u(s,t)\xi, \Phi_u(s,t)\xi') = \omega_{\mathrm{std}}(\xi,\xi')$ for all $s \geq s_0$, $t \in [0,1]$ and $\xi, \xi' \in \mathbb{R}^{2n}$,

- complex linear, i.e. $J_t(u(s,t))\Phi_u(s,t)\xi = \Phi_u(s,t)J_{\mathrm{std}}\xi$ for all $s \geq s_0$, $t \in [0,1]$ and $\xi \in \mathbb{R}^{2n}$,
- and trivializes the Lagrangians, i.e. $T_{u(s,k)}L_k = \Phi_u(s,k)\mathbb{R}^n$ for all $k = 0,1$ and $s \geq s_0$.

Similarly we construct Φ_u over $(-\infty,-s_0]\times[0,1]$. Then we extend Φ_u over $[-s_0,s_0]\times[0,1]$ such that it is symplectic and complex linear (satisfies the first two properties) but not necessarily trivializes the Lagrangians. In fact a trivialization which satisfies all three properties might not exist. We define the matrix valued function $S : \mathbb{R}\times[0,1] \to \mathbb{R}^{2n\times 2n}$ by

$$\Phi_u\left(\partial_s\xi + J_{\mathrm{std}}\partial_t\xi + S\xi\right) = D_u\Phi_u\xi \ ,$$

for all smooth $\xi : \mathbb{R}\times[0,1] \to \mathbb{R}^{2n}$. Set $x_- = u(-\infty,\cdot)$ and $x_+ = u(\infty,\cdot)$. We also define $\Phi_{x_+}(t) = \mathrm{d}\varphi^t_{H_+}\Phi_+(t,x(t)) : \mathbb{R}^{2n} \to T_{x_+(t)}M$ and similarly Φ_{x_-}. The matrix is asymptotic to the paths $\sigma_-,\sigma_+ : [0,1] \to \mathbb{R}^{2n\times 2n}$ given by

$$\Phi_{x_\pm}\left(J_{\mathrm{std}}\partial_t\xi + \sigma_\pm\xi\right) = J(x_\pm)(\nabla_t - \nabla X_{H_\pm}(x_\pm))\Phi_{x_\pm}\xi \, ,$$

for all smooth $\xi : [0,1] \to \mathbb{R}^{2n}$. By assumption u has μ-decay. We conclude that $S(-s,\cdot)$ converges to σ_- and $S(s,\cdot)$ to σ_+ as s tends to ∞ and moreover that S has μ-decay (cf. Lemma 4.4.1). We define the paths of linear Lagrangians $F = (F_0,F_1) : \mathbb{R} \to \mathcal{L}(n)\times\mathcal{L}(n)$ via

$$F_0(s) = \Phi_u(s,0)^{-1}T_{u(s,0)}L_0, \qquad F_1(s) = \Phi_u(s,1)^{-1}T_{u(s,1)}L_1 \ .$$

By construction the path of Lagrangians $F = (F_0,F_1)$ is asymptotically constant. In particular the pair (F,S) is admissible in the sense of Definition 6.2.1 and has the asymptotic operators $A_- = A_{\sigma_-}$ and $A_+ = A_{\sigma_+}$. With notation from Section 6.2 (in particular Definition (6.2.2)) we have isomorphisms of Banach spaces

$$H^{1,p;\delta}_{F,W}(\Sigma,\mathbb{R}^{2n}) \xrightarrow{\cong} T_u\mathcal{B}^{1,p;\delta}(C_-,C_+), \qquad L^{p;\delta}(\Sigma,\mathbb{R}^{2n}) \xrightarrow{\cong} \mathcal{E}^{p;\delta}_u \ ,$$

both given by $\xi \mapsto ((s,t) \mapsto \Phi_u(s,t)\xi(s,t))$ where $W = (\ker A_-, \ker A_+)$. Via these isomorphism the operator D_u is conjugated to the operator

$$D : H^{1,p;\delta}_{F,W}(\Sigma,\mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma,\mathbb{R}^{2n}), \qquad \xi \mapsto \partial_s\xi + J_{\mathrm{std}}\partial_t\xi + S\xi \ .$$

The asymptotic operators A_- and A_+ are conjugated to the Hessians A_{x_-} and A_{x_+} respectively, in particular have the same spectrum. The claim that D and hence also D_u is well-defined and Fredholm now follows by Lemma 6.2.5 using the fact that S has μ-decay. It remains to check the index formula. By Lemma 6.2.6 the index of D is given by

$$\mu(\Psi_+\mathbb{R}^n,\mathbb{R}^n) + \mu(F_0,F_1) - \mu(\Psi_-\mathbb{R}^n,\mathbb{R}^n) + \frac{1}{2}\dim\ker A_- + \frac{1}{2}\dim\ker A_+$$

where $\Psi_\pm : [0,1] \to Sp(2n)$ are the fundamental solutions for $\sigma_\pm$ (cf. equation (6.2.10)). We claim that for all $t \in [0,1]$

$$\Psi_\pm(t) = \Phi_\pm(t)^{-1}\mathrm{d}\varphi^t_{H_\pm}\Phi_\pm(0) \, , \tag{6.1.9}$$

in which $t \mapsto \varphi^t_{H_\pm}$ denotes the Hamiltonian flow. Denote by $e_1, \ldots, e_{2n}$ the standard basis of $\mathbb{R}^{2n}$ and for each $i = 1, \ldots, 2n$ and $t \in [0,1]$ we define the vector

$$\xi_i(t) := \Phi_\pm(t)^{-1} \mathrm{d}\varphi^1_{H_\pm} \Phi_\pm(0) e_i \in \mathbb{R}^{2n} .$$

By definition of $\sigma_\pm$ we have using the fact that ∇ is torsion-free and $\partial_t x_\pm = X_{H_\pm}$

$$\Phi_\pm\big(J_{\mathrm{std}} \partial_t \xi_i + \sigma_\pm \xi_i\big) = J(x_\pm)\big(\nabla_t \mathrm{d}\varphi_{H_\pm} \Phi_\pm(0) e_i + \nabla_{\mathrm{d}\varphi_{H_\pm} \Phi_\pm(0) e_i} X_{H_\pm}\big) = 0 .$$

We see that the function $\xi_i(t)$ satisfies the same ordinary differential equation as $\Psi_\pm(t) e_i$. This implies (6.1.9) and so

$$\Psi_\pm(t) \mathbb{R}^n = \Phi_\pm(t) \mathrm{d}\varphi^t_{H_\pm} T_{x_\pm(0)} L_0 .$$

By definition of the Viterbo index we conclude

$$\mu(\Psi_+ \mathbb{R}^n, \mathbb{R}^n) + \mu(F_0, F_1) - \mu(\Psi_- \mathbb{R}^n, \mathbb{R}^n) = \mu_{\mathrm{Vit}}(u) .$$

Since the asymptotic operators $A_\pm$ are conjugated to the Hessian at $x_\pm$ whose kernel is given by $T_{x_\pm} C_\pm$ by Lemma 3.2.13, we obtain

$$\dim \ker A_\pm = \dim C_\pm .$$

Obviously $\operatorname{ind} D = \operatorname{ind} D_u$ and the last two equations plugged into the index formula for D gives the result. $\square$

6.2. Linear Theory

Denote by $\mathcal{L}(n)$ the space of linear Lagrangian subspaces in $\big(\mathbb{R}^{2n}, \omega_{\mathrm{std}}\big)$ and abbreviate $\Sigma = \mathbb{R} \times [0,1]$. Fix smooth maps $F : \mathbb{R} \to \mathcal{L}(n) \times \mathcal{L}(n)$ and $S : \Sigma \to \mathbb{R}^{2n \times 2n}$. In this section we study a differential operator

$$D\xi = \partial_s \xi + J_{\mathrm{std}} \partial_t \xi + S \cdot \xi ,$$

defined on some Banach space of maps $\xi : \Sigma \to \mathbb{R}^{2n}$ satisfying the boundary conditions

$$(\xi(s,0), \xi(s,1)) \in F(s), \qquad \forall\, s \in \mathbb{R} . \tag{6.2.1}$$

The precise definition of the Banach space is given below. We proof that D is a Fredholm operator and compute its index. We now give more details.

Definition 6.2.1. Let (F, S) be a pair of smooth maps $F : \mathbb{R} \to \mathcal{L}(n) \times \mathcal{L}(n)$ and $S : \Sigma \to \mathbb{R}^{2n \times 2n}$ (F, S). We call (F, S) *admissible* if

(i) F is asymptotically constant, i.e. there exists $s_0 > 0$ and $\Lambda_-, \Lambda_+ \in \mathcal{L}(n) \times \mathcal{L}(n)$ such that $F(-s) = \Lambda_-$ and $F(s) = \Lambda_+$ for all $s \geq s_0$ and

(ii) there exists paths of symmetric matrices $\sigma_-, \sigma_+ : [0,1] \to \mathrm{Sym}(2n) \subset \mathbb{R}^{2n\times 2n}$ such that $\lim_{s\to\pm\infty} S(s,\cdot) = \sigma_\pm$ uniformly.

Fix constants $\delta \in \mathbb{R}$, $p > 1$ and a pair of finite dimensional subspaces $W_-, W_+ \subset L^p([0,1],\mathbb{R}^{2n})$. We define the Banach space

$$H^{1,p;\delta}_{F,W}(\Sigma,\mathbb{R}^{2n}) \subset H^{1,p}_{\mathrm{loc}}(\Sigma,\mathbb{R}^{2n})\,, \tag{6.2.2}$$

as the space of all functions $\xi : \Sigma \to \mathbb{R}^{2n}$ such that

(i) ξ is of regularity $H^{1,p}_{\mathrm{loc}}$,

(ii) ξ satisfies the boundary condition (6.2.1),

(iii) there exists $\xi_- \in W_-$ and $\xi_+ \in W_+$ such that the following norm is bounded

$$\|\xi\|_{1,p;\delta} := \|\xi_-\|_{L^p} + \|\xi_+\|_{L^p} + \|(\xi-\xi_-)\kappa_\delta\|_{H^{1,p}(\Sigma_-)} + \|(\xi-\xi_+)\kappa_\delta\|_{H^{1,p}(\Sigma_+)}\ ,$$

with weight-function $\kappa_\delta(s) = e^{\delta|s|}$.

Secondly we define $L^{p;\delta}(\Sigma,\mathbb{R}^{2n})$ as the Banach space of all $\eta \in L^p_{\mathrm{loc}}(\Sigma,\mathbb{R}^{2n})$ which are bounded with respect to the norm

$$\|\eta\|_{p;\delta} := \|\eta\kappa_\delta\|_{L^p(\Sigma)}$$

If $W_- = 0$ and $W_+ = 0$ are trivial spaces, we abbreviate (6.2.2) by $H^{1,p;\delta}_F(\Sigma,\mathbb{R}^{2n})$. If moreover the weight $\delta = 0$ vanishes we abbreviate the space by $H^{1,p}_F(\Sigma,\mathbb{R}^{2n})$. We are ready to give a precise definition of the operator under consideration

$$D : H^{1,p;\delta}_{F,W}(\Sigma,\mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma,\mathbb{R}^{2n}), \qquad \xi \mapsto \partial_s\xi + J_{\mathrm{std}}\partial_t\xi + S\cdot\xi\,. \tag{6.2.3}$$

Secondly we define the *asymptotic operators.* To a pair of Lagrangians $\Lambda \in \mathcal{L}(n)\times\mathcal{L}(n)$ and a path $\sigma : [0,1] \to \mathbb{R}^{2n\times 2n}$ of symmetric matrices, we associate the Banach space

$$H^{1,2}_\Lambda([0,1],\mathbb{R}^{2n}) := \{\xi \in H^{1,2}([0,1],\mathbb{R}^{2n}) \mid (\xi(0),\xi(1)) \in \Lambda\}\,,$$

and the operator

$$A : H^{1,2}_\Lambda([0,1],\mathbb{R}^{2n}) \to L^2([0,1],\mathbb{R}^{2n}), \qquad \xi \mapsto J_{\mathrm{std}}\partial_t\xi + \sigma\cdot\xi\,. \tag{6.2.4}$$

Let (F,S) be an admissible pair such that $\Lambda_\pm = F(\pm\infty)$ and $\sigma_\pm = S(\pm\infty)$. We define the *asymptotic operators for* (F,S) as the operators A_- and A_+ given by

$$A_\pm = J_{\mathrm{std}}\partial_t + \sigma_\pm : H^{1,2}_{\Lambda_\pm}([0,1],\mathbb{R}^{2n}) \to L^2([0,1],\mathbb{R}^{2n})\,. \tag{6.2.5}$$

The following lemma is crucial for the study of the operator D.

Lemma 6.2.2. *The operator A given in* (6.2.4) *is Fredholm of index zero. Considered as an unbounded operator with dense domain acting in the Hilbert space $L^2([0,1],\mathbb{R}^{2n})$ the operator A is self-adjoint with spectrum consisting only of eigenvalues.*

Proof. We have the estimate

$$\|\xi\|_{1,2} \le c(\|A\xi\|_2 + \|\xi\|_2)\,.$$

for some constant c, which shows that A is a semi-Fredholm operator. That A is self-adjoint is proved in [33, Lmm. 4.3]. Consequently $\operatorname{coker} A = \ker A^* = \ker A$ is finite dimensional, which implies that A is Fredholm of index zero. $\square$

Formal adjoint Given an admissible pair (F,S) and some $q>1$. We define the operator $D^*_{F,S} : H^{1,q}_F(\Sigma,\mathbb{R}^{2n}) \to L^q(\Sigma,\mathbb{R}^{2n})$ by

$$(D^*_{F,S}\xi)(s,t) = -\partial_s\xi(s,t) + J_{\mathrm{std}}\partial_t\xi(s,t) + S^T(s,t)\xi(s,t)\,. \tag{6.2.6}$$

The next lemma states that the operators $D_{F,S}$ and $D^*_{F,S}$ are formally adjoint.

Lemma 6.2.3. *Given an admissible pair (F,S). Assume that $1 = 1/p + 1/q$ and consider the operators $D = D_{F,S}$ and $D^* = D^*_{F,S}$. For all $\xi \in H^{1,p}_F(\Sigma,\mathbb{R}^{2n})$ and $\eta \in H^{1,q}_F(\Sigma,\mathbb{R}^{2n})$ we have*

$$\int_\Sigma \langle D\xi,\eta\rangle \mathrm{d}s\mathrm{d}t = \int_\Sigma \langle \xi, D^*\eta\rangle \mathrm{d}s\mathrm{d}t\,. \tag{6.2.7}$$

Proof. By partial integration we have with $s\in\mathbb{R}$ fixed

$$\int_0^1 \langle \xi, J_{\mathrm{std}}\partial_t\eta\rangle \mathrm{d}t = \int_0^1 \langle J_{\mathrm{std}}\partial_t\xi,\eta\rangle \mathrm{d}t\ ,$$

because $\langle J_{\mathrm{std}}\xi,\eta\rangle = \omega_{\mathrm{std}}(\xi,\eta)$ vanishes for $t=0,1$ after the Lagrangian boundary condition. Again by partial integration for ∂_s and the fact that $\|\xi(s,\cdot)\|_{L^2}$ and $\|\eta(s,\cdot)\|_{L^2}$ vanish as s tends to $\pm\infty$,

$$\int_\Sigma \langle \partial_s\xi,\eta\rangle \mathrm{d}s\mathrm{d}t = \int_\Sigma \langle \xi,\partial_s\eta\rangle \mathrm{d}s\mathrm{d}t\ .$$

We compute

$$\begin{aligned}\langle \xi, D^*\eta\rangle_{L^2} &= -\langle \xi,\partial_s\eta\rangle_{L^2} + \langle \xi, J_{\mathrm{std}}\partial_t\eta\rangle_{L^2} + \langle \xi, S^T\eta\rangle_{L^2}\\ &= \langle \partial_s\xi,\eta\rangle_{L^2} + \langle J_{\mathrm{std}}\partial_t\xi,\eta\rangle_{L^2} + \langle S\xi,\eta\rangle_{L^2} = \langle D\xi,\eta\rangle_{L^2}\ ,\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the inner product on $L^2(\Sigma,\mathbb{R}^{2n})$. $\square$

6.2.1. Fredholm property

To show that the operator given in (6.2.3) is Fredholm, we follow [66, Section 2] and [68, Section 3]. In these two sources the authors considers the perturbed Cauchy-Riemann operator defined on the cylinder instead of the strip with boundary values. However the proofs go through with almost no change.

Lemma 6.2.4. *Let (F, S) be an admissible pair with asymptotic operators A_- and A_+. If $-\delta$ and δ is not a spectral value of A_- and A_+ respectively then the operator*

$$D = \partial_s + J_{\mathrm{std}}\partial_t + S : H^{1,p;\delta}_F(\Sigma, \mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma, \mathbb{R}^{2n})\,,$$

is Fredholm.

Proof. Assume first that $\delta = 0$, i.e. the operators A_- and A_+ are invertible. Then proof is completely analogous to the proof of [66, Thm. 2.2]. Note that in order to prove the precursors [66, Lmm. 2.4] all is necessary that the operators A_- and A_+ are invertible self-adjoint operators. This fact is established in Lemma 6.2.2. To prove the statement for $\delta \neq 0$, pick a smooth function $\kappa_\delta : \mathbb{R} \to \mathbb{R}$ such that $\kappa_\delta(s) = e^{\delta|s|}$ for all $|s| \geq 1$. We have isomorphisms

$$H^{1,p;\delta}_F(\Sigma, \mathbb{R}^{2n}) \xrightarrow{\cong} H^{1,p}_F(\Sigma, \mathbb{R}^{2n}), \qquad L^{p;\delta}(\Sigma, \mathbb{R}^{2n}) \xrightarrow{\cong} L^p(\Sigma, \mathbb{R}^{2n})\,,$$

both given by sending ξ to the function $\kappa_\delta(s)\xi(s,t)$. Conjugating the operator D with these isomorphisms gives the operator $D^\delta = D - (\partial_s\kappa_\delta)/\kappa_\delta$, which has asymptotic operators $A_- + \delta$ and $A_+ - \delta$ respectively. By assumption these are invertible and thus by our first remark D^δ is Fredholm, hence D is Fredholm too. □

Lemma 6.2.5. *With the same assumptions as Lemma 6.2.4 suppose additionally that S has μ-decay for some $\mu > \delta$. Set $W = (\ker A_-, \ker A_+)$, then the operator*

$$D = \partial_s + J_{\mathrm{std}}\partial_t + S : H^{1,p;\delta}_{F;W}(\Sigma, \mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma, \mathbb{R}^{2n})\,, \tag{6.2.8}$$

is bounded and Fredholm.

Proof. Given $\xi \in H^{1,p;\delta}_{F;W}(\Sigma, \mathbb{R}^{2n})$ with limits $\xi_\pm = \xi(\pm\infty) \in \ker A_\pm$. By assumption $J_{\mathrm{std}}\partial_t\xi_+ + \sigma_+\xi_+ = 0$ and $\partial_s\xi_+ \equiv 0$. Hence with the decay property of S we conclude that for all s large enough

$$\begin{aligned} |(\partial_s + J_{\mathrm{std}}\partial_t + S)\xi| &\leq |(\partial_s + J_{\mathrm{std}}\partial_t + S)(\xi - \xi_+)| + |(S - \sigma_+)\xi_+| \\ &\leq |\mathrm{d}(\xi - \xi_+)| + O(1)|\xi - \xi_+| + O(e^{-\mu s})|\xi_+|\,. \end{aligned} \tag{6.2.9}$$

We have a similar estimate for the negative end. If we multiply with $e^{\delta|s|}$ and integrate over Σ we obtain

$$\|D\xi\|^p_{p;\delta} \leq O(1)\,\|\xi\|_{1,p;\delta} + O(1)\int_0^\infty e^{-(\mu-\delta)s}\,ds)\,\|\xi\|_{1,p;\delta}\;.$$

We conclude that the operator D is bounded. Restricted to the finite co-dimensional subspace

$$H^{1,p;\delta}_F(\Sigma, \mathbb{R}^{2n}) \subset H^{1,p;\delta}_{F;W}(\Sigma, \mathbb{R}^{2n})$$

the operator D is Fredholm by Lemma 6.2.4. This shows the claim. □

6.2.2. Index

In this section we compute the index of the Fredholm operator D in terms of the Robbin-Salamon index for paths. This index is defined in [63] and corresponds to the spectral flow of the family of self-adjoint operators $A(s)$ as given in (6.2.4). We give here a quick introduction and review the basic properties.

Let $F = (F_0, F_1) : [a, b] \to \mathcal{L}(n) \times \mathcal{L}(n)$ be a path of pairs of linear Lagrangian spaces. Assume that $F_1(s) = \Lambda$ is constant, we define the *crossing form* $\Gamma(F_0, \Lambda; s)$ as a quadratic form on $F_0(s) \cap \Lambda$ given by

$$\Gamma(F_0, \Lambda; s)v := \left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=s} \omega_{\mathrm{std}}(v, w(\sigma)) ,$$

where $v \in F_0(s) \cap \Lambda$ and $w : (s - \varepsilon, s + \varepsilon) \to J_{\mathrm{std}}F(s)$ is any differentiable map such that $w(\sigma) + v \in F(\sigma)$ for all $\sigma \in (s - \varepsilon, s + \varepsilon)$. The proof that $\Gamma(F_0, \Lambda; s)$ is well-defined is given in [63, Theorem 1.1 (1)]. In the case when F_1 is not constant we define the quadratic form on $F_0(s) \cap F_1(s)$ via

$$\Gamma(F_0, F_1; s) := \Gamma(F_0, F_1(s); s) - \Gamma(F_1, F_0(s); s) .$$

A *crossing* $s \in [a, b]$ is a time where $F_0(s) \cap F_1(s)$ is non-trivial. A crossing is called *regular* if $\Gamma(F_0, F_1; s)$ is non-degenerate. If $F = (F_0, F_1)$ has only regular crossings the *Robbin-Salamon index* of F is defined by

$$\mu(F_0, F_1) := \frac{1}{2} \operatorname{sign} \Gamma(F_0, F_1; a) + \sum_{a<s<b} \operatorname{sign} \Gamma(F_0, F_1; s) + \frac{1}{2} \operatorname{sign} \Gamma(F_0, F_1; b) ,$$

where sign denotes the signature, that is the number of positive eigenvalues minus the number of negative eigenvalues. The sum is finite because regular crossings are isolated. The Robbin-Salamon index for an arbitrary path $F = (F_0, F_1)$ is defined by the index of a perturbation that fixes the endpoints and has only regular crossings. As proven in [63, Theorem 2.3] the index enjoys the following properties.

Naturality For any path $\Psi : [a, b] \to Sp(2n)$ we have

$$\mu(\Psi F_0, \Psi F_1) = \mu(F_0, F_1) .$$

Homotopy The Robbin-Salamon index is invariant under homotopies which fix the endpoints.

Zero If $F_0(s) \cap F_1(s)$ is of constant dimension for all $s \in [a, b]$, then $\mu(F_0, F_1) = 0$.

Direct sum If $F = F' \oplus F''$, then

$$\mu(F_0' \oplus F_0'', F_1' \oplus F_1'') = \mu(F_0', F_1') + \mu(F_0'', F_1'') .$$

Concatenation If $F = F' \# F''$, then

$$\mu(F_0, F_1) = \mu(F_0', F_1') + \mu(F_0'', F_1'') .$$

Localization If $F_0(s) = \mathbb{R}^n \times \{0\}$ and $F_1(s) = \mathrm{Gr}(B(s))$ for a path $B : [a,b] \to \mathbb{R}^{n\times n}$ of symmetric matrices, then the Robbin-Salamon index is given by

$$\mu(F_0, F_1) = \frac{1}{2}\operatorname{sign} B(b) - \frac{1}{2}\operatorname{sign} B(a)\,.$$

Given a path $\sigma : [0,1] \to \mathbb{R}^{2n\times 2n}$ of symmetric matrices, the *fundamental solution for σ* is a path of symplectic matrices $\Psi : [0,1] \to Sp(2n)$ given as the unique solution of

$$J_{\mathrm{std}}\partial_t\Psi + \sigma\Psi = 0, \qquad \Psi(0) = \mathbb{1}\,. \tag{6.2.10}$$

Recall that $\iota(A) \geq 0$ denotes the spectral gap around zero of a self-adjoint operator A (cf. equation (B.1.1)).

Lemma 6.2.6. *Let (F,S) be an admissible pair with asymptotic operators A_- and A_+. For all δ with $0 < \delta < \min\{\iota(A_-), \iota(A_+)\}$ the index of the operator*

$$D = \partial_s + J_{\mathrm{std}}\partial_t + S : H^{1,p;\delta}_F(\Sigma, \mathbb{R}^{2n}) \to L^2(\Sigma, \mathbb{R}^{2n})$$

is is given by

$$\mu(\Psi^+\Lambda_0^+, \Lambda_1^+) + \mu(F_0, F_1) - \mu(\Psi^-\Lambda_0^-, \Lambda_1^-) - \frac{1}{2}\dim C_- - \frac{1}{2}\dim C_+\,, \tag{6.2.11}$$

where $\Lambda^\pm = F(\pm\infty)$, $C_\pm = \ker A_\pm$ and $\Psi^\pm$ are the fundamental solutions of $S(\pm\infty)$.

Proof. Assume for the moment that the asymptotic operators A_- and A_+ are injective. We claim that the formula (6.2.11) holds for $\delta = 0$ and any $p > 1$. For $p = 2$ this is proven in [64, Theorem 7.42] (note that [64] use different sign convention). It remains to show that the index does not depend on $p > 1$. For the case of the perturbed Cauchy-Riemann operator defined on the cylinder, the claim is proven in [68, Prp. 3.1.26]. The arguments easily adapt to our situation. Denote by D_p the operator

$$\partial_s + J_{\mathrm{std}}\partial_t + S : H^{1,p}_F(\Sigma, \mathbb{R}^{2n}) \to L^p(\Sigma, \mathbb{R}^{2n})\,.$$

The claim follows, if we show $\ker D_p \cong \ker D_2$ and $\operatorname{coker} D_p \cong \operatorname{coker} D_2$. Since the index is invariant under homotopies of Fredholm operators, we assume without loss of generality that S is asymptotically constant, i.e. there exists a constant s_0 such that $S(\pm s, t) = \sigma_\pm(t)$ for all $s \geq s_0$ and $t \in [0,1]$. By elliptic regularity (cf. [53, Prp. B.4.6]) we know every element in the kernel of D_p is smooth. In order to show $\ker D_p \cong \ker D_2$ it suffices to show that $\|\xi\|_{1,2}$ is finite for all $\xi \in \ker D_p$. To show that we deduce the following exponential decay condition: there exists constants c and ι such that for all $|s| \geq s_0$ and $t \in [0,1]$

$$|\xi(s,t)| + |\mathrm{d}\xi(s,t)| \leq c e^{-\iota|s|}\,. \tag{6.2.12}$$

By analogy we only deduce this inequality for the positive end. Denote the Hilbert space

$$H = L^2([0,1], \mathbb{R}^{2n})\,,$$

equipped with the standard norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. Consider the positive asymptotic operator $A_+ = J_{\mathrm{std}}\partial_t + \sigma_+$ defined on $V = \{\xi \in H^{1,2}([0,1],\mathbb{R}^{2n}) \mid \xi(0),\xi(1) \in \mathbb{R}^n\}$. By abuse of notation think of $\xi : \mathbb{R} \to H$, $s \mapsto \xi(s)(t) = \xi(s,t)$ as a path in H. Since $\xi \in \ker D_p$ the path solves the differential equation for all $s \geq s_0$,

$$\partial_s\xi(s) + A_+\xi(s) = 0\,.$$

Define the function $g : [s_0,\infty) \to \mathbb{R}$, $g(s) := \frac{1}{2}\|\xi(s)\|^2$. By assumption the asymptotic operator A_+ is injective. We have shown in Lemma 6.2.2 that A_+ is unbounded self-adjoint and has a closed range. With Lemma B.1.1 we have $\|A_+\xi(s)\| \geq \iota\|\xi(s)\|$ for all $s \geq s_0$ where $\iota = \iota(A_+)$. We compute

$$\partial_s\partial_s g(s) = -\partial_s\langle A_+\xi,\xi\rangle = -2\langle A_+\xi,\partial_s\xi\rangle = 2\|A_+\xi\|^2 \geq 2\iota^2\|\xi\|^2 = 4\iota^2 g(s)\;, \qquad (6.2.13)$$

For any $s \geq 0$ set $\xi_s := \xi(s+\cdot)$. The Sobolev embedding $H^{1,2} \hookrightarrow C^0$ for functions with one-dimensional domain $[-1,+1]$ implies that we have for all $s \geq s_0+2$,

$$\|\xi(s)\| \leq O(1)\left(\int_{-1}^{+1}\|\xi_s(\sigma)\|^2 + (\partial_s\|\xi_s(\sigma)\|)^2\,\mathrm{d}\sigma\right)^{1/2}$$

We conclude via the Rellich embedding $H^{1,2} \hookrightarrow L^p$ for functions on $\Sigma_{-1}^1 := [-1,+1]\times[0,1]$ and elliptic regularity for D (cf. [53, Prop. B.4.6])

$$\|\xi(s)\| \leq O(1)\|\xi_s\|_{1,2;\Sigma_{-1}^1} \leq O(1)\|\xi_s\|_{p;\Sigma_{-1}^1} \leq O(1)\|\xi_s\|_{1,p;\Sigma_{-2}^2}\,.$$

The norm $\|\xi\|_{1,p}$ on $\mathbb{R}\times[0,1]$ is finite and we conclude that

$$\lim_{s\to\infty} g(s) \leq O(1)\lim_{s\to\infty}\|\xi\|^2_{1,p;\Sigma_{s-2}^{s+2}} = 0\,. \qquad (6.2.14)$$

Define the functions $g_0,\psi : [s_0,\infty) \to \mathbb{R}$ by

$$g_0(s) := g(s_0)e^{-2\iota(s-s_0)}, \qquad \psi(s) := g(s) - g_0(s)\,.$$

From (6.2.13) we have $\ddot{\psi} \geq 4\iota^2\psi$. By (6.2.14) we have $\psi(s_0) = 0$ and $\psi(s) \to 0$. Hence the maximum of ψ on $[s_0,\infty)$ can not be strictly positive, which implies for all $s \geq s_0$ that $\psi(s) \leq 0$ or equivalently that

$$g(s) \leq g_0(s)\,.$$

Last inequality, elliptic regularity for D and Sobolev embeddings show

$$|\xi(s,t)| + |\mathrm{d}\xi(s,t)| \leq O(1)\|\xi_s\|_{3,2;\Sigma_{-2}^2} \leq O(1)\|\xi_s\|_{2;\Sigma_{-2}^2} \leq O(1)e^{-\iota s}\,.$$

Therefore (6.2.12) and $\xi \in H^{1,2}(\Sigma)$. On the other hand given any $\xi \in \ker D_2$ we conclude analogously that $\xi \in H_F^{1,p}$, thus

$$\ker D_p \cong \ker D_2\,.$$

With Lemma 6.2.3 any element $\eta \in \operatorname{coker} D_p$ is identified with $\eta \in H^{1,q}_F(\Sigma, \mathbb{R}^{2n})$ satisfying $D^*\eta = 0$ where $1/p + 1/q = 1$ and $D^* = -\partial_s + J_{\mathrm{std}}\partial_t + S^T$. As before $\eta \in H^{1,2}_F(\Sigma)$ and so

$$\operatorname{coker} D_p \cong \operatorname{coker} D_2\ .$$

This shows formula (6.2.11) for the case $\delta = 0$ and $A_\pm$ injective.

To show the formula for $\delta \neq 0$ we reduce to the previous case. Choose a smooth function κ_δ such that $\kappa_\delta(s) = e^{\delta|s|}$ for all $|s| \geq 1$. As explained in the proof of Lemma 6.2.4 we consider the conjugated operator $D' = D + \partial_s\kappa_\delta/\kappa_\delta$, with asymptotic operators $A'_\pm = J_{\mathrm{std}}\partial_t + \sigma_\pm \mp \delta$. Since the operators $A'_\pm$ are invertible and by the last step, the operator D' has the index

$$\mu\big(\Psi^+_\delta\Lambda^+_0, \Lambda^+_1\big) + \mu(F_0, F_1) - \mu\big(\Psi^-_{-\delta}\Lambda^-_0, \Lambda^-_1\big)\ ,$$

where $\Psi^+_\delta, \Psi^-_{-\delta} : [0,1] \to Sp(2n)$ are given as the unique solution of

$$J_{\mathrm{std}}\partial_t\Psi^\pm_{\pm\delta} + \big(\sigma_\pm \mp \delta\big)\Psi^\pm_{\pm\delta} = 0, \qquad \Psi^\pm_{\pm\delta}(0) = \mathbb{1}\ .$$

Then the index formula for D follows by the last equality and Lemma 6.2.8. □

Corollary 6.2.7. *With assumptions of Lemma 6.2.6. Assume additionally that S has μ-decay for some $\mu > 0$ and set $W = (\ker A_-, \ker A_+)$. For any constant δ such that $0 < \delta < \min\{\iota(A_-), \iota(A_+), \mu\}$ the index of the operator*

$$D = \partial_s + J_{\mathrm{std}}\partial_t + S : H^{1,p;\delta}_{F;W}(\Sigma, \mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma, \mathbb{R}^{2n})\,,$$

is given by

$$\mu(\Psi^+\Lambda^+_0, \Lambda^+_1) + \mu(F_0, F_1) - \mu(\Psi^-\Lambda^-_0, \Lambda^-_1) + \frac{1}{2}\dim\ker A_- + \frac{1}{2}\dim\ker A_+\ .$$

Proof. This follows immediately from Lemma 6.2.6 since the codimension of the space $H^{1,p;\delta}_F(\Sigma, \mathbb{R}^{2n})$ as a subspace of $H^{1,p;\delta}_{F;W}(\Sigma, \mathbb{R}^{2n})$ is $\dim\ker A_- + \dim\ker A_+$. □

Lemma 6.2.8. *Given a path of symmetric matrices $\sigma : [0,1] \to \mathbb{R}^{2n\times 2n}$ and a pair of Lagrangians $\Lambda = (\Lambda_0, \Lambda_1) \in \mathcal{L}(n) \times \mathcal{L}(n)$. Consider the operator $A = J_{\mathrm{std}}\partial_t + \sigma : H_\Lambda \to L^2([0,2], \mathbb{R}^{2n})$. For all δ with $0 < \delta < \iota(A)$ we have*

$$\mu(\Psi_\delta\Lambda_0, \Lambda_1) = \mu(\Psi_0\Lambda_0, \Lambda_1) - \frac{1}{2}\dim\ker A,$$
$$\mu(\Psi_{-\delta}\Lambda_0, \Lambda_1) = \mu(\Psi_0\Lambda_0, \Lambda_1) + \frac{1}{2}\dim\ker A\ ,$$

where Ψ_ρ is the fundamental solution of $\sigma - \rho\mathbb{1}$ for each $\rho = -\delta, 0, \delta$.

Proof. For every $\rho \in [-\delta, \delta]$ consider Ψ_ρ as the fundamental solution of $\sigma - \rho\mathbb{1}$. The function $(\rho, t) \mapsto \Psi(\rho, t) = \Psi_\rho(t)$ is smooth in both variables. Hence $F(\rho, t) := \Psi(\rho, t)\Lambda_0$ defines a homotopy with fixed endpoints of $\Psi(\delta, \cdot)\Lambda_0$ to the concatenation of the paths

$\rho \mapsto \Psi(\delta-\rho,0)\Lambda_0$, $t \mapsto \Psi(0,t)\Lambda_0$ and $\rho \mapsto \Psi(\rho,t)\Lambda_0$. By the axioms of the Robbin-Salamon index we have

$$\mu(\Psi(\delta,\cdot)\Lambda_0,\Lambda_1) = -\mu(\Psi(\cdot,0)\Lambda_0,\Lambda_1) + \mu(\Psi(0,\cdot)\Lambda_0,\Lambda_1) + \mu(\Psi(\cdot,1)\Lambda_0,\Lambda_1)\ .$$

Since $\Psi(\cdot,0) = \mathbb{1}$ we have $\mu(\Psi(\cdot,0)\Lambda_0,\Lambda_1) = \mu(\Lambda_0,\Lambda_1) = 0$. We claim that the crossings of $\rho \mapsto (\Psi(\rho,1)\Lambda_0,\Lambda_1)$ agree with eigenvalues of A. Indeed, let $\rho \in [0,\delta]$ be a crossing, then there exists a non-trivial $w = \Psi(\rho,1)v \in \Psi(\rho,1)\Lambda_0 \cap \Lambda_1$. Define $\xi(t) := \Psi(\rho,t)v$. By construction we have $\xi(0) \in \Lambda_0$, $\xi(1) \in \Lambda_1$ and $J_{\mathrm{std}}\partial_t\xi + \sigma\xi = \rho\xi$. This shows that ξ is an eigenvector of A with eigenvalue ρ. By assumption there are no eigenvalues in $(0,\delta]$ and hence the only crossing occurs for $\rho = 0$.

We compute the crossing form $\Gamma(\Psi(\cdot,1)\Lambda_0,\Lambda_1;0)$. Differentiate the identity $J_{\mathrm{std}}\partial_t\Psi + (\sigma-\rho)\Psi = 0$ by ∂_ρ to obtain

$$\Psi(\rho,t) = J_{\mathrm{std}}\partial_\rho\partial_t\Psi(\rho,t) + \big(\sigma(t)-\rho\big)\partial_\rho\Psi(\rho,t)\ .$$

Using the last equation for $\rho = 0$ we compute

$$\begin{aligned}\langle\xi,\xi\rangle = \langle\Psi v,\Psi v\rangle &= \langle\Psi v, J_{\mathrm{std}}\partial_\rho\partial_t\Psi v + \sigma\partial_\rho\Psi v\rangle\\ &= \langle\Psi v, J_{\mathrm{std}}\partial_\rho\partial_t\Psi v\rangle + \langle\sigma\Psi v,\partial_\rho\Psi v\rangle\\ &= -\langle J_{\mathrm{std}}\Psi v,\partial_\rho\partial_t\Psi v\rangle - \langle J_{\mathrm{std}}\partial_t\Psi,\partial_\rho\Psi v\rangle\\ &= -\partial_t\langle J_{\mathrm{std}}\Psi v,\partial_\rho\Psi v\rangle = -\partial_t\omega_{\mathrm{std}}(\Psi v,\partial_\rho\Psi v)\ .\end{aligned}$$

Integrating the last equation shows (note that $\partial_\rho\Psi(\rho,0) = 0$)

$$\int_0^1\langle\xi,\xi\rangle\mathrm{d}t = -\omega_{\mathrm{std}}(\Psi(0,1)v,\partial_\rho\big|_{\rho=0}\Psi(0,1)v) = -\Gamma(\Psi(\cdot,1)\Lambda_0,\Lambda_1;0)w\ .$$

The last identity is established in [63, Theorem 1.1 (2)]. We see that the crossing form is negative definite and defined on the space $\ker A$. By the definition of the Robbin-Salamon index we conclude

$$\mu(\Psi(\cdot,1)\Lambda_0,\Lambda_1) = \frac{1}{2}\dim\ker A\ .$$

This shows the claim. $\square$

7. Transversality

In terms of last chapter we show that the Fredholm section is transverse to the zero section for a generic choice of the almost complex structures. Consequently every connected component of the moduli space of perturbed holomorphic strips with Lagrangian boundary conditions is a manifold. In the non-degenerated case of transversely intersecting Lagrangians this was solved originally by Floer and Hofer in [31]. The generalization for the degenerate case of cleanly intersecting Lagrangians was treated by Frauenfelder in [33]. Besides recapitulating these ideas we prove some additional transversality results for the evaluation map based on ideas of Seidel from [70]. Transversality is achieved by allowing the almost complex structure to explicitly depend on the domain and is based on the existence of regular points.

7.1. Setup

Let (M, ω) be a symplectic manifold and $L_0, L_1 \subset M$ be two Lagrangian submanifolds. For any admissible vector field X and almost complex structure J we denote by $\mathcal{M}_J$ the space of all (J, X)-holomorphic strips (cf. Section 5.1 for definitions). For any such strip $u \in \mathcal{M}_J$ the arc $u(s, \cdot)$ is asymptotic to perturbed intersection points $\mathcal{I}_-$ and $\mathcal{I}_+$ for $s \to -\infty$ or $s \to +\infty$ respectively. Given smooth maps $\varphi_- : W_- \to \mathcal{I}_-$ and $\varphi_+ : W_+ \to \mathcal{I}_+$ we define

$$\widetilde{\mathcal{M}}(W_-, W_+; J, X) := \left\{ (w_-, u, w_+) \;\middle|\; \begin{array}{l} \varphi_-(w_-) = u(-\infty) \\ \varphi_+(w_+) = u(+\infty) \end{array} \right\}, \tag{7.1.1}$$

as a subspace of $W_- \times \mathcal{M}_J \times W_+$. Let $D_{u,J}$ denote the vertical differential of the Cauchy-Riemann-Floer operator (cf. equation (6.1.6)).

Definition 7.1.1. We way that J is *regular for X* if $D_{u,J}$ is surjective for all $u \in \mathcal{M}_J$. Moreover J is *regular for X and φ* if additionally (7.1.1) is cut-out transversely, i.e.

$$\{(\xi(-\infty), \xi(\infty)) \mid \xi \in \ker D_{u,J}\} + \operatorname{im} \mathrm{d}_w \varphi = T_{u(-\infty)}\mathcal{I}_- \oplus T_{u(\infty)}\mathcal{I}_+ \,, \tag{7.1.2}$$

for all $(w_-, w_+, u) \in \widetilde{\mathcal{M}}(W_-, W_+; J, X)$.

If J is regular then each connected component of the space (7.1.1) is a manifold by the implicit function theorem (cf. [53, Thm. A.3.3]). We now show that regular almost complex structures exist in abundance. We split up the argument depending whether J is chosen to be $\mathbb{R}$-invariant or not.

7.2. $\mathbb{R}$-dependent structures

Let s_0 be a constant such that such that $X(-s,\cdot) = X_{H_-}$ and $X(s,\cdot) = X_{H_+}$ for all $s \geq s_0$. Fix a constant $s_1 > s_0$ and two paths of almost complex structures $J_-, J_+ : [0,1] \to \mathrm{End}(TM,\omega)$. We look for regular structures in the space

$$\mathcal{J} := \{J \in C^\infty(\mathbb{R} \times [0,1], \mathrm{End}(TM,\omega)) \mid J(\pm s,\cdot) = J_\pm \ \forall\ s \geq s_1\}\,.$$

A subset of a topological space is *comeager* if it is a countable intersection of open and dense sets.

Theorem 7.2.1. *The subset of almost complex structures which are regular for X and φ is comeager in $\mathcal{J}$.*

Proof. For the proof we follow the original approach of Floer using the ε-norms combined with the argument of Taubes as described in [53, Section 3]. Except for the part of transversality with respect to the evaluation the theorem is proven in [33, Theorem 4.10].

Set $W = W_- \times W_+$ and let $W = \bigcup_{k\in\mathbb{N}} W_k$ be an exhaustion by compact subsets $W_k \subset W$ with $W_k \subset W_{k+1}$ for all $k \in \mathbb{N}$ and denote by φ_k the restriction of φ to W_k. Fix constants $p > 2$ and $\mu > 0$ small enough. For $k \in \mathbb{N}$ we define $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ to be the space J with the property that for any (J,X)-holomorphic strip u such that

$$|\partial_s u(s,t)| \leq k e^{-\mu|s|} \tag{7.2.1}$$

for all $s \in \mathbb{R}$ and $t \in [0,1]$ it holds that

(i) the operator $D_{u,J}$ is surjective

(ii) for all $w \in W_k$ with $(u(-\infty), u(\infty)) = \varphi(w)$ we have (7.1.2).

It suffices to show that $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ is dense and open for all $k \in \mathbb{N}$ because if that is true we write $\mathcal{J}_{\mathrm{reg}} = \bigcap_{k\in\mathbb{N}} \mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ as a countable intersection of open and dense sets.

Step 1. The subset $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ is open for all $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. We show that the complement of $\mathcal{J}_{\mathrm{reg},k}$ is closed. Take a sequence $J_\nu \in \mathcal{J} \setminus \mathcal{J}_{\mathrm{reg},k}$ such that J_ν converges to J with respect to the C^∞-topology. Because $J_\nu \notin \mathcal{J}_{\mathrm{reg},k}$ there exists a sequence $(u_\nu)_{\nu\in\mathbb{N}}$ of (J_ν, X)-holomorphic strips such that for each $\nu \in \mathbb{N}$ we have (7.2.1) and at least one of the following holds

(i) D_{u_ν,J_ν} is not surjective,

(ii) there exists $w_\nu \in W_k$ with $(u_\nu(-\infty), u_\nu(\infty)) = \varphi(w_\nu)$ and

$$\{(\xi(-\infty), \xi(\infty)) \mid \xi \in \ker D_{u_\nu,J_\nu}\} + \mathrm{im}\, \mathrm{d}_{w_\nu}\varphi \subsetneq T_{u_\nu(-\infty)}\mathcal{I}_- \oplus T_{u_\nu(\infty)}\mathcal{I}_+\,.$$

Since the gradient of (u_ν) is uniformly bounded, a subsequence, still denoted by (u_ν), converges to a (J,X)-holomorphic map u in C^∞_{loc} (cf. Lemma 5.2.1). We also have with (7.2.1)

$$\lim_{s\to\infty}\lim_{\nu\to\infty} E(u_\nu;[s,\infty)\times[0,1]) \le \lim_{s\to\infty} k^2\int_s^\infty e^{-2\mu s}\mathrm{d}s = 0\,.$$

Similarly for the negative end. This shows that $E(u_\nu)\to E(u)$. Provided with the convergence of the energy we conclude that u_ν converges to u uniformly (cf. Lemma 5.3.1). By the mean-value inequality and exponential decay of the energy (cf. Lemma 4.3.2) we have an uniform constant c such that for all s large enough

$$|\partial_s u(s,t)|^2 \le cE(u;[s,\infty)\times[0,1])) \le k^2 e^{-2\mu s}\ .$$

Similar for the negative end. This shows that u satisfies (7.2.1) since on compact subsets we have C^1-convergence.

We distinguish two cases. In the first case we assume that after passing to a subsequence we have that D_{u_ν,J_ν} is not surjective all $\nu\in\mathbb{N}$. Let $\Pi_u^{u_\nu}$ be the parallel transport operator defined in equation (6.1.4). Lemma A.3.10 shows that $D_\nu := \Pi^u_{u_\nu} D_{u_\nu,J_\nu}\Pi_u^{u_\nu}$ converges to $D := D_{u,J}$ in the operator norm. Since by assumption D_ν is not surjective for all $\nu\in\mathbb{N}$ and surjectivity is an open condition this shows that D is not surjective. Hence $J\notin\mathcal{J}_{\mathrm{reg},k}$ and we are finished in that case.

Assume in the second case by passing to a subsequence that for all $\nu\in\mathbb{N}$ it holds that the operator D_{u_ν,J_ν} is surjective and there exists $\zeta_\nu = (\zeta_\nu^-,\zeta_\nu^+)\in T_{u_\nu(-\infty)}\mathcal{I}_-\oplus T_{u_\nu(\infty)}\mathcal{I}_+$ such that $|\zeta_\nu^-|+|\zeta_\nu^+| = 1$, $\zeta_\nu\perp \mathrm{im}\,\mathrm{d}_{w_\nu}\varphi$ and $\langle\zeta_\nu^-,\xi(-\infty)\rangle + \langle\zeta_\nu^+,\xi(\infty)\rangle = 0$ for all $\xi\in\ker D_{u_\nu,J_\nu}$. Since W_k is compact we assume by passing to another subsequence that $w_\nu\to w\in W$ and ζ_ν converges to a non-vanishing $\zeta = (\zeta^-,\zeta^+)$ such that $\zeta\perp\mathrm{im}\,\mathrm{d}_w\varphi$. It remains to show that for all $\xi\in\ker D_{u,J}$ we have

$$\langle\zeta^-,\xi(-\infty)\rangle + \langle\zeta^+,\xi(\infty)\rangle = 0\,. \tag{7.2.2}$$

Let Q be a right-inverse of D. For ν large enough the kernel of D_ν is transverse to the image of Q and since the operators D_ν and D are both surjective with same index their kernels have the same dimension. In particular for every $\xi\in\ker D$ there exists a unique $\xi_\nu\in\ker D_\nu$ such that $\xi-\xi_\nu\in\mathrm{im}\,Q$. With $Q\eta_\nu = \xi-\xi_\nu$ and norm $\|\cdot\|$ either $\|\cdot\|_{1,p;\delta}$ or $\|\cdot\|_{p;\delta}$ respectively we have

$$\begin{aligned}\|\xi-\xi_\nu\| = \|Q\eta_\nu\| \le O(1)\,\|\eta_\nu\| \le \|DQ\eta_\nu\| = \|D(\xi-\xi_\nu)\| = \\ = \|(D-D_\nu)\xi_\nu\| \le o(1)\,\|\xi_\nu\| \le o(1) + o(1)\,\|\xi-\xi_\nu\|\ .\end{aligned}$$

This shows that $\|\xi-\xi_\nu\|\to 0$. Since $\xi_\nu(\pm\infty)\perp\zeta_\nu^\pm$

$$\begin{aligned}&|\langle\xi(-\infty),\zeta^-\rangle + \langle\xi(\infty),\zeta^+\rangle|\\ &\quad\le \|\xi-\xi_\nu\| + \left|\langle\xi_\nu(-\infty),\zeta^- - \zeta_\nu^-\rangle + \langle\xi_\nu(-\infty),\zeta^+ - \zeta_\nu^+\rangle\right|\\ &\qquad\le \|\xi-\xi_\nu\| + \|\xi_\nu\|\left(\left|\zeta^- - \zeta_\nu^-\right| + \left|\zeta^+ - \zeta_\nu^+\right|\right)\to 0\,.\end{aligned}$$

This shows (7.2.2) and hence the claim in the second case.

Step 2. Fix connected components $C_- \subset \mathcal{I}_-$ and $C_+ \subset \mathcal{I}_+$. There exists a dense subspace $\mathcal{J}' \subset \mathcal{J}$ such that the *universal moduli space*

$$\widetilde{\mathcal{M}}(C_-, C_+; \mathcal{J}', X) = \{(u, J) \mid u \in \widetilde{\mathcal{M}}(C_-, C_+; J, X),\ J \in \mathcal{J}'\}$$

is a Banach manifold and the evaluation map $(u, J) \mapsto (u(-\infty), u(\infty))$ is a submersion.

By Lemma 7.5.5 there exists a dense subspace $\mathcal{J}' \subset \mathcal{J}$ which is a separable Banach space. It suffices to see that for all $(u, J) \in \widetilde{\mathcal{M}}(C_-, C_+; \mathcal{J}', X)$ the operator is surjective

$$D^{\mathrm{univ}}_{u,J} : T_u\mathcal{B}^{1,p;\delta}(C_-, C_+) \oplus T_J\mathcal{J}' \to \mathcal{E}^{p;\delta}_u(C_-, C_+),\ (\xi, Y) \mapsto D_{u,J}\xi + Y(\partial_t u - X)\,.$$

Because $D_{u,J}$ is Fredholm the operator $D^{\mathrm{univ}}_{u,J}$ has a closed range. Take η in the cokernel, which is identified with an element in the Banach space $\mathcal{E}^{q,-\delta}_u$ where $1/p + 1/q = 1$ such that for all $\xi \in T_u\mathcal{B}^{1,p;\delta}$ and $Y \in T_J\mathcal{J}'$ we have

$$\int_\Sigma \langle \eta, D_u\xi\rangle \mathrm{d}s\mathrm{d}t = 0, \qquad \int_\Sigma \langle \eta, Y(\partial_t u - X)\rangle \mathrm{d}s\mathrm{d}t = 0\,. \tag{7.2.3}$$

The first equation implies that η is smooth after elliptic regularity. We claim that by the second equation η vanishes. Given a point $(s,t) \in [s_0, s_1] \times [0,1]$ such that $\partial_s u(s,t) \neq 0$ and assume by contradiction that $\eta(s,t) \neq 0$. Using an explicit formula (given in [53, Lemma 3.2.2]) and a cut-off function we find Y supported in a small neighborhood of $(s, t, u(s,t))$ such that

$$\int_\Sigma \langle \eta, Y(\partial_t u - X)\rangle \mathrm{d}s\mathrm{d}t > 0\,.$$

This is in contradiction to the second equation of (7.2.3) and shows that $\eta(s,t) = 0$ for all points (s,t) with $\partial_s u(s,t) \neq 0$. Since these points are dense (cf. Lemma 7.4.2) we conclude that η vanishes restricted to $[s_0, s_1] \times [0,1]$ and by unique continuation we conclude that η vanishes altogether. This shows that universal moduli space is a Banach submanifold.

We claim that the operator $D^{\mathrm{univ}}_{u,J} + \mathrm{d}_u ev$ is surjective for all (u, J) in the universal moduli space. We use an idea from [70, Lemma 2.5]. As above $D^{\mathrm{univ}}_{u,J} + \mathrm{d}_u ev$ has a closed range. Take (η, ζ^-, ζ^+) in the cokernel. We have (7.2.3) and

$$\langle \xi(-\infty), \zeta^-\rangle + \langle \xi(\infty), \zeta^+\rangle = 0\,,$$

for all $\xi \in T_u\mathcal{B}^{1,p;\delta}$. With (7.2.3) we conclude again that η vanishes and by the last identity we show that ζ^- and ζ^+ vanishes because we find $\xi \in T_u\mathcal{B}^{1,p;\delta}$ such that $\xi(\pm\infty) = \zeta^\pm$. This shows that $D^{\mathrm{univ}}_{u,J} + \mathrm{d}_u ev$ is surjective . Hence given any $\zeta^- \in T_{u(-\infty)}C_-$ and $\zeta^+ \in T_{u(\infty)}C_+$ there exists $\xi \in T_u\mathcal{B}^{1,p;\delta}$ and $Y \in T_J\mathcal{J}'$ such that

$$D_{u,J}\xi + Y(\partial_t u - X) = 0, \qquad \xi(-\infty) = \zeta^-, \qquad \xi(\infty) = \zeta^+\,.$$

We see that (ξ, Y) is an element in the tangent space of $\widetilde{\mathcal{M}}(C_-, C_+; \mathcal{J}', X)$ and by the second equation that $\mathrm{d}_u ev(\xi) = (\xi(-\infty), \xi(\infty)) = \zeta$. This shows the claim.

Step 3. We show that the subset $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ is dense for all $k \in \mathbb{N}$.

By the last step $\widetilde{\mathcal{M}}(W;\mathcal{J}',X) := \{(u,J,w) \mid u \in \widetilde{\mathcal{M}}(C_-,C_+;J,X),\ J \in \mathcal{J}',\ ev(u) = \varphi(w)\}$ is a Banach manifold with tangent space at a point (u,J,w) given by the set of triples (ξ,Y,v) such that

$$D_{u,J}\xi + Y(\partial_t u - X) = 0, \qquad (\xi(-\infty),\xi(\infty)) = \mathrm{d}_w\varphi(v)\,.$$

Abbreviate $C := T_{u(-\infty)}C_- \oplus T_{u(\infty)}C_+$ and let $\pi : C \to C/\operatorname{im}\mathrm{d}_w\varphi$ the canonical projection. We conclude that (ξ,Y) lies in the kernel of the operator

$$(\xi,Y) \mapsto (D_{u,J}\xi + Y(\partial_t u - X), \pi(\xi(-\infty),\xi(\infty))) \in \mathcal{E}_u \oplus C/\operatorname{im}\mathrm{d}_w\varphi\,.$$

By [53, Lemma A.3.6] we have that $J \in \mathcal{J}'$ is a regular value of the projection $\mathrm{pr}_2 : \widetilde{\mathcal{M}}(W;\mathcal{J}',X) \to \mathcal{J}'$ if and only if the map $\xi \mapsto (D_{u,J}\xi, \pi(\xi(-\infty),\xi(\infty)))$ is surjective. Hence let J be a regular value of pr_2 and $[\zeta] \in C/\operatorname{im}\mathrm{d}_w\varphi$ we find $\xi \in T_u\mathcal{B}$ such that

$$D_{u,J}\xi = 0, \qquad \pi(\xi(-\infty),\xi(\infty)) = [\zeta]\,.$$

We conclude that $\mathrm{regv}(\mathrm{pr}_2) \subset \mathcal{J}_{\mathrm{reg},k}$. Since after Sards theorem the space of regular values is comeager and by Baires theorem every complete metric space has the property that a comeager subset is dense, the inclusion $\mathrm{regv}(\mathrm{pr}_2) \subset \mathcal{J}'$ is dense. Since $\mathcal{J}' \subset \mathcal{J}$ is dense by construction we conclude that $\mathrm{regv}(\mathrm{pr}_2) \subset \mathcal{J}$ is dense. Thus $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ is dense. □

7.2.1. Glued structures

Given two admissible vector fields X_0 and X_1 such that $X_0(s,\cdot) = X_1(-s,\cdot)$ for all $s \geq s_0$. For every $R \geq R_0$ we define the *glued vector field* $X_R := X_0\#_R X_1$ via

$$X_R(s,\cdot) := \begin{cases} X_0(s+2R,\cdot) & \text{if } s \leq 0 \\ X_1(s-2R,\cdot) & \text{if } s \geq 0\,. \end{cases} \tag{7.2.4}$$

Similarly given two admissible almost complex structures J_0 and J_1 such that $J_0(s,\cdot) = J_1(-s,\cdot)$ for all s large enough, we define for all $R \geq R_0$ the *glued almost complex structure* $J_R := J_0\#_R J_1$ via

$$J_R(s,\cdot) := \begin{cases} J_0(s+2R,\cdot) & \text{if } s \leq 0 \\ J_1(s-2R,\cdot) & \text{if } s \geq 0\,. \end{cases} \tag{7.2.5}$$

In this section $\mathcal{M}_J$ denotes the space of pairs (u,R) where u is a (J_R,X_R)-holomorphic strip and $R \geq R_0$. As above the arcs $u(s,\cdot)$ converge to perturbed intersection points $\mathcal{I}_-$ and $\mathcal{I}_+$ for $s \to -\infty$ and $s \to +\infty$ respectively. Given smooth maps $\varphi_- : W_- \to \mathcal{I}_-$ and $\varphi_+ : W_+ \to \mathcal{I}_+$ we denote the space

$$\widetilde{\mathcal{M}}(W_-,W_+;J,X) := \left\{ (w_-,u,R,w_+) \;\middle|\; \begin{matrix} u(-\infty) = \varphi_-(w_-) \\ u(+\infty) = \varphi_+(w_+) \end{matrix} \right\}, \tag{7.2.6}$$

as subspace of $W_- \times \mathcal{M}_J \times W_+$. For all $(u,R) \in \mathcal{M}_J$ we consider the operator

$$\widehat{D}_{u,R} : T_u\mathcal{B}^{1,p;\delta} \oplus \mathbb{R} \to \mathcal{E}_u^{p;\delta}, \qquad (\xi,\theta) \mapsto D_{u,R}\xi + \theta\,\eta_R\,,$$

with $\eta_R := (\partial_R J)\,(\partial_t u - X_R) - J_R(\partial_R X_R)$ and $D_{u,R}$ is the vertical differential of the Cauchy-Riemann-Floer operator (i.e. the operator (6.1.6) with $J = J_R$ and $X = X_R$).

Definition 7.2.2. The homotopy $J = (J_R)_{R \geq R_0}$ is called *regular for* $X = (X_R)_{R\geq R_0}$ if $\widehat{D}_{u,R}$ is surjective for all $(u,R) \in \mathcal{M}_J$. Moreover J is called *regular for* X *and* (φ_-, φ_+) if additionally (7.2.6) is cut-out transversely, i.e.

$$\{(\xi(-\infty), \xi(\infty)) \mid (\xi,\eta) \in \ker \widehat{D}_{u,R}\} + \operatorname{im} \mathrm{d}_w\varphi = T_{u(-\infty)}\mathcal{I}_- \times T_{u(\infty)}\mathcal{I}_+\,.$$

for all $(w_-, w_+, u, R) \in \widetilde{\mathcal{M}}(W_-, W_+; J, X)$.

Remark 7.2.3. This is of course *not* equivalent to demand that for all $R \geq R_0$ the structure J_R is regular for X_R and (φ_-, φ_+).

Fix paths of almost complex structures J_∞^-, J_∞ and J_∞^+ and some $s_1 > s_0$. We denote by $\mathcal{J}_{\mathrm{adm}}$ the space of admissible almost complex structures (cf. Definition 5.1.1). We search for regular structures in the space

$$\mathcal{J} := \left\{ (J_0, J_1) \in \mathcal{J}_{\mathrm{adm}} \times \mathcal{J}_{\mathrm{adm}} \;\middle|\; \forall s \geq s_1 : \begin{array}{l} J_\infty^- = J_0(-s,\cdot)\,, \\ J_\infty = J_0(s,\cdot) = J_1(-s,\cdot)\,, \\ J_\infty = J_1(s,\cdot) \end{array} \right\}.$$

Theorem 7.2.4. *The subspace of almost complex structures which are regular for X and φ is comeager in $\mathcal{J}$.*

Proof. Since we apply the same principle ideas from the proof of Theorem 7.2.1 we give just sketch. Let $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}$ be the subset of regular pairs. Fix constants $p > 2$ and $\mu > 0$ small enough. For any $k \in \mathbb{N}$ we define $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ consisting of all pairs $(J_0, J_1) \in \mathcal{J}$ with the property that for all $R \geq R_0$ with $R \leq k$ the operator $\widehat{D}_{u,R}$ is surjective for all (J_R, X_R)-holomorphic curves u which satisfy additionally $|\partial_s u(s,t)| \leq k e^{-\mu|s|}$ for all $(s,t) \in \Sigma$ and for all $w \in W_k$ with $\varphi(w) = (u(-\infty), u(\infty))$ we have $\{(\xi(-\infty), \xi(\infty)) \mid (\xi,\theta) \in \ker \widehat{D}_{u,R}\} + \operatorname{im} \mathrm{d}_w\varphi = T_{u(-\infty)}\mathcal{I}_- \oplus T_{u(\infty)}\mathcal{I}_+$. By the same arguments as in the first step of the proof of Theorem 7.2.1 we show that $\mathcal{J}_{\mathrm{reg},k}$ is open for all $k \in \mathbb{N}$.

We show that the corresponding universal moduli space is a Banach manifold and the evaluation map is a submersion. As above we construct a separable Banach space $\mathcal{J}'$ such that $\mathcal{J}' \subset \mathcal{J}$ is dense. Define the universal moduli space

$$\widetilde{\mathcal{M}}(C_-, C_+; \mathcal{J}', X) := \{(u, J, R) \mid u \in \widetilde{\mathcal{M}}(C_-, C_+; J_R, X_R),\ J \in \mathcal{J}'\}\,.$$

The vertical differential of the perturbed Cauchy-Riemann operator is

$$\widehat{D}_{u,R}^{\mathrm{univ}} : T_u\mathcal{B}^{1,p;\delta} \oplus T_J\mathcal{J}' \oplus \mathbb{R} \to \mathcal{E}_u^{p;\delta}, \qquad (\xi, Y_0, Y_1, \theta) \mapsto \widehat{D}_{u,R}(\xi,\theta) + Y_R\big(\partial_t u - X_R\big)\,,$$

where $Y_R : \Sigma \times M \to \mathrm{End}(TM)$ is defined by

$$Y_R(s,\cdot) := \begin{cases} Y_0(s+2R,\cdot) & \text{if } s \leq 0 \\ Y_1(s-2R,\cdot) & \text{if } s \geq 0\,. \end{cases}$$

Let $D^{\mathrm{univ}}_{u,R}$ denote the restriction of $\widehat{D}^{\mathrm{univ}}_{u,R}$ to the subspace $T_u\mathcal{B}^{1,p;\delta} \oplus T_J\mathcal{J}'$. We claim that $D^{\mathrm{univ}}_{u,R}$ is surjective. Take any element $\eta \in \mathcal{E}^{p;\delta}_u$ in the annihilator of the image. Then for all $\xi \in T_u\mathcal{B}$ and $(Y_0,Y_1) \in T_J\mathcal{J}'$ we have

$$\int_\Sigma \langle D_{u,R}\xi, \eta\rangle = 0, \qquad \int_\Sigma \langle Y_R(\partial_t u - X_R), \eta\rangle = 0\,.$$

The first equation shows that η is smooth as in the proof of Theorem 7.2.1. Choose a point $(s,t) \in [2R+s_0, 2R+s_1]\times[0,1]$ such that $\partial_s u(s,t) \neq 0$ and suppose by contradiction that $\eta(s,t) \neq 0$. We find an infinitesimal almost complex structure $Y = (0, Y_1) \in T_J\mathcal{J}'$ supported in a small neighborhood about $(s-2R, t, u(s,t))$ such that $\int_\Sigma \langle Y_R J_R \partial_s u, \eta)\rangle > 0$ in contradiction the the second equation. Hence $\eta(s,t)$ vanishes on points $(s,t) \in [2R+s_0, 2R+s_1]\times[0,1]$ with $\partial_s u(s,t) \neq 0$. Because such points are dense in $[2R+s_0, 2R+s_1]\times[0,1]$ we conclude by continuity that η restricted to $[2R+R_0, 2R+R_1]\times[0,1]$ vanishes and by unique continuation we see that η vanishes everywhere. The rest of the proof follows word by word from the proof of Theorem 7.2.1. □

7.2.2. Homotopies

Given homotopies $X = (X_R)_{R\in[a,b]}$ and $J = (J_R)_{R\in[a,b]}$ of admissible vector fields and almost complex structures respectively we denote by $\widetilde{\mathcal{M}}(W_-, W_+; J, X)$ the space of all (J_R, X_R)-holomorphic maps, similarly to (7.2.6). For technical reasons we require that $X_R(-s,\cdot) = X_{H_-}$ and $X_R(s,\cdot) = X_{H_+}$ for all $s \geq s_0$ and two fixed Hamiltonians H_- and H_+ which do not depend on R. We say that $J = (J_R)_{R\in[a,b]}$ is *regular for* $X = (X_R)_{R\in[a,b]}$ *and* φ similarly to Definition 7.2.2. Given two admissible almost complex structures J_a, J_b such that $J_- := J_a(-s,\cdot) = J_b(-s,\cdot)$ and $J_+ := J_a(s,\cdot) = J_b(s,\cdot)$ for all $s \geq s_1$. We search for regular almost complex structures in the space $\mathcal{J}(J_a, J_b)$ which is the space of smooth homotopies $(J_R)_{R\in[a,b]}$ from J_a to J_b such that $J_R(-s,\cdot) = J_-$ and $J_R(s,\cdot) = J_+$ for all $s \geq s_1$ and $R \in [a,b]$.

Theorem 7.2.5. *If $s_1 > s_0$ the subspace structures which are regular for X and (φ_-, φ_+) in $\mathcal{J}(J_a, J_b)$ is comeager.*

Proof. The proof is completely analogous to the proof of Theorem 7.2.1 and Theorem 7.2.4. Note that the space of infinitesimal almost complex structures is given by section supported in the compact cube $[a,b]\times[-s_1,s_1]\times[0,1]$ and hence the resulting Banach spaces are separable. □

7.3. $\mathbb{R}$-invariant structures

Let $X = X_H$ be the Hamiltonian vector field for some clean Hamiltonian $H \in C^\infty([0,1]\times M)$. In this section we construct regular structures in a set of almost complex structures

$$\mathcal{J} := C^\infty([0,1], \mathrm{End}(TM, \omega)) .$$

For any $J \in \mathcal{J}$ we denote by $\mathcal{M}_J$ the space of (J, X)-holomorphic strips and by $\mathcal{I}$ the space of perturbed intersection points of H. We also give a transversality result of the evaluation of tuples of (J, X)-holomorphic maps given by

$$\begin{gathered} ev : \mathcal{M}_J^m \to \mathcal{I}^{2m}, \\ (u_1, \dots, u_m) \to (u_1(-\infty), u_1(\infty), u_2(-\infty), \dots, u_m(-\infty), u_m(\infty)) . \end{gathered} \tag{7.3.1}$$

The difficulty here lies in the fact that we need to perturbed J simultaneously for the curves u_i and u_j, which is obviously not possible if u_i and u_j have the exact same image. For that reason we define the notion of a *distinct tuple*.

Definition 7.3.1. A tuple $(u_1, \dots, u_m)$ of maps $\Sigma \to M$ is called *distinct*, if for all $i \neq j$ and $a \in \mathbb{R}$ we have $u_i \neq u_j(a + \cdot, \cdot)$.

Remark 7.3.2. Distinct tuples should not be confused with the stronger notion of *absolutely distinct* tuples as defined in [11]. The transversality theory in [11] is more difficult since the authors achieve transversality for domain-independent almost complex structures.

Definition 7.3.3. Given a smooth map $\varphi : W \to \mathcal{I}_H(L_0, L_1)^{2m}$. The almost complex structure $J \in \mathcal{J}$ is *regular for X and φ* if J is regular for X and φ transverse to the evaluation map (7.3.1) restricted to the space of distinct tuples.

Theorem 7.3.4. *The subspace of J which are regular for X and φ is comeager in $\mathcal{J}$.*

Proof. For $\mu > 0$ we define $\mathcal{J}^\mu \subset \mathcal{J}$ as the open subspace of all J with $\mu < \iota(J, H)$. Choose an exhaustion $W = \bigcup_k W_k$ by compact subsets W_k such that $W_k \subset W_{k+1}$ for all $k \in \mathbb{N}$. For $\mu > 0$ and $k \in \mathbb{N}$ we denote by $\mathcal{J}^\mu_{\mathrm{reg},k} \subset \mathcal{J}^\mu$ the space of all $J \in \mathcal{J}^\mu$ such that the operator $D_{u,J}$ is surjective for all (J, X)-holomorphic strips u which satisfy (7.2.1) and for all distinct tuples $(u_1, \dots, u_m)$ of (J, X)-holomorphic strips which satisfy (7.2.1) and $w \in W$ such that $\varphi(w) = ev(u)$ we have that the image of $\mathrm{d}_w\varphi$ is a complement of

$$\{(\xi_1(-\infty), \xi_1(\infty), \xi_2(-\infty), \dots, \xi_m(\infty)) \mid \xi_j \in \ker D_{u_j,J},\ j = 1, \dots, m\} . \tag{7.3.2}$$

First we that $\mathcal{J}^\mu_{\mathrm{reg},k} \subset \mathcal{J}^\mu$ is open as in the proof of Theorem 7.2.1. To show that that $\mathcal{J}^\mu_{\mathrm{reg},k} \subset \mathcal{J}^\mu$ is dense we proceed as follows. Let $\mathcal{J}' \subset \mathcal{J}$ be the dense subspace which is a separable Banach manifold. Then $\mathcal{J}'^\mu := \mathcal{J}' \cap \mathcal{J}^\mu$ is also separable Banach manifold which is dense in $\mathcal{J}^\mu$. Let $C = (C_1, C_2, \dots, C_{2m})$ be a tuple of connected components in $\mathcal{I}_H(L_0, L_1)$. Abbreviate $\mathcal{B} := \mathcal{B}^{1,p;\delta}(C_1, C_2) \times \mathcal{B}^{1,p;\delta}(C_3, C_4) \times \dots \mathcal{B}^{1,p;\delta}(C_{2m-1}, C_{2m})$. Define the universal moduli space

$$\widetilde{\mathcal{M}}(C; \mathcal{J}'^\mu, X) \subset \mathcal{B} \times \mathcal{J}'^\mu ,$$

to be the space of $(u_1, \dots, u_m, J)$ where $J \in \mathcal{J}'^\mu$ and $(u_1, \dots, u_m)$ is a distinct tuple of (J, X)-holomorphic strips. We want to show that $\widetilde{\mathcal{M}}(C; \mathcal{J}'^\mu, X)$ is a Banach manifold and that the evaluation map is a submersion

$$ev : \widetilde{\mathcal{M}}(C; \mathcal{J}'^\mu, X) \to C_1 \times C_2 \times \cdots \times C_m, \qquad (u_1, \dots, u_m, J) \mapsto ev(u_1, \dots, u_m) .$$

For $(u, J) = (u_1, \dots, u_m, J) \in \widetilde{\mathcal{M}}(C; J'^\mu, X)$ with $(x_1, x_2, \dots, x_{2m}) = ev(u)$ consider the operator

$$\begin{gathered} T_u\mathcal{B} \oplus T_J\mathcal{J}' \to \mathcal{E}^{p;\delta}_{u_1} \oplus \mathcal{E}^{p;\delta}_{u_2} \oplus \dots \mathcal{E}^{p;\delta}_{u_m} \oplus T_{x_1}C_1 \oplus T_{x_2}C_2 \oplus \cdots \oplus T_{x_{2m}}C_{2m} \\ (\xi_1, \xi_2, \dots, \xi_m, Y) \mapsto (D^{\mathrm{univ}}_{u_1,J}(\xi_1, Y), D^{\mathrm{univ}}_{u_2,J}(\xi_2, Y), \dots, D^{\mathrm{univ}}_{u_m,J}(\xi_m, Y), \mathrm{d}_u ev(\xi)). \end{gathered}$$

We claim that the operator is surjective. It suffices to show that the cokernel is trivial. Given an element $(\eta, \zeta) = (\eta_1, \dots, \eta_m, \zeta_1, \dots, \zeta_{2m})$ in the cokernel. For any $\xi = (\xi_1, \dots, \xi_m) \in T_u\mathcal{B}$ and $Y \in T_J\mathcal{J}'^\mu$ the following terms vanish

$$\int \langle D_{u_j,J}\xi_j, \eta_j\rangle \mathrm{d}s\mathrm{d}t\,, \qquad \sum_{j=1}^m \int \langle Y\big(\partial_t u_j - X\big), \eta_j\rangle \mathrm{d}s\mathrm{d}t\,,$$

as well as $\langle \xi_j(-\infty), \zeta_{2j-1}\rangle + \langle \xi_j(\infty), \zeta_{2j}\rangle$ for all $j = 1, \dots, m$. By the first term we see that η_j is smooth for all $j = 1, \dots, m$. By the last term we see that ζ vanishes since there exists ξ with $\zeta = \mathrm{d}_u ev(\xi)$. To prove that also η vanishes choose a regular point $(s, t) \in \mathcal{R}(u_1, \dots, u_m)$ (cf. Definition 7.4.1). Then for any $j = 1, \dots, m$ we find Y supported in a small neighborhood of $(t, u_j(s, t))$ such that

$$\sum_{j=1}^m \int \langle Y\,(\partial_t u_j - X), \eta_j\rangle = \int \langle Y\,(\partial_t u_j - X), \eta_j\rangle > 0\,,$$

in contradiction to the second equation. By Proposition 7.4.3 regular points are dense. Hence η vanishes. We conclude that $\mathcal{J}^\mu_{\mathrm{reg},k} \subset \mathcal{J}^\mu$ is dense. The union

$$\mathcal{J}_{\mathrm{reg},k} := \bigcup_{\mu > 0} \mathcal{J}^\mu_{\mathrm{reg},k} \subset \mathcal{J}\,,$$

is open. It is also dense because for any $J \in \mathcal{J}$, there exists $\mu > 0$ such that $J \in \mathcal{J}^\mu$ and we find $J_\nu \in \mathcal{J}^\mu_{\mathrm{reg},k} \subset \mathcal{J}_{\mathrm{reg},k}$ converging to J. This shows that $\mathcal{J}_{\mathrm{reg},k} \subset \mathcal{J}$ is dense and open. Hence $\mathcal{J}_{\mathrm{reg}} = \bigcap_k \mathcal{J}_{\mathrm{reg},k}$ is comeager. □

7.4. Regular points

Fix $X = X_H$ be a Hamiltonian vector field for a clean Hamiltonian $H \in C^\infty([0,1] \times M)$ and $J : [0, 1] \to \mathrm{End}(TM, \omega)$ be a path of almost complex structures. We abbreviate the strip $\Sigma = \mathbb{R} \times [0, 1]$. Recall that a tuple of maps $(u_1, \dots, u_m)$ is *distinct* if $u_i = u_j \circ \tau_a$ for some $a \in \mathbb{R}$ implies that $j = i$.

Definition 7.4.1. Let $(u_1, \ldots, u_m)$ be a tuple of finite energy (J, X)-holomorphic strips with boundary in (L_0, L_1). A point $(s, t) \in \Sigma$ is a *regular point for* $(u_1, \ldots, u_m)$ if

(i) $\partial_s u_j(s, t) \neq 0$ for all $j = 1, \ldots, m$

(ii) $u_i(s, t) \neq \lim_{s' \to \pm\infty} u_j(s', t)$ for all $i, j = 1, \ldots, m$

(iii) for all $s' \in \mathbb{R}$ we have: $u_i(s, t) = u_j(s', t) \iff s' = s$ and $j = i$.

We denote this set of points by $\mathcal{R}(u_1, \ldots, u_m) \subset \Sigma$.

Lemma 7.4.2. *Let u be a (J, X)-holomorphic strip such that $\partial_s u \not\equiv 0$, then the set $C(u) := \{(s, t) \in \mathbb{R} \times [0, 1] \mid \partial_s u(s, t) = 0\}$ is finite.*

Proof. By a change of variables we assume $H = 0$ and that L_0 and L_1 intersect cleanly (see Lemma 3.2.4). By the asymptotic analysis we know that there exists $s_0 \in \mathbb{R}$ such that $\partial_s u(s, t) \neq 0$ for all $|s| \geq s_0$ and $t \in [0, 1]$ (cf. Corollary 4.1.3). By [31, Lemma 2.3] the set of critical points is discrete. This shows the claim. □

Proposition 7.4.3. *Given a distinct tuple $(u_1, \ldots, u_m)$ of (J, X)-holomorphic strips, then $\mathcal{R}(u_1, \ldots, u_m) \subset \Sigma$ is open and dense.*

Proof. The proof goes along the lines of [31, Theorem 4.3] or [33, Theorem 4.9]. Without loss of generality we assume that $H \equiv 0$ and L_0, L_1 intersect cleanly (cf. Lemma 3.2.4). We abbreviate the points $u_j(\pm\infty) := \lim_{s \to \pm\infty} u(s, \cdot) \in L_0 \cap L_1$ for $j = 1, \ldots, m$ and $\mathcal{R} := \mathcal{R}(u_1, \ldots, u_m)$.

Step 1. We show that $\mathcal{R} \subset \Sigma$ is open.

By contradiction let $(s_\nu, t_\nu) \subset \Sigma \setminus \mathcal{R}$ be a sequence such that $\lim_{\nu \to \infty}(s_\nu, t_\nu) = (s, t) \in \mathcal{R}$. Hence for all $\nu \in \mathbb{N}$ at least one of the following statements holds

(i) $\partial_s u_j(s_\nu, t_\nu) = 0$ for some $j = 1, \ldots, m$

(ii) $u_j(s_\nu, t_\nu) = u_i(-\infty)$ or $u_j(s_\nu, t_\nu) = u_i(\infty)$ for some $i, j = 1, \ldots, m$,

(iii) $u_i(s_\nu, t_\nu) = u_j(s'_\nu, t_\nu)$ for some $s'_\nu \in \mathbb{R}$ and $i \neq j$

(iv) $u_j(s_\nu, t_\nu) = u_j(s'_\nu, t_\nu)$ for some $s'_\nu \neq s_\nu$ and $j = 1, \ldots, m$

In the first case we argue by continuity that $\partial_s u_j(s, t) = 0$ in contradiction to $(s, t) \in \mathcal{R}$. Similarly we exclude the second case. Suppose that the third statement holds after passing to a subsequence for all $\nu \in \mathbb{N}$. If (s'_ν) is unbounded then without loss of generality we have $s'_\nu \to \infty$ hence $u_j(\infty) \leftarrow u_j(s'_\nu, t_\nu) = u_i(s_\nu, t_\nu) \to u_i(s, t)$, which contradicts the fact that (s, t) is a regular point. If (s'_ν) is bounded, then after possibly passing to a subsequence we have $s'_\nu \to s'$ and $u_j(s', t) \leftarrow u_j(s'_\nu, t_\nu) = u_i(s_\nu, t_\nu) \to u_i(s, t)$, hence $u_j(s', t) = u_i(s, t)$, which again contradicts the fact that (s, t) is a regular point. Suppose finally that the last case holds after passing to a subsequence for all $\nu \in \mathbb{N}$. If (s'_ν) is unbounded, then without loss of generality $s'_\nu \to \infty$ and we obtain $u_j(s, t) \leftarrow u_j(s_\nu, t_\nu) = u_j(s'_\nu, t_\nu) \to u_j(\infty)$. This shows that $u_j(s, t) = u_j(\infty)$ in contradiction to

$(s,t) \in \mathcal{R}$. If on the other hand (s'_ν) is bounded then $s'_\nu \to s'$ without loss of generality. If $s' \neq s$ we conclude that $u_j(s,t) = u_j(s',t)$ in contradiction to the fact that (s,t) is regular and if $s' = s$ we conclude that $\partial_s u_j(s,t) = 0$ which again contradicts the fact that (s,t) is regular. We conclude that $\mathcal{R}$ is open.

Step 2. We show that $\mathcal{R} \subset \Sigma$ is dense under the additional assumption that $m = 1$.

Write $u = u_1$. Given any point $(s_1,t_1) \in \Sigma$ and $\varepsilon > 0$. We have to show that there exists a regular point in the ball $B_\varepsilon(s_1,t_1)$. Let $\mathrm{regv}(u) \subset M$ be the space of regular values of u. Since by Lemma 7.4.2 the set of critical points is finite we assume after possibly replacing (s_1,t_1) by a point which is ε-close and decreasing ε, that for all $(s,t) \in B_\varepsilon(s_1,t_1)$ we have

$$u(s,t) \in \mathrm{regv}(u), \qquad\qquad u(s,t) \neq u(\pm\infty). \tag{7.4.1}$$

We claim that this implies that for all $(s,t) \in B_\varepsilon(s_1,t_1)$ the set $u^{-1}(u(s,t))$ is finite. Indeed, assume by contradiction that we find a sequence $(s_\nu,t_\nu) \subset \Sigma$ consisting of distinct points and $u(s_\nu,t_\nu) = u(s,t)$ for some $(s,t) \in B_\varepsilon(s_1,t_1)$ and all $\nu \in \mathbb{N}$. If (s_ν) is unbounded then after possibly passing to a subsequence we have $s_\nu \to \pm\infty$ and $u(s,t) = u(s_\nu,t_\nu) = u(\pm\infty)$ in contradiction to (7.4.1) and if (s_ν) is bounded then after possibly passing to a subsequence we have $s_\nu \to s'$ and $t_\nu \to t'$, which shows that $u(s_\nu,t_\nu) = u(s',t') = u(s,t)$ for all $\nu \in \mathbb{N}$ and hence $\mathrm{d}u(s',t') = 0$ in contradiction to $u(s,t) \in \mathrm{regv}(u)$. Now define

$$u^{-1}(u(s_1,t_1)) \cap \mathbb{R} \times \{t_1\} = \{(s_1,t_1),(s_2,t_1),\dots,(s_\ell,t_1)\}\,. \tag{7.4.2}$$

For $\delta > 0$ we define

$$F_\delta := \{(s,t) \in \Sigma \mid \exists\ (s',t) \in B_\delta(s_1,t_1),\ u(s,t) = u(s',t)\}\,.$$

We claim that for all $r > 0$ there exist $\delta > 0$ such that

$$F_\delta \subset B_r(s_1,t_1) \cup B_r(s_2,t_1) \cup \dots \cup B_r(s_\ell,t_1)\,. \tag{7.4.3}$$

If not then we find $r > 0$ and sequences $(s'_\nu),(s_\nu) \subset \mathbb{R}$, $(t_\nu) \subset [0,1]$ with $u(s'_\nu,t_\nu) = u(s_\nu,t_\nu)$, $(s_\nu,t_\nu) \to (s_1,t_1)$ and $(s'_\nu,t_\nu) \notin B_r(s_j,t_1)$ for any $j = 1,\dots,\ell$. If (s'_ν) is unbounded we find a diverging subsequence $s'_\nu \to \pm\infty$ and we conclude $u(s_1,t_1) \leftarrow u(s_\nu,t_\nu) = u(s'_\nu,t_\nu) \to u(\pm\infty)$ in contradiction to $u(s_1,t_1) \neq u(\pm\infty)$. On the other hand if (s'_ν) is bounded by possibly passing to a subsequence we assume without loss of generality that $s'_\nu \to s'$, and $u(s',t_1) \leftarrow u(s'_\nu,t_\nu) = u(s_\nu,t_\nu) \to u(s_1,t_1)$. Hence $u(s',t_1) = u(s_1,t_1)$ and by (7.4.2) we have $s' \in \{s_1,\dots,s_\ell\}$. But this is in contradiction to $s' \notin B_{r/2}(s_j,t_1)$ for all $j = 1,\dots,\ell$. We conclude (7.4.3).

By possibly decreasing ε again we assume that u restricted to $B_\varepsilon(s_j,t_1)$ is an embedding for all $j = 1,\dots,\ell$ and for all $i,j = 1,\dots,\ell$ with $i \neq j$ we have

$$(s_j - \varepsilon, s_j + \varepsilon) \cap (s_i - \varepsilon, s_i + \varepsilon) = \emptyset. \tag{7.4.4}$$

Note that by (7.4.1) the map u is already an immersion restricted to $B_\varepsilon(s_j, t_1)$. Choose $\delta < \varepsilon$ such that (7.4.3) holds for $r = \varepsilon$. We assume that $\ell \geq 2$ because otherwise $(s_1, t_1) \in \mathcal{R}(u)$ and we are finished. Let $\mathrm{cl}\big(A\big)$ denote the closure of any subset $A \subset \Sigma$. For $j = 2, \dots, \ell$ we define

$$\Sigma_j := \{(s,t) \in \mathrm{cl}\big(B_{\delta/2}(s_1,t_1)\big) \mid \exists\ (s',t) \in B_\varepsilon(s_j,t_1),\ u(s,t) = u(s',t)\} .$$

By (7.4.3) we obtain the same set when replacing $B_\varepsilon(s_j, t_1)$ with $\mathrm{cl}\big(B_\varepsilon(s_j, t_1)\big)$ in the definition, which implies that Σ_j is closed for all $j = 2, \dots, \ell$. Again by (7.4.3) we have

$$\mathrm{cl}\big(B_{\delta/2}(s_1,t_1)\big) = \mathrm{cl}\big(\mathcal{R}(u) \cap B_{\delta/2}(s_1,t_1)\big) \cup \Sigma_2 \cup \Sigma_3 \cup \cdots \cup \Sigma_\ell .$$

Suppose by contradiction that $\mathcal{R}(u) \cap B_{\delta/2}(s_1, t_1) = \emptyset$. Since for all $j = 2, \dots, \ell$ the set Σ_j is closed, there must exists $j_0 = 2, \dots, \ell$ such that Σ_{j_0} contains an open subset. We assume without loss of generality that there exists $\rho > 0$ and $(\hat{s}_1, \hat{t}_1) \in B_{\delta/2}(s_1, t_1)$ such that $B_\rho(\hat{s}_1, \hat{t}_1) \subset \Sigma_2$. By possibly making ρ even smaller we assume that $B_\varepsilon(s_2, t_1) \cap B_\rho(\hat{s}_1, \hat{t}_1) = \emptyset$. Define

$$\Omega := u^{-1}(u(B_\rho(\hat{s}_1,\hat{t}_1)) \cap B_\varepsilon(s_2,t_1) \subset \Sigma ,$$

which is an open subset because u restricted to $B_\rho(\hat{s}_1, \hat{t}_1)$ is an embedding. We have the diffeomorphism

$$\phi := u_2^{-1} \circ u_\rho : B_\rho(\hat{s}_1,\hat{t}_1) \xrightarrow{\sim} \Omega ,$$

where u_2 and u_ρ denotes the map u restricted to $B_\varepsilon(s_2, t_1)$ and $B_\rho(\hat{s}_1, \hat{t}_1)$ respectively. In particular for all $(s, t) \in B_\rho(\hat{s}_1, \hat{t}_1)$ there exists uniquely $(s'', t'') = \phi(s, t) \in \Omega$ such that $u(s'', t'') = u(s, t)$. On the other hand by construction there exists $(s', t) \in B_\varepsilon(s_2, t_1)$ such that $u(s, t) = u(s', t)$. This implies that $(s', t) \in \Omega$ and by uniqueness $(s', t) = (s'', t'')$. We see that $\phi(s, t) = (\kappa(s, t), t)$ for some map $\kappa : B_\rho(\hat{s}_1, \hat{t}_1) \to \mathbb{R}$ or equivalently $u(\kappa(s, t), t) = u(s, t)$. Since u is J-holomorphic we compute

$$0 = \partial_s u + J\partial_t u = \partial_s u \partial_s \kappa + J(\partial_s u \partial_t \kappa + \partial_t u) = \partial_s u\,(\partial_s \kappa - 1) + \partial_t u\,\partial_t \kappa . \tag{7.4.5}$$

Since u restricted to $B_\rho(\hat{s}_1, \hat{t}_1)$ is an immersion, we see that $\partial_t \kappa \equiv 0$ and $\partial_s \kappa \equiv 1$. This implies that there exists $a \in \mathbb{R}$ such that $\kappa(s, t) = \kappa(s) = s + a$. We claim that $a \neq 0$. Assume by contradiction that $a = 0$, then we have $\kappa(\hat{s}_1) = \hat{s}_1 \in (s_2 - \varepsilon, s_2 + \varepsilon)$ and $\hat{s}_1 \in (s_1 - \varepsilon, s_1 + \varepsilon)$. But after (7.4.4) the sets $(s_2 - \varepsilon, s_2 + \varepsilon)$ and $(s_1 - \varepsilon, s_1 + \varepsilon)$ have an empty intersection. We have deduced that $u(s + a, t) = u(s, t)$ for all $(s, t) \in B_\rho(\hat{s}_1, \hat{t}_1)$ and by unique continuation we have $u \equiv u \circ \tau_a$ with $a \neq 0$. This contradicts the fact that the energy of u is finite.

Step 3. We proof that $\mathcal{R}$ is dense with any m.

Given a point $(s_1, t_1) \in \Sigma$ and $\varepsilon > 0$. By possibly replacing (s_1, t_1) with a point which is ε-close and decreasing ε we assume that for all $(s, t) \in B_\varepsilon(s_1, t_1)$ and $i, j = 1, \dots, \ell$ we have

$$u_i(s,t) \in \mathrm{regv}(u_j), \qquad u_i(s,t) \neq u_j(\pm\infty) . \tag{7.4.6}$$

We claim that the set $u_j^{-1}(u_i(s,t))$ is finite for all $(s,t) \in B_\varepsilon(s_1,t_1)$ and $i,j = 1,\dots,m$. Suppose by contradiction that there exists i,j and a sequence (s_ν,t_ν) of distinct points such that $u_j(s_\nu,t_\nu) = u_i(s,t)$. If (s_ν) is unbounded, then without loss of generality $s_\nu \to \pm\infty$, $t_\nu \to t$ and hence $u_j(\pm\infty) = u_j(s_\nu,t_\nu) = u_i(s,t)$ contradicting (7.4.6). If (s_ν) is bounded, then without loss of generality $s_\nu \to s'$, $t_\nu \to t$, $u_j(s',t) = u_j(s_\nu,t_\nu) = u_i(s,t)$ and hence $\partial_s u_j(s',t) = 0$. This contradicts the fact that $u_i(s,t) \in \mathrm{regv}(u_j)$.

By the last step and yet again moving (s_1,t_1) and decreasing ε, we assume without loss of generality that $B_\varepsilon(s_1,t_1) \subset \mathcal{R}(u_j)$ for all $j = 1,\dots,m$. Define

$$\bigcup_{1\le i,j\le m} u_i^{-1}(u_j(s_1,t_1)) \cap \mathbb{R}\times\{t_1\} = \{(s_1,t_1),(s_2,t_1),\dots,(s_\ell,t_1)\}\,.$$

For $\delta > 0$ define

$$F_\delta := \{(s,t) \in \Sigma \mid \exists\ (s',t) \in B_\delta(s_1,t_1) \text{ with } u_i(s',t) = u_j(s,t) \text{ for some } i,j\}\,.$$

By the same argument as in last step we conclude that for all $r > 0$ there exists $\delta > 0$ such that

$$F_\delta \subset B_r(s_1,t_1) \cup B_r(s_2,t_1) \cup \dots \cup B_r(s_\ell,t_1)\,. \tag{7.4.7}$$

Fix $\delta < \varepsilon$ such that (7.4.7) holds for $r = \varepsilon$. For $k = 1,\dots,\ell$ and $i \neq j$ we define

$$\Sigma_{i,j,k} := \{(s,t) \in \mathrm{cl}\big(B_{\delta/2}(s_1,t_1)\big) \mid \exists\ (s',t) \in B_\varepsilon(s_k,t_1) \text{ with } u_i(s',t) = u_j(s,t)\}\,.$$

By (7.4.7) and the assumption that $B_\varepsilon(s_1,t_1) \subset \mathcal{R}(u_j)$ for all $j = 1,\dots,m$ we have

$$\mathrm{cl}\big(B_{\delta/2}(s_1,t_1)\big) = \mathrm{cl}\big(\mathcal{R}(u_1,\dots,u_m) \cap B_{\delta/2}(s_1,t_1)\big) \cup \bigcup\nolimits_{i,j,k} \Sigma_{i,j,k}\,.$$

Arguing indirectly assume that $\mathcal{R}(u_1,\dots,u_m) \cap B_{\delta/2}(s_1,t_1) = \emptyset$. Without loss of generality there exists $\rho > 0$ and $(\hat{s}_1,\hat{t}_1) \in B_{\delta/2}(s_1,t_1)$ such that $B_\rho(\hat{s}_1,\hat{t}_1) \subset \Sigma_{2,1,k}$ for some $k = 1,\dots,\ell$. Define the open subset

$$\Omega := u_1^{-1}\big(u_2(B_\rho(\hat{s}_1,\hat{t}_1))\big) \cap B_\varepsilon(s_k,t_1) \subset \Sigma.$$

Hence for all $(s,t) \in B_\rho(\hat{s}_1,\hat{t}_1)$ there exists uniquely $(s'',t'') \in \Omega$ such that $u_1(s'',t'') = u_2(s,t)$. On the other hand since $B_\rho(\hat{s}_1,\hat{t}_1) \subset \Sigma_{2,1,k}$ there exists $(s',t) \in B_\varepsilon(s_k,t_1)$ such that $u_1(s',t) = u_2(s,t)$. This implies that $(s',t) \in \Omega$ and by uniqueness $(s'',t'') = (s',t)$. We conclude that there exists a map $\kappa : \mathcal{B}_\rho(\hat{s}_1,\hat{t}_1) \to \mathbb{R}$ such that $u_1(\kappa(s,t),t) = u_2(s,t)$ for all $(s,t) \in B_\rho(\hat{s}_1,\hat{t}_1)$. Since both u_1 and u_2 are J-holomorphic we conclude by a computation similar to (7.4.5) that $\kappa(s,t) = \kappa(s) = s + a$ for some $a \in \mathbb{R}$. Hence $u_1(s+a,t) = u_2(s,t)$ for all $(s,t) \in B_\rho(\hat{s}_1,\hat{t}_1)$ and by unique continuation $u_1 \equiv u_2 \circ \tau_a$ in contradiction to the fact that the tuple $(u_1,\dots,u_m)$ is distinct. $\square$

7.5. Floer's ε-norm

Fix some $p > 1$, $\Omega \subset \mathbb{R}^\ell$ a subset with Lipschitz type boundary and a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers $\varepsilon_k > 0$. Given a smooth function with compact support $f \in C_0^\infty(\Omega)$ we define the norm

$$\|f\|_\varepsilon := \sum_{k\geq 0} \varepsilon_k \|f\|_{H^{k,p}(\Omega)} ,$$

and the subspace $C_0^\varepsilon(\Omega) \subset C_0^\infty(\Omega)$ by

$$C_0^\varepsilon(\Omega) = \{f \in C_0^\infty(\Omega) \mid \|f\|_\varepsilon < \infty\} .$$

Floer originally used C^k-norms instead of Sobolev norms, but after the Sobolev embedding theorem the norm defined here is equivalent. We have chosen this approach because it suits better when considering domains with boundary.

Lemma 7.5.1. *If Ω is bounded then the space $C_0^\varepsilon(\Omega)$ with the topology induced by the norm $\|\cdot\|_\varepsilon$ is a complete and separable space. In particular $(C_0^\varepsilon(\Omega), \|\cdot\|_\varepsilon)$ is a separable Banach space.*

Proof. See [68, Lemma 4.2.7] and [68, Lemma 4.2.9]. □

Clearly $C_0^\varepsilon(\Omega) \subset C_0^\infty(\Omega)$ is continuous. The next lemma states that for certain sequences this inclusion is dense. It is a slight generalization of [68, Lemma 4.2.8] allowing boundary values.

Lemma 7.5.2. *Given $\ell \in \mathbb{N}$ there exists a sequence (ε_k) such that the inclusion $C_0^\varepsilon(\Omega) \subset C_0^\infty(\Omega)$ is dense for all subsets $\Omega \subset \mathbb{R}^\ell$ with Lipschitz type boundary.*

Proof. Fix some $p > 1$ and denote by $B_r \subset \mathbb{R}^\ell$ the ball of radius $r > 0$ centered at the origin. Choose a smooth function $\rho : \mathbb{R}^\ell \to [0, 1]$ with $\operatorname{supp}\rho \subset B_1$ and $\int_{\mathbb{R}^\ell} \rho = 1$. Then set $\rho_\delta(x) = \rho(x/\delta)$ for $\delta > 0$. Note that we have $\operatorname{supp}\rho_\delta \subset B_\delta(0)$ and $\partial^\alpha\rho_\delta = \delta^{-k}\partial^\alpha\rho$ with $k = |\alpha|$. Define

$$\varepsilon_k := (a_k k^k)^{-1}, \qquad a_k := \|\rho\|_{H^{k,p}} .$$

Now let $\varepsilon > 0$ and $f \in C_0^\infty(\Omega)$ be any given smooth function with compact support. Fix $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon$. Using cut-off functions we find $g \in H_0^{m,p}(\mathbb{R}^\ell)$ such that $\|g - f\|_{H^{m,p}(\Omega)} \leq \varepsilon/4$ (see [53, Exercise B.1.3]). Secondly we find $\delta > 0$ such that the smooth and compactly supported function $h = \rho_\delta * g$ satisfies $\|g - h\|_{H^{m,p}(\mathbb{R}^\ell)} < \varepsilon/4$. Then we have

$$\begin{aligned} \operatorname{dist}(f, h)_{C^\infty(\Omega)} &= \sum_{k\geq 0} \frac{\|f-h\|_{k,p;\Omega}}{1 + \|f-h\|_{k,p;\Omega}} 2^{-(k+1)} \leq \|f-h\|_{m,p;\Omega} + 2^{-(m+1)} \\ &\leq \|f-g\|_{m,p;\Omega} + \|g-h\|_{m,p;\Omega} + 2^{-(m+1)} \leq \varepsilon . \end{aligned}$$

This shows that h lies in the ε-ball about the function f in the C^∞-topology. It remains to show that $h \in C^\varepsilon(\Omega)$. Indeed, by Young's inequality we have

$$\varepsilon_k \|h\|_{H^{k,p}} = \varepsilon_k \|g * \rho_\delta\|_{H^{k,p}} \leq \varepsilon_k \|g\|_{L^1} \|\rho_\delta\|_{H^{k,p}} \leq \varepsilon_k a_k \delta^{-k} \|g\|_{L^1} \leq 2^{-k} \|g\|_{L^1} ,$$

for every $k > 2\delta^{-1}$. This shows that the ε-norm of h is finite or equivalently that $h \in C^\varepsilon(\Omega)$. $\square$

Let $E \to M$ be any vector bundle over a compact Riemannian manifold with boundary and corners. Choose a connection ∇ and a Riemannian metric on E. This induces connections and a metric on $E \otimes F$, where F is any tensor bundle over M. Let vol_M be a volume form. Define the norm

$$\|\xi\|_p := \left(\int_M |\xi|^p \,\mathrm{vol}_M \right)^{1/p} ,$$

and recursively for all $k \in \mathbb{N}$ define the norms

$$\|\xi\|_{0,p} := \|\xi\|_p , \qquad \|\xi\|_{k,p} := \|\nabla\xi\|_{k-1,p} .$$

Definition 7.5.3. Let $p > 1$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence. We define the space $\Gamma^\varepsilon(E) \subset \Gamma^\infty(E)$ to be the space of all smooth sections ξ which are bounded in the norm

$$\|\xi\|_\varepsilon = \sum_{k\geq 0} \varepsilon_k \|\xi\|_{k,p} .$$

Proposition 7.5.4. *Suppose that M is compact with boundary and corners. The space $\Gamma^\varepsilon(E)$ with norm $\|\cdot\|_\varepsilon$ is a separable Banach space and there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that the inclusion $\Gamma^\varepsilon(E) \subset \Gamma^\infty(E)$ is dense for the C^∞-topology.*

Proof. Choose a local trivialization of E over charts of M which are adapted to the boundary ∂M and an associated partition of unity. Then the norm $\|\xi\|_\varepsilon$ of any section $\xi \in \Gamma(E)$ is equivalent to the finite sum of the ε-norm of its local representatives. Then the claim follows from Lemma 7.5.1 and 7.5.2. $\square$

Lemma 7.5.5. *Fix paths J_-, $J_+ : [0,1] \to \mathrm{End}(TM, \omega)$. For any $s_1 > 0$ consider the space $\mathcal{J} := \{J \in C^\infty(\mathbb{R} \times [0,1], \mathrm{End}(TM, \omega) \mid J(\pm s, \cdot) = J_\pm \forall\ s \geq s_1\}$. There exists a dense subspace $\mathcal{J}' \subset \mathcal{J}$ which is a separable Banach manifold. The same holds for $\mathcal{J} := C^\infty([0,1], \mathrm{End}(TM, \omega)$.*

Proof. For any $J \in \mathcal{J}$ we define the linear bundle $S_J \to \Sigma \times M$ where the fibre of S_J over a point $(s,t,p) \in \Sigma \times M$ is given by linear maps $Y \in \mathrm{End}(T_pM)$ such that

$$YJ(s,t,p) + J(s,t,p)Y = 0, \qquad \omega_p(Y\xi, \xi') + \omega_p(\xi, Y\xi') = 0 ,$$

for all vectors $\xi, \xi' \in T_pM$. The tangent space $T_J\mathcal{J}$ is given by smooth sections $Y \in \Gamma(S_J)$ with support contained in $[-s_1, s_1] \times [0,1] \times M$. Fix any $J_0 \in \mathcal{J}$, we identify the space $\mathcal{J}$ with the C^0-unit ball in the space smooth sections $Y \in \Gamma(S_{J_0})$ with support in $[-s_1, s_1] \times [0,1] \times M$, by $J \mapsto Y_J := (J + J_0)^{-1}(J - J_0)$. The inverse is $Y \mapsto J_0(1-Y)^{-1}(1+Y)$. For further details see [68, Section 4.2]. By Proposition 7.5.4 there exists a sequence $\varepsilon := (\varepsilon_\ell)_{\ell \in \mathbb{R}}$ such that the subspace is dense

$$\mathcal{J}' = \{J \in \mathcal{J} \mid \|Y_J\|_\varepsilon < \infty\} \subset \mathcal{J}\,.$$

By the same lemma we see that the space $\mathcal{J}'$ is identified with an open subset in a separable Banach space. $\square$

8. Gluing

All algebraic statements for Floer homology in this work are based on a gluing result of holomorphic strips, which is in a sense the converse for the Floer-Gromov breaking phenomenon. Originally the problem has been addressed and solved by Floer in the series of papers [25], [27] and [28] under the assumption that the holomorphic strips have boundary in two transversely intersecting Lagrangians. The generalization for the degenerate case in which both Lagrangians are equal was worked out by Fukaya, Oh, Ohta and Ono [36, Chapter 7]. In this chapter we give a further generalization of the gluing theorem for holomorphic strips with boundary on two cleanly intersecting Lagrangians. Our approach is not new and was previously sketched out by Frauenfelder in [33, Chapter 4.7]. Since we need precise statements for the construction of coherent orientations we give a complete proof here. We follow closely the lines of [36] as well as the gluing results of [5] and [11]. Very recently another approach by Simčević has been developed in [71] using completely different methods of interpolation theory. At the end of the chapter we also give a small generalization of a gluing result in [1] which is for classical Morse theory.

8.1. Setup and main statement

Let (M,ω) be a symplectic manifold and $L_0, L_1 \subset M$ closed Lagrangian submanifolds. Fix admissible vector fields X_0, X_1 and admissible almost complex structure J_0, J_1 (cf. Definition 5.1.1) such that $X_0(s,\cdot) = X_1(-s,\cdot)$ and $J_0(s,\cdot) = J_1(-s,\cdot)$ for all s large enough. Abbreviate by $\mathcal{M}_k$ the moduli space of all (J_k, X_k)-holomorphic strips modulo reparametrization. We denote by

$$\mathcal{M}^1(W_-, W_+) := \left\{ (u_0,u_1) \in \mathcal{M}_0 \times \mathcal{M}_1 \;\middle|\; \begin{array}{l} u_0(-\infty) \in W_- \\ u_0(\infty) = u_1(-\infty) \\ u_1(\infty) \in W_+ \end{array} \right\}, \tag{8.1.1}$$

for some fixed submanifolds W_- and W_+ in the space of perturbed intersection points (cf. Section 7.1). We distinguish three cases and define (J,X)

(A) both (J_0,X_0) and (J_1,X_1) are $\mathbb{R}$-invariant, $(J,X) := (J_0,X_0) = (J_1,X_1)$

(B) either $(J,X) := (J_0,X_0)$ or $(J,X) := (J_1,X_1)$ is $\mathbb{R}$-dependent,

(C) both (J_0,X_0) and (J_1,X_1) are $\mathbb{R}$-dependent, then $(J,X) = (J_R,X_R)_{R\geq R_0}$ with $J_R = J_0\#_R J_1$ and $X_R = X_0\#_R X_1$ (cf. Section 7.2.1)

Mostly the arguments are the same for these three cases and we only distinguish them at parts where it is necessary. We glue a pair $(u_0, u_1) \in \mathcal{M}^1(W_-, W_+)$ at the point $u_0(\infty) = u_1(-\infty)$ to obtain a family of strips in the space (cf. equation (7.1.1) or (7.2.6))

$$\mathcal{M}(W_-, W_+) := \widetilde{\mathcal{M}}(W_-, W_+; J, X)/\sim \ . \tag{8.1.2}$$

We say that J_0 and J_1 are *regular* if (cf. Definitions 7.1.1 and 7.2.2)

- J_k is regular for X_k for $k = 0, 1$,
- in case (C), the glued structure (J_R) is regular for (X_R) and
- the spaces $\mathcal{M}^1(W_-, W_+)$ and $\mathcal{M}(W_-, W_+)$ are cut-out transversely.

Consequently each connected component of the above spaces is a manifold and as usual we denote with the subscrip $[d]$ the union of all d-dimensional components.

Theorem 8.1.1. *Assume that the almost complex structures J_0 and J_1 are regular. Given a pair $u = (u_0, u_1) \in \mathcal{M}^1(W_-, W_+)_{[0]}$. There exists R_0 and a continuous map*

$$\mathcal{G}_u : [R_0, \infty) \to \mathcal{M}(W_-, W_+)_{[1]}\,, \qquad R \mapsto w_R\,,$$

such that

(i) (w_R) Floer-Gromov converges to u as $R \to \infty$,

(ii) given a sequence $(w^\nu) \subset \mathcal{M}(W_-, W_+)_{[1]}$ which Floer-Gromov converges to u, then w^ν lies in the image of the map $\mathcal{G}_u$ for all but finitely many ν.

Moreover with orientations given in Lemma 8.7.1, the space

$$\overline{\mathcal{M}}(W_-, W_+)_{[1]} := \mathcal{M}(W_-, W_+)_{[1]} \sqcup \mathcal{M}^1(W_-, W_+)_{[0]}\,,$$

is an oriented manifold with oriented boundary $(-1) \cdot \mathcal{M}^1(W_-, W_+)_{[0]}$ if (X_1, J_1) is $\mathbb{R}$-invariant and $\mathcal{M}^1(W_-, W_+)_{[0]}$ otherwise.

Proof. The proof covers the rest of the chapter. Here we give an overview of the principal arguments. Basically we follow the standard gluing procedure, which we quickly recall now. Fix a rigid pair $(u_0, u_1) \in \mathcal{M}^1(W_-, W_+)_{[0]}$ and a large enough gluing parameter $R \geq R_0$. We denote the glued structures $J_R := J_0 \#_R J_1$ and $X_R := X_0 \#_R X_1$ (cf. equations (7.2.5) and (7.2.4)). We define the *preglued map* u_R using cut-off functions and then roughly speaking solve the equation $\partial_s w + J_R(w)(\partial_t w - X_R(w)) = 0$ for w in a neighborhood of u_R using the Newton-Picard theorem. More precisely given a small vector field ξ along u_R and write the map w as $w(s,t) = \exp_{u_R(s,t)} \xi(s,t)$ with respect to some exponential function associated to an axillary Levi-Civita connection. Then w is (J_R, X_R)-holomorphic if and only if ξ is a zero of a non-linear map $\mathcal{F}_R$ defined on an open ball in a Banach space of sections of u_R^*TM (cf. equation (8.4.1)). Since we work with degenerated asymptotics which require exponential weights, the Sobolev norms which we work with have adapted weights that depend on the gluing parameter (cf. Section 8.2). In the assumptions of the Newton-Picard theorem we need a bound on the right-inverse of the differential of $\mathcal{F}_R$ at zero, denoted D_R, which does not depend on R. The right inverse Q_R is constructed in (8.3.15) and the uniform bound is established in Corollary 8.3.5. Moreover we need a quadratic estimate (cf. Lemma 8.4.1). Then all (J_R, X_R)-holomorphic strips in a neighborhood of u_R are modeled on the kernel of D_R, i.e. for each element $\xi' \in \ker D_R$ small enough there exists an unique element $\xi'' := \sigma_R(\xi) \in \operatorname{im} Q_R$ such that $(s,t) \mapsto \exp_{u_R(s,t)}(\xi'(s,t) + \xi''(s,t))$ is a (J_R, X_R)-holomorphic strip and any w close enough to u_R is of that form (cf. Lemma 8.4.2). In particular the map $v_R := \exp_{u_R} \sigma_R(0)$ is (J_R, X_R)-holomorphic. We define the gluing map $\mathcal{G}_u(R) = w_R$ where

- in case (A) $w_R = [v_R]$ the equivalence class modulo reparametrizations,
- in case (B) for all $(s,t) \in \mathbb{R} \times [0,1]$ we define

$$w_R(s,t) = \begin{cases} v_R(s-2R,t) & \text{if } (J_1, X_1) \text{ is } \mathbb{R}\text{-invariant} \\ v_R(s+2R,t) & \text{if } (J_0, X_0) \text{ is } \mathbb{R}\text{-invariant}\,. \end{cases}$$

- in case (C) $w_R = v_R$.

That the gluing map is continuous is proven in Lemma 8.5.1, that it is asymptotically surjective is proven in Lemma 8.6.2 and the statement about the orientations is proven in Proposition 8.7.4. □

8.2. Pregluing

In this section we introduce the Sobolev framework. The main ideas in this chapter are straight-forward generalizations of the methods of [36, Chapter 7.1]. We assume for simplicity that $X_0 \equiv 0$, $X_1 \equiv 0$ and W_-, W_+ lie on different connected components. Choose an auxiliary metric on M such that W_-, W_+, L_0 and L_1 are totally geodesic (cf. Lemma 6.1.6). All norms, parallel transport and exponential maps in the following sections are induced by this metric. For the general case where X_0, $X_1 \not\equiv 0$ or W_- and W_+ lie on the same connected component, we need to work with metrics that depend on the domain as explained in the proof of Lemma 6.1.5.

Preglued strip From now that the pair $u = (u_0, u_1) \in \mathcal{M}^1(W_-, W_+)_{[0]}$ is fixed. In case (A) or (B), the maps u_0 and u_1 are unparametrized. We choose parametrizations and still denote the maps with the same symbol by abuse of notation. Due to exponential decay (see Theorem 4.1.1) there exists an intersection point $p = u_0(\infty) = u_1(-\infty) \in C$, a constant $s_0 \geq 0$ and two maps $\zeta_0 : [s_0, \infty) \times [0,1] \to T_pM$, $\zeta_1 : (-\infty, -s_0] \times [0,1] \to T_pM$ such that for all $s \geq s_0$ and $t \in [0,1]$ we have

$$u_0(s,t) = \exp_p \zeta_0(s,t), \qquad u_1(-s,t) = \exp_p \zeta_1(-s,t)\ .$$

We fix once and for all smooth cut-off functions

$$\beta^-,\ \beta^+ : \mathbb{R} \times [0,1] \to [0,1], \qquad \beta^-(-s,t) = \beta^+(s,t) = \begin{cases} 1 & \text{if } s \geq 1 \\ 0 & \text{if } s \leq 0\,. \end{cases} \tag{8.2.1}$$

For any $R \geq s_0$ large enough we define the *preglued strip* $u_R : \mathbb{R} \times [0,1] \to M$ via

$$u_R(s,t) = \begin{cases} u_1(s-2R,t) & \text{if } s \geq 1 \\ \text{see below} & \text{if } -1 \leq s \leq 1 \\ u_0(s+2R,t) & \text{if } s \leq -1\,, \end{cases} \tag{8.2.2}$$

and if $-1 \leq s \leq 1$ we use the interpolation

$$u_R(s,t) := \exp_p(\beta^-(s,t)\zeta_0(s+2R,t) + \beta^+(s,t)\zeta_1(s-2R,t))\,.$$

We frequently use the following decay property of the preglued map u_R in the neck region.

Lemma 8.2.1. *There exists constants c, R_0 and ι such that for all $R \geq R_0$ and $\mu < \iota$ we have*

$$|\mathrm{d}u_R(s,t)| + \mathrm{dist}\,(u_R(s,t), u_R(0,0)) \leq ce^{-\mu(2R-|s|)}\,.$$

for all $(s,t) \in [-2R, 2R] \times [0,1]$.

Proof. Set $p := u_R(0,0)$. By Proposition 6.1.9 the maps u_0 and u_1 have μ-decay. If $s \leq -1$ we have by definition $u_R = u_0 \circ \tau_{-2R}$ and the claim follows since u_0 has μ-decay. Similar for $s \geq 1$. If $|s| \leq 1$ and R is large enough $u_R(s,t)$ is close to p for all $t \in [0,1]$. By Corollary A.1.2 we have as $R \to \infty$

$$\begin{aligned}
&|\mathrm{d}u_R| + \mathrm{dist}\,(u_R(s,t), p) \\
&\leq O(1)\left(\left|\nabla\left(\beta^-\zeta_0 \circ \tau_{-2R} + \beta^+\zeta_1 \circ \tau_{2R}\right)\right| + |\zeta_0 \circ \tau_{-2R}| + |\zeta_1 \circ \tau_{2R}|\right) \\
&\leq O(1)\left(|\nabla\zeta_0 \circ \tau_{-2R}| + |\nabla\zeta_1 \circ \tau_{2R}| + |\zeta_0 \circ \tau_{-2R}| + |\zeta_1 \circ \tau_{2R}|\right) \\
&\leq O(1)\left(|\mathrm{d}u_0 \circ \tau_{-2R}| + |\mathrm{d}u_1 \circ \tau_{2R}| + \mathrm{dist}\,(u_0 \circ \tau_{-2R}, p) + \mathrm{dist}\,(u_1 \circ \tau_{2R}, p)\right) \\
&\leq O(1)e^{-\mu(2R-|s|)}\,.
\end{aligned}$$

This proves the lemma. □

Linear pregluing and breaking Choose $p > 2$, $\delta > 0$ and abbreviate

- $H_0 := T_{u_0}\mathcal{B}^{1,p;\delta}$ and $H_1 := T_{u_1}\mathcal{B}^{1,p;\delta}$,
- $L_0 := \mathcal{E}^{p;\delta}_{u_0}$ and $L_1 := \mathcal{E}^{p;\delta}_{u_1}$,
- $H_{01} \subset H_0 \oplus H_1$ consisting of pairs (ξ_0, ξ_1) such that $\xi_0(\infty) = \xi_1(-\infty)$,
- $H_R := T_{u_R}\mathcal{B}^{1,p;\delta}$ and $L_R := \mathcal{E}^{p;\delta}_{u_R}$ for any $R \geq s_0$.

Define the *linear pregluing operator* $\Theta_R : H_{01} \to H_R$, $(\xi_0, \xi_1) \mapsto \xi_R$ with

$$\xi_R(s,t) = \begin{cases} \xi_1(s-2R,t) & \text{if } s \geq R\,, \\ \text{see below} & \text{if } s \in [-R,R], \\ \xi_0(s+2R,t) & \text{if } s \leq -R\,. \end{cases} \tag{8.2.3}$$

and if $-R \leq s \leq R$ we use define (omitting the arguments for convenience)

$$\xi_R = \widehat{\Pi}_p^{u_R}\bar{\xi} + \beta^+_{-R}\left(\Pi^{u_R}_{u_1\circ\tau_{2R}}\xi_1 \circ \tau_{2R} - \widehat{\Pi}_p^{u_R}\bar{\xi}\right) + \beta^-_R\left(\Pi^{u_R}_{u_0\circ\tau_{-2R}}\xi_0 \circ \tau_{-2R} - \widehat{\Pi}_p^{u_R}\bar{\xi}\right)\,,$$

with notations $\bar{\xi} := \xi_0(\infty) = \xi_1(-\infty)$, $\tau_R : \Sigma \to \Sigma$, $(s,t) \mapsto (s-R,t)$, $\beta^+_{-R} = \beta^+ \circ \tau_{-R}$, $\beta^-_R = \beta^- \circ \tau_R$ and the parallel transport maps Π, $\widehat{\Pi}$ as given in (6.1.1). Finally define the *breaking operator* $\Xi_R : L_R \to L_0 \oplus L_1$, $\eta \mapsto (\eta_{0,R}, \eta_{1,R})$ via

$$\begin{aligned} \eta_{1,R}(s,t) &= \begin{cases} \eta(s+2R,t) & \text{if } s \geq -2R\,, \\ \text{see below} & \text{if } -2R-1 \leq s \leq -2R \\ 0 & \text{if } s \leq -2R-1\,, \end{cases} \\ \eta_{0,R}(s,t) &= \begin{cases} 0 & \text{if } s \geq 2R\,, \\ \text{see below} & \text{if } 2R-1 \leq s \leq 2R\,, \\ \eta(s-2R,t) & \text{if } s \leq 2R-1\,. \end{cases} \end{aligned} \tag{8.2.4}$$

For the interpolation we just use parallel transport. We do not need to use cut-off functions because the maps are only supposed to be of regularity L^p_{loc}. More precisely for $2R-1 \leq s \leq 2R$ and $t \in [0,1]$ we define

$$\eta_{0,R}(s,t) = \Pi^{u_0(s,t)}_{u_R(s-2R,t)}\eta(s-2R,t)\,,$$

and for $-2R-1 \leq s \leq -2R$ and $t \in [0,1]$ we define

$$\eta_{1,R}(s,t) = \Pi^{u_1(s,t)}_{u_R(s+2R,t)}\eta(s+2R,t)\,.$$

We now show that these constructions are uniformly continuous with respect to an adapted norm.

Adapted norms For $R > 0$ we define a weight function $\gamma_{\delta,R} : \mathbb{R} \to \mathbb{R}$

$$\gamma_{\delta,R}(s) = \begin{cases} e^{-\delta(2R+s)} & \text{if } s < -2R \\ e^{\delta(2R-|s|)} & \text{if } |s| < 2R \\ e^{\delta(s-2R)} & \text{if } s > 2R\,. \end{cases}$$

Given a curve $u \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$, we define weighted norms for all vector fields $\eta \in \mathcal{E}^{p;\delta}_u$

$$\|\eta\|_{p;\delta,R} := \Big(\int_\Sigma |\eta|^p\,\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t\Big)^{1/p}\,,$$

and for all vector fields $\xi \in T_u\mathcal{B}^{1,p;\delta}(C_-,C_+)$ we define the norm $\|\xi\|_{1,p;\delta,R}$ via

$$\begin{aligned}
&\Big(|\xi(0,0)|^p + \|\xi(-\infty)\|^p + \|\xi(-\infty)\|^p + \\
&+ \int_{\Sigma^{-2R}_{-\infty}} \left(\left|\xi - \widehat{\Pi}^u_{u(-\infty)}\xi(-\infty)\right|^p + \left|\nabla\left(\xi - \widehat{\Pi}^u_{u(-\infty)}\xi(-\infty)\right)\right|^p\right)\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t \\
&+ \int_{\Sigma^{2R}_{-2R}} \left(\left|\xi - \widehat{\Pi}^u_{u(0,0)}\xi(0,0)\right|^p + \left|\nabla\left(\xi - \widehat{\Pi}^u_{u(0,0)}\xi(0,0)\right)\right|^p\right)\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t \\
&+ \int_{\Sigma^{\infty}_{2R}} \left(\left|\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\right|^p + \left|\nabla\left(\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\right)\right|^p\right)\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t\Big)^{1/p} .
\end{aligned} \tag{8.2.5}$$

It is straight-forward to check that that for a fixed R these define equivalent norms (see [5, Lemma 5.8])

8.3. A uniform bounded right inverse

For the remaining statements to hold true the decay parameter $\delta > 0$ must be sufficiently small. The bound on δ depends on the spectral gap of the asymptotic operators given in (3.2.13). More precisely, we assume for the rest of the section:

$$2\delta < \iota, \qquad \iota := \min\{\iota(J^-_\infty), \iota(J_\infty), \iota(J^+_\infty)\}, \tag{8.3.1}$$

in which J^-_∞, J_∞ and J^+_∞ are paths of almost complex structures such that $J_0(-s,\cdot) = J^-_\infty$, $J_0(s,\cdot) = J_1(-s,\cdot) = J_\infty$ and $J_1(s,\cdot) = J^+_\infty$ for s large enough.

Lemma 8.3.1. *There exists constants c and R_0 such that for all $(\xi_0,\xi_1) \in H_{01}$ and $R \geq R_0$*

$$\|\Theta_R(\xi_0,\xi_1)\|_{1,p;\delta,R} \leq c\left(\|\xi_0\|_{1,p;\delta} + \|\xi_1\|_{1,p;\delta}\right) .$$

Proof. We follow the proof of [11, Prp. 4.7.5]. Fix $(\xi_0,\xi_1) \in H_{01}$ and denote $\xi_R := \Theta_R(\xi_0,\xi_1)$ and $p := u_R(0,0)$. By definition we have

$$\begin{aligned}
\|\xi_R\|^p_{1,p;\delta,R} = &\left\| \xi_0|_{\Sigma^0_{-\infty}} \right\|^p_{1,p;\delta} + \left\| \xi_1|_{\Sigma^\infty_0} \right\|^p_{1,p;\delta} + |\xi_R(0,0)|^p \\
&+ \int_{\Sigma^{2R}_{-2R}} \left(\left|\xi_R - \widehat{\Pi}^{u_R}_p\xi_R(0,0)\right|^p + \left|\nabla\left(\xi_R - \widehat{\Pi}^{u_R}_p\xi_R(0,0)\right)\right|^p\right)\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t .
\end{aligned} \tag{8.3.2}$$

Lets concentrate on the last summand. We deduce a pointwise estimate of

$$\xi_R - \widehat{\Pi}^{u_R}_p\xi_R(0,0) = \left(\xi_R - \widehat{\Pi}^{u_R}_p\bar{\xi}\right) + \widehat{\Pi}^{u_R}_p\left(\bar{\xi} - \xi_R(0,0)\right) , \tag{8.3.3}$$

and its covariant derivative. Abbreviate $u_{0,R} := u_0\circ\tau_{-2R}$, $u_{1,R} = u_1\circ\tau_{2R}$, $\xi_{0,R} := \xi_0\circ\tau_{-2R}$ and $\xi_{1,R} := \xi_1 \circ \tau_{2R}$. By definition of ξ_R, the first summand of the right hand side of (8.3.3) equals

$$\beta^-_R\left(\Pi^{u_R}_{u_{0,R}}\xi_{0,R} - \widehat{\Pi}^{u_R}_p\bar{\xi}\right) + \beta^+_{-R}\left(\Pi^{u_R}_{u_{1,R}}\xi_{1,R} - \widehat{\Pi}^{u_R}_p\bar{\xi}\right) . \tag{8.3.4}$$

Focusing now on the first summand of (8.3.4) and taking into account the support of the cut-off function we have to estimate the integral over the smaller strip $[-2R, R] \times [0,1]$ of the norm of

$$\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \widehat{\Pi}^{u_R}_p \bar{\xi} \,, \tag{8.3.5}$$

and its covariant derivative, since β^-_R vanishes on $[R, 2R] \times [0,1]$. Now on $[-2R, -1] \times [0,1]$ we have that $u_R = u_0 \circ \tau_{-2R}$ by Definition (8.2.2). Hence by substitution $s \mapsto s + 2R$ we have

$$\begin{aligned}
&\int_{\Sigma^{-1}_{-2R}} \left(\left|\xi_{0,R} - \widehat{\Pi}^{u_{0,R}}_p \bar{\xi}\right|^p + \left|\nabla\left(\xi_{0,R} - \widehat{\Pi}^{u_{0,R}}_p \bar{\xi}\right)\right|^p \right) e^{p\delta(2R-|s|)} \mathrm{d}s\mathrm{d}t \\
&\qquad = \int_{\Sigma^{2R-1}_0} \left(\left|\xi_0 - \widehat{\Pi}^{u_0}_p \bar{\xi}\right|^p + \left|\nabla\left(\xi_0 - \widehat{\Pi}^{u_0}_p \bar{\xi}\right)\right|^p \right) e^{p\delta s} \mathrm{d}s\mathrm{d}t \le \|\xi_0\|_{1,p;\delta} \,. \quad (8.3.6)
\end{aligned}$$

We estimate the same term on $[-1, R] \times [0,1]$. If R is large enough the distance of $u_R(s,t)$ to p is less then one third the injectivity radius for every $(s,t) \in [-1, R] \times [0,1]$. Hence without loss of generality we replace $\widehat{\Pi}$ by Π in the formula (8.3.5) and continue

$$\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_p \bar{\xi} = \Pi^{u_R}_{u_{0,R}} \left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right) + \left(\Pi^{u_R}_{u_{0,R}} \Pi^{u_{0,R}}_p \bar{\xi} - \Pi^{u_R}_p \bar{\xi}\right) \,. \tag{8.3.7}$$

For the first summand on the right-hand side we estimate using Corollary A.2.4 and Lemma 8.2.1

$$\begin{aligned}
\left|\Pi^{u_R}_{u_{0,R}} \left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right)\right| &= \left|\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right| \\
\left|\nabla\left(\Pi^{u_R}_{u_{0,R}} \left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right)\right)\right| &\le \left|\nabla\left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right)\right| + O(1) \left|\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right| \,.
\end{aligned}$$

For the second summand on the right-hand side of (8.3.7) we estimate using Corollary A.2.3 and Corollary A.2.4

$$\begin{aligned}
\left|\Pi^{u_R}_{u_{0,R}} \Pi^{u_{0,R}}_p \bar{\xi} - \Pi^{u_R}_p \bar{\xi}\right| &\le O\big(\operatorname{dist}(u_R, p) + \operatorname{dist}(u_{0,R}, p)\big) \left|\bar{\xi}\right| \\
\left|\nabla\left(\Pi^{u_R}_{u_{0,R}} \Pi^{u_{0,R}}_p \bar{\xi} - \Pi^{u_R}_p \bar{\xi}\right)\right| &\le O\big(|\mathrm{d}u_R| + |\mathrm{d}u_{0,R}|\big) \left|\bar{\xi}\right| \,.
\end{aligned}$$

In particular we see that both quantities are bounded by $O(\omega)\left|\bar{\xi}\right|$ with $\omega(s) = e^{-\mu(2R-|s|)}$. Use the last two estimates and the identity (8.3.7) to show

$$\begin{aligned}
\left|\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_p \bar{\xi}\right| &\le \left|\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right| + O(\omega) \left|\bar{\xi}\right| \,, \\
\left|\nabla(\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_p \bar{\xi})\right| &\le O(1) \left|\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right| + \left|\nabla\left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right)\right| + O(\omega) \left|\bar{\xi}\right| \,.
\end{aligned}$$

Integrating these pointwise estimates gives

$$\begin{aligned}
&\int_{\Sigma^R_{-1}} \left(\left|\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_p \bar{\xi}\right|^p + \left|\nabla\left(\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_p \bar{\xi}\right)\right|^p \right) e^{p\delta(2R-|s|)} \mathrm{d}s\mathrm{d}t \\
&\qquad \le O(1) \int_{\Sigma^{4R}_{2R-1}} \left(\left|\xi_0 - \Pi^{u_0}_p \bar{\xi}\right|^p + \left|\nabla\left(\xi_0 - \Pi^{u_0}_p \bar{\xi}\right)\right|^p \right) e^{p\delta|s|} \mathrm{d}s\mathrm{d}t + \\
&\qquad\qquad + O\big(e^{-2pR(\mu-\delta)}\big) \left|\bar{\xi}\right|^p \int_{-1}^{R} e^{p(\delta-\mu)|s|} \mathrm{d}s \,.
\end{aligned}$$

To show that the factor with $\left|\bar{\xi}\right|^p$ in the last summand is uniformly bounded we compute directly assuming without loss of generality that $R \geq 1$

$$\int_{-1}^{R} e^{p(\mu-\delta)|s|} \mathrm{d}s \leq 2 \int_{0}^{R} e^{p(\mu-\delta)s} \mathrm{d}s = \frac{2}{p(\mu-\delta)} \left(e^{pR(\mu-\delta)} - 1 \right) \leq O(e^{pR(\mu-\delta)}) \,.$$

The last estimate and estimate (8.3.6) shows that the integral

$$\int_{\Sigma_{-2R}^{2R}} \left(\left|\Pi_{u_{0,R}}^{u_R} \xi_{0,R} - \widehat{\Pi}_p^{u_R} \bar{\xi}\right|^p + \left|\nabla \left(\Pi_{u_{0,R}}^{u_R} \xi_{0,R} - \widehat{\Pi}_p^{u_R} \bar{\xi} \right)\right|^p \right) \gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t \,.$$

is bounded by $O(1) \left\|\xi_0\right\|_{1,p;\delta}^p$. Similarly we proceed with the second term of the summand (8.3.4) and find that the integral

$$\int_{\Sigma_{-2R}^{2R}} \left(\left|\xi_R - \widehat{\Pi}_p^{u_R} \bar{\xi}\right|^p + \left|\nabla \left(\xi_R - \widehat{\Pi}_p^{u_R} \bar{\xi} \right)\right|^p \right) \gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t \,,$$

is bounded by $O(1)(\left\|\xi_0\right\|_{1,p;\delta} + \left\|\xi_1\right\|_{1,p;\delta})^p$. For the last term of (8.3.3) we use the fact that $u_R(0,t) = p$, $\bar{\xi} = \xi_0(\infty) = \xi_1(-\infty)$, Lemma A.3.5 to show

$$\left|\xi_R(0,0) - \bar{\xi}\right| \leq \left|\xi_1(-2R,0) - \Pi_p^{u_1(-2R,0)} \xi_1(\infty)\right| + \left|\xi_0(2R,0) - \Pi_p^{u_0(2R,0)} \xi_0(\infty)\right|$$

and with Corollary A.2.4

$$\left|\nabla \widehat{\Pi}_p^{u_R} \left(\bar{\xi} - \xi_R(0,0) \right)\right| \leq O\big(|\mathrm{d}u_R| \big) \left|\bar{\xi} - \xi_R(0,0)\right| \,.$$

We conclude that both quantities are bounded by $O\big(e^{-2\delta R}\big)(\left\|\xi_0\right\|_{1,p;\delta} + \left\|\xi_1\right\|_{1,p;\delta})$ and after intergation we have

$$\begin{aligned} &\int_{\Sigma_{-2R}^{2R}} \left(\left|\widehat{\Pi}_p^{u_R} \bar{\xi} - \xi_R(0,0)\right|^p + \left|\nabla \left(\widehat{\Pi}_p^{u_R} \bar{\xi} - \xi_R(0,0) \right)\right|^p \right) \gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t \\ &\qquad \leq O(1) \left(\left\|\xi_0\right\|_{1,p;\delta}^p + \left\|\xi_1\right\|_{1,p;\delta}^p \right) \int_{-2R}^{2R} e^{-p\delta|s|} \mathrm{d}s \leq O(1) \left(\left\|\xi_0\right\|_{1,p;\delta} + \left\|\xi_1\right\|_{1,p;\delta} \right)^p \,. \end{aligned}$$

Now the claim follows from the last four estimates plugged into (8.3.2). □

Lemma 8.3.2. *For all R and for all $\eta \in L_R$ we have*

$$\left\|\eta_{0,R}\right\|_{p;\delta}^p + \left\|\eta_{1,R}\right\|_{p;\delta}^p = \left\|\eta\right\|_{p;\delta,R}^p \,,$$

where $(\eta_{0,R}, \eta_{1,R}) = \Xi_R(\eta)$.

Proof. Given any $\eta \in L_R$, by definition of the norm and (8.2.4) we have

$$\left\|\eta\right\|_{p;\delta,R}^p = \left\| \eta_0|_{\Sigma_{-\infty}^0} \right\|_{p;\delta}^p + \left\| \eta_1|_{\Sigma_0^\infty} \right\|_{p;\delta}^p + \int_{\Sigma_{-2R}^{2R}} |\eta|^p \gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t \,.$$

Again using the definition of the norm we compute

$$
\begin{aligned}
\int_{\Sigma_{-2R}^{2R}} |\eta|^p \, \gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t &= \int_{-2R}^{0} \int_0^1 |\eta|^p \, e^{p\delta(2R+s)} \mathrm{d}t\mathrm{d}s + \int_0^{2R} \int_0^1 |\eta|^p \, e^{p\delta(2R-s)} \mathrm{d}t\mathrm{d}s \\
&= \int_0^{2R} \int_0^1 |\eta_{0,R}|^p \, e^{p\delta s} \mathrm{d}t\mathrm{d}s + \int_{-2R}^{0} \int_0^1 |\eta_{1,R}|^p \, e^{-p\delta s} \mathrm{d}t\mathrm{d}s \\
&= \int_0^{\infty} \int_0^1 |\eta_{0,R}|^p \, e^{p\delta s} \mathrm{d}t\mathrm{d}s + \int_{-\infty}^{0} \int_0^1 |\eta_{1,R}| \, e^{-p\delta s} \mathrm{d}t\mathrm{d}s \, .
\end{aligned}
$$

Since $\eta_{0,R}$ vanishes for $s \geq 2R$ and $\eta_{1,R}$ for $s \leq -2R$. Inserting the identity back into the first equation gives the results. □

Denote the linearized Cauchy-Riemann operators $D_0 = D_{u_0} : H_0 \to L_0$ and $D_1 = D_{u_1} : H_1 \to L_1$ (cf. equation (6.1.6)). We define the restricted operators $D_{01} = D_0 \oplus D_1|_{H_{01}}$ and $D'_{01} = D_0 \oplus D_1|_{H'_{01}}$, in which $H'_{01} \subset H_{01}$ is the subspace of pairs (ξ_0, ξ_1) such that $\xi_0(-\infty) \in T_{p_-}W_-$ and $\xi_1(\infty) \in T_{p_+}W_+$, where $p_- = u_0(-\infty)$ and $p_+ = u_1(\infty)$.

Lemma 8.3.3. *The operator $D'_{01} : H'_{01} \to L_0 \oplus L_1$ is surjective and has a bounded linear right inverse.*

Proof. See [36, 7.1.20], [33, corollary 4.14] or [5, Lemma 4.9]. Define the subspaces

$$
H'_0 := \{\xi \in H_0 \mid \xi(-\infty) \in T_{p_-}W_-\}, \qquad H'_1 := \{\xi \in H_1 \mid \xi(\infty) \in T_{p_+}W_+\} \, .
$$

Further define the restrictions $D'_0 := D_0|_{H'_0}$ and $D'_1 := D_1|_{H'_1}$. By assumption the almost complex structures J_0 and J_1 are regular, which implies that the operator is surjective

$$
\ker D'_0 \oplus \ker D'_1 \to T_pC, \qquad (\xi_0, \xi_1) \mapsto \xi_0(\infty) - \xi_1(-\infty) \, . \tag{8.3.8}
$$

Given $(\eta_0, \eta_1) \in L_0 \oplus L_1$ we choose lifts $(\xi'_0, \xi'_1) \in H'_0 \oplus H'_1$ such that $D_0\xi'_0 = \eta_0$ and $D_1\xi'_1 = \eta_1$. Since (8.3.8) is surjective we find $(\xi''_0, \xi''_1) \in \ker D'_0 \oplus \ker D'_1$ such that $\xi''_0(\infty) - \xi''_1(-\infty) = \xi'_0(\infty) - \xi'_1(-\infty)$. Then the pair $(\xi_0, \xi_1) := (\xi'_0 - \xi''_0, \xi'_1 - \xi''_1)$ lies in H'_{01} and is a preimage of (η_0, η_1) under the map D_{01}.

We have the inclusion $\ker D'_{01} \subset \ker D_0 \oplus \ker D_1$. Since D_0 and D_1 are Fredholm $\ker D'_{01}$ is finite dimensional and by the Hahn-Banach theorem we find a closed linear complement $H_{01}^{\perp}$ in H'_{01}. Restricted to $H_{01}^{\perp}$ the operator D'_{01} is invertible and hence there exists a bounded inverse $Q'_{01} : L_{01} \to H_{01}^{\perp} \subset H'_{01}$. □

Approximate right inverse Let $Q'_{01} : L_0 \oplus L_1 \to H'_{01}$ be a bounded right inverse of D'_{01} which exists by Lemma 8.3.3. Let $D_R : H_R \to L_R$ be the linearized Cauchy-Riemann-Floer operator at u_R. Moreover define the restricted operator $D'_R := D_R|_{H'_R}$ where $H'_R \subset H_R$ is the space of $\xi \in H_R$ such that $\xi(-\infty) \in T_{p_-}W_-$ and $\xi(\infty) \in T_{p_+}W_+$. By construction the linear pregluing operator Θ_R sends the subspace H'_{01} to H'_R. We define the operator

$$
\widetilde{Q}_R = \Theta_R \circ Q'_{01} \circ \Xi_R : L_R \to H'_R \, .
$$

The next lemma shows that $\widetilde{Q}_R$ is an uniformly bounded approximate right inverse of D_R for every R sufficiently large.

Lemma 8.3.4. *There exist constants c and R_0 such that for all $R \geq R_0$ and $\eta \in L_R$ we have*
$$\left\|\widetilde{Q}_R \eta\right\|_{1,p;\delta,R} \leq c \left\|\eta\right\|_{p;\delta,R}, \qquad \left\|D_R \widetilde{Q}_R \eta - \eta\right\|_{p;\delta,R} \leq c e^{-\delta R} \left\|\eta\right\|_{p;\delta,R}.$$

Proof. The first estimate follows directly by Lemma 8.3.1 and 8.3.2. We show the second estimate we follow [36, Lemma 7.1.32]. Fix any $\eta \in L_R$ and abbreviate
$$\xi_R = \widetilde{Q}_R \eta, \qquad (\xi_0, \xi_1) = \left(Q'_{01} \circ \Xi_R\right) \eta,$$
and moreover
$$\begin{aligned} u_{0,R} &= u_0 \circ \tau_{-2R}, & u_{1,R} &= u_1 \circ \tau_{2R} \\ \xi_{0,R} &= \xi_0 \circ \tau_{-2R}, & \xi_{1,R} &= \xi_1 \circ \tau_{2R}. \end{aligned}$$
By construction we have (recall that $\Sigma^a_{-\infty} = (-\infty, a] \times [0,1]$ and $\Sigma^\infty_a = [a, \infty) \times [0,1]$ for any $a \in \mathbb{R}$)
$$\Pi^{u_R}_{u_{0,R}} D_{u_{0,R}} \xi_{0,R} = \begin{cases} 0 & \text{on } \Sigma^\infty_0 \\ \eta & \text{on } \Sigma^0_{-\infty} \end{cases}, \qquad \Pi^{u_R}_{u_{1,R}} D_{u_{1,R}} \xi_{1,R} = \begin{cases} \eta & \text{on } \Sigma^\infty_0 \\ 0 & \text{on } \Sigma^0_{-\infty} \end{cases}, \tag{8.3.9}$$
and
$$\begin{aligned} u_R\big|_{\Sigma^{-R}_{-\infty}} &= u_{0,R}\big|_{\Sigma^{-R}_{-\infty}}, & u_R\big|_{\Sigma^\infty_R} &= u_{1,R}\big|_{\Sigma^\infty_R} \\ \xi_R\big|_{\Sigma^{-R}_{-\infty}} &= \xi_{0,R}\big|_{\Sigma^{-R}_{-\infty}}, & \xi_R\big|_{\Sigma^\infty_R} &= \xi_{1,R}\big|_{\Sigma^\infty_R}. \end{aligned}$$
Since the operators are local we have
$$D_R \xi_R\big|_{\Sigma^{-R}_{-\infty}} = D_{u_{0,R}} \xi_{0,R}\big|_{\Sigma^{-R}_{-\infty}} = \eta\big|_{\Sigma^{-R}_{-\infty}}, \qquad D_R \xi_R\big|_{\Sigma^\infty_R} = D_{u_{1,R}} \xi_{1,R}\big|_{\Sigma^\infty_R} = \eta\big|_{\Sigma^\infty_R}.$$
This shows that $D_R \xi_R - \eta$ is supported in $[-R, R] \times [0,1]$. According to (8.3.9) and taking into account the the support of the cut-off functions we have
$$\eta = \beta^-_R \Pi^{u_R}_{u_{0,R}} D_{u_{0,R}} \xi_{0,R} + \beta^+_{-R} \Pi^{u_R}_{u_{1,R}} D_{u_{1,R}} \xi_{1,R}.$$
and by a zero addition
$$\begin{aligned} &D_R \xi_R - \eta \\ &= \left(1 - \beta^-_R - \beta^+_{-R}\right) D_R \Pi^{u_R}_p \bar{\xi} + \\ &\quad + \left(\partial_s \beta^-_R\right) \left(\Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_p \bar{\xi}\right) + \beta^-_R \left(D_R \Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_{u_{0,R}} D_{u_{0,R}} \xi_{0,R}\right) \\ &\quad + \left(\partial_s \beta^+_{-R}\right) \left(\Pi^{u_R}_{u_{1,R}} \xi_{1,R} - \Pi^{u_R}_p \bar{\xi}\right) + \beta^+_{-R} \left(D_R \Pi^{u_R}_{u_{1,R}} \xi_{1,R} - \Pi^{u_R}_{u_{1,R}} D_{u_{1,R}} \xi_{1,R}\right). \end{aligned} \tag{8.3.10}$$
Focusing on the third summand of the right hand side without the factor β^-_R. After a zero addition we obtain
$$\begin{aligned} D_R \Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_R}_{u_{0,R}} D_{u_{0,R}} \xi_{0,R} &= D_R \Pi^{u_R}_{u_{0,R}} \left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right) - \\ &- \Pi^{u_R}_{u_{0,R}} D_{u_{0,R}} \left(\xi_{0,R} - \Pi^{u_{0,R}}_p \bar{\xi}\right) + D_R \Pi^{u_R}_{u_{0,R}} \Pi^{u_{0,R}}_p \bar{\xi} - \Pi^{u_R}_{u_{0,R}} D_{u_{0,R}} \Pi^{u_{0,R}}_p \bar{\xi}. \end{aligned} \tag{8.3.11}$$

By Lemma A.3.1 and Lemma 8.2.1 we find $\mu > 2\delta$ such that

$$\left|D_R\Pi^{u_R}_{u_{0,R}}\left(\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right)-\Pi^{u_R}_{u_{0,R}}D_{u_{0,R}}\left(\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right)\right|$$
$$\leq O(1)e^{-\mu(2R-|s|)}\left(\left|\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right|+\left|\nabla\left(\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right)\right|\right) ,$$

and

$$\left|D_R\Pi^{u_R}_{u_{0,R}}\Pi_p^{u_{0,R}}\bar\xi-\Pi^{u_R}_{u_{0,R}}D_{u_{0,R}}\Pi_p^{u_{0,R}}\bar\xi\right|\leq c_1e^{-\mu(2R-|s|)}\left|\bar\xi\right| ,$$

for all $|s|\leq R$ and $t\in[0,1]$. Hence by (8.3.11) we have

$$\left|\beta^-_R\left(D_R\Pi^{u_R}_{u_{0,R}}\xi_{0,R}-\Pi^{u_R}_{u_{0,R}}D_{u_{0,R}}\xi_{0,R}\right)\right|$$
$$\leq c_1e^{-\mu(2R-|s|)}\left(\left|\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right|+\left|\nabla\left(\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right)\right|+\left|\bar\xi\right|\right) .$$

For $p\in\mathcal{B}^{1,p;\delta}(C,C)$ considered as a constant function we have $D_p\bar\xi=0$ and by lemmas A.3.1 and 8.2.1 again we have a constant c_2 such that

$$\left|\left(1-\beta^-_R-\beta^+_{-R}\right)D_R\Pi_p^{u_R}\bar\xi\right|\leq\left|D_R\Pi_p^{u_R}\bar\xi\right|\leq c_2e^{-\mu(2R-|s|)}\left|\bar\xi\right| ,$$

for all $|s|\leq R$ and $t\in[0,1]$. Integrating the point-wise estimate gives

$$\begin{aligned}
&\int_{\Sigma^R_{-R}}\left|\beta^-_R\left(D_R\Pi^{u_R}_{u_{0,R}}\xi_{0,R}-\Pi^{u_R}_{u_{0,R}}D_{u_{0,R}}\xi_{0,R}\right)\right|^p\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t\\
&\leq 3^pc_1^pe^{-p\mu R}\int_{\Sigma^R_{-R}}\left(\left|\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right|^p+\left|\nabla\left(\xi_{0,R}-\Pi_p^{u_{0,R}}\bar\xi\right)\right|^p\right)e^{p\delta(2R+s)}\mathrm{d}s\mathrm{d}t\\
&\qquad+3^pc_1^pe^{-p(\mu-\delta)2R}\left|\bar\xi\right|^p\int_{-R}^Re^{p(\mu-\delta)|s|}\mathrm{d}s\\
&\leq\left(3^pc_1^pe^{-p\mu R}+\frac{3^{p+1}c_1^p}{p(\mu-\delta)}e^{-p(\mu-\delta)R}\right)\|\xi_0\|^p_{1,p;\delta} ,
\end{aligned}$$

where in the last line we used

$$e^{-p(\mu-\delta)R}\int_{-R}^Re^{p(\mu-\delta)|s|}\mathrm{d}s=\frac{2e^{-p(\mu-\delta)R}}{p(\mu-\delta)}\left(e^{p(\mu-\delta)R}-1\right)\leq\frac{3}{p(\mu-\delta)} .$$

Since $\delta<\mu/2$ the last estimate shows

$$\int_{\Sigma^R_{-R}}\left|\beta^-_R\left(D_R\Pi^{u_R}_{u_{0,R}}\xi_{0,R}-\Pi^{u_R}_{u_{0,R}}D_{u_{0,R}}\xi_{0,R}\right)\right|^p\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t\leq O(1)e^{-\delta R}\|\xi_0\|^p_{1,p;\delta} . \tag{8.3.12}$$

Along the same lines we show that

$$\begin{aligned}
&\int_{\Sigma^R_{-R}}\left|\beta^+_{-R}\left(D_R\Pi^{u_R}_{u_{1,R}}\xi_{1,R}-\Pi^{u_R}_{u_{1,R}}D_{u_{1,R}}\xi_{1,R}\right)\right|^p\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t &&\leq O(1)e^{-\delta R}\|\xi_1\|^p_{1,p;\delta}\\
&\int_{\Sigma^R_{-R}}\left|\left(1-\beta^-_R-\beta^+_{-R}\right)D_R\Pi_p^{u_R}\bar\xi\right|^p\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t &&\leq O(1)e^{-\delta R}\left|\bar\xi\right| .
\end{aligned} \tag{8.3.13}$$

Focusing now on the second term on the right hand side of (8.3.10) without the factor $\partial_s\beta_R^-$ we have

$$\Pi_{u_{0,R}}^{u_R}\xi_{0,R} - \Pi_p^{u_R}\bar{\xi} = \Pi_{u_{0,R}}^{u_R}\left(\xi_{0,R} - \Pi_p^{u_{0,R}}\bar{\xi}\right) + \left(\Pi_{u_{0,R}}^{u_R}\Pi_p^{u_{0,R}}\bar{\xi} - \Pi_p^{u_R}\bar{\xi}\right) .$$

By Lemma A.3.5 and Corollary A.2.3

$$\begin{aligned}\left|\Pi_{u_{0,R}}^{u_R}\xi_{0,R} - \Pi_p^{u_R}\bar{\xi}\right| &\leq \left|\xi_{0,R} - \Pi_p^{u_{0,R}}\bar{\xi}\right| + \left|\Pi_{u_{0,R}}^{u_R}\Pi_p^{u_{0,R}}\bar{\xi} - \Pi_p^{u_R}\bar{\xi}\right| \\ &\leq O(1)e^{-\delta R}\left\|\xi_0\right\|_{1,p;\delta} + O(1)e^{-\mu(2R-|s|)}\left|\bar{\xi}\right| \leq O(1)e^{-\delta R}\left\|\xi_0\right\|_{1,p;\delta} ,\end{aligned}$$

for all $|s| \leq R$. Since the support of $\partial_s\beta_R^-$ is in $[R-1,R]\times[0,1]$ and $\left|\partial_s\beta_R^-\right| < 2$ for all R we have

$$\int_{\Sigma_{-R}^R}\left|\partial_s\beta_R^-\left(\Pi_{u_{0,R}}^{u_R}\xi_{0,R} - \Pi_p^{u_R}\bar{\xi}\right)\right|^p\gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t \leq O(1)e^{-\delta R}\left\|\xi_0\right\|_{1,p;\delta}^p .$$

By a completely symmetric argument

$$\int_{\Sigma_{-R}^R}\left|\partial_s\beta_{-R}^+\left(\Pi_{u_{1,R}}^{u_R}\xi_{1,R} - \Pi_p^{u_R}\bar{\xi}\right)\right|^p\gamma_{\delta,R}^p \mathrm{d}s\mathrm{d}t \leq O(1)e^{-\delta R}\left\|\xi_1\right\|_{1,p;\delta}^p .$$

Denote $(\eta_{0,R},\eta_{1,R}) = \Xi_R\eta$. Since Q_{01} is bounded and by Lemma 8.3.2

$$\left\|\xi_0\right\|_{1,p;\delta} + \left\|\xi_1\right\|_{1,p;\delta} \leq O(1)\left(\left\|\eta_{0,R}\right\|_{p;\delta} + \left\|\eta_{1,R}\right\|_{p;\delta}\right) = O(1)\left\|\eta\right\|_{p;\delta,R}$$

By the identity (8.3.10) as well as (8.3.12), (8.3.13) and the last three estimates we have

$$\left\|D_R\xi_R - \eta\right\|_{p;\delta,R} \leq O(1)e^{-\delta R}\left(\left\|\xi_0\right\|_{1,p;\delta} + \left\|\xi_1\right\|_{1,p;\delta}\right) \leq O(1)e^{-\delta R}\left\|\eta\right\|_{p;\delta,R} .$$

This shows the claim. □

Corollary 8.3.5. *There exists uniform constants c and R_0 and for all $R \geq R_0$ there exists an operator $Q_R : L_R \to H_R'$, which is a right inverse for D_R' and we have for all $\eta \in L_R$*

$$\left\|Q_R\eta\right\|_{1,p;\delta,R} \leq c\left\|\eta\right\|_{p;\delta,R} . \tag{8.3.14}$$

Proof. Let R_0 and c denote the constants from Lemma 8.3.4. By possibly increasing R_0 we assume that $ce^{-\delta R_0} < 1/2$. With Lemma 8.3.4 the composition $D_R \circ \widetilde{Q}_R$ is invertible for all $R \geq R_0$ and we define

$$Q_R := \widetilde{Q}_R\left(D_R\widetilde{Q}_R\right)^{-1} = \widetilde{Q}_R\sum_{k=0}^{\infty}(1 - D_R\widetilde{Q}_R)^k . \tag{8.3.15}$$

Given for some $\eta \in L_R$ we have $\left\|Q_R\eta\right\|_{1,p;\delta,R} \leq c\left\|\eta\right\|_{p;\delta,R}\sum_{k=0}^{\infty}2^{-k} = 2c\left\|\eta\right\|_{p;\delta,R}$. □

8.4. Quadratic estimate

We build up the quadratic estimate which is needed to run the Newton-Picard theorem. Fix some $\varepsilon > 0$ and denote by $H'_R(\varepsilon) \subset H'_R$ the ball of all ξ with L^∞-norm strictly smaller than ε. Define the non-linear map

$$\mathcal{F}_R : H'_R(\varepsilon) \to L_R, \qquad \xi \mapsto \Pi^{u_R}_{u_\xi}\big(\partial_s u_\xi + J_R(u_\xi)\,(\partial_t - X_R(u_\xi))\,\big)\,, \tag{8.4.1}$$

with $u_\xi := \exp_{u_R} \xi$. By the special choice of the metric and definition of the space H'_R, we have $u_\xi(\pm\infty) \in W_\pm$. In particular if ξ is a zero of $\mathcal{F}_R$, then u_ξ is an element of $\widetilde{\mathcal{M}}(W_-, W_+)$.

Lemma 8.4.1 (Quadratic estimate)**.** *There exists constants R_0, ε and c such that we have the following uniform bounds. For all $R \geq R_0$ it holds*

$$\|\mathcal{F}_R(0)\|_{p;\delta,R} \leq ce^{-2\delta R}\,. \tag{8.4.2}$$

If $\xi, \xi' \in T_{u_R}\mathcal{B}^{1,p;\delta}(C_-, C_+)$ such that $\|\xi\|_{L^\infty} < \varepsilon$ then

$$\big\|\mathrm{d}\mathcal{F}_R(\xi)\xi' - D_R\xi'\big\|_{p;\delta,R} \leq c\,\|\xi\|_{1,p;\delta,R}\,\big\|\xi'\big\|_{1,p;\delta,R}\,. \tag{8.4.3}$$

Proof. We show estimate (8.4.2). Since u_0 is (J_0, X_0)-holomorphic, u_1 is (J_1, X_1) holomorphic and by definition of u_R and the glued structures (J_R, X_R) we have that $\mathcal{F}_R(0) = \overline{\partial}_{J_R,X_R} u_R$ is supported in $[-1, 1] \times [0, 1]$ and moreover for all $s \in [-1, 1]$ and $t \in [0, 1]$ we have

$$(\overline{\partial}_{J_R,X_R} u_R)(s,t) = \partial_s u_R(s,t) + J_\infty(t, u_R(s,t))(\partial_t u_R(s,t) - X_H(t, u_R(s,t)))\,.$$

Since J_∞ and X_H is uniformly bounded and by the decay of the preglued map in the neck region (cf. Lemma 8.2.1) we have

$$\int_{\Sigma^1_{-1}} \big|\overline{\partial}_{J_R,X_R} u_R\big|^p\, e^{p\delta(2R-|s|)}\mathrm{d}s\mathrm{d}t \leq O\big(e^{-2Rp(\mu-\delta)}(1 - e^{p(\mu-\delta)})\big) = O\big(e^{-2Rp\delta}\big)\,.$$

We show (8.4.3). We have $\mathrm{d}\mathcal{F}_R(0) = D_R$. Integrate the pointwise estimate from Lemma A.3.2 to obtain

$$\begin{aligned}
&\big\|\mathrm{d}\mathcal{F}_R(\xi)\xi' - D_R\xi'\big\|_{p;\delta,R} \\
&\qquad \leq O\big(\,\big\|\xi'\big\|_\infty \|\xi\|_\infty \|\mathrm{d}u_R\|_{p;\delta,R} + \|\nabla\xi\|_{p;\delta,R}\big\|\xi'\big\|_\infty + \|\xi\|_\infty \big\|\nabla\xi'\big\|_{p;\delta,R}\big)\,.
\end{aligned} \tag{8.4.4}$$

The norm $\|J_R\|_{C^2}$ appearing in A.3.2 is independent of R. By definition and Lemma 8.2.1 we have

$$\begin{aligned}
\|\mathrm{d}u_R\|^p_{p;\delta,R} &= \big\|\,\mathrm{d}u_0|_{\Sigma^0_{-\infty}}\big\|^p_{p;\delta} + \big\|\,\mathrm{d}u_1|_{\Sigma^\infty_0}\big\|^p_{p;\delta} + \int_{\Sigma^{2R}_{-2R}} |\mathrm{d}u_R|^p\,\gamma^p_{\delta,R}\mathrm{d}s\mathrm{d}t \\
&= \big\|\,\mathrm{d}u_0|_{\Sigma^0_{-\infty}}\big\|^p_{p;\delta} + \big\|\,\mathrm{d}u_1|_{\Sigma^\infty_0}\big\|^p_{p;\delta} + O(1)\int_0^{2R} e^{p(\delta-\mu)(2R-s)}\mathrm{d}s \leq O(1)\,.
\end{aligned}$$

We obtain (8.4.3) by plugging the last estimate and the estimates stated in Lemma A.3.4 into (8.4.4). This finishes the proof. ∎

We come to the key result of this section. For some small number $\varepsilon > 0$, we denote by $\ker_\varepsilon D_R \subset \ker D_R$ all elements in the kernel with norm smaller than ε.

Lemma 8.4.2. *There exists constants ε and R_0 such that for all $R \geq R_0$ there exists a map*

$$\sigma_R : \ker_\varepsilon D_R \longrightarrow \operatorname{im} Q_R\ ,$$

which satisfies the following properties

(i) for all $\xi \in \ker_\varepsilon D_R$ the map $\exp_{u_R}(\xi + \sigma_R(\xi))$ is (J_R, X_R)-holomorphic,

(ii) for each $R \geq R_0$ the map σ_R is differentiable and we have a constant $c > 0$ such that for all $R \geq R_0$

$$\|\sigma_R\|_{C^1} \leq c e^{-2\delta R}\,,$$

(iii) for every $\xi' \in T_{u_R}\mathcal{B}$ such that $\exp_{u_R} \xi'$ is (J_R, X_R)-holomorphic and satisfies $\|\xi'\|_{1,p;\delta,R} < \varepsilon$ there exist $\xi \in \ker_e D_R$ such that $\xi' = \xi + \sigma_R(\xi)$.

Proof. This is a direct consequence of the Newton-Picard theorem provided the quadratic estimate of the non-linear map given in Lemma 8.4.1 and the uniform bound on the right inverse established in Corollary 8.3.5. Set $\mathcal{N}_R(\xi) := \mathcal{F}_R(\xi) - \mathcal{F}_R(0) - D_R\xi$ for all $\xi \in T_{u_R}\mathcal{B}$. Let ε and R_0 be the constants from Lemma 8.4.1. Given $\xi_0, \xi_1 \in T_{u_R}\mathcal{B}$. By the mean-value theorem there exists $\theta \in [0,1]$ such that

$$\mathcal{F}_R(\xi_0) - \mathcal{F}_R(\xi_1) = \mathrm{d}\mathcal{F}_R\big(\theta\xi_0 + (1-\theta)\xi_1\big)(\xi_0 - \xi_1)\ .$$

If $\|\xi_0\|_{1,p;\delta} + \|\xi_1\|_{1,p;\delta} \leq \varepsilon$ we conclude using (8.4.3) that there are constants c_1, c_2 such that (where for convenience we have dropped the subindex of the norms since they are clear from the context)

$$\begin{aligned}
\|Q_R\mathcal{N}_R(\xi_0) - Q_R\mathcal{N}_R(\xi_1)\| &\leq c_1 \|\mathcal{N}_R(\xi_0) - \mathcal{N}_R(\xi_1)\| \\
&= c_1 \|\mathcal{F}_R(\xi_0) - \mathcal{F}_R(\xi_1) - D_R(\xi - \xi_1)\| \\
&= c_1 \big\|\mathrm{d}\mathcal{F}_R\big(\theta\xi_0 + (1-\theta)\xi_1\big)(\xi_0 - \xi_1) - D_R(\xi_0 - \xi_1)\big\| \\
&\leq c_1 c_2 \|\theta\xi_0 + (1-\theta)\xi_1\|\ \|\xi_0 - \xi_1\| \\
&\leq c_1 c_2 \big(\|\xi_0\| + \|\xi_1\|\big)\ \|\xi_0 - \xi_1\| \\
&\leq c_1 c_2 \varepsilon\ \|\xi_0 - \xi_1\|\ ,
\end{aligned}$$

and using (8.4.2) we find another constant c_3 such that

$$\|Q_R\mathcal{F}_R(0)\|_{1,p;\delta,R} \leq c_1 \|\mathcal{F}_R(0)\|_{p;\delta,R} \leq c_1 c_3 e^{-2\delta R}\ .$$

By [30, Proposition 24] and after possibly making R_0 bigger and ε smaller we conclude that for all $R \geq R_0$ there exists a map σ_R satisfying all three properties. More precisely we must have ε so small that $c_1 c_2 \varepsilon \leq 1/4$ and R_0 so large that $c_1 c_3 e^{-2\delta R_0} \leq \varepsilon/2$. $\square$

8.5. Continuity of the gluing map

The only non-trivial issue is continuity of the solution maps with respect to the gluing parameter, which essentially reduces to a question of continuity of the family of right-inverses. We denote by $\ker \mathcal{D}$ the vector-bundle over the base $[R_0, \infty)$ with fibre $\ker D_R$ and $\ker_\varepsilon \mathcal{D} \subset \ker \mathcal{D}$ the disk bundle with fibre $\ker_\varepsilon D_R$.

Lemma 8.5.1. *There exists constants R_0 and ε such that the map*

$$\sigma : \ker_\varepsilon \mathcal{D}|_{[R_0,\infty)} \to T\mathcal{B}^{1,p;\delta}, \qquad (R, \xi) \mapsto \sigma_R(\xi)\,,$$

is continuous.

Proof. We follow the proof of [5, Prp. 5.5]. Let R_0 and ε denote the constants from Lemma 8.4.2, which we possible have to increase (resp. decrease) as explained later in the proof. Given sequences (ξ_ν) and (R_ν) such that $R_\nu \geq R_0$ and $\xi_\nu \in \ker_\varepsilon D_{R_\nu}$ for all ν. Suppose that $R_\nu \to R$ and $\Pi_\nu \xi_\nu \to \xi \in \ker_\varepsilon D_R$, where we write $\Pi_\nu := \Pi_{u_{R_\nu}}^{u_R}$ for the parallel transport map. We also abbreviate $\sigma_\nu := \sigma_{R_\nu}$, $u := u_R$, and $u_\nu := u_{R_\nu}$. We have to show that $\lim_{\nu\to\infty} \Pi_\nu \sigma_\nu(\xi_\nu) = \sigma_R(\xi)$. Arguing indirectly we assume that there exists a subsequence $(\nu_k) \subset (\nu)$ such that

$$\lim_{k\to\infty} \|\Pi_{\nu_k} \sigma_{\nu_k}(\xi_{\nu_k}) - \sigma_R(\xi)\|_{1,p;\delta} > 0\,. \tag{8.5.1}$$

Without loss of generality we assume that $\nu_k = k$ for all $k \in \mathbb{N}$. We now build up a contradiction to (8.5.1) in the following three steps.

Step 1. Define $w_\nu := \exp_{u_\nu}(\xi_\nu + \sigma_\nu(\xi_\nu))$. A subsequence of w_ν Gromov converges to a J_R-holomorphic strip $w : \Sigma \to M$.

By the first property of the solution map we know that w_ν is J_{R_ν}-holomorphic. By the second property and general bounds for the derivative of the exponential map (cf. Corollary A.1.2)

$$|\mathrm{d}w_\nu| \leq O\left(|\mathrm{d}u_\nu| + |\nabla\xi_\nu| + |\nabla\sigma_\nu(\xi_\nu)|\right) \leq O\Big(|\mathrm{d}u_0| + |\mathrm{d}u_1| + |\nabla\xi| + e^{-\delta R}\Big)\,.$$

In particular we conclude that the gradient of w_ν is uniformy bounded. By local compactness we conclude the existence of w such that $w_\nu \to w$ in C^∞_{loc} (cf. Lemma 5.2.1). In remains to control the convergence on the ends. Denote $p_+ := u_\nu(\infty) = u(\infty)$. Choose large constants s_0, ν_0 and estimate for all $s \geq s_0$ and $\nu \geq \nu_0$ using exponential decay for u (cf. Theorem 4.1.1), omitting (s,t) whenever convenient

$$\begin{aligned}
\operatorname{dist}(w_\nu(s,t), p_+) &\leq \operatorname{dist}(w_\nu, u_\nu) + \operatorname{dist}(u_\nu, u) + \operatorname{dist}(u, p_+)\\
&\leq |\xi_\nu| + \|\sigma_\nu(\xi_\nu)\|_\infty + o(1) + O(e^{-\mu s})\\
&\leq \varepsilon + O\big(e^{-\delta R_0}\big) + o(1) + O(e^{-\mu s})\,.
\end{aligned}$$

After possibly decreasing ε and increasing R_0, s_0 and ν_0 the right-hand side is smaller than the diameter of a ball about p_+ which lies completely in the Pozniak neighborhood

U_{Poz} for all $s \geq s_0$ and $\nu \geq \nu_0$. Hence the image of w_ν restricted to $\Sigma_{s_0}^\infty$ lies in U_{Poz}, where the symplectic form is exact $\omega = \mathrm{d}\lambda$ and λ vanishes on $L_0 \cap U_{\mathrm{Poz}}$ and $L_1 \cap U_{\mathrm{Poz}}$. By exactness and C^∞_{loc}-convergence we conclude

$$\int_{\Sigma_{s_0}^\infty} w_\nu^* \omega = \int_0^1 w_\nu\big|_{s=s_0}^* \lambda \to \int_0^1 w\big|_{s=s_0}^* \lambda = \int_{\Sigma_{s_0}^\infty} w^* \omega \,.$$

By convergence of the energy we have C^0-convergence on the end, i.e. w_ν converges to w in $C^0(\Sigma_{s_0}^\infty)$ (cf. Lemma 5.3.1). We proceed similarly for the negative end to show that w_ν converges to w in $C^0(\Sigma_{-\infty}^{-s_0})$ and hence $w_\nu \to w$ in C^0 and $E(w_\nu) \to E(w)$.

Step 2. There exists a vector field $\xi'' \in \Gamma(u^*TM)$ such that $\exp_u \xi'' = w$ and moreover we have

$$\lim_{\nu \to \infty} \left\| \Pi_\nu \sigma_\nu(\xi_\nu) + \xi - \xi'' \right\|_{1,p;\delta} = 0 \,.$$

By the last step the vector field $\zeta_\nu := \exp_w^{-1} w_\nu$ is well-defined for all ν sufficiently large. We estimate for any $(s,t) \in \Sigma$ omitting the arguments s and t for convenience

$$\begin{aligned} \operatorname{dist}(u,w) &\leq \operatorname{dist}(u,u_\nu) + \operatorname{dist}(u_\nu,w_\nu) + \operatorname{dist}(w_\nu,w) \\ &\leq \operatorname{dist}(u,u_\nu) + |\xi_\nu + \sigma_\nu(\xi_\nu)| + \|\zeta_\nu\|_\infty \leq \varepsilon + O\big(e^{-\delta R_0}\big) + o(1)\,, \end{aligned}$$

since by the last step $\|\zeta_\nu\|_\infty \to 0$. Hence $\operatorname{dist}(u,w) \leq \varepsilon + O\big(e^{-\delta R_0}\big)$ and after possibly decreasing ε and increasing R_0 again we assume that the distance from $u(s,t)$ to $w(s,t)$ is smaller than the injectivity radius for any $(s,t) \in \Sigma$. In particular the vector field $\xi'' := \exp_u^{-1} w$ is well-defined. Because the strips u and w are elements of $B^{1,p;\delta}(C_-,C_+)$ Lemma A.3.8 shows that the norm $\|\xi''\|_{1,p;\delta}$ is finite. By construction it holds that $\sigma_\nu(\xi_\nu) = \exp_{u_\nu}^{-1} w_\nu - \xi_\nu$ and we estimate

$$\begin{aligned} \left\|\Pi_\nu \sigma_\nu(\xi_\nu) + \xi - \xi''\right\|_{1,p;\delta} &= \left\|\Pi_\nu \exp_{u_\nu}^{-1} w_\nu - \Pi_\nu \xi_\nu + \xi - \exp_u^{-1} w\right\|_{1,p;\delta} \\ &\leq \left\|\Pi_\nu \exp_{u_\nu}^{-1} w_\nu - \exp_u^{-1} w\right\|_{1,p;\delta} + \|\Pi_\nu \xi_\nu - \xi\|_{1,p;\delta} \,. \end{aligned}$$

To show the claim it remains to see that the first summand on the right-hand converges to zero as ν tends to ∞. Define the points $q_\nu := w_\nu(\infty)$ and $q := w(\infty)$ as well as the vector field

$$\xi_\nu^+ := \Pi_{u_\nu}^u \exp_{u_\nu}^{-1} w_\nu - \exp_u^{-1} w - \widehat{\Pi}_{p_+}^u \left(\exp_{p_+}^{-1} q_\nu - \exp_{p_+}^{-1} q\right) \in \Gamma(u^*TM)\,.$$

We use corollaries A.1.2 and A.1.3 to estimate the norm of ξ_ν^+ by

$$\left|\Pi_{u_\nu}^u \exp_{u_\nu}^{-1} w_\nu - \exp_u^{-1} w_\nu\right| + \left|\exp_u^{-1} w_\nu - \exp_u^{-1} w\right| + \left|\exp_{p_+}^{-1} q_\nu - \exp_{p_+}^{-1} q\right|$$

and conclude that

$$\left|\xi_\nu^+\right| \leq O(\operatorname{dist}(u_\nu,u) + \operatorname{dist}(w_\nu,w) + \operatorname{dist}(q_\nu,q))\,.$$

With Corollary A.2.4 using the notation $\zeta'_\nu := \exp_u^{-1} u_\nu \in \Gamma(u^*TM)$ we bound the norm of $\nabla\xi_\nu^+$ with

$$\left|\left(\nabla\Pi_{u_\nu}^u - \Pi_{u_\nu}^u\nabla\right)\exp_{u_\nu}^{-1} w_\nu\right| + \left|\Pi_{u_\nu}^u \nabla\exp_{u_\nu}^{-1} w_\nu - \nabla\exp_u^{-1} w_\nu\right| + \\ + \left|\nabla\exp_u^{-1} w_\nu - \nabla\exp_u^{-1} w\right| + \left|\nabla\widehat{\Pi}_{p_+}^u\left(\exp_{p_+}^{-1} q_\nu - \exp_{p_+}^{-1} q\right)\right|$$

Hence we conclude that

$$\left|\nabla\zeta^+\right| \le O(1)\Big(\mathrm{dist}\,(u,u_\nu)\,(|\mathrm{d}u| + |\mathrm{d}u_\nu|) + \left|\nabla\zeta'_\nu\right| + |\zeta_\nu|\,(|\mathrm{d}u| + |\mathrm{d}w|) + \\ |\nabla\zeta_\nu| + |\mathrm{d}u|\,\mathrm{dist}\,(q_\nu,q)\Big)\,.$$

The last two estimates and using C^∞_{loc} converges of w_ν to w we conclude that for a fixed s we have

$$\lim_{\nu\to\infty}\left\|\xi_\nu^+\right\|_{C^1(\Sigma_0^s)} = 0\,. \tag{8.5.2}$$

Choose μ such that $2\delta < \mu < \iota$ with ι as defined in (8.3.1) and let $\varepsilon_0 = \varepsilon_0(\mu)$ be the associated constant from Lemma 4.3.2. Now choose s_0 large enough such that $E(w,\Sigma_{s_0}^\infty) < \varepsilon_0/2$. By convergence of the energy as established in the last step there exists $\nu_0 \in \mathbb{N}$ such that $E(w_\nu,\Sigma_{s_0}^\infty) < \varepsilon_0$ for all $\nu \ge \nu_0$. Thus the assumptions of Lemma 4.3.2 are met and there exists constant c_1 independent of ν such that

$$\mathrm{dist}\,(w_\nu, w_\nu(\infty)) + |\mathrm{d}w_\nu| \le c_1 e^{-\mu s}, \qquad \forall\ s \ge s_0\,.$$

Without loss of generality we assume that the same holds with w_ν replaced by u, w and u_ν. Then the previous estimates show that there exists a constant c_2 such that for all $s \ge s_0$ and ν large enough

$$\left|\nabla\xi_\nu^+\right| \le c_2 e^{-\mu s}\,. \tag{8.5.3}$$

Fix $t \in [0,1]$ and ν for the moment and define the function $f : [s_0,\infty) \to \mathbb{R}$ by $f(s) := |\xi_\nu^+(s,t)|$. We claim that $\lim_{s\to\infty} f(s) = 0$. For s large enough we replace $\widehat{\Pi}$ with Π in the formula for ξ^+ and estimate $f(s)$ with

$$\left|\exp_{u_\nu}^{-1} w_\nu - \Pi_{p_+}^{u_\nu}\exp_{p_+}^{-1} q_\nu\right| + \left|\left(\Pi_{p_+}^{u_\nu} - \Pi_u^{u_\nu}\Pi_{p_+}^u\right)\exp_{p_+}^{-1} q_\nu\right| + \left|\exp_u^{-1} w - \Pi_{p_+}^u\exp_{p_+}^{-1} q\right|.$$

The first and the last summand converge to zero as s tends to ∞ by Lemma A.3.8 and so does the second summand after Corollary A.2.3. Hence

$$f(s) = \int_s^\infty -\partial_\sigma f(\sigma)\mathrm{d}\sigma \le \int_s^\infty \left|\nabla\xi_\nu^+\right|\mathrm{d}\sigma \le c_2\int_s^\infty e^{-\mu\sigma}\mathrm{d}\sigma \le \frac{c_2}{\mu}e^{-\mu s}\,.$$

Using the last estimate and (8.5.3) we see that there exists an universal constant c_3 such that for all $s \ge s_0$

$$\int_{\Sigma_0^\infty}\left(\left|\xi_\nu^+\right|^p + \left|\nabla\xi_\nu^+\right|^p\right)\kappa_\delta\mathrm{d}s\mathrm{d}t \le \\ \le \left\|\xi_\nu^+\right\|_{C^1(\Sigma_0^s)}^p\int_0^s e^{\delta s}\mathrm{d}s + (c_2^p + (c_2/\mu)^p)\int_s^\infty e^{-(\mu-\delta)ps}\mathrm{d}s \le \\ \le c_3 e^{\delta s}\left\|\xi_\nu^+\right\|_{C^1(\Sigma_0^s)}^p + c_3 e^{-(\mu-\delta)ps}\,.$$

According to (8.5.2) the right-hand side converges to $c_3 e^{-(\mu-\delta)ps}$ as $\nu \to \infty$ and since s was chosen freely, we see that left-hand side converges to zero as $\nu \to \infty$. Similar we proceed with the negative end to show that

$$\left\|\Pi_\nu \exp_{u_\nu}^{-1} w_\nu - \exp_u^{-1} w\right\|_{1,p;\delta} \to 0\,,$$

This shows the claim.

Step 3. We show that $\|\Pi_\nu \sigma_\nu(\xi_\nu) - \sigma_R(\xi)\|_{1,p;\delta} \to 0$ as $\nu \to \infty$ in contradiction to (8.5.1).

By the last step, we see that $\Pi_\nu \sigma_\nu(\xi_\nu)$ converges to $\xi' := \xi'' - \xi$. Define $\eta' := D_R \xi'$ and $\eta'_\nu := D_{R_\nu} \sigma_\nu(\xi_\nu)$, then using Corollary A.3.6 and A.3.7

$$\begin{aligned}
\left\|\eta' - \Pi_\nu \eta'_\nu\right\|_{p;\delta} &= \left\|D_R \xi' - \Pi_\nu D_{R_\nu} \sigma_\nu(\xi_\nu)\right\|_{p;\delta} \\
&\leq \left\|D_R \xi' - D_R \Pi_\nu \sigma_\nu(\xi_\nu)\right\|_{p;\delta} + \|\Pi_\nu D_{R_\nu} \sigma_\nu(\xi_\nu) - D_R \Pi_\nu \sigma_\nu(\xi_\nu)\|_{p;\delta} \\
&\leq O(1) \left\|\xi' - \Pi_\nu \sigma_\nu(\xi_\nu)\right\|_{1,p;\delta} + o(1) = o(1)\;.
\end{aligned}$$

Since Q_{R_ν} is a right inverse to D_{R_ν} and $\sigma_\nu(\xi_\nu) \in \operatorname{im} Q_{R_\nu}$ we have $Q_{R_\nu} \eta'_\nu = \sigma_\nu(\xi_\nu)$ and using the fact that $R \mapsto Q_R \eta'$ is continuous for a fixed η' (see Lemma 8.5.2) we have (omitting the subscripts of the norms for convenience)

$$\begin{aligned}
\left\|Q_R \eta' - \Pi_\nu \sigma_\nu(\xi_\nu)\right\| &= \left\|Q_R \eta' - \Pi_\nu Q_{R_\nu} \eta'_\nu\right\| \\
&\leq \left\|\Pi_\nu Q_{R_\nu} \Pi_\nu^{-1} \eta' - \Pi_\nu Q_{R_\nu} \eta'_\nu\right\| + \left\|Q_R \eta' - \Pi_\nu Q_{R_\nu} \Pi_\nu^{-1} \eta'\right\| \\
&\leq O(1) \left\|\eta' - \Pi_\nu \eta'_\nu\right\| + o(1) = o(1)\;.
\end{aligned}$$

Hence $\xi' = Q_R \eta'$ and from $\mathcal{F}_R(\xi + \xi') = 0$ it follows that there exists $\xi_0 \in \ker D_R$ such that

$$\xi + \xi' = \xi_0 + \sigma_R(\xi_0)\;.$$

We have the splitting $T_{u_R}\mathcal{B} = \ker D_R \oplus \operatorname{im} Q_R$. Sine $\xi', \sigma_R(\xi_0) \in \operatorname{im} Q_R$ we conclude that $\xi_0 = \xi$ and $\xi' = \sigma_R(\xi_0) = \sigma_R(\xi)$. In particular

$$\|\Pi_\nu \sigma_\nu(\xi_\nu) - \sigma_R(\xi)\|_{1,p;\delta} \to 0\,,$$

contradicting (8.5.1) and proving the lemma. □

Lemma 8.5.2. *Fix $\eta \in L_R$ and given a sequence $R_\nu \to R$ then*

$$\lim_{\nu \to \infty} \left\|Q_{R^\nu} \Pi_{u_R}^{u_{R_\nu}} \eta - \Pi_{u_R}^{u_{R_\nu}} Q_R \eta\right\|_{1,p;\delta} = 0\,.$$

Proof. Abbreviate the norm $\|\cdot\| := \|\cdot\|_{1,p;\delta}$, the operators $D := D_R$, $D_\nu := D_{R_\nu}$, $Q := Q_R$, $Q_\nu := Q_{R_\nu}$, $\widetilde{Q} := \widetilde{Q}_R$, $\widetilde{Q}_\nu := \widetilde{Q}_{R_\nu}$, $\Pi_\nu := \Pi_{u_R}^{u_{R_\nu}}$ and the vector $\eta_j := (1 - D\widetilde{Q})^j \eta$

for all $j = 0, \dots, k$. We estimate using dominated convergence

$$\begin{aligned}
\lim_{\nu\to\infty} \left\|(\Pi_\nu Q - Q_\nu \Pi_\nu)\eta\right\| &\leq \lim_{\nu\to\infty} \sum_{k\geq 0} \left\| \Pi_\nu \widetilde{Q}(1 - D\widetilde{Q})^k \eta - \widetilde{Q}_\nu (1 - D_\nu \widetilde{Q}_\nu)^k \Pi_\nu \eta \right\| \\
&= \sum_{k\geq 0} \lim_{\nu\to\infty} \left\| \Pi_\nu \widetilde{Q}(1 - D\widetilde{Q})^k \eta - \widetilde{Q}_\nu (1 - D_\nu \widetilde{Q}_\nu)^k \Pi_\nu \eta \right\| \\
&\leq O(1) \sum_{k\geq 0} \sum_{j=0}^{k} \lim_{\nu\to\infty} \left\|(\Pi_\nu \widetilde{Q} - \widetilde{Q}_\nu \Pi_\nu)\eta_j\right\| ,
\end{aligned}$$

where the last inequality follows from Corollary A.3.6. According to the preceding consideration we see that it suffices to show the lemma for the corresponding approximate right-inverses.

We have by definition with $\xi := Q_{01}\Xi_R \eta$

$$\left\|(\Pi_\nu \widetilde{Q} - \widetilde{Q}_\nu \Pi_\nu)\eta\right\|_{1,p;\delta} \leq \left\|(\Pi_\nu \Theta_R - \Theta_{R_\nu})\xi\right\|_{1,p;\delta} + \left\|(\Xi_R - \Xi_{R_\nu}\Pi_\nu)\eta\right\|_{L_0\oplus L_1} . \tag{8.5.4}$$

We show that both terms on the right-hand side converge to zero separately. In order to control the second term we define the paths of vector fields $\tilde{\eta}_0 : \mathbb{R} \to \Gamma(u_0^* TM)$, $\rho \mapsto \tilde{\eta}_{0,\rho}$ and $\tilde{\eta}_1 : \mathbb{R} \to \Gamma(u_1^* TM)$, $\rho \mapsto \tilde{\eta}_{1,\rho}$ where

$$\begin{aligned}
\tilde{\eta}_{0,\rho} &:= \Pi^{u_0}_{u_R\circ\tau_{2R}} \eta \circ \tau_{2R} - \Pi^{u_0}_{u_{R+\rho}\circ\tau_{2(R+\rho)}} \Pi^{u_{R+\rho}\circ\tau_{2(R+\rho)}}_{u_R\circ\tau_{2(R+\rho)}} \eta \circ \tau_{2(R+\rho)} \\
\tilde{\eta}_{1,\rho} &:= \Pi^{u_1}_{u_R\circ\tau_{-2R}} \eta \circ \tau_{-2R} - \Pi^{u_1}_{u_{R+\rho}\circ\tau_{-2(R+\rho)}} \Pi^{u_{R+\rho}\circ\tau_{-2(R+\rho)}}_{u_R\circ\tau_{-2(R+\rho)}} \eta \circ \tau_{-2(R+\rho)} .
\end{aligned}$$

We assume for the moment that η is smooth and compactly supported. We have with a standard result on the derivative of parallel transport maps (cf. Corollary A.2.4) that norm of $\partial_\rho \tilde{\eta}_{0,\rho}$ is bounded by

$$O(1) \left(\left(\left|\partial_\rho u_{R+\rho} \circ \tau_{2(R+\rho)}\right| + \left|\partial_\rho u_R \circ \tau_{2(R+\rho)}\right| \right) \left|\eta \circ \tau_{2(R+\rho)}\right| + \left|\nabla_\rho \eta \circ \tau_{2(R+\rho)}\right| \right) .$$

We conclude in particular that the $\partial_\rho \tilde{\eta}_{0,\rho}$ is uniformly bounded. Obviously $\tilde{\eta}_{0,0} \equiv 0$. By the mean-value theorem and the last estimate we have $|\tilde{\eta}_{0,\rho}| \leq O(|\rho|)$. Similarly we have $|\tilde{\eta}_{1,\rho}| \leq O(|\rho|)$. Therefore with $\rho_\nu = R_\nu - R$

$$\|(\Xi_R - \Xi_{R_\nu}\Pi_\nu)\eta\|^p_{L_0\oplus L_1} \leq O(1) \int_\Sigma |\widetilde{\eta}_{0,\rho_\nu}|^p + |\widetilde{\eta}_{1,\rho_\nu}|^p \,\mathrm{d}s\mathrm{d}t \leq O(|\rho_\nu|^p) .$$

Thus the second term in (8.5.4) converges to zero as ν tends to ∞ if η is smooth and compactly supported. If η is not smooth or not compactly supported we find arbitrarily close $\eta' \in L_R$ which is smooth and compactly supported such that

$$\left\|(\Xi_R - \Xi_{R_\nu}\Pi_\nu)(\eta - \eta')\right\| \leq \left\|\Xi_R(\eta - \eta')\right\| + \left\|\Xi_{R_\nu}\Pi_\nu(\eta - \eta')\right\| \leq O(1) \left\|\eta - \eta'\right\| .$$

In particular we assume that η' is choosen such that the right-hand side is smaller than some arbitrary $\varepsilon > 0$. Using the above we conclude

$$\lim_{\nu\to\infty} \|(\Xi_R - \Xi_{R_\nu}\Pi_\nu)\eta\| \leq \lim_{\nu\to\infty} \left\|(\Xi_R - \Xi_{R_\nu}\Pi_\nu)\eta'\right\| + \left\|(\Xi_R - \Xi_{R_\nu}\Pi_\nu)(\eta - \eta')\right\| < \varepsilon .$$

This shows that the second term in (8.5.4) converges to zero for any η.

We now show that the first term in (8.5.4) converges to zero for any fixed $\xi = (\xi_0, \xi_1) \in H_{01}$. By the same argument we assume without loss of generality that ξ_0 and ξ_1 are smooth and compactly supported derivative. We define vector fields $\xi_R := \Theta_R \xi$, and $\xi_{R_\nu} := \Theta_{R_\nu} \xi$. By the definition of the interpolation (8.2.3) we have the point-wise estimate of the norm of difference $\Pi_\nu \xi_R - \xi_{R_\nu}$ by

$$
\begin{aligned}
&\left|(1-\beta^-_R - \beta^+_{-R})\big(\Pi^{u_{R_\nu}}_{u_R} \widehat{\Pi}^{u_R}_p - \widehat{\Pi}^{u_{R_\nu}}_p\big)\bar{\xi}\right| + \\
&+ \left|\big(\beta^-_R - \beta^-_{R_\nu}\big)\widehat{\Pi}^{u_{R_\nu}}_p \bar{\xi}\right| + \left|\big(\beta^+_{-R} - \beta^+_{-R_\nu}\big)\widehat{\Pi}^{u_{R_\nu}}_p \bar{\xi}\right| + \\
&+ \left|\Pi^{u_{R_\nu}}_{u_R} \Pi^{u_R}_{u_{0,R}} \xi_{0,R} - \Pi^{u_{R_\nu}}_{u_{0,R_\nu}} \xi_{0,R_\nu}\right| + \left|\Pi^{u_{R_\nu}}_{u_R} \Pi^{u_R}_{u_{1,R}} \xi_{1,R} - \Pi^{u_{R_\nu}}_{u_{1,R_\nu}} \xi_{1,R_\nu}\right| + \\
&+ \left|(\beta^-_R - \beta^-_{R_\nu})\Pi^{u_R}_{u_{0,R}} \xi_{0,R}\right| + \left|(\beta^+_{-R} - \beta^+_{R_\nu})\Pi^{u_R}_{u_{1,R}} \xi_{1,R}\right|
\end{aligned}
$$

Using again the mean value theorem we show that

$$
\left|\Pi_\nu \xi_R - \xi_{R_\nu}\right| \leq O(1)\left|R - R_\nu\right| .
$$

We deduce the same estimate for the norm of $\nabla(\Pi_\nu \xi_R - \xi_{R_\nu})$ and conclude as above that the first term of (8.5.4) converges to zero. This shows the claim. □

Remark 8.5.3. An estimate similar to Corollary A.3.6 does not hold for the right-inverse. In particular $R \mapsto Q_R$ is not a continuous path of operators! The failure of uniform continuity is due to the fact that the definition of Q_R involves a shift-operator. For our purposes pointwise continuity suffices. It does not however, if we were to prove higher regularity of the gluing map. Then one would need a more sophisticated analytical setup, as for example the theory of polyfolds.

8.6. Surjectivity of the gluing map

In this section we show that the gluing map is asymptotically surjective. First we need an auxiliary lemma.

Lemma 8.6.1. *Given a sequence $[(w_\nu)] \subset \mathcal{M}(W_-, W_+)_{[1]}$ which Floer-Gromov converges to $[(u_0, u_1)] \in \mathcal{M}^1(W_-, W_+)_{[0]}$. There exists a vector field ξ_ν along u_{R_ν} and a constant $a_\nu \in \mathbb{R}$ such that $w_\nu = \exp_{u_{R_\nu}} \xi_\nu \circ \tau_{a_\nu}$ for all but finitely many $\nu \in \mathbb{N}$ and $\lim_{\nu\to\infty} \|\xi_\nu\|_{1,p;\delta,R^\nu} = 0$.*

Proof. We follow the proof of [5, Lemma 10.12]. By Gromov convergence there are two sequences $(b_\nu), (c_\nu) \subset \mathbb{R}$ such that $w_\nu \circ \tau_{b_\nu} \to u_0$ and $w_\nu \circ \tau_{c_\nu} \to u_1$ in C^∞_{loc}. We define $a_\nu := 1/2(b_\nu + c_\nu)$ and $2R_\nu := 1/2(c_\nu - b_\nu)$. Set

$$
v_\nu := w_\nu \circ \tau_{-a_\nu} .
$$

Then v_ν is (J_{R_ν}, X_{R_ν})-holomorphic with respect to the glued structures $J_{R_\nu} = J_0 \#_{R_\nu} J_1$ and $X_{R_\nu} = X_0 \#_{R_\nu} X_1$ and satisfies $v_\nu \circ \tau_{2R_\nu} \to u_0$ and $v_\nu \circ \tau_{-2R_\nu} \to u_1$ in C^∞_{loc}.

Step 1. We have $\lim_{\nu\to\infty}\sup_{(s,t)\in\Sigma}\mathrm{dist}\,(v_\nu(s,t),u_{R_\nu}(s,t))=0$.

By Floer-Gromov convergence we have convergence the energy $E(v_\nu)\to E(u_0)+E(u_1)=:E$. For any ε_0 there exists $s_0=s_0(\varepsilon_0)$ large enough such that

$$E(u_0;\Sigma_{-s_0}^{s_0})+E(u_1;\Sigma_{-s_0}^{s_0})\geq E-\varepsilon_0/2\,.$$

By C^∞_{loc}-convergence on the compact set $\Sigma_{-s_0}^{s_0}$ this implies that there exists ν_0 such that for all $\nu\geq\nu_0$ we have

$$E(v_\nu;\Sigma_{-\infty}^{-2R_\nu-s_0})+E(v_\nu;\Sigma_{-2R_\nu+s_0}^{2R_\nu-s_0})+E(v_\nu;\Sigma_{2R_\nu+s_0}^{\infty})<\varepsilon_0\,. \tag{8.6.1}$$

Now assume that $\varepsilon_0=\varepsilon_0(\mu)$ is the constant given in Lemma 4.3.2 for some $\mu>0$ with $2\delta<\mu<\iota$ where ι as given in (8.3.1). Hence there exists an uniform constant c_1 which is independent of ν such that for all $s\geq s_0+1$, $\nu\geq\nu_0$ and $t\in[0,1]$ we have the decay estimates

$$\begin{aligned}
&\forall\ |\sigma|\leq 2R_\nu-s: &&\mathrm{dist}\,(w_\nu(-2R_\nu+s,t),w_\nu(\sigma,t))\leq c_1e^{-\mu s}\\
&\forall\ \sigma\leq -2R_\nu-s: &&\mathrm{dist}\,(w_\nu(-2R_\nu-s,t),w_\nu(\sigma,t))\leq c_1e^{-\mu s}\\
&\forall\ \sigma\geq 2R_\nu+s: &&\mathrm{dist}\,(w_\nu(2R_\nu+s,t),w_\nu(\sigma,t))\leq c_1e^{-\mu s}\,.
\end{aligned}$$

These are all proven using Lemma 4.3.2. For the first inequality we have used (4.3.8) of Lemma 4.3.2 with $b=-a=2R_\nu-s_0$, $\sigma'=-2R_\nu+s$, $\sigma=\sigma$ and we have replaced the s in that estimate by $s-s_0$.

We now use these estimates to prove the claim. Abbreviate $p:=u_0(\infty)=u_1(-\infty)$ and $u_\nu:=u_{R_\nu}$. We estimate for all $|\sigma|\leq 2R_\nu-s$

$$\begin{aligned}
&\mathrm{dist}\,(v_\nu(\sigma,t),u_\nu(\sigma,t))\leq\\
&\qquad\leq\mathrm{dist}\,(v_\nu(\sigma,t),v_\nu(-2R_\nu+s,t))+\mathrm{dist}\,(v_\nu(-2R_\nu+s,t),u_0(s,t))+\\
&\qquad\qquad+\mathrm{dist}\,(u_0(s,t),p)+\mathrm{dist}\,(p,u_\nu(\sigma,t))\leq O(e^{-\mu s})+o(1)\,,
\end{aligned}$$

in which we have used the decay for $u_\nu=u_{R_\nu}$ as given in Lemma 8.2.1 and the fact that $e^{-\mu(2R_\nu-|\sigma|)}\leq e^{-\mu s}$. Abbreviate $p_+:=u_1(\infty)$ and estimate for all $\sigma\geq 2R_\nu+s$ the distance of $v_\nu(\sigma,t)$ to $u_\nu(\sigma,t)$ by

$$\begin{aligned}
&\mathrm{dist}\,(v_\nu(\sigma,t),v_\nu(2R_\nu+s,t))+\mathrm{dist}\,(v_\nu(2R_\nu+s,t),u_1(s,t))+\\
&\qquad+\mathrm{dist}\,(u_1(s,t),p_+)+\mathrm{dist}\,(p_+,u_\nu(\sigma,t))
\end{aligned}$$

We conclude that all terms are bounded by $O(e^{-\mu s})+o(1)$. Now abbreviate $p_-:=u_0(-\infty)$ and estimate for all $\sigma\leq-2R_\nu-s$ the distance from $v_\nu(\sigma,t)$ to $u_\nu(\sigma,t)$ by

$$\begin{aligned}
&\mathrm{dist}\,(v_\nu(\sigma,t),v_\nu(-2R_\nu-s,t))+\mathrm{dist}\,(v_\nu(-2R_\nu-s,t),u_0(-s,t))+\\
&\qquad+\mathrm{dist}\,(u_0(-s,t),p_-)+\mathrm{dist}\,(p_-,u_\nu(\sigma,t))\,.
\end{aligned}$$

We see again that all terms are bounded by $O(e^{-\mu s}) + o(1)$. Combining the above with C^∞_{loc} convergence we have for any $\sigma \in \mathbb{R}$

$$\begin{aligned}\sup_\Sigma \operatorname{dist}(v_\nu, u_\nu) &\leq \sup_{\Sigma^s_{-s}} \operatorname{dist}(v_\nu \circ \tau_{2R_\nu}, u_0) + \\ &\quad + \sup_{\Sigma^s_{-s}} \operatorname{dist}(v_\nu \circ \tau_{-2R_\nu}, u_1) + O(e^{-\mu s}) + o(1) \leq O(e^{-\mu s}) + o(1)\,.\end{aligned}$$

Now the right-hand side converges to $O(e^{-\mu s})$ and since s was chosen freely we conclude that the left-hand side converges to zero. This shows the claim.

Step 2. We have $v_\nu = \exp_{u_\nu} \xi_\nu$ for some vector field ξ_ν along u_ν and the norm $\|\xi_\nu\|_{1,p;\delta}$ converges to zero as $\nu \to \infty$.

Because of the last step the vector field ξ_ν is well-defined and the norm $\|\xi_\nu\|_{L^\infty}$ converges to zero. We claim that there exists an uniform constant c_2 such that for all $s \geq s_0 + 1$, $t \in [0,1]$ and $\nu \geq \nu_0$ the following estimate holds omitting the arguments σ and t whenever convenient

$$\begin{aligned}
|\sigma| \leq 2R_\nu - s: &\quad \left|\xi_\nu - \widehat{\Pi}^{u_\nu}_p \xi_\nu(0)\right| + \left|\nabla\big(\xi_\nu - \widehat{\Pi}^{u_\nu}_p \xi_\nu(0)\big)\right| \leq c_2 e^{-\mu(2R_\nu - |\sigma|)}\,. \\
\sigma \geq 2R_\nu + s: &\quad \left|\xi_\nu - \widehat{\Pi}^{u_\nu}_{p_+} \xi_\nu(\infty)\right| + \left|\nabla\big(\xi_\nu - \widehat{\Pi}^{u_\nu}_{p_+} \xi_\nu(\infty)\big)\right| \leq c_2 e^{-\mu(\sigma - 2R_\nu)} \\
\sigma \leq -2R_\nu - s: &\quad \left|\xi_\nu - \widehat{\Pi}^{u_\nu}_{p_-} \xi_\nu(-\infty)\right| + \left|\nabla\big(\xi_\nu - \widehat{\Pi}^{u_\nu}_{p_-} \xi_\nu(-\infty)\big)\right| \leq c_2 e^{-\mu(|\sigma| - 2R_\nu)}
\end{aligned} \tag{8.6.2}$$

By analogy we will only deduce the first estimate. First of all we assume without loss of generality after possibly increasing s that the distance from $u_\nu(\sigma,t)$ to p is small enough to replace $\widehat{\Pi}$ with Π in the formula. Since the exponential function is uniformly Lipschitz (see Corollary A.1.3) and the distance of parallel geodesics is uniformly bounded by the distance of their starting points (see Corollary A.1.4) we estimate for all $|\sigma| \leq 2R_\nu - s$

$$\begin{aligned}\left|\xi_\nu - \Pi^{u_\nu}_p \xi_\nu(0)\right| &\leq O(\operatorname{dist}(v_\nu, v_\nu(0)) + \operatorname{dist}\big(\exp_p \xi_\nu(0), \exp_{u_\nu} \Pi^{u_\nu}_p \xi_\nu(0)\big)) \\ &\leq O(\operatorname{dist}(v_\nu, v_\nu(0)) + \operatorname{dist}(u_\nu, p)) \leq e^{-\mu(2R_\nu - |\sigma|)}\,.\end{aligned}$$

For the last inequality we have used the decay of $u_\nu = u_{R_\nu}$ (cf. Lemma 8.2.1) and of v_ν as given by Lemma 4.3.2 which is applicable because the energy of v_ν restricted to $\Sigma^{2R_\nu - s}_{-2R_\nu + s}$ is small (cf. equation (8.6.1)). We deduce the estimate for the covariant derivative using Corollary A.2.4 to commute ∇ with Π and Corollary A.1.2 to control the covariant derivative of ξ_ν by the differential of u_ν and v_ν.

$$\left|\nabla(\xi_\nu - \Pi^{u_\nu}_p \xi_\nu(0))\right| \leq |\nabla \xi_\nu| + \left|\nabla \Pi^{u_\nu}_p \xi_\nu(0)\right| \leq O(|\mathrm{d}u_\nu| + |\mathrm{d}v_\nu|) \leq O(e^{-\mu(2R_\nu - |\sigma|)})\,.$$

Using the point-wise estimate (8.6.2) we estimate the norm on the neck

$$\begin{aligned}
&\int_{\Sigma_{-2R_\nu}^{2R_\nu}} \Big(\big|\xi_\nu - \widehat{\Pi}_p^{u_\nu}\xi_\nu(0)\big|^p + \big|\nabla(\xi_\nu - \widehat{\Pi}_p^{u_\nu}\xi_\nu(0))\big|^p\Big)\, e^{p\delta(2R_\nu - |\sigma|)}\mathrm{d}\sigma\mathrm{d}t \leq \\
&\leq O(1)\int_0^{2R_\nu - s} e^{-p(\mu-\delta)(2R_\nu-\sigma)}\mathrm{d}\sigma + \\
&\qquad + O\big(\,\|\xi_\nu\|^p_{\Sigma_{-2R_\nu}^{-2R_\nu+s}} + \|\xi_\nu\|^p_{\Sigma_{2R_\nu-s}^{2R_\nu}} + |\xi_\nu(0)|^p\,\big)\int_{2R_\nu-s}^{2R_\nu} e^{p\delta(2R_\nu-\sigma)}\mathrm{d}\sigma \\
&\leq O(e^{-\mu s}) + o(1)\,,
\end{aligned}$$

on the positive end

$$\begin{aligned}
&\int_{\Sigma_{2R_\nu}^{\infty}} \Big(\big|\xi_\nu - \widehat{\Pi}_{p+}^{u_\nu}\xi_\nu(\infty)\big|^p + \big|\nabla(\xi_\nu - \widehat{\Pi}_p^{u_\nu}\xi_\nu(\infty))\big|^p\Big)\, e^{p\delta(\sigma - 2R_\nu)}\mathrm{d}\sigma\mathrm{d}t \leq \\
&\qquad + O(1)\int_s^{\infty} e^{-p(\mu-\delta)\sigma}\mathrm{d}\sigma + o(1)\int_0^s e^{p\delta s}\mathrm{d}s \leq O(e^{-\mu s}) + o(1)\,,
\end{aligned}$$

and similarly for the negative end. The last estimates amount to

$$\|\xi_\nu\|_{1,p;\gamma_{\delta,R_\nu}} \leq O(e^{-\mu s}) + o(1)\,,$$

and in particular show that $\lim_{\nu\to\infty}\|\xi_\nu\|_{1,p;\gamma_{\delta,R_\nu}} \leq O(e^{-\mu s})$. Now since s was chosen freely we see that the limit must vanish. □

Lemma 8.6.2. *With the same assumptions as Lemma 8.6.1, then w_ν lies in the image of the gluing map for all but finitely many ν.*

Proof. The case (C) directly follows from Lemma 8.6.1 because in that case $\ker D_R$ is zero dimensional and by the uniqueness property the solution map we have $w_\nu = \exp_{u_{R_\nu}}\sigma_{R_\nu}(0) = \mathcal{G}(R_\nu)$ for all but finitely many ν. For case (B) set $\epsilon := +1$ if (J_0, X_0) is flow-dependent and $e := -1$ if (J_1, X_1) is flow-dependent. We define for some $\varepsilon > 0$ small and R_0 large enough,

$$\mathcal{M}_{\varepsilon,R_0} := \{w = (\exp_{u_R}\xi)\circ\tau_{2\epsilon R} \mid R \geq R_0,\ \|\xi\|_{1,p;\delta,R} < \varepsilon\} \cap \mathcal{M}(W_-, W_+)\,.$$

In Lemma 8.6.1 we show that $w_\nu \in \mathcal{M}_{\varepsilon,R_0}$ for all but finitely many $\nu \in \mathbb{N}$. We claim that $\mathcal{M}_{\varepsilon,R_0}$ is path-connected. Indeed given two elements $w, w' \in \mathcal{M}_{\varepsilon,R_0}$, by the third property of the solution map (cf. Lemma 8.4.2) there exists constants $R, R' \geq R_0$ and $\zeta \in \ker D_R$, $\zeta' \in \ker D_{R'}$ such that

$$w = \exp_{u_R}(\zeta + \sigma_R(\zeta))\circ\tau_{2\epsilon R}, \qquad w' = \exp_{u_{R'}}(\zeta' + \sigma_{R'}(\zeta'))\circ\tau_{2\epsilon R}\,.$$

We connect w to $w_R = \mathcal{G}(R)$ via $[0,1] \ni \theta \mapsto \exp_{u_R}(\theta\zeta + \sigma_R(\theta\zeta))\circ\tau_{2\epsilon R}$. Similar we connect w' to $w_{R'}$. Assuming without loss of generality that $R < R'$, we connect w_R to $w_{R'}$ via $[R, R'] \ni r \mapsto \mathcal{G}(r)$. We identify the connected one-dimensional space

$\mathcal{M}_{\varepsilon,R_0}$ with an half-infinite interval $[0,\infty)$ such that under the identification the strips w_ν converges to ∞. After the identification the gluing map is an unbounded continuous map $[R_0,\infty) \to [0,\infty)$ and its image by the intermediate value theorem contains all but finitely many elements of w_ν.

In case (A) we define similarly $\widetilde{\mathcal{M}}_{\varepsilon,R_0} := \{w = \exp_{u_R} \xi \mid R_0 \leq R,\ \|\xi\|_{1,p;\delta} < \varepsilon\} \cap \widetilde{\mathcal{M}}(W_-,W_+)$ and by $\mathcal{M}_{\varepsilon,R_0} \subset \mathcal{M}(W_-,W_+)$ the image under the quotient map. Again the space is a connected one-dimensional manifold containing by Lemma 8.6.1 all but finally many strips w_ν. We argue as in case (B). $\square$

8.7. Degree of the gluing map

Fix a relative spin structure for (L_0,L_1) and denote by $\mathcal{O}$ the associated double cover (cf. Definitions 9.3.1 and 9.3.4). Let C_-, $C_+ \subset L_0 \cap L_1$ denote the connected components of W_-, W_+ respectively.

Lemma 8.7.1. *If $W_- \subset C_-$ is equipped with an $\mathcal{O}^\vee$-orientation and $W_+ \subset C_+$ is equipped with an $\mathcal{O}$-orientation then the spaces $\mathcal{M}(W_-,W_+)$ and $\mathcal{M}^1(W_-,W_+)$ have an induced orientation.*

Proof. Abbreviate $\widetilde{\mathcal{M}}(C_-,C_+) := \widetilde{\mathcal{M}}(C_-,C_+;J,X)$ equipped with obvious evaluations into C_- and C_+. By Theorem 9.3.6 and Lemma 9.1.3 the fibre product

$$W_- \times_{C_-} \widetilde{\mathcal{M}}(C_-,C_+) \times_{C_+} W_+\,,$$

carries a canonical orientation and its quotient space $\mathcal{M}(W_-,W_+)$ carries the induced orientation by (9.1.8). Abbreviate the spaces

$$\widetilde{\mathcal{M}}(C_-,C) := \widetilde{\mathcal{M}}(C_-,C;J_0,X_0) \quad \text{and} \quad \widetilde{\mathcal{M}}(C,W_+) := \widetilde{\mathcal{M}}(C,C_+;J_1,X_1)\,,$$

equipped with obvious evaluations into C_-, C and C_+. As above the fibre product

$$W_- \times_{C_-} \widetilde{\mathcal{M}}(C_-,C) \times_C \widetilde{\mathcal{M}}(C,C_+) \times_{C_+} W_+\,,$$

carries an induced orientation and hence its quotient $\mathcal{M}^1(W_-,W_+)$ too. $\square$

Lemma 8.7.2. *Linear gluing $P_R\Theta_R : \ker D_{01} \to \ker D_R$ is orientation preserving.*

Proof. Consider the intersection points $p_- = u_0(-\infty)$, $p = u_0(\infty) = u_1(-\infty)$ and $p_+ = u_1(\infty)$ with caps D_-, D and D_+ respectively. By associativity of linear gluing we have a commutative diagram.

$$\begin{tikzcd}
|D_-| \otimes |C| \otimes |D_{01}| \arrow[r] \arrow[d, "\mathrm{id}\otimes|P_R\Theta_R|"] & |D_-| \otimes |D_0| \otimes |D_1| \arrow[r] & |C_-| \otimes |D| \otimes |D_1| \arrow[d] \\
|D_-| \otimes |C| \otimes |D_R| \arrow[rr] & & |C_-| \otimes |C| \otimes |D_+|
\end{tikzcd}$$

By definition of gluing the orientation on D_{01} is induced by following the diagram from the down-right to the top-left corner and the orientation of D_R is by definition given by going from the down-right to the down-left corner. $\square$

Corollary 8.7.3. *The restriction* $P_R\Theta_R|_{\ker D'_{01}} : \ker D'_{01} \to \ker D'_R$ *is orientation preserving.*

Proof. By definition the orientation of D'_{01} and D'_R are given by (9.1.1) on the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \ker D'_{01} \arrow[r] \arrow[d, "P_R\Theta_R|_{\ker D'_{01}}"] & \ker D_{01} \oplus W_- \oplus W_+ \arrow[r] \arrow[d, "P_R\Theta_R \oplus \mathrm{id}"] & C_- \oplus C_+ \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \ker D'_R \arrow[r] & D_R \oplus W_- \oplus W_+ \arrow[r] & C_- \oplus C_+ \arrow[r] & 0
\end{tikzcd}$$

The claim follows by naturality of (9.1.1). □

Proposition 8.7.4. *The space* $\overline{\mathcal{M}}(W_-, W_+)_{[1]}$ *has the structure of a one dimensional manifold with boundary. With orientations from Lemma 8.7.1 the oriented boundary is*

- $\mathcal{M}^1(W_-, W_+)_{[0]}$ *if* (J_0, X_0) *is* $\mathbb{R}$*-invariant and*
- $(-1) \cdot \mathcal{M}^1(W_-, W_+)_{[0]}$ *if* (J_1, X_1) *is* $\mathbb{R}$*-invariant.*

Proof. The space $\overline{\mathcal{M}}(W_-, W_+)_{[1]}$ is a manifold with boundary using the gluing map as chart map for a boundary point. It remains to show the statement about the degree. We treat each case separately.

Step 1. We prove the proposition in case (C).

The tangent space at some $(R, u) \in \mathcal{M}(W_-, W_+)$ is given as the kernel of the operator $\widehat{D}_u : \mathbb{R} \oplus T_u\mathcal{B}'$, $(\theta, \xi) \mapsto D_u\xi + \theta\eta_R$ with

$$\eta_R = (\partial_R J_R(u))(\partial_t u - X_R(u)) - J_R(u)(\partial_R X_R(u))\,.$$

Here $\mathcal{B}' \subset \mathcal{B}(C_-, C_+)$ is the subspace of all u with $u(-\infty) \in W_-$ and $u(\infty) \in W_+$. We assume without loss of generality that D_u is surjective and hence an isomorphism when restricted to $T_u\mathcal{B}'$. We conclude that there exists a unique $\xi_R \in T_u\mathcal{B}'$ such that $D_u\xi_R = \eta_R$. The vector $(1, -\xi_R) \in \ker \widehat{D}_u$ is pointing outward has has the same orientation as the sign of the isomorphism D_u, which by parallel transport is the same as the sign of the isomorphism D'_R as considered above in Corollary 8.7.3. We conclude that the sign of D'_R is the same as the sign of D'_{01}, which by definition is the sign of (u_0, u_1). This shows the claim.

Step 2. We prove the proposition in case (B).

The orientation on $\mathcal{M}(W_-, W_+)_{[1]}$ induces a total order on each connected component. We have to distinguish the two sub case when (J_0, X_0) is $\mathbb{R}$-invariant or (J_1, X_1) is $\mathbb{R}$-invariant. By analogy we only treat the case where (J_0, X_0) is $\mathbb{R}$-invariant. Fix a point $([u_0], u_1) \in \mathcal{M}^1(W_-, W_+)_{[0]}$. By surjectivity of gluing there exists one connected component of $\mathcal{M}(W_-, W_+)_{[1]}$ containing a sequence which converges to $([u_0], u_1)$. Let $(w_\nu)_{\nu\in\mathbb{N}}$ be such a sequence which is monotone with respect to the total order. By surjectivity of gluing we have for all ν large enough

$$w_\nu \circ \tau_{2R_\nu} = \exp_{u_\nu} \xi_\nu, \qquad \xi_\nu := \sigma_\nu(0) \in \operatorname{im} Q_{R_\nu}\,.$$

Choosing the sequence fine enough we assume without loss of generality there exists $\zeta_\nu \in \ker D'_{R_\nu}$ such that

$$w_{\nu+1} \circ \tau_{2R_\nu} = \exp_{u_\nu}(\zeta_\nu + \eta_\nu) \qquad \eta_\nu := \sigma_\nu(\zeta_\nu) \in \operatorname{im} Q_{R_\nu}\,. \tag{8.7.1}$$

Let o_∂ the orientation of $([u_0], u_1)$ as a boundary point of $\mathcal{M}(W_-, W_+)_{[1]}$ and o be the orientation as an element of the oriented space $\mathcal{M}^1(W_-, W_+)_{[0]}$. We claim that using the orientations of D'_ν given by Lemma 8.7.1 we have

$$o_\partial = +1 \iff w_\nu < w_{\nu+1} \iff \zeta_\nu \text{ is pos.} \iff o = +1\,. \tag{8.7.2}$$

This clearly shows the assertion and we are left to deduce all equivalences. The first equivalence is a definition. The second equivalence is also a definition, since we have by the properties of the solution map (cf. Lemma 8.4.2) an orientation preserving path from w_ν to $w_{\nu+1}$ via $[0,1] \ni \theta \mapsto \exp_{u_\nu}(\theta\zeta_\nu + \sigma(\theta\zeta_\nu)) \circ \tau_{-2R_\nu}$. We show the third equivalence. For $R > 0$ we define $u^\tau_R := u_R \circ \tau_{-2R}$, $\xi^\tau_\nu := \xi_\nu \circ \tau_{-2R_\nu}$, $\eta^\tau_\nu := \eta_\nu \circ \tau_{-2R_\nu}$ and for $\theta \in [0,1]$

$$u^\tau_{\nu,\theta} := u^\tau_{R_\nu + \theta(R_{\nu+1} - R_\nu)}, \qquad \xi^\tau_{\nu,\theta} := \Pi_\nu(\theta)\xi^\tau_{\nu+1}, \qquad \chi^\tau_{\nu,\theta} := \exp^{-1}_{u^\tau_\nu} \exp_{u^\tau_{\nu,\theta}} \xi^\tau_{\nu,\theta}\,,$$

in which $\Pi_\nu(\theta)$ denotes the parallel transport from $u^\tau_{\nu+1}$ to $u^\tau_{\nu,\theta}$ along the path $[\theta, 1] \ni \tau \mapsto u^\tau_{\nu,\tau}$. By construction and (8.7.1) we have

$$u^\tau_{\nu,1} = u^\tau_{\nu+1}, \qquad u^\tau_{\nu,0} = u^\tau_\nu, \qquad \zeta^\tau_\nu + \eta^\tau_\nu = \chi^\tau_{\nu,1}, \qquad \chi^\tau_{\nu,0} = \Pi_\nu(0)\xi^\tau_{\nu+1}\,.$$

The path $\theta \mapsto \chi^\tau_{\nu,\theta}$ is differentiable and by the mean value theorem we conclude that there exists $\theta_\nu \in [0,1]$ such that

$$\zeta^\tau_\nu + \eta^\tau_\nu = \chi^\tau_{\nu,1} = \Pi_\nu(0)\xi^\tau_{\nu+1} + \rho_\nu \cdot (\partial_\theta \chi^\tau_{\nu,\theta})|_{\theta=\theta_\nu}, \qquad \rho_\nu := R_{\nu+1} - R_\nu\,.$$

We apply $\tau_{-2\epsilon R_\nu}$ and subtract $\kappa_\nu := \partial_\theta u^\tau_{\nu,\theta}|_{\theta=0} \circ \tau_{-2R_\nu}$ on both sides

$$\zeta_\nu - \kappa_\nu = \Pi_\nu(0)\xi_\nu - \eta_\nu + \rho_\nu \cdot (\partial_\theta \chi^\tau_{\nu,\theta}|_{\theta=\theta_\nu} - \partial_\theta u^\tau_{\nu,\theta}|_{\theta=0}) \circ \tau_{-2R_\nu}\,.$$

By the property of the solution map the correction terms ξ_ν and η_ν converge to zero uniformly (cf. Lemma 8.4.2). Moreover by Corollary A.1.2 the last term is uniformly bounded by $O(\rho_\nu^2)$. Thus we have the pointwise estimate

$$|\zeta_\nu - \kappa_\nu| \le o(1) + O(\rho_\nu^2)\,.$$

Since u_0 is holomorphic and non-constant the preglued strip u_R has an end which is non-constant and holomorphic. By the asymptotic behavior (cf. Theorem 4.1.2) we conclude that

$$\rho_\nu \le O(1)\mathrm{dist}\,(u_\nu, u_{\nu+1}) \le O(1)\mathrm{dist}\,(u_\nu, w_{\nu+1}) + o(1) \le O(1)|\zeta_\nu| + o(1)\,.$$

Fix constants $s_0 < s_1$ and define the strips $\Sigma_\nu := [s_0 - 2R_\nu, s_1 - 2R_\nu] \times [0,1]$. We have with uniform constants

$$\|Q_\nu D_\nu \kappa_\nu\|_{C^0(\Sigma_\nu)} \le O(1)\,\|Q_\nu D_\nu \kappa_\nu\|_{1,p;\delta,R} \le O(1)\,\|D_\nu \kappa_\nu\|_{p;\delta} \le o(1)\,.$$

Combining the last estimates we have for all $(s,t) \in \Sigma_\nu$

$$|\zeta_\nu - P_\nu \kappa_\nu| \leq |\zeta_\nu - \kappa_\nu| + |Q_\nu D_\nu \kappa_\nu| \leq O(1)|\zeta_\nu| + o(1)\,.$$

Let $\varepsilon > 0$ be some sufficiently small number and assume that the constants $s_0 < s_1$ are chosen such that $|\partial_s u_0(s,t)| > \varepsilon$ for all $(s,t) \in [s_0, s_1] \times [0,1]$. Hence by construction we have for all $(s,t) \in \Sigma_\nu$ and all ν sufficiently large

$$|P_\nu \kappa_\nu| = |\kappa_\nu| + o(1) \geq \varepsilon/2\,.$$

Now define $\alpha_\nu \in \mathbb{R}$ via $\alpha_\nu P_\nu \kappa_\nu = \zeta_\nu$ uniquely since $\ker D_{R_\nu}$ is one dimensional. We have with the above estimates

$$|1 - \alpha_\nu| = |(1-\alpha_\nu) P_\nu \kappa_\nu| / |P_\nu \kappa_\nu| \leq 2/\varepsilon \cdot |\zeta_\nu - P_\nu \kappa_\nu| \leq O(1)|\zeta_\nu| + o(1)\,.$$

If ν is large enough and $|\zeta_\nu|$ small enough (by choosing the sequence fine enough), we have $|1-\alpha_\nu| < 1/2$ in particular α_ν is positive. We conclude that ζ_ν has the same orientation as $P_\nu \kappa_\nu$. A direct computation shows that $\kappa_\nu = 4 R_\nu \Theta_\nu \partial_s u_0$ for all $(s,t) \in \Sigma_\nu$ which togeher with Lemma 8.7.2 shows that ζ_ν has the same orientation as $\partial_s u_0$ as claimed.

Step 3. We prove the proposition in case (A).

Similarly to the last step let $([w_\nu]) \subset \mathcal{M}(W_-, W_+)_{[0]}$ be a strictly monotone sequence converging to $([u_0],[u_1])$. We assume after a possible reparametrization and by surjectivity of gluing that $w_\nu = \exp_{u_\nu} \xi_\nu$ for $u_\nu = u_{R_\nu}$ and $\xi_\nu \in \operatorname{im} Q_\nu$. Choosing the sequence fine enough we have for all $\nu \in \mathbb{N}$ large enough

$$w_{\nu+1} = \exp_{u_\nu}(\zeta_\nu + \eta_\nu), \qquad \zeta_\nu \in \ker D_\nu, \quad \eta_\nu \in \operatorname{im} Q_\nu\,.$$

Define the path from w_ν to $w_{\nu+1}$ via $[0,1] \ni \theta \mapsto w_{\nu,\theta} := \exp_{u_\nu}(\theta\zeta_\nu + \sigma_\nu(\theta\zeta_\nu))$. Let $o_\partial \in \{\pm 1\}$ denote the orientation of $([u_0],[u_1])$ as a boundary point of $\mathcal{M}(W_-, W_+)$ and $o \in \{\pm 1\}$ as a point of the oriented space $\mathcal{M}^1(W_-, W_+)$. We claim that we have the following equivalences with orientations on $D'_{w_{\nu,\theta}}$, D'_{R_ν} and D'_{01} given by Lemma 8.7.1,

$$\begin{aligned} o_\partial = +1 &\iff ([w_\nu]) \text{ is incr.} \\ &\iff \partial_s w_{\nu,\theta} \wedge \partial_\theta w_{\nu,\theta} \in \det D'_{w_{\nu,\theta}} \text{ is pos.} \\ &\iff P_\nu \partial_s u_\nu \wedge P_\nu \kappa_\nu \in \det D'_{R_\nu} \text{ is pos.} \qquad \kappa_\nu := (\partial_R u_R)|_{R=R_\nu} \\ &\iff \partial_s u_1 \wedge \partial_s u_0 \in \det D'_{01} \text{ is pos.} \\ &\iff o = -1\,. \end{aligned} \tag{8.7.3}$$

This clearly implies the assertion of the proposition. We show (8.7.3). The first and the last equivalence is a definition. The second equivalence also clear by definition of the quotient orientation (cf. equation (9.1.8)). We claim that the third equivalence of (8.7.3) follows if we find a norm $\|\cdot\|_\nu$ on $\det D_{R_\nu}$ such that there exists an uniform constant $\delta > 0$ with

$$\|P_\nu \partial_s u_\nu \wedge P_\nu \kappa_\nu\|_\nu > \delta \tag{8.7.4}$$

and if $\Pi_{\nu,\theta}$ denotes the parallel transport from $w_{\nu,\theta}$ to u_ν we have

$$\|P_\nu \partial_s u_\nu \wedge P_\nu \kappa_\nu - P_\nu \Pi_{\nu,\theta}\partial_s w_{\nu,\theta} \wedge P_\nu \Pi_{\nu,\theta}\partial_\theta w_{\nu,\theta}\|_\nu \le O(\|\zeta_\nu\|_{C^1}) + o(1). \tag{8.7.5}$$

Indeed, suppose that (8.7.4) and (8.7.5) is true. Then define $\alpha_\nu \in \mathbb{R}$ by

$$\alpha_\nu P_\nu \partial_s u_\nu \wedge P_\nu \kappa_\nu = P_\nu \Pi_{\nu,\theta}\partial_s w_{\nu,\theta} \wedge P_\nu \Pi_{\nu,\theta}\partial_\theta w_{\nu,\theta}\,.$$

Using (8.7.4) and (8.7.5) we find an uniform constant $c > 0$ such that for all ν sufficiently large

$$|1-\alpha_\nu| = \frac{\|(1-\alpha_\nu)P_\nu \partial_s u_\nu \wedge P_\nu \kappa_\nu\|_\nu}{\|P_\nu \partial_s u_\nu \wedge P_\nu \kappa_\nu\|_\nu} \le \frac{c\,\|\zeta_\nu\|_{C^1} + \delta/3}{\delta}\,.$$

Choosing the sequence $([w_\nu])$ fine enough we assume without loss of generality that $\|\zeta_\nu\|_{C^1} < \delta/3c$. We conclude that $|1-\alpha_\nu| \le 2/3$ and hence α_ν is positive. Thus the third equivalence of (8.7.3) follows (using the fact that the operator $P_\nu \Pi_{\nu,\theta}$ is orientation preserving).

It remains to find a norm $\|\cdot\|_\nu$ such that (8.7.4) and (8.7.5) holds. Fix constants $s_0 < s_1$ and consider the Hilbert space $H_\nu := L^2(\Sigma_\nu)$ on the domain $\Sigma_\nu = \Sigma_{\nu,-} \cup \Sigma_{\nu,+}$ with $\Sigma_- = [s_0 - 2R_\nu, s_1 - 2R_\nu] \times [0,1]$ and $\Sigma_{\nu,+} = [s_0 + 2R_\nu, s_1 + 2R_\nu] \times [0,1]$. Abbreviate $\|\cdot\|_\nu = \|\cdot\|_{L^2(\Sigma_\nu)}$ and $\langle\cdot,\cdot\rangle_\nu = \langle\cdot,\cdot\rangle_{L^2(\Sigma_\nu)}$ the standard norm and the scalar product on H_ν. We consider the norm on $\Lambda^2 H_\nu$ given by

$$\|\xi \wedge \xi'\|_\nu := (\|\xi\|_\nu^2\,\|\xi'\|_\nu^2 - \langle \xi, \xi'\rangle_\nu^2)^{1/2}\,.$$

Using the Cauchy-Schwarz inequality we have $\|\xi \wedge \xi'\| \le 2\,\|\xi\|\,\|\xi'\|$ for all $\xi, \xi' \in H_\nu$. With Corollary A.1.2 we obtain

$$\|\Pi_{\nu,\theta}\partial_s w_{\nu,\theta} - P_\nu \partial_s u_\nu\|_\nu \le o(1) + O(\|\zeta_\nu\|_{C^1}).$$

In the last step we show that

$$\|\Pi_{\nu,\theta}\partial_\theta w_{\nu,\theta} - P_\nu \kappa_\nu\|_\nu \le o(1) + O(\|\zeta_\nu\|_{C^0})\,.$$

This shows (8.7.5).

We show (8.7.4). Choose a small constant $\varepsilon > 0$ and assume that $s_0 < s_1$ are such that $|\partial_s u_0(s,t)| > \varepsilon$ and $|\partial_s u_1(s,t)| > \varepsilon$ for all $(s,t) \in [s_0, s_1] \times [0,1]$. Abbreviate the norm $\|\cdot\| == \|\cdot\|_{L^2([s_0,s_1]\times[0,1])}$. By construction $\|\partial_s u_\nu\|_\nu^2 = \|\kappa_\nu\|_\nu^2 = \|\partial_s u_0\|^2 + \|\partial_s u_1\|^2$ and $\langle \partial_s u_\nu, \kappa_\nu\rangle = \|\partial_s u_0\|^2 - \|\partial_s u_1\|^2$. We compute

$$\begin{aligned}
\|P_\nu \kappa_\nu \wedge P_\nu \partial_s u_\nu\|_\nu^2 &= \|P_\nu \kappa_\nu\|^2\,\|P_\nu \partial_s u_\nu\|^2 - \langle P_\nu \kappa_\nu, P_\nu \partial_s u_\nu\rangle^2 \\
&= \|\kappa_\nu\|^2\,\|\partial_s u_\nu\|^2 - \langle \kappa_\nu, \partial_s u_\nu\rangle^2 + o(1) \\
&= (\|\partial_s u_0\|^2 + \|\partial_s u_1\|^2)^2 - (\|\partial_s u_0\|^2 - \|\partial_s u_1\|^2)^2 + o(1) \\
&= 4\,\|\partial_s u_0\|^2\,\|\partial_s u_1\|^2 + o(1) \ge 2(s_1 - s_0)^4 \varepsilon^4 =: \delta^2 > 0\,.
\end{aligned}$$

This shows (8.7.4) and hence the third equivalence of (8.7.3).

Finally the fourth equivalence follows because by construction

$$\|P_\nu \Theta_\nu(\partial_s u_0, \partial_s u_1) - P_\nu \partial_s u_\nu\|_\nu \to 0, \qquad \|P_\nu \Theta_\nu(\partial_s u_0, -\partial_s u_1) - 1/2 P_\nu \kappa_\nu\|_\nu \to 0\,.$$

We conclude with Corollary 8.7.3 that $\partial_s u_0 \wedge \partial_s u_1 \in \Lambda^2 \ker D'_{01}$ has the same orientation as $P_\nu \kappa_\nu \wedge P_\nu \partial_s u_\nu \in \Lambda^2 \ker D'_R$. $\square$

8.8. Morse gluing

In the section we describe a gluing result for Morse trajectories, which is a little more general than the gluing result in [1]. We have included a complete proof, since we have not found it in the literature. The methods remain the same and rely on known results about hyperbolic dynamics, in particular the graph transform theorem.

Let $f : C \to \mathbb{R}$ be a Morse function and denote by $\psi : \mathbb{R} \times C \to C$, $\psi^a = \psi(a, \cdot)$ the negative gradient flow of f with respect the some fixed Riemannian metric. Given sub-manifolds $W_-, W_+ \subset C$ we define the space

$$W_- \times_\psi W_+ := \{(R, w_-, w_+) \mid \psi^R(w_-) = w_+\} \subset \mathbb{R} \times W_- \times W_+ \,, \tag{8.8.1}$$

which is the space of finite length flow-lines from W_- to W_+. Standard compactness shows that as a finite length flow-line gets longer and longer it approaches a broken flow-line, which generically is a pair

$$w^\infty = (w_-^\infty, w_+^\infty) \in (W_- \cap W^s(p)) \times (W^u(p) \cap W_+) \,, \tag{8.8.2}$$

for some critical point $p \in \operatorname{crit} f$. The next lemma shows that this process is reversible, i.e. any a broken flow-line can be glued together to obtain a family of finite length flow-lines.

For the orientation statement we assume that W_- is oriented and W_+ is cooriented. The space (8.8.1) is cut-out transversely, if for all points we have the exact sequence

$$0 \longrightarrow T_{(R,w_-,w_+)} W_- \times_\psi W_+ \longrightarrow \mathbb{R} \oplus T_{w_-} W_- \xrightarrow{\mathrm{d}\psi} T_{w_+} C / T_{w_+} W_+ \longrightarrow 0 \,. \tag{8.8.3}$$

From the sequence we obtain via (9.1.1) an orientation on the space $W_- \times_\psi W_+$ provided with the fixed orientations. If $\dim W_- + \dim W_+ = \dim C$ and (8.8.2) is cut-out transversely, the space (8.8.2) is zero-dimensional. Moreover if $W^u(p)$ is oriented then $W^s(u)$ is cooriented and by (9.1.6) we have signs $\varepsilon_- := \operatorname{sign}(w_-^\infty)$ and $\varepsilon_+ := \operatorname{sign}(w_+^\infty)$. The product $\varepsilon := \varepsilon_- \, \varepsilon_+$ does not depend on the choice of the orientation of $W^u(p)$. If W_- is everywhere transverse to the gradient of f we write $\operatorname{grad} f \pitchfork W_-$ and define the manifold $\widetilde{W}_- := \mathbb{R} \times W_-$ embedded into C via the flow ψ and oriented by

$$-\operatorname{grad}_w f \mathbb{R} \oplus T_w W_- \cong T_w \widetilde{W}_- \,. \tag{8.8.4}$$

Similarly if W_+ is everywhere transverse to the gradient of f we write $\operatorname{grad} f \pitchfork W_+$ and define the manifold $\widetilde{W}_+ := \mathbb{R} \times W_+$ embedded into C via the flow and cooriented by

$$-\operatorname{grad}_w f \mathbb{R} \oplus T_w C_+ / T_w \widetilde{W}_+ \cong T_w C_+ / T_w W_+ \,. \tag{8.8.5}$$

We obtain orientations of $\widetilde{W}_- \cap W_+$ and $W_- \cap \widetilde{W}_+$ via (9.1.6).

Lemma 8.8.1 (Morse gluing)**.** *Assume that* (8.8.1) *and* (8.8.2) *are cut-out transversely and* $\dim W_- + \dim W_+ = \dim C$*. For any element* (w_-^∞, w_+^∞) *of the space* (8.8.2) *there exists an injective immersion*

$$[R_0, \infty) \to W_- \times_\psi W_+, \qquad R \mapsto (R, w_-^R, w_+^R) \,, \tag{8.8.6}$$

such that

(i) $\lim_{R\to\infty} w_\pm^R = w_\pm^\infty$,

(ii) *there exists* $\delta > 0$ *such that for any* $(R, w_-, w_+) \in W_- \times_\psi W_+$ *with* $\operatorname{dist}\big(w_-, w_-^\infty\big) + \operatorname{dist}\big(w_+, w_+^\infty\big) < \delta$ *we have* $(w_-, w_+) = (w_-^R, w_+^R)$,

(iii) *suppose that* W_- *is oriented and* W_+ *is cooriented, then*

a) the orientation of the vector $(1, \partial_R w_-^R, \partial_R w_+^R) \in W_- \times_\psi W_+$ *is* ε,

b) if $\operatorname{grad} f \pitchfork W_-$ *the orientation of the vector* $\partial_R w_+^R \in \widetilde{W}_- \cap W_+$ *is* ε,

c) if $\operatorname{grad} f \pitchfork W_+$ *the orientation of the vector* $\partial_R w_-^R \in W_- \cap \widetilde{W}_+$ *is* $-\varepsilon$.

where $\varepsilon = \operatorname{sign}(w_-^\infty) \operatorname{sign}(w_+^\infty)$ *and the spaces are oriented as described above.*

Proof. Let $B_\rho(w) \subset C$ denote an open ball with radius $\rho > 0$ centered at $w \in C$. Without loss of generality we replace W_- with $W_- \cap B_\rho(w_-^\infty)$ and W_+ with $W_+ \cap B_\rho(w_+^\infty)$ for some sufficiently small $\rho > 0$. How small ρ needs to be is explained throughout the course of the proof.

By assumption $w_-^\infty \in W_- \cap W^s(p)$ and $w_+^\infty \in W^u(p) \cap W_+$. We identify a neighborhood of the critical point p in C with a neighborhood of 0 in a vector space H identifying p with 0. The splitting of the spectrum of the Hessian of f at p into negative and positive part induces a splitting of H denoted $H^u \oplus H^s$. By $H^u(r)$ (resp. $H^s(r)$) we denote the closed r-ball centered at 0 of the linear space H^u (resp. H^s). We set $Q(r) := H^u(r) \times H^s(r)$, which after an identification of $H^u \times H^s$ with $H^u \oplus H^s$ is a subset of H and also a neighborhood of p in C.

Step 1. We claim that for sufficiently small ρ, there exists positive constants R_0, r_0 and continuous paths $\sigma : [R_0, \infty) \to C^0(H^u(r_0), H^s(r_0))$, $R \mapsto \sigma_R$ and $\tau : [R_0, \infty) \to C^0(H^s(r_0), H^u(r_0))$, $R \mapsto \tau_R$ such that for all $R \geq R_0$ the functions σ_R and τ_R are Lipschitz with constant $\theta < 1$ and moreover

(i) $W_-^R := \psi^R(W_-) \cap Q(r_0)$ is the graph of σ_R for all $R \geq R_0$,

(ii) σ_R uniformly converges to σ_∞ as $R \to \infty$ and $\operatorname{graph} \sigma_\infty = W^u(p) \cap Q(r_0)$,

(iii) $W_+^{-R} := \psi^{-R}(W_+) \cap Q(r_0)$ is the graph of τ_R for all $R \geq R_0$,

(iv) τ_R uniformly converges to τ_∞ as $R \to \infty$ and $\operatorname{graph} \tau_\infty = W^s(p) \cap Q(r_0)$.

A proof is given in [1, Lmm. 11.2] up to a small issue that W_- (resp. W_+) is assumed to be an open subset of the unstable (resp. stable) manifold of some critical point of index $\mu(p) + 1$ (resp. $\mu(p) - 1$) intersected with a level set. Nevertheless the proof goes through without any change for general W_- and W_+ provided that we have the splitting

$$T_{w_-^\infty} W_- \oplus T_{w_-^\infty} W^s(p) = T_{w_-^\infty} C \quad \text{and} \quad T_{w_+^\infty} W^u(p) \oplus T_{w_+^\infty} W_+ = T_{w_+^\infty} C\,,$$

which holds by transversality and the assumption $\dim W_- + \dim W_+ = \dim C$.

Step 2. We define (8.8.6)

For all $R_1, R_2 \geq R_0$ the spaces $W_-^{R_1}$ and $W_+^{-R_2}$ have an unique intersection point inside $Q(r_0)$. Indeed, given any $(x, y) \in W_-^{R_1} \cap W_+^{-R_2} \in H^u \oplus H^s$ we conclude with step 1 that $(x, y) = (x, \sigma_{R_1}(x)) = (\tau_{R_2}(y), y)$. Hence x is the unique fixed point of the contraction $\tau_{R_2} \circ \sigma_{R_1}$ and y is the unique fixed point of the contraction $\sigma_{R_1} \circ \tau_{R_2}$. By abuse of notation we denote the intersection point (x, y) by $W_-^{R_1} \cap W_+^{-R_2}$. For all $R \geq 2R_0$ we define the map (8.8.6) via $R \mapsto (R, w_-^R, w_+^R)$ with $w_-^R := \psi^{-R/2}(W_-^{R/2} \cap W_+^{-R/2})$ and $w_+^R := \psi^{R/2}(W_-^{R/2} \cap W_+^{-R/2})$. It remains to check the properties.

Step 3. We show (i).

By construction the orthogonal projection onto H^u of the point $\psi^{R_0} w_-^R = W_-^{R_0} \cap W_+^{R_0 - R} \in H^u \oplus H^s$ is the fixed point of the contraction $\tau_{R-R_0} \circ \sigma_{R_0}$. Since τ_{R-R_0} converges to τ_∞ uniformly as $R \to \infty$ the fixed point converges to the fixed point of $\tau_\infty \circ \sigma_{R_0}$. Similarly we show that the orthogonal projection onto H^s of $\psi^{R_0} w_-^R$ converges to the fixed point of $\sigma_{R_0} \circ \tau_\infty$. In other words $\psi^{R_0} w_-^R$ converges to the unique intersection point $W_-^{R_0} \cap W^s(p)$ as $R \to \infty$ because $W^s(p) \cap Q(r_0)$ is the graph of τ_∞. After possibly making $\rho > 0$ smaller again we assume that w_-^∞ is the only point in $W_- \cap W^s(p)$. Hence $\psi^{R_0} w_-^R \to \psi^{R_0} w_-^\infty$ and thus $w_-^R \to w_-^\infty$. Completely analogous we argue that $w_+^R \to w_+^\infty$.

Step 4. We show (ii).

A standard Morse compactness argument shows that there exists $\delta > 0$ such that for any $w \in B_\delta(w_-^\infty)$ with $\psi^R w \in B_\delta(w_+^\infty)$ for some R we must have $\psi^{R/2} w \in Q(r_0)$. In particular if $(R, w_-, w_+) \in W_- \times_\psi W_+$ such that $w_- \in B_\delta(w_-^\infty)$ and $w_+ = \psi^R w_- \in B_\delta(w_+^\infty)$, then $\psi^{R/2} w_- \in Q(r_0)$. By uniqueness of the intersection point we conclude that $\psi^{R/2} w_- = W_-^{R/2} \cap W_+^{-R/2}$, hence $w_- = w_-^R$ and $w_+ = w_+^R$.

Step 5. We equip $[0, \infty] := [0, \infty) \cup \{\infty\}$ with the topology of $[0, \pi/2]$ induced by the bijection $[0, \pi/2] \cong [0, \infty]$, $s \mapsto \tan(s)$ and $\pi/2 \mapsto \infty$. Let $\mathrm{Gr}(H)$ denote the space of linear subspaces in H. There exists continuous maps

$$\Omega_- : [0, \infty]^2 \to \mathrm{Gr}(H)\,, \tag{8.8.7}$$

such that for all $R \in [0, \infty]$ we have

(*a*) $\Omega_-(R, 0) = T_{w_-^R} W_-$,

(*b*) $\Omega_-(R, \infty) = T_{\psi^{-R}(w_+^\infty)} W^u(p)$,

(*c*) $\Omega_-(0, R) \oplus T_{w_+^R} W_+ = H$,

(*d*) $\Omega_-(\infty, R) \oplus T_{\psi^R(w_-^\infty)} W^s(p) = H$.

Gradient flow lines $R \mapsto \psi^R(w_-^\infty)$ and $R \mapsto \psi^{-R}(w_+^\infty)$ extend to continuous functions on $[0, \infty]$. By Step 3 the paths $R \mapsto w_-^R$ and $R \mapsto w_+^R$ also extend to continuous functions. Write $\Omega_-(R_1, R_2) \subset H$ as the graph of a linear map $S(R_1, R_2) : H^u \to H^s$ of norm < 1. On two sides of the quadrilateral $[0, \infty]^2$ the map $(R_1, R_2) \mapsto S(R_1, R_2)$ is already determined by the conditions (*a*) and (*b*). For these sides the condition on the norm is satisfied since by Step 1 the spaces $W_- \cap Q(r_0)$ and $W^u(p) \cap Q(r_0)$ are graphs of maps with Lipschitz-constant < 1. As the space of linear maps with norm < 1 is convex we extend the map S to $[0,\infty]^2$ uniquely up to homotopy. Because the norm of $S(0,R)$ is < 1 and $W_+ \cap Q(r_0)$ and $W^s(p) \cap Q(r_0)$ are graphs of a map with Lipschitz-constant < 1 the conditions (c) and (d) are automatically satisfied.

Step 6. We show (iii) case a).

Without loss of generality assume $R_0 = 0$ (if not replace W_- by $\psi^{R_0/2} W_-$ and W_+ by $\psi^{-R_0/2} W_+$). Abbreviate $W := W_- \times_\psi W_+$. Set $w = w_-^0 = w_+^0 \in W_- \cap W_+$. By sequence (8.8.3) the tangent space of W at $(0, w, w)$ is identified with the kernel of the map

$$\phi : \mathbb{R} \oplus T_w W_- \longrightarrow T_w C / T_w W_+, \quad (\theta, \xi^-) \mapsto \xi^- - \theta \, \mathrm{grad}_w f \, .$$

By definition the orientation o_W of W is given by

$$o_W \wedge o_+ = o_{\mathbb{R}} \wedge o_- \, ,$$

in which $o_{\mathbb{R}}$ is the standard orientation of $\mathbb{R}$, o_- is the fixed orientation of $T_w W_-$ and o_+ is the orientation of a linear complement of the kernel of ϕ such that $\phi_*(o_+)$ is the fixed orientation of $T_w C / T_w W_+$. The kernel of ϕ is one-dimensional. The sign $\delta \in \{\pm 1\}$ of the vector $\xi := (1, \partial_R w_-^R, \partial_R w_+^R)$ is given by

$$\xi = \delta \, o_W \, ,$$

where by abuse of notation we denote by ξ also the orientation on W induced by ξ. We see directly that $T_w W_-$ is a linear complement of ξ in $\mathbb{R} \oplus T_w W_-$ and moreover

$$\xi \wedge o_- = o_{\mathbb{R}} \wedge o_- \, .$$

Since ϕ maps the subspace $T_w W_- \subset \mathbb{R} \oplus T_w W_-$ isomorphically onto $T_w C / T_w W_+$ there exists $\alpha \in \{\pm 1\}$ such that

$$o_+ = \alpha \, o_- \, .$$

Collecting the last four equations we have

$$\delta \, o_W \wedge o_+ = \xi \wedge o_+ = \alpha \, \xi \wedge o_- = \alpha \, o_{\mathbb{R}} \wedge o_- = \alpha \, o_W \wedge o_+ \, .$$

It suffices to show that $\alpha = \varepsilon = \mathrm{sign}(w_-^\infty) \, \mathrm{sign}(w_+^\infty)$. We view Ω_- which is constructed in Step 5 as a vector bundle over $[0,\infty]^2$. Because the base is contractible the vector bundle is orientable and an orientation is determined by an orientation of a fibre. By condition (a) the bundle Ω_- is oriented by W_-, also denoted o_-. Let o_p be an orientation of $W^u(p)$. By condition (b) the bundle is oriented by o_p also denoted by o_p. Finally by condition (c) the bundle Ω_- is oriented by o_+, also denoted o_+. Abbreviate $\varepsilon_- := \mathrm{sign}(w_-^\infty)$ and $\varepsilon_+ := \mathrm{sign}(w_+^\infty)$ defined above. By definition $\alpha \, o_- = o_+$, $\varepsilon_- \, o_- = o_p$ and $\varepsilon_+ \, o_p = o_+$. Putting these three equations together shows (iii) case a).

Step 7. For all $R \geq R_0$ we have

(a) $\partial_R w_-^R$ is the projection of $\operatorname{grad}_{w_-^R} f$ onto $T_{w_-^R} W_-$ along $T_{w_-^R}\psi^{-R}W_+$,

(b) $\partial_R w_+^R$ is the projection of $-\operatorname{grad}_{w_+^R} f$ onto $T_{w_+^R} W_+$ along $T_{w_+^R}\psi^{R}W_-$.

Without loss of generality $R=0$. We identify a neighborhood of $w := w_-^0 = w_+^0$ with an open ball in H identifying w with zero and such that the gradient vector field of f is constant under the identification. Write $-\operatorname{grad} f = v$ for some vector $v \in H$. We have a splitting $H = H_- \oplus H_+$ where $H_\pm = T_w W_\pm$. The space W_- is given as a graph $\varphi : H_- \to H_+$ with $\mathrm{d}\varphi(0) = 0$ and W_+ is given as a graph $\phi : H_+ \to H_-$ with $\mathrm{d}\phi(0) = 0$. Write $v = (v_-, v_+) \in H_- \oplus H_+$. An intersection point $(x,y) \in \psi^R W_- \cap W_+$ satisfies

$$(x + Rv_-, \varphi(x) + Rv_+) = (\phi(y), y)\,.$$

In particular y is a fixed point of the map $\theta_R(y) := \varphi(\phi(y) - Rv_-) + Rv_+$. Up to possibly considering a smaller neighborhood θ_R is a contraction for all R small enough and hence the unique fixed point $y_0(R)$ depends smoothly on R. The corresponding intersection point is $w_+^R = (\phi(y_0(R)), y_0(R))$. By deriving the equation $\theta_R(y_0(R)) = y_0(R)$ by R shows that $\partial_R w_+^R = (0, v_+)$. Similarly we show $\partial_R w_-^R = (-v_-, 0)$.

Step 8. We show (iii) case c).

Let $\Omega_- : [0,\infty] \to \mathrm{Gr}(H)$ be the map (8.8.7). For each $R \in [0,\infty]$ consider the vector $\xi(R) \in \Omega_-(0,R)$ which is defined to be the projection of $-\operatorname{grad} f$ at w_+^R onto $\Omega_-(0,R)$ along W_+. The vector $\xi(R)$ is well-defined by property (c) of Ω_- and we have that $\xi(R)\mathbb{R} = \Omega_-(0,R) \cap T_{w_+^R}\widetilde{W}_+$. Let $\chi(R) \subset \Omega_-(0,R)$ be a linear complement of $\xi(R)$ which depends continuously on R. The space $\xi(R)$ is oriented via the coorientation of $\widetilde{W}_+$ and the canonical identification

$$\chi(R) \cong T_{w_+^R} C_+ / T_{w_+^R}\widetilde{W}_+\,.$$

We view χ as a vector bundle over $[0,\infty]$ and denote the orientation by $\widetilde{o}_+$, which by (8.8.5) is uniquely determined by $-\operatorname{grad} f \wedge \widetilde{o}_+ = o_+$. As above we have orientations o_p and o_- of Ω_- induced by $W^u(p)$ and W_- respectively. By Step 8.8 we have $\partial_R w_-^R|_{R=0} = -\xi(0)$. By definition of the orientation of $\widetilde{W}_- \cap W_+$ the sign $\delta \in \{\pm 1\}$ of $\partial_R w_-^R$ is defined by $-\delta\,\xi \wedge \widetilde{o}_+ = o_-$. We still have $\varepsilon_- o_- = o_p$ and $\varepsilon_+ o_p = o_+ = \xi \wedge \widetilde{o}_+$. We conclude that $-\delta = \varepsilon_-\varepsilon_+ = \varepsilon$.

Step 9. We show (iii) case b).

Similar to step 5 we show that there exists a continuous map

$$\Omega_+ : [0,\infty]^2 \to \mathrm{Gr}(H)\,,$$

such that for all $R \in [0,\infty]$ we have

(a) $\Omega_+(R,0) \oplus T_{w_-^R} W_- = H$,

(b) $\Omega_+(R,\infty) \oplus T_{\psi^{-R}(w_+^\infty)} W^u(p) = H$,

(c) $\Omega_+(0,R) = T_{w_+^R} W_+$,

(d) $\Omega_+(\infty,R) = T_{\psi^R(w_-^\infty)} W^s(p)$.

For all $R \in [0,\infty]$ let $\xi(R) \in \Omega_+(R,0)$ be the projection of the negative gradient $-\operatorname{grad} f$ at w_-^R onto $\Omega_+(R,0)$ along W_-. Let $\widetilde{o}_-$ be the orientation of $\widetilde{W}_-$, which by (8.8.4) is determined as $\widetilde{o}_- = -\operatorname{grad} f \wedge o_- = \xi \wedge o_-$. By Step 8.8 the degree $\delta \in \{\pm 1\}$ of $\partial_R w_+^R \in \widetilde{W}_- \cap W_+$ is defined by the requirement $\delta\, \xi \wedge o_+ = \widetilde{o}_-$. We still have $\varepsilon_- o_- = o_p$ and $\varepsilon_+ o_p = o_+$ with coorientations o_p and o_+ of Ω_+ induced by an orientation of $W^u(p)$ and a coorientation of W_+. We conclude that $\delta = \varepsilon_- \varepsilon_+$ as claimed. □

9. Orientations

We construct orientations for the moduli space of holomorphic strips with boundary on cleanly intersecting Lagrangians using relative spin structures. In principle this has been established by Fukaya et al. in [36, §8]. Because our setup is a bit different we repeat these ideas here using a slightly different language. In particular we use a different but equivalent notion of relative spin structures due to Wehrheim&Woodward [74].

9.1. Preliminaries and notation

Determinant To any real finite dimensional vector space X of dimension $n \geq 0$, we associate the *determinant* denoted by

$$\det X := \Lambda^n X \,.$$

The choice of a basis of X gives an isomorphism $\det X \cong \mathbb{R}$. By definition the determinant of the zero dimensional space is a fixed copy of $\mathbb{R}$. It is well-known that given an exact sequence of finite dimensional real vector spaces

$$0 \to X_1 \to X_2 \to X_3 \to \cdots \to X_k \to 0\,,$$

we obtain an natural isomorphism

$$\bigotimes_{j \text{ odd}} \det X_j \cong \bigotimes_{j \text{ even}} \det X_j \,. \tag{9.1.1}$$

The word "natural" means that an isomorphism between exact sequences gives rise to a commuting square (cf. [2, §5]).

Orientation Torsor To a finite dimensional vector space X of dimension $n \geq 0$, we associate the set

$$|X| := (\Lambda^n X \setminus \{0\})/\mathbb{R}^+ \,.$$

Here $\mathbb{R}^+$ is the group of positive real numbers acting freely on $\Lambda^n X \setminus \{0\}$ by scalar multiplication. The set $|X|$ has two elements and choosing a basis picks one of the two elements. We call $|X|$ the *orientation torsor of* X and the elements of $|X|$ are called *orientations*. We say that X is *oriented*, if an element of $|X|$ is chosen. If X is zero dimensional then we have a canonical identification $|X| = \{\pm 1\}$.

Let $\mathbb{Z}_2$ denote the group with two elements. The group $\mathbb{Z}_2$ acts freely and transitively on $|X|$ with action of the non-trivial element induced by multiplication of -1 on $\Lambda^n X$. Given two vector spaces X and X' we define

$$|X| \otimes |X'| := (|X| \times |X'|)/\mathbb{Z}_2 \,,$$

with quotient taken with respect to the diagonal action $(-1) \cdot (o, o') = (-o, -o')$. The space $|X| \otimes |X'|$ has again two elements and a free and transitive $\mathbb{Z}_2$-action induced by $(-1) \cdot [o, o'] := [-o, o'] = [o, -o']$, where $[o, o']$ denotes the equivalence class of (o, o') in $|X| \otimes |X'|$. It is easy to check that we have a natural isomorphism $|X| \otimes (|X'| \otimes |X''|) \cong (|X| \otimes |X'|) \otimes |X''|$ for any three vector space X, X' and X''. Thus we do not specify parenthesis for iterated products. We also define the *dual torsor*

$$|X|^\vee := \mathrm{Hom}_{\mathbb{Z}_2}(|X|, \mathbb{Z}_2) \,,$$

consisting of all $\mathbb{Z}_2$-equivariant maps to $\mathbb{Z}_2$. We have a natural isomorphism $|X| \otimes |X|^\vee \cong \mathbb{Z}_2$. For two finite dimensional vector spaces X, Y we have natural isomorphism

$$|X \oplus Y| \cong |X| \otimes |Y| \,. \tag{9.1.2}$$

Commuting the factors is natural with respect action with

$$(-1)^{\dim X \dim Y} \,. \tag{9.1.3}$$

Fibre products For finite dimensional vector spaces X, Y and Z and linear maps $\varphi : X \to Z$, $\psi : Y \to Z$, we define the *fibre product*

$$X \;{}_{\varphi}{\times}_{\psi}\; Y := X \times_Z Y := \{(x, y) \mid \varphi(x) = \psi(y)\} \subset X \oplus Y \,.$$

We say that a fibre product is *transverse* if the sequence is exact

$$0 \longrightarrow X \times_Z Y \longrightarrow X \oplus Y \xrightarrow{\varphi - \psi} Z \longrightarrow 0 \;. \tag{9.1.4}$$

We obtain via (9.1.1) and (9.1.2) the canonical isomorphism

$$|X \times_Z Y| \cong |X| \otimes |Z|^\vee \otimes |Y| \,. \tag{9.1.5}$$

The order of the factors leads to the following associativity property of the orientation of fibre products.

Lemma 9.1.1. *Given oriented vector spaces X_0, X_{01}, X_1, Z_0 and Z_1 as well as maps $\varphi_0 : X_0 \to Z_0$, $\psi = (\psi_0, \psi_1) : X_{01} \to Z_0 \oplus Z_1$ and $\varphi_1 : X_1 \to Z_1$. Provided that the fibre products are transverse we have*

$$X_0 \times_{Z_0} (X_{01} \times_{Z_1} X_1) = (X_0 \times_{Z_0} X_{01}) \times_{Z_1} X_1$$

where the equality holds as oriented subspaces of $X_0 \oplus X_{01} \oplus X_1$.

Proof. This is a slight generalization of [36, Lmm. 8.2.3] since we do not require that the maps are surjective (we do however require that their respective differences are surjective). Obviously both fibre products define the subspace

$$Y = \{(x_0, x_{01}, x_1) \mid \varphi_0(x_0) = \psi_0(x_{01}),\ \psi_1(x_{01}) = \varphi_1(x_1)\} \subset X_0 \oplus X_{01} \oplus X_1 .$$

It remains to check that the induced orientations on Y agree. We denote the oriented spaces $Y := X_0 \times_{Z_0} (X_{01} \times_{Z_1} X_1)$ and $Y' := (X_0 \times_{Z_0} X_{01}) \times_{Z_1} X_1$. Define the fibre products $Y_0 := X_0 \times_{Z_0} X_{01}$ and $Y_1 := X_{01} \times_{Z_1} X_1$. We identify Z_0 and Z_1 with subspaces of $X_0 \oplus X_{01}$ and $X_{01} \oplus X_1$ using right-inverses to $\phi_0 = \varphi_0 - \psi_0$ and $\phi_1 = \psi_1 - \varphi_1$ respectively. Moreover we identify Z_0 and Z_1 with subspaces $Z_0' \subset X_0 \oplus Y_1$ and $Z_1' \subset Y_0 \oplus X_1$ using right-inverses of the restriction of ϕ_0 and ϕ_1 respectively. We use small letters x_0, y_0, etc. to denote the dimensions of the spaces X_0, Y_0, etc. By definition we have the oriented isomorphisms

$$Y \oplus Z_0' \cong (-1)^{z_0 y_1} X_0 \oplus Y_1, \qquad Y_1 \oplus Z_1 \cong (-1)^{z_1 x_1} X_{01} \oplus X_1 .$$

Hence as subspaces of $X = X_0 \oplus X_{01} \oplus X_1$

$$Y \oplus Z_0' \oplus Z_1 \cong (-1)^{z_0 y_1} X_0 \oplus Y_1 \oplus Z_1 \cong (-1)^{z_0 y_1 + z_1 x_1} X .$$

On the other hand we have similarly

$$Y' \oplus Z_0 \oplus Z_1' \cong (-1)^{z_0 z_1 + z_1 x_1} Y_0 \oplus X_1 \oplus Z_0 \cong (-1)^{z_1 x_1 + z_0 x_1 + z_0 x_{01} + z_0 z_1} X .$$

By transversality we have $y_1 = x_{01} + x_1 - z_1$. By direct verification we see that in the last two isomorphisms the coefficient on the right-hand side is the same. Since the space of linear complements is contractible there exists a homotopy from $Z_0 \oplus Z_1'$ to $Z_0' \oplus Z_1$. We conclude that the orientation of Y and Y' is the same. □

A special case of a fibre product is obtained if $X, Y \subset Z$ are subspaces and the maps φ and ψ are inclusions. Then the fibre product is isomorphic to the intersection $X \cap Y$. If $X + Y = Z$ we have the exact sequence

$$0 \longrightarrow X \cap Y \longrightarrow X \longrightarrow Z/Y \longrightarrow 0 , \tag{9.1.6}$$

which is usually used to orient the intersection of two vector spaces provided with an orientation of X and a coorientation of Y (i.e. an orientation of Z/Y). The next lemma shows that the orientation on the intersection seen as a fibre product agrees with this orientation.

Lemma 9.1.2. *Given oriented vector spaces $X, Y \subset Z$ such that $X + Y = Z$. If $X \times_Z Y$ and $X \cap Y$ are oriented via* (9.1.5) *and* (9.1.6) *respectively then the projection to the first factor $X \times_Z Y \to X \cap Y$ is orientation preserving.*

Proof. The space Z/Y is oriented by the canonical sequence $0 \to Y \to Z \to Z/Y \to 0$. Abbreviate $W := X \cap Y$ equipped with orientation given by the sequence (9.1.6). Let $\rho : Z \to X \oplus Y$ be a right inverse to $X \oplus Y \to Z$, $(x, y) \mapsto x - y$. We need to check that the determinant of the isomorphism $W \oplus Z \to X \oplus Y$, $(w, z) \mapsto w + \rho(z)$ has sign $(-1)^{\dim Y \dim Z}$. Pick a linear complement $\overline{X}$ of W inside X. We choose an orientation on $\overline{X}$ such that $W \oplus \overline{X} = X$ is orientation preserving. If W is oriented via (9.1.6) then $Y \oplus \overline{X} = Z$ is orientation preserving. Define the right-inverse $\rho : Z = Y \oplus \overline{X} \mapsto X \oplus Y$, $(y, x) \mapsto x - y$. Then the sign of $W \oplus Z \to X \oplus Y$, $(w, y, x) \mapsto w + x - y$ is $(-1)^{\dim Y \dim Z}$. □

Vector bundles and manifolds All previous observations extend directly to the category of manifolds and finite rank vector bundles. In particular if $\pi : E \to X$ is a finite rank vector bundle over a locally path-connected space X we define the *orientation cover* $|E| \to X$ as the double cover with fibre over x given by $|E_x|$ with vector space $E_x = \pi^{-1}(x)$. The vector bundle E is *orientable* if there exists a section of $|E|$, which happens if and only if the *first Stiefel-Whitney class* $w_1(E) \in H^1(X, \mathbb{Z}_2)$ vanishes (cf. [49, Thm. II.1.2]). If $w_1(E) = 0$, then $|E|$ has exactly two sections. We say that an orientable vector bundle E is *oriented*, if a section of $|E|$ is chosen.

If X is a finite dimensional manifold, we abbreviate by $|X| = |TX|$ the *orientation cover of* X. We say that X is *oriented* if a section of $|X|$ is chosen. If $X = \{x\}$ is a point the space $|X|$ is canonically identified with $\{\pm 1\}$ and an orientation of x is denoted by $\operatorname{sign} x \in \{\pm 1\}$. If X is a manifold with boundary ∂X, then an orientation of the interior induces a canonical orientation on ∂X by demanding that for all $x \in \partial X$ and outward pointing vectors $\xi_{\text{out}} \in T_x X$ the isomorphism is orientation preserving

$$T_x \partial X \oplus \xi_{\text{out}} \mathbb{R} \cong T_x X\,. \tag{9.1.7}$$

Given smooth smooth maps $\varphi : X \to Z$, $\psi : Y \to Z$ between smooth finite dimensional manifolds X, Y and Z. We define the *fibre product*

$$X\ {}_{\varphi}\!\times_{\psi} Y = X \times_Z Y = \{(x, y) \mid \varphi(x) = \psi(y)\} \subset X \times Y\,.$$

If the maps are evident from the context we simply denote the fibre product by $X \times_Z Y$. We say that the fibre product is cut-out transversely if at each point $(x, y) \in X \times_Z Y$ the differentials $\mathrm{d}_x\varphi$ and $\mathrm{d}_y\psi$ are transverse in the sense above. If so the fibre product is a manifold with tangent space at (x, y) give by the fibre product of $\mathrm{d}_x\varphi$ with $\mathrm{d}_y\psi$. Let $\mathcal{O} \to Z$ be a double cover. An *$\mathcal{O}$-orientation on* $\varphi : X \to Z$ is a section of $|X| \otimes \varphi^*\mathcal{O}$. Similar an *$\mathcal{O}$-coorientation on* $\psi : Y \to Z$ is a section of $|Y| \otimes \psi^*|Z|^\vee \otimes \psi^*\mathcal{O}$.

Lemma 9.1.3. *An $\mathcal{O}$-orientation on φ and an $\mathcal{O}^\vee$-coorientation on ψ induce an orientation on the transverse fibre product $X \times_Z Y$.*

Proof. The tangent space of $X \times_Z Y$ is the fibre product of $\mathrm{d}_x\varphi$ with $\mathrm{d}_y\psi$. Let $z = \varphi(x) = \psi(y)$ and pick orientations of $\mathcal{O}_z$ and $T_z Z$. We obtain orientations of $T_x X$ and $T_y Y$ using the sections and an orientation on the fibre product via (9.1.5). It is easy to check that the orientation on the fibre product is independent of choices. □

If G is a Lie group acting freely on the manifold $\widetilde{X}$, then the quotient $X := \widetilde{X}/G$ is a manifold. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G. We obtain an exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow T_x\widetilde{X} \longrightarrow T_{[x]}X \longrightarrow 0\,, \tag{9.1.8}$$

which is natural with respect to homotopies. Hence if $\widetilde{X}$ and G are oriented we obtain a canonical orientation on X via (9.1.1) and (9.1.8).

The determinant bundle over the space of Fredholm operators Let X, Y be Banach spaces and denote $\mathcal{F}(X,Y)$ the space of Fredholm operators from X to Y, equipped with the induced topology as a subspace of the bounded linear operators from X to Y. We define the *determinant line bundle*, denoted $\det(X,Y)$, as line bundle on $\mathcal{F}(X,Y)$ with fibre over D given by

$$\det D := \det(\ker D) \otimes \det(\operatorname{coker} D)^\vee\,. \tag{9.1.9}$$

The fibre $\det D$ is called *determinant line*. Although in general the dimension of the kernel and the cokernel is not constant as D varies continuously in $\mathcal{F}(X,Y)$, we have the following fact.

Proposition 9.1.4. *The space* $\det(X,Y)$ *is a locally trivial line bundle.*

A proof is given in [53, Theorem A.2.2] or [2, §7]. A simple observation shows that if $X \cong Y$ and X is an infinite dimensional Hilbert space the determinant line bundle is not orientable (cf. [53, Exercise A.2.5]). Also we denote by $|D| = |\det D|$ the *orientation torsor* and we call the elements of $|D|$ the *orientations of* D. We say that D is *oriented* if an element of $|D|$ is chosen. If D is an isomorphism the orientation torsor $|D|$ is canonically identified with $\{\pm 1\}$ and an orientation of D is denoted by $\operatorname{sign} D \in \{\pm 1\}$.

9.2. Spin structures and relative spin structures

We recall the notion of a spin structure and a relative spin structure. The definition which we give is due to Wehrheim-Woodward (cf. [74, §3]).

Čech cohomology We give basic definitions which are taken from [74, §3] and [16, §5, 10]. Let X be a manifold and G a topological group which is not necessarily abelian. For $k \in \mathbb{N}_0$ and an open cover $\mathcal{U} = \{U_\alpha \subset X \mid \alpha \in I\}$ with totally ordered index set I a *Čech k-cochain with values in G* is a tuple of continuous maps

$$\mathfrak{p} = (\mathfrak{p}_{\alpha_0\alpha_1\ldots\alpha_k} : U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \to G)_{\alpha_0\alpha_1\ldots\alpha_k}$$

indexed over all strictly ordered subsets in I with $k+1$ elements. We write $C^k(\mathcal{U}, G)$ for the space of all such tuples. The space $C^k(\mathcal{U}, G)$ is a group with group law given by pointwise multiplication and neutral element given by the cocycle $\mathbb{1}$, which is the function mapping each point to the neutral element in G. We define the *Čech differential*

$$\mathrm{d} : C^k(\mathcal{U}, G) \to C^{k+1}(\mathcal{U}, G), \quad (\mathrm{d}\mathfrak{p})_{\alpha_0\alpha_1\ldots\alpha_{k+1}} = \prod_{j=0}^{k+1} \mathfrak{p}^{(-1)^j}_{\alpha_0\alpha_1\ldots\hat{\alpha}_j\ldots\alpha_{k+1}} ,$$

and the *Čech k-cocycles*,

$$Z^k(\mathcal{U}, G) := \{\mathfrak{p} \in C^k(\mathcal{U}, G) \mid \mathrm{d}\mathfrak{p} = \mathbb{1}\} .$$

The group $C^0(\mathcal{U}, G)$ acts from the left on $Z^1(\mathcal{U}, G)$ by $(\mathfrak{h}.\mathfrak{p})_{\alpha\beta} = h_\alpha\, \mathfrak{p}_{\alpha\beta}\, h_\beta^{-1}$. We define the *Čech cohomology groups* by

$$H^1(\mathcal{U}, G) := C^0(\mathcal{U}, G) \setminus Z^1(\mathcal{U}, G), \qquad H^0(\mathcal{U}, G) := Z^0(\mathcal{U}, G) .$$

If G is abelian then for any $k \in \mathbb{N}_0$ the Čech k-cochains are abelian groups and together with the Čech differential form a cochain complex, which allows us to define the Čech cohomology groups, denoted $H^k(\mathcal{U}, G)$, for all $k \in \mathbb{N}_0$.

A *refinement $\mathcal{V}$ of $\mathcal{U}$* is a cover $\mathcal{V} = \{V_{\alpha'} \subset X \mid \alpha' \in I'\}$ such that for all indices $\alpha \in I$ there exists $\alpha' \in I'$ with $V_{\alpha'} \subset U_\alpha$. We have natural restriction maps

$$C^k(\mathcal{U}, G) \to C^k(\mathcal{V}, G), \qquad \mathfrak{p}_{\alpha_0\alpha_1\ldots\alpha_k} \mapsto \mathfrak{p}_{\alpha'_0\alpha'_1\ldots\alpha'_k} .$$

On the right-hand side it might be that the indices are not strictly ordered. To allow indices of any order we use the convention $\mathfrak{p}_{\alpha_0\alpha_1\ldots\alpha_k} = \mathfrak{p}^{\operatorname{sign}\sigma}_{\alpha_{\sigma(0)}\alpha_{\sigma(1)}\ldots\alpha_{\sigma(k)}}$ for any permutation σ of $k+1$ elements and $\mathfrak{p}_{\alpha_0\alpha_1\ldots\alpha_k} = \mathbb{1}$ if any two indices are the same ([16, p. 93]).

A *good cover $\mathcal{U}$* is a cover such that all multiple intersections are contractible. One shows that for any good cover $\mathcal{U}$ the group $H^*(\mathcal{U}, G)$ does not depend on $\mathcal{U}$ up to canonical isomorphism and in that case we denote the group by $H^*(X, G)$. Moreover if G is abelian and equipped with a discrete topology this group is canonically isomorphic to the usual cohomology groups with coefficients in G, so there is no ambiguity in the notation. Any open cover on a manifold has a refinement which is a good cover (cf. [16, p. 43]) and by restricting we always assume that cochains are given with respect to a good cover.

As Čech cochains are basically maps, we naturally define the pull-back with respect to continuous maps $\varphi : X \to Y$ and the push-forward with respect to continuous group homomorphisms $\tau : G \to H$. To define the push-forward we put

$$\tau_* : C^k(\mathcal{U}, G) \to C^k(\mathcal{U}, H), \qquad (\tau_*\mathfrak{p})_{\alpha_0\alpha_1\ldots\alpha_k} = \tau \circ \mathfrak{p}_{\alpha_0\alpha_1\ldots\alpha_k} .$$

For the pull-back we define

$$\varphi^* : C^k(\mathcal{U}_Y, G) \to C^k(\mathcal{U}_X, G), \qquad (\varphi^*\mathfrak{p})_{\alpha_0\alpha_1\ldots\alpha_k} = \mathfrak{p}_{\alpha_0\alpha_1\ldots\alpha_k} \circ \varphi ,$$

where $\mathcal{U}_Y = \{U_\alpha \subset Y \mid \alpha \in I\}$ is an open cover of Y and $\mathcal{U}_X = \varphi^*\mathcal{U}_Y := \{\varphi^{-1}(U_\alpha) \subset X \mid \alpha \in I\}$ is the pull-back cover.

Spin structures For $n \geq 2$ the spin group $Spin(n)$ is by definition the non-trivial double cover of the special orthogonal group $SO(n)$. We denote by $\tau : Spin(n) \to SO(n)$ the covering map and identify the kernel of τ with $\mathbb{Z}_2$. Write the action of an element $h \in \mathbb{Z}_2$ on $g \in Spin(n)$ via $(-1)^h g$. Let X be a manifold and $\pi : E \to X$ an oriented finite rank vector bundle equipped with a Riemannian structure. We denote by $SO(E)$ the *oriented orthonormal frame bundle*, i.e. the principle bundle over X with fibre over the point $x \in X$ given by all oriented orthonormal bases in the vector space $E_x := \pi^{-1}(x)$. The transition maps for local trivializations of $SO(E)$ over the open sets of a good cover $\mathcal{U}_X = \{U_\alpha \subset X\}_{\alpha \in I}$ define a Čech cocycle $\mathfrak{f} \in Z^1(\mathcal{U}_X, SO(n))$. If a cocycle $\mathfrak{f}$ arises in such a way for some local trivializations we say that $\mathfrak{f}$ *represents* $SO(E)$.

Definition 9.2.1. A *spin structure on* E is a Čech cocycle $\mathfrak{p} \in Z^1(\mathcal{U}_X, Spin(n))$ such that $\tau_*\mathfrak{p}$ represents E. We call two spin structures $\mathfrak{p}$ and $\mathfrak{p}'$ isomorphic, if there exists a cochain $\mathfrak{h} \in C^0(\mathcal{U}_X, \mathbb{Z}_2)$ such that $\mathfrak{h}.\mathfrak{p} = \mathfrak{p}'$.

Remark 9.2.2. Classically a spin structure on E is a $Spin(n)$-bundle $P \to X$ together with a double cover $\rho : P \to SO(E)$ such that $\rho(g.p) = \tau(g).\rho(p)$ for all $g \in Spin(n)$ and $p \in P$ (cf. [49, Dfn. 1.3]). A spin structure $\mathfrak{p}$ in the above sense is formed by the transition maps from a local trivialization of P. Conversely, given a spin structure $\mathfrak{p}$ as above, we obtain a principle $Spin(n)$-bundle P by gluing. Since $\tau_*\mathfrak{p}$ represents E the map $P \to X$ lifts to a double cover $\rho : P \to SO(E)$ with the required property. See also [74, Prop. 3.1.3].

Not every oriented vector bundle admits a spin structure. The topological obstruction is given by the *second Stiefel-Whitney class* $w_2(E) \in H^2(X, \mathbb{Z}_2)$. The class $w_2(E)$ is defined as follows: Let $\mathfrak{f}$ be a cocycle representing $SO(E)$. We find $\mathfrak{p} \in C^1(\mathcal{U}_X, Spin(n))$ such that $\tau_*\mathfrak{p} = \mathfrak{f}$. Then $\mathrm{d}\mathfrak{p}$ is a cocycle with values in $\mathbb{Z}_2$ and $w_2(E) = [\mathrm{d}\mathfrak{p}]$ is its cohomology class (cf. [49, page 83]). If $w_2(E) = 0$, the bundle E admits a spin structure and moreover a free and transitive action of $H^1(X, \mathbb{Z}_2)$ on the isomorphism class of spin structures on E. Consequently the space of isomorphism classes of spin structures on E is an affine space, which is (non-canonically) isomorphic to $H^1(X, \mathbb{Z}_2)$ (cf. [49, Thm. II.1.7]). If X is an oriented Riemannian manifold, a *spin structure of* X is a spin structure of its tangent bundle and we call X *spin* if it admits a spin structure or equivalently if $w_2(TX) = 0$.

Relative spin structures Given a smooth map $\phi : X \to Y$ between smooth manifolds and an oriented finite rank vector bundle $\pi : E \to X$ equipped with a Riemannian structure. Fix good covers $\mathcal{U}_X$ and $\mathcal{U}_Y$ of X and Y respectively such that $\mathcal{U}_X$ is a refinement of the pull-back cover $\phi^*\mathcal{U}_Y$.

Definition 9.2.3. A *spin structure of* E *relative to* ϕ is a pair $(\mathfrak{p}, \mathfrak{w})$ consisting of a cochain $\mathfrak{p} \in C^1(\mathcal{U}_X, Spin(n))$ and a cocycle $\mathfrak{w} \in Z^2(\mathcal{U}_Y, \mathbb{Z}_2)$ such that $\tau_*\mathfrak{p}$ is a cocycle representing E and we have $\mathrm{d}\mathfrak{p} = \phi^*\mathfrak{w}$. Two relative spin structures $(\mathfrak{p}, \mathfrak{w})$ and $(\mathfrak{p}', \mathfrak{w}')$ are *isomorphic*, if there exists cochains $\mathfrak{h} \in C^0(\mathcal{U}_X, \mathbb{Z}_2)$ and $\mathfrak{k} \in C^1(\mathcal{U}_Y, \mathbb{Z}_2)$ such that $\mathfrak{h}.\mathfrak{p} = \mathfrak{p}'$ and $\mathfrak{k}.\mathfrak{w} = \mathrm{d}\mathfrak{k} + \mathfrak{w} = \mathfrak{w}'$. The cohomology class $w := [\mathfrak{w}] \in H^2(Y, \mathbb{Z}_2)$ is called the *background class*.

For example an ordinary spin structure on E is a special case of a relative spin structure with trivial cochain $\mathfrak{w}$. That our definition of a relative spin structure is equivalent to [36, Def. 8.1.2] is proven in [74, Prop. 3.1.15]. Let $\phi^* : C^*(Y;\mathbb{Z}_2) \to C^*(X;\mathbb{Z}_2)$ be the pull-back. The *cone of* ϕ^*, denoted $C^*(\phi;\mathbb{Z}_2)$, is the complex $C^*(X;\mathbb{Z}_2) \oplus C^{*+1}(Y;\mathbb{Z}_2)$ equipped with boundary operator $\mathrm{d}(\mathfrak{h},\mathfrak{k}) = (\mathrm{d}\mathfrak{h} + \phi^*\mathfrak{k}, \mathrm{d}\mathfrak{k})$. Here we have used the more familiar notation of writing the group law in $\mathbb{Z}_2$ additively. The space of cocycles is denoted by $Z^*(\phi;\mathbb{Z}_2)$ and the homology by $H^*(\phi;\mathbb{Z}_2)$. The next proposition is proven in [74, Prop. 3.1.13].

Proposition 9.2.4. *A bundle $E \to X$ admits a spin structure relative to $\phi : X \to Y$ if and only if there exists a class $w \in H^2(Y,\mathbb{Z}_2)$ such that $\phi^* w = w_2(E)$. If so, $H^1(\phi^*;\mathbb{Z}_2)$ acts freely and transitively on the set of isomorphism classes of relative spin structures via $[\mathfrak{h},\mathfrak{k}].[\mathfrak{p},\mathfrak{w}] = [(-1)^{\mathfrak{h}}\,\mathfrak{p}, \mathfrak{k}+\mathfrak{w}]$ for each $(\mathfrak{h},\mathfrak{k}) \in Z^1(\phi^*;\mathbb{Z}_2)$.*

Bundles over strips The purpose of relative spin structures is to keep track of homotopy classes of trivializations for bundles over the boundary of a strip. Abbreviate by $\Sigma = \mathbb{R} \times [0,1]$ the strip with boundary $\partial\Sigma = \mathbb{R} \times \{0,1\}$.

Lemma 9.2.5. *Given a vector bundle $F \to \partial\Sigma$ with fixed trivializations Φ_- and Φ_+ of the restrictions $F|_{(-\infty,-s_0]\times\{0,1\}}$ and $F|_{[s_0,\infty)\times\{0,1\}}$ respectively. A relative spin structure of F relative to the inclusion $\partial\Sigma \subset \Sigma$ induces a homotopy class of trivializations of $F \oplus \mathbb{R}$ which agree with Φ_- and Φ_+ over the ends.*

Proof. See [74, Prop. 3.1.15, Prop. 3.3.1] for the same statement for compact surfaces with boundary.

Let $(\mathfrak{p},\mathfrak{w})$ be the relative spin structure defined with respect to open covers $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\partial\Sigma}$. Since Σ is contractible the cycle $\mathfrak{w}$ is exact and we find $\mathfrak{v} \in C^1(\mathcal{U}_\Sigma,\mathbb{Z}_2)$ such that $\mathrm{d}\mathfrak{v} = \mathfrak{w}$. Fix $k = 0,1$ and we denote by $\mathfrak{v}_k := \mathfrak{v}|_{\mathbb{R}\times\{k\}}$, $\mathfrak{p}_k := \mathfrak{p}|_{\mathbb{R}\times\{k\}}$ and $\mathfrak{w}_k := \mathfrak{w}|_{\mathbb{R}\times\{k\}}$ the pull-back of the cochains to the boundary, which we identify with cochains on $\mathbb{R}$ with respect to an open cover $\mathcal{U}_\mathbb{R}$. Consider the cochain

$$\hat{\mathfrak{p}}_k := (\mathfrak{p}_k, -\mathfrak{v}_k) \in C^1(\mathcal{U}_\mathbb{R}, Spin(n) \times \mathbb{Z}_2)\,.$$

Let $Spin(n) \times_{\mathbb{Z}_2} \mathbb{Z}_2$ denote the quotient of $Spin(n) \times \mathbb{Z}_2$ by the anti-diagonal action of $\mathbb{Z}_2$, which is a Lie group because $\mathbb{Z}_2$ acts by central elements. The boundary of $\hat{\mathfrak{p}}_k$ is $(\mathfrak{w}_k, -\mathfrak{w}_k)$ hence the push-forward of $\hat{\mathfrak{p}}_k$ to a chain with values in $Spin(n) \times_{\mathbb{Z}_2} \mathbb{Z}_2$ is a cocycle, which we denote by $\bar{\mathfrak{p}}_k$. By assumption the push-forward of $\hat{\mathfrak{p}}_k$ to a $SO(n)\times\{\mathbb{1}\}$-chain is $(\mathfrak{f}_k, \mathbb{1})$, where $\mathfrak{f}_k$ is the cocycle obtained from a trivialization of $SO(F_k)$. By the homotopy lifting principle we have the commutative diagram in which the vertical arrows are double covers and horizontal arrows are inclusions

$$\begin{tikzcd} Spin(n) \times_{\mathbb{Z}_2} \mathbb{Z}_2 \arrow[r] \arrow[d, "2:1"] & Spin(n+1) \arrow[d, "2:1"] \\ SO(n) \times \{\mathbb{1}\} \arrow[r] & SO(n+1)\,. \end{tikzcd}$$

The push-forward of $\bar{\mathfrak{p}}_k$ along the inclusion $Spin(n) \times_{\mathbb{Z}_2} \mathbb{Z}_2 \hookrightarrow Spin(n+1)$ is denoted by $\check{\mathfrak{p}}_k$. By commutativity we conclude that the push-forward of $\check{\mathfrak{p}}_k$ to $SO(n+1)$ is $(\mathfrak{f}_k, \mathbb{1})$. Thus $\check{\mathfrak{p}}_k$ is a spin structure of $F_k \oplus \mathbb{R}$. By gluing (cf. Remark 9.2.2) we obtain $Spin(n+1)$-bundles P_k over $\mathbb{R}$ and maps $P_k \to SO(F_k \oplus \mathbb{R})$, which are non-trivial double covers on each fibre. Using the trivializations Φ_- and Φ_+ we identify the fibre of P_k over s for $|s| \geq s_0$ with $Spin(n+1)$. Because the spin group is connected there exists a section of P_k which is the identity element over $(-\infty, -s_0]$ and $[s_0, \infty)$. The push-forward of the section to $SO(F_k \oplus \mathbb{R})$ gives the trivialization. Since the spin group is simply connected any two choices of the section of P_k are homotopic through a homotopy that fixes the endpoints. Hence the trivialization does not depend on the choices up to homotopy. □

9.3. Orientation of caps

For ordinary Morse theory an orientation on the moduli space of Morse trajectories is given once an orientation of the space of unstable directions for each critical point is fixed. Unfortunately for the moduli spaces of holomorphic strips with boundary on Lagrangians in clean intersection the situation is not so simple. In fact already for Morse-Bott functions on finite dimensional manifolds, the space of Morse trajectories is not necessarily orientable anymore. However it still holds locally that the orientation of the tangent space of the moduli space of Morse trajectories at any Morse trajectory is given canonically in terms of the orientations of the unstable directions of the critical points which the trajectory connects. If the Lagrangians are relatively spin the situation is similar for the moduli space of holomorphic strips where orientation of the caps take the role of the orientation of the unstable directions.

Given a symplectic manifold M and Lagrangians submanifolds $L_0, L_1 \subset M$ such that there exists a relative spin structure on $TL_0 \sqcup TL_1$ relative to $L_0 \sqcup L_1 \to M$ (cf. Definition 9.2.3). We repeat the definition adapted to the context.

Definition 9.3.1. A *relative spin structure for* (L_0, L_1) is a triple $(\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{w})$ such that $\mathfrak{w}$ is a $\mathbb{Z}_2$-cocycle on M, $\mathfrak{p}_k$ is a $Spin(n)$-cochain on L_k, $\tau_* \mathfrak{p}_k$ represents TL_k and $\mathrm{d}\mathfrak{p}_k = \mathfrak{w}|_{L_k}$ for $k = 0, 1$.

Let $X = X_H$ be the Hamiltonian vector field of a clean Hamiltonian $H \in C^\infty([0,1] \times M)$ and $J : [0,1] \to \mathrm{End}(TM, \omega)$ be a path of almost complex structures. The following discussion easily generalizes when X and J are admissible in the sense of Definition 5.1.1. However for the sake of simplicity we only consider the case of $\mathbb{R}$-invariant structures. Use the short-hand notation $\mathcal{I} := \mathcal{I}_H(L_0, L_1)$ for the perturbed intersection points. Choose a constant $\varepsilon > 0$ and consider the space of strips (cf. Definition 6.1.8 for the definition of $C^{\infty;\varepsilon}$-regularity)

$$\mathcal{B} := \{u \in C^{\infty;\varepsilon}(\bar{\mathbb{R}} \times [0,1], M) \mid u(\cdot, k) \subset L_k \text{ for } k = 0, 1,\ u(\pm\infty) \in \mathcal{I}\}\,.$$

Fix an element $x_* \in \mathcal{I}$ once and for all. A *cap of* $x \in \mathcal{I}$ is an element $u \in \mathcal{B}$ such that $u(-\infty) = x_*$ and $u(\infty) = x$. We denote by $\mathcal{B}(x_*, x) \subset \mathcal{B}$ the subspace of all caps of x and by $\mathcal{B}(x_*)$ the space of all caps.

Remark 9.3.2. The space $\mathcal{B}(x_*)$ replaces the space $\tilde{\mathcal{I}}(R_h)$ from [36, §8.8].

By Theorem 6.1.10 we see if ε is small enough then for all $u \in \mathcal{B}$ the linearized Cauchy-Riemann operator D_u is Fredholm and we denote by $|D_u| = |\det D_u|$ the corresponding orientation torsor.

Lemma 9.3.3. *Given $u \in \mathcal{B}$ and caps $u_-, u_+ \in \mathcal{B}(x_*)$ such that $x_- := u(-\infty) = u_-(\infty)$ and $x_+ := u(\infty) = u_+(\infty)$. A relative spin structure for the pair (L_0, L_1) induces an isomorphism*

$$|D_{u_-}| \otimes |D_u| \cong |D_{u_+}| \otimes |T_{x_-}\mathcal{I}|\,, \tag{9.3.1}$$

which is natural with respect to homotopies, i.e. given homotopies $(u^\tau)_{\tau\in[a,b]} \subset \mathcal{B}$ and $(u^\tau_-)_{\tau\in[a,b]}$, $(u^\tau_+)_{\tau\in[a,b]} \subset \mathcal{B}(x_*)$ such that $x^\tau_- := u^\tau(-\infty) = u^\tau_-(\infty)$ and $x^\tau_+ := u^\tau(\infty) = u^\tau_+(\infty)$ we have have the commutative diagram

$$\begin{tikzcd}
|D_{u^a_-}| \otimes |D_{u^a}| \arrow[r] \arrow[d] & |D_{u^a_+}| \otimes |T_{x^a_-}\mathcal{I}| \arrow[d] \\
|D_{u^b_-}| \otimes |D_{u^b}| \arrow[r] & |D_{u^b_+}| \otimes |T_{x^b_-}\mathcal{I}|\,,
\end{tikzcd}$$

in which the horizontal maps are by (9.3.1) *and the vertical are induced by the homotopy.*

Proof. Disclaimer: The construction of the isomorphism involves choices. These choices are unique up to homotopy and hence the isomorphism on the orientations does not depend on these choices. We will not specifically mention this every time.

Consider trivializations Φ_u, Φ_{u_-} and Φ_{u_+} of u^*TM, u_-^*TM and u_+^*TM respectively with properties listed in the proof of Theorem 6.1.10 and such that $\Phi_u(-\infty) = \Phi_{u_-}(\infty)$, $\Phi_u(\infty) = \Phi_{u_+}(\infty)$ and $\Phi_{u_-}(-\infty) = \Phi_{u_+}(-\infty) = \Phi_*$, where Φ_* is a trivialization of $(x_*)^*TM$ which is fixed once and for all. As explained further in the proof using the trivializations we obtain maps S, S_-, $S_+ : \bar{\mathbb{R}} \times [0,1] \to \mathbb{R}^{2n\times 2n}$ and F, F_-, $F_+ : \mathbb{R} \to \mathcal{L}(n) \times \mathcal{L}(n)$ such that (F,S), (F_-,S_-) and (F_+,S_+) are admissible and the operators D_u, D_{u_-} and D_{u_+} are conjugated to $D_{F,S}$, D_{F_-,S_-} and D_{F_+,S_+} respectively. By construction we have asymptotics $\sigma_- := S(-\infty) = S_-(\infty)$, $\sigma_+ := S(\infty) = S_+(\infty)$ and $\sigma_* := S_-(-\infty) = S_+(-\infty)$, where σ_* is a path which is fixed once and for all. In particular σ_* does not depend on the maps u, u_- and u_+. Using the isomorphism the kernel of the operator A_{σ_-} is conjugated to $T_{x_-}\mathcal{I}$ (cf. Lemma 3.2.13). Via pull-back we have relative spin structures and by Lemma 9.2.5 trivializations of F, F_- and F_+ which are standard over the ends. Using the trivializations in Lemma 9.4.3 we obtain integers μ_-, μ, $\mu_+ \in \mathbb{Z}$ and isomorphisms of $|D_u|$, $|D_{u_-}|$ and $|D_{u_+}|$ with $|\mu.\sigma_- : \sigma_+|$, $|\mu_-.\sigma_* : \sigma_-|$ and $|\mu_+.\sigma_* : \sigma_+|$ respectively. For any $\nu \in \mathbb{Z}$ we fix once and for all an orientation in $|\sigma_* : \nu.\sigma_*|$. The claim follows by linear orientation gluing Lemma 9.4.1 and the canonical isomorphism (9.4.1). Indeed we have

$$|D_{u_+}| \cong |\sigma_* : \mu_+.\sigma_*| \otimes |\mu_+.\sigma_* : \sigma_+| \cong |\sigma_* : \sigma_+| \otimes |\sigma_* : \sigma_*|\,,$$

and plugging this isomorphism in the next

$$\begin{aligned}|D_{u_-}| \otimes |D_u| &\cong |\sigma_* : (\mu_- + \mu).\sigma_*| \otimes |\mu_-.\sigma_* : \sigma_-| \otimes |\mu.\sigma_- : \sigma_+| \\ &\cong |\sigma_* : (\mu_- + \mu).\sigma_*| \otimes |(\mu_- + \mu).\sigma_* : \mu.\sigma_-| \otimes |\mu.\sigma_- : \sigma_+| \\ &\cong |\sigma_* : \sigma_+| \otimes |\sigma_* : \sigma_*| \otimes |\sigma_- : \sigma_-| \\ &\cong |\sigma_* : \sigma_+| \otimes |\sigma_* : \sigma_*| \otimes |T_{x_-}\mathcal{I}| \\ &\cong |D_{u_+}| \otimes |T_{x_-}\mathcal{I}| \,.\end{aligned}$$

This shows (9.3.1). Since all isomorphisms are natural with respect to homotopies we also have the commutative diagram. □

In particular if $u \in \mathcal{B}$ is such that $u(s, \cdot) = x$ for all $s \in \mathbb{R}$, we have canonically $|D_u| \cong |T_x\mathcal{I}|$ and we conclude that a relative spin structure induces a canonical isomorphism for any two caps u_-, $u_+ \in \mathcal{B}(x_*, x)$

$$|D_{u_-}| \cong |D_{u_+}| \,, \tag{9.3.2}$$

which is natural with respect to homotopies. Thus the following double cover is well-defined.

Definition 9.3.4. Given a relative spin structure for (L_0, L_1). We define the double cover $\mathcal{O} \to \mathcal{I}$ with fibre over $x \in \mathcal{I}$ given by

$$\mathcal{O}_x := \bigsqcup_{u \in \mathcal{B}(x_*, x)} |D_u| \Big/ \sim$$

in which two elements $o \in |D_u|$ and $o' \in |D_{u'}|$ are equivalent if they are identified by the isomorphism (9.3.2).

Remark 9.3.5. If we pull-back the double cover $\mathcal{O}$ to $\mathcal{B}(x_*)$ along the fibration $\mathcal{B}(x_*) \to \mathcal{I}$, $u \mapsto u(\infty)$ it is isomorphic to the double cover $\mathcal{B}(x_*)^+ \to \mathcal{B}(x_*)$ with fibre over u given by $|D_u|$ (cf. [36, Prp. 8.8.1]). Using notation of [36, §8.8] the cover $\mathcal{O}$ is the orientation bundle of Θ.

We come to the main result of the chapter. Given connected components C_-, $C_+ \subset \mathcal{I}$. We denote by $\widetilde{\mathcal{M}}(C_-, C_+; J, X)$ the space of (J, X)-holomorphic strips u such that $u(-\infty) \in C_-$ and $u(\infty) \in C_+$ which comes equipped with the evaluation map

$$ev = (ev_-, ev_+) : \widetilde{\mathcal{M}}(C_-, C_+; J, X) \to C_- \times C_+, \qquad u \mapsto (u(-\infty), u(\infty)) \,.$$

For more details see Section 7.1.

Theorem 9.3.6. *Assume (L_0, L_1) is equipped with a relative spin structure and let $\mathcal{O}$ be the associated double cover (cf. Definition 9.3.4). Given connected components $C_-, C_+ \subset \mathcal{I}_H(L_0, L_1)$ and suppose that J is regular for $X = X_H$. For any $u \in \widetilde{\mathcal{M}}(C_-, C_+; J, X)$ connecting $x_- = u(-\infty)$ to $x_+ = u(\infty)$ we have the canonical isomorphism*

$$|\widetilde{\mathcal{M}}(C_-, C_+; J, X)|_u \cong \mathcal{O}_{x_-}^\vee \otimes \mathcal{O}_{x_+} \otimes |\mathcal{I}|_{x_-} \,,$$

which is natural with respect to homotopies.

Proof. See [36, Proposition 8.8.6]. The result easily follows from Lemma 9.3.3. By assumption the linearized Cauchy-Riemann operator D_u is surjective and the tangent bundle of $\widetilde{\mathcal{M}}(C_-, C_+; J, X)$ at u is the kernel of D_u. We conclude $|T_u\widetilde{\mathcal{M}}(C_-, C_+; J, X)| = |D_u|$. Choose caps u_- and u_+ of $x_- := u(-\infty)$ and $x_+ := u(\infty)$ respectively. The fibers $\mathcal{O}_{x_-}$ and $\mathcal{O}_{x_+}$ are represented by orientations of D_{u_-} and D_{u_+} respectively. Then with Lemma 9.3.3 we have the natural isomorphism $|D_u| \cong |D_{u_-}|^\vee \otimes |D_{u_+}| \otimes |T_{x_-}C_-|$. □

Corollary 9.3.7. *Given a smooth map $\varphi_- : W_- \to C_-$ such that J is regular for X and φ. An $\mathcal{O}^\vee$-orientation for φ_- induces an $\mathcal{O}^\vee$-orientation for*

$$ev_+ : \widetilde{\mathcal{M}}(W_-, C_+; J, X) \to C_+, \qquad (w, u) \mapsto u(\infty)\,.$$

Proof. Using the canonical orientation of the tangent space on the fibre product (cf. equation (9.1.5)) and Theorem 9.3.6. □

9.4. Linear theory

We proof the orientation gluing for Cauchy-Riemann operators with degenerated asymptotics defined on strips. This generalizes the orientation gluing for Cauchy-Riemann operators the cylinder with non-degenerated asymptotics as was established in [24, Section 3].

Fix a constant $\delta > 0$, we denote by $\mathcal{A}$ the space of paths $\sigma : [0,1] \to \mathbb{R}^{2n\times 2n}$ such that $\sigma(t)$ is symmetric for all $t \in [0,1]$ and the operator (cf. equation (6.2.4))

$$A_\sigma : H^{1,2}_\Lambda([0,1], \mathbb{R}^{2n}) \to L^2([0,1], \mathbb{R}^{2n}), \qquad \xi \mapsto J_{\mathrm{std}}\partial_t \xi + \sigma \cdot \xi\,,$$

with $\Lambda = (\mathbb{R}^n, \mathbb{R}^n)$, has spectral gap $\iota(A_\sigma) > \delta$ (cf. equation B.1.1). Fix two paths $\sigma_-, \sigma_+ \in \mathcal{A}$ and a constant ε such that $\delta < \varepsilon < \min\{\iota(A_{\sigma_-}, \iota(A_{\sigma_+})\}$. We define

$$\mathcal{D}(\sigma_-, \sigma_+) := \{S \in C^{\infty;\varepsilon}(\bar{\mathbb{R}} \times [0,1], \mathbb{R}^{2n\times 2n}) \mid S(\pm\infty, \cdot) = \sigma_\pm\}\,.$$

To any $S \in \mathcal{D}(\sigma_-, \sigma_+)$ and $F : \mathbb{R} \to \mathcal{L}(n) \times \mathcal{L}(n)$ such that (F, S) is admissible (cf. Definition 6.2.1) we associate the operator (cf. equation (6.2.2) and (6.2.8))

$$D_{F,S} : H^{1,2;\delta}_{F;W}(\Sigma, \mathbb{R}^{2n}) \to L^{2;\delta}(\Sigma, \mathbb{R}^{2n}), \qquad \xi \mapsto \partial_s\xi + J_{\mathrm{std}}\partial_t\xi + S\xi\,,$$

with $W = (\ker A_{\sigma_-}, \ker A_{\sigma_+})$. In case when F is such that $F(s) = (\mathbb{R}^n, \mathbb{R}^n)$ for all $s \in \mathbb{R}$ we simply write D_S. By Lemma 6.2.5 the operator $D_{F,S}$ is Fredholm and we denote by $\det D_{F,S}$ and $|D_{F,S}|$ the associated determinant line and orientation torsor respectively. We identify $\mathcal{D}(\sigma_-, \sigma_+)$ with an open subset in the space of Fredholm operators, via $S \mapsto D_S$. Moreover it is easily seen that $\mathcal{D}(\sigma_-, \sigma_+)$ is convex, hence the pull-back of the determinant line bundle to $\mathcal{D}(\sigma_-, \sigma_+)$ is orientable. We denote by

$$|\sigma_-, \sigma_+| \tag{9.4.1}$$

the space of the two possible orientations of the determinant line bundle over $\mathcal{D}(\sigma_-, \sigma_+)$, i.e. the space of sections in the double cover over $\mathcal{D}(\sigma_-, \sigma_+)$ where the fibre over S is given by the two possible orientations of the Fredholm operator D_S.

For any $\mu \in \mathbb{Z}$ and $t \in [0,1]$ define the unitary matrix

$$\phi_\mu(t) = \begin{pmatrix} e^{\pi i \mu t} & 0 \\ 0 & \mathbb{1} \end{pmatrix} . \tag{9.4.2}$$

There is an action of $\mathbb{Z}$ on $\mathcal{A}$ given by $\mu.\sigma = \phi_{-\mu}\, \sigma\, \phi_\mu + J_{\mathrm{std}}\, \phi_{-\mu} \partial_t \phi_\mu$. Alternatively the action is defined for the associated operator by $A_{\mu.\sigma} = \phi_{-\mu}\, A_\sigma\, \phi_\mu$. This easily shows that the spectrum of A_σ is not changed under the action, which implies that the spectral gap is left invariant. Moreover for all $\mu \in \mathbb{Z}$ we have $\phi_{-\mu}\, D_S\, \phi_\mu = D_{\phi_{-\mu}\, S\, \phi_\mu + J_{\mathrm{std}} \phi_{-\mu} \partial_t \phi_\mu}$ and thus a canonical isomorphism for all $\mu \in \mathbb{Z}$

$$|\mu.\sigma_-, \mu.\sigma_+| \cong |\sigma_-, \sigma_+| . \tag{9.4.3}$$

We now state the main lemma.

Lemma 9.4.1 (Orientation gluing)**.** *Given $\sigma_-, \sigma, \sigma_+ \in \mathcal{A}$, there exists an isomorphism*

$$|\sigma_-, \sigma| \otimes |\sigma, \sigma_+| \longrightarrow |\sigma, \sigma| \otimes |\sigma_-, \sigma_+| , \tag{9.4.4}$$

which is natural with respect to homotopies, i.e. given homotopies $(\sigma_-^\tau), (\sigma^\tau)$ and (σ_+^τ) in $\mathcal{A}$, then there exists a commutative diagram,

$$\begin{tikzcd} \left|\sigma_-^0, \sigma^0\right| \otimes \left|\sigma^0, \sigma_+^0\right| \arrow[r] \arrow[d] & \left|\sigma^0, \sigma^0\right| \otimes \left|\sigma_-^0, \sigma_+^0\right| \arrow[d] \\ \left|\sigma_-^1, \sigma^1\right| \otimes \left|\sigma^1, \sigma_+^1\right| \arrow[r] & \left|\sigma^1, \sigma^1\right| \otimes \left|\sigma_-^1, \sigma_+^1\right| , \end{tikzcd}$$

where the horizontal isomorphism is induced by (9.4.4) *and the vertical by homotopies.*

Proof. Choose $S_0 \in \mathcal{D}(\sigma_-, \sigma)$ and $S_1 \in \mathcal{D}(\sigma, \sigma_+)$ with $S_0(s, \cdot) = S_1(-s, \cdot)$ for all $s \geq 2$. For any $R \geq 2$, we consider the glued map $S_0 \#_R S_1$ as given in (9.4.6). We obtain an isomorphisms $|D_0| \otimes |D_1| \cong |\ker A_\sigma| \otimes |D_R|$, constructed in Lemma 9.4.9 below. Note that $\ker A_\sigma$ is the same as $\ker D_\sigma$ and D_σ is surjective, hence $|\ker A_\sigma| = |D_\sigma|$. Extend the isomorphism uniquely to obtain (9.4.4) via the homotopy lifting principle. We explain, why the isomorphism does not depend on the choice of S_0, S_1: Choose another elements $S_0' \in \mathcal{D}(\sigma_-, \sigma)$ and $S_1' \in \mathcal{D}(\sigma, \sigma_+)$. These are joined to S_0 and S_1 via a homotopy $(S_0^\tau)_{\tau \in [0,1]}$ and $(S_1^\tau)_{\tau \in [0,1]}$ respectively. By naturality of the gluing construction, the obtained isomorphism on the orientations is the same (cf. Lemma 9.4.11). We argue similarly to show that the isomorphism is natural with respect to homotopies of σ_-, σ and σ_+. $\square$

Lemma 9.4.2. *The isomorphism* (9.4.4) *is associative. More precisely, given paths* $\sigma_0, \ldots, \sigma_3 \in \mathcal{A}$*, then we have a commuting square*

$$\begin{tikzcd}
|\sigma_0, \sigma_1| \otimes |\sigma_1, \sigma_2| \otimes |\sigma_2, \sigma_3| \arrow[r] \arrow[d] & |\sigma_1, \sigma_1| \otimes |\sigma_0, \sigma_2| \otimes |\sigma_2, \sigma_3| \arrow[d] \\
|\sigma_0, \sigma_1| \otimes |\sigma_2, \sigma_2| \otimes |\sigma_1, \sigma_3| \arrow[r] & |\sigma_1, \sigma_1| \otimes |\sigma_2, \sigma_2| \otimes |\sigma_0, \sigma_3| .
\end{tikzcd}$$

in which all but the lower horizontal map is given by the gluing map (9.4.4) *and the lower horizontal map is given by commuting the factors and the gluing map* (9.4.4)*.*

Proof. See [74, Lmm 2.4.2] or [22, Lemma 3.5]. □

Path of Lagrangians

We show that a stable trivialization of F induces an orientation of $D_{F,S}$ up to data which only depends on the asymptotics and the index of F. We view F as a bundle over $\mathbb{R} \times \{0,1\}$ with fibre over $(s,k) \in \mathbb{R} \times \{0,1\}$ given by $F_k(s)$. Suppose that $F_k(s) = F_k(-s) = \mathbb{R}^n$ for all $s \geq s_0$, $k = 0,1$. An *admissible trivialization* is a trivialization of F given by a special orthogonal frame which is standard over $(-\infty, -s_0]$ and $[s_0, \infty)$.

Lemma 9.4.3. *Given* $S \in \mathcal{D}(\sigma_-, \sigma_+)$ *and* $F : \mathbb{R} \to \mathcal{L}(n) \times \mathcal{L}(n)$ *such that* (F, S) *is admissible. Let* μ *be the Robbin-Salamon index of* F*. An admissible trivialization of* $F \oplus \mathbb{R}$ *induces an isomorphism*

$$|D_{F,S}| \cong |\mu.\sigma_-, \sigma_+| , \tag{9.4.5}$$

which is natural with respect to homotopies, i.e. given homotopies $(S^\tau)_\tau$ *and* $(F^\tau)_\tau$ *where* $\tau \in [a,b]$ *and such that* $S^\tau \in \mathcal{D}(\sigma_-^\tau, \sigma_+^\tau)$ *with* $\sigma_\pm^\tau = S^\tau(\pm\infty, \cdot)$ *and* (S^τ, F^τ) *is admissible for all* $\tau \in [a,b]$*, then an admissible trivialization of* $F \oplus \mathbb{R}$ *gives the commutative diagram*

$$\begin{tikzcd}
|D_{F^a,S^a}| \arrow[r] \arrow[d] & |\mu.\sigma_-^a, \sigma_+^a| \arrow[d] \\
|D_{F^b,S^b}| \arrow[r] & |\mu.\sigma_-^b, \sigma_+^b| ,
\end{tikzcd}$$

in which the horizontal isomorphism is induced by (9.4.5) *and the vertical is induced homotopies.*

Proof. For $k = 0,1$ a trivialization of F_k is given by a frame $e_1^k, \ldots, e_{n+1}^k : \mathbb{R} \to \mathbb{R}^{2n+2}$ of $F_k \oplus \mathbb{R}$ which is standard over $(-\infty, -s_0]$ and $[s_0, \infty)$. For $s \in \mathbb{R}$ define the unitary matrix $\Psi_k(s)$ with column vectors $e_1^k(s), \ldots, e_{n+1}^k(s)$ and $J_{\mathrm{std}} e_1^k(s), \ldots, J_{\mathrm{std}} e_{n+1}^k(s)$. Thus $F_k(s) \oplus \mathbb{R} = \Psi_k(s)\mathbb{R}^{n+1}$ for all $s \in \mathbb{R}$. The Robbin-Salamon index of F is an integer, since F starts and ends at $(\mathbb{R}^n, \mathbb{R}^n)$. With ϕ_μ as given in (9.4.2) the concatenation $\phi_\mu \# \Psi_0$ has the same Malsov index as Ψ_1. By [63, Thm. 4.1] we conclude that the paths are homotopic with fixed endpoints, which implies the existence of a map $\Psi : \Sigma \to U(n+1)$ such that

- $\Psi(s,k)\mathbb{R}^{n+1} = F_k(s) \oplus \mathbb{R} \subset \mathbb{C}^{n+1}$ for all $s \in \mathbb{R}$ and $k = 0, 1$,
- $\Psi(s,t) = \phi_\mu(t)$ for all $s \leq -s_0$ and $t \in [0,1]$,
- $\Psi(s,t) = \mathbb{1}$ for all $s \geq s_0$ and $t \in [0,1]$.

To see that Ψ exists uniquely up to homotopy, let Ψ' be another choice. Along the boundary of $[-s_0, s_0] \times [0,1]$ we identify Ψ and Ψ' to obtain an map from S^2 to $U(n+1)$. Since the unitary group is two-connected we find a homotopy from Ψ to Ψ'.

Now define the map $S_\Psi := \Psi^{-1}\, S\, \Psi + \Psi^{-1}\partial_s\Psi + J_{\mathrm{std}}\Psi^{-1}\partial_t\Psi$ and $F' = (F_0', F_1')$ with $F_k'(s) = F_k(s) \oplus \mathbb{R}$ for $k = 0, 1$. The operators $D_{F',S}$ and D_{S_Ψ} are conjugated by Ψ. The kernel of $D_{F',S}$ splits into $\ker D_{F,S} \oplus \mathbb{R}$ where $\mathbb{R}$ is given by the space of constant maps $\xi : \mathbb{R} \times [0,1] \to \mathbb{R}^{2n} \oplus \mathbb{R}^2$, $\xi(s,t) = (0,a)$ for some $a \in \mathbb{R}$. Moreover the cokernel of $D_{F',S}$ and $D_{F,S}$ are the same. Fixing the standard orientation on $\mathbb{R}$, then Ψ induces an isomorphism between the orientation torsors of $D_{F,S}$ and D_{S_Ψ}. By construction $S_\Psi(-\infty,\cdot) = \mu.\sigma_-$ and $S_\Psi(\infty,\cdot) = \sigma_+$. This shows (9.4.5).

We show naturality. Using the trivialization of F we define Ψ as above on the six sides of the cuboid $[a,b] \times [-s_0, s_0] \times [0,1]$ such that $\Psi(\tau, -s_0, t) = \phi_\mu(t)$ and $\Psi(\tau, s_0, t) = \mathbb{1}$ for all $\tau \in [a,b]$ and $t \in [0,1]$. Note that the by the homotopy axiom the Robbin-Salamon index of F^τ is independent of τ. Because the unitary group is two connected the map extends to a map defined on the cuboid $\Psi : [a,b] \times [-s_0, s_0] \times [0,1] \to U(n+1)$, $\Psi^\tau = \Psi(\tau, \cdot)$. In particular the orientation torsor of $D_{F,S}$ is isomorphic to $D_{S_{\Psi^\tau}}$ by conjugation with Ψ^τ. Since $S_{\Psi^\tau} \in \mathcal{D}(\mu.\sigma_-^\tau, \sigma_+^\tau)$ for all $\tau \in [a,b]$ the claim follows by the homotopy lifting principle. □

Linear gluing

Given $S_0 \in \mathcal{D}(\sigma_-, \sigma)$ and $S_1 \in \mathcal{D}(\sigma, \sigma_+)$ such that $S_0(s,\cdot) = S_1(-s,\cdot) = \sigma$ for all $s \geq 1$. For each $R \geq 1$ we define

$$S_R := S_0 \#_R S_1 = \begin{cases} S_{0,R} = S_0 \circ \tau_{-2R} & \text{if } s \leq -R \\ S_0(\infty) = S_1(-\infty) & \text{if } |s| \leq R \\ S_{1,R} = S_1 \circ \tau_{2R} & \text{if } s \geq R\,, \end{cases} \tag{9.4.6}$$

where $\tau_{2R} : \Sigma \to \Sigma$ denotes the translation $\tau_R(s,t) = (s - 2R, t)$. We abbreviate

- the asymptotic operators $A_- := A_{\sigma_-}$, $A := A_\sigma$ and $A_+ := A_{\sigma_+}$,
- their kernels $C_- := \ker A_-$, $C := \ker A$ and $C_+ := \ker A_+$,
- the operators $D_0 := D_{S_0}$, $D_1 := D_{S_1}$ and $D_R := D_{S_R}$ which are defined on the Banach spaces H_0, H_1 and H_R with target L_0, L_1 and L_R respectively,
- the restricted operators $D_{01} := D_0 \oplus D_1|_{H_{01}} : H_{01} \to L_{01}$ where $H_{01} \subset H_0 \oplus H_1$ consists of functions (ξ_0, ξ_1) such that $\xi_0(\infty) = \xi_1(-\infty)$ and $L_{01} := L_0 \oplus L_1$.

Lemma 9.4.4. *We have* $\operatorname{ind} D_{01} = \operatorname{ind} D_R$.

Proof. We denote by $H_R^{\mathrm{red}} \subset H_R$ and $H_{01}^{\mathrm{red}} \subset H_{01}$ the subspaces of functions with vanishing asymptotics, i.e. $\xi \in H_R$ with $\xi(\pm\infty) = 0$ and $(\xi_0, \xi_1) \in H_{01}$ with $\xi_0(\pm\infty) = \xi_1(\pm\infty) = 0$. Secondly denote D_R^{red} and D_{01}^{red} the operators D_R and D_{01} restricted to the spaces H_R^{red} and H_{01}^{red} respectively. We have

$$\begin{aligned} \operatorname{ind} D_{01} &= \operatorname{ind} D_{01}^{\mathrm{red}} + \dim C_- + \dim C + \dim C_+ \,, \\ \operatorname{ind} D_R &= \operatorname{ind} D_R^{\mathrm{red}} + \dim C_- + \dim \ker C_+ \,. \end{aligned} \tag{9.4.7}$$

The indices of the reduced Fredholm operators are computed in Lemma 6.2.6. We have

$$\begin{aligned} \operatorname{ind} D_{01}^{\mathrm{red}} &= \operatorname{ind} D_0^{\mathrm{red}} + \operatorname{ind} D_1^{\mathrm{red}} \\ &= \mu(\Psi\mathbb{R}^n, \mathbb{R}^n) - \mu(\Psi_-\mathbb{R}^n, \mathbb{R}^n) - \frac{1}{2}\dim C_- - \frac{1}{2}\dim C \\ &\quad + \mu(\Psi_+\mathbb{R}^n, \mathbb{R}^n) - \mu(\Psi\mathbb{R}^n, \mathbb{R}^n) - \frac{1}{2}\dim C - \frac{1}{2}\dim C_+ \,, \end{aligned}$$

where Ψ_-, Ψ and Ψ_+ are the fundamental solutions of σ_-, σ and σ_+ respectively. Thus

$$\operatorname{ind} D_{01}^{\mathrm{red}} = \mu(\Psi_+\mathbb{R}^n, \mathbb{R}^n) - \mu(\Psi_-\mathbb{R}^n, \mathbb{R}^n) - \frac{1}{2}\dim C_- - \dim C - \frac{1}{2}\dim C_+ \,.$$

On the other hand we have

$$\operatorname{ind} D_R^{\mathrm{red}} = \mu(\Psi_+\mathbb{R}^n, \mathbb{R}^n) - \mu(\Psi_-\mathbb{R}^n, \mathbb{R}^n) - \frac{1}{2}\dim C_- - \frac{1}{2}\dim C_+ \,.$$

Plugging the last two equations in (9.4.7) proves the lemma. □

Adapted norms For $R > 0$, we define a weight function $\gamma : \mathbb{R} \to \mathbb{R}$

$$\gamma(s) = \begin{cases} e^{-\delta(2R+s)} & \text{if } s < -2R \\ e^{\delta(2R-|s|)} & \text{if } |s| < 2R \\ e^{\delta(s-2R)} & \text{if } s > 2R \,. \end{cases}$$

For any $\eta \in L_R := L^{2;\delta}(\Sigma, \mathbb{R}^{2n})$ we defined the weighted norm

$$\|\eta\|_{L_R} := \left(\int_\Sigma |\eta|^2 \, \gamma_{\delta,R}^2 \mathrm{d}s\mathrm{d}t \right)^{1/2}$$

and for any $\xi \in H_R$ with $\xi_\pm = \xi(\pm\infty)$ and $\bar{\xi} = P\xi(0,\cdot)$, where P denotes the orthogonal projector to $\ker A$, we define the norm

$$\begin{aligned} \|\xi\|_{H_R} := \|\xi_-\| + \|\bar{\xi}\| + \|\xi_+\| + \\ + \|(\xi - \xi_-)\gamma\|_{1,2;\Sigma_{-\infty}^{-2R}} + \|(\xi - \bar{\xi})\gamma\|_{1,2;\Sigma_{-2R}^{2R}} + \|(\xi - \xi_+)\gamma\|_{1,2;\Sigma_{2R}^{\infty}} \,. \end{aligned}$$

It is easy to see that these define equivalent norms for every fixed $R \geq 1$.

Pregluing and breaking Fix once and for all a cut-off function β^+ and β^- given in (8.2.1). Further denote $\beta_R^- = \beta^- \circ \tau_R$ and $\beta_R^+ = \beta^+ \circ \tau_R$ we define the *linear pregluing operator* via

$$\Theta_R : H_{01} \to H_R, \qquad (\xi_0, \xi_1) \mapsto \bar{\xi} + \beta_{-R}^+(\xi_1 \circ \tau_{2R} - \bar{\xi}) + \beta_R^-(\xi_0 \circ \tau_{-2R} - \bar{\xi}),$$

with $\bar{\xi} = \xi_0(\infty) = \xi_1(-\infty)$ and the *breaking operator* $\Xi_R : L_R \to L_0 \oplus L_1$, $\eta \mapsto (\eta_{0,R}, \eta_{1,R})$ in which

$$\eta_{0,R}(s,t) = \begin{cases} 0 & \text{for } s \geq 2R \\ \eta(s - 2R, t) & \text{for } s \leq 2R \end{cases}$$

$$\eta_{1,R}(s,t) = \begin{cases} \eta(s + 2R, t) & \text{for } s \geq -2R \\ 0 & \text{for } s \leq -2R\,. \end{cases}$$

We define another *linear pregluing operator* $\Omega_R : L_0 \oplus L_1 \to L_R$, $(\eta_0, \eta_1) \mapsto \eta_R$ in which

$$\eta_R = \begin{cases} \eta_0(s + 2R, t) & \text{for } s \leq 0 \\ \eta_1(s - 2R, t) & \text{for } s \geq 0\,. \end{cases}$$

It is easily seen that Ξ_R is a right-inverse for Ω_R.

Lemma 9.4.5. *There exists constants c and R_0 such that for all $(\xi_0, \xi_1) \in H_{01}$ and $R \geq R_0$ we have*

$$\|\Theta_R(\xi_0, \xi_1)\|_{H_R} \leq c \left(\|\xi_0\|_{1,2;\delta} + \|\xi_1\|_{1,2;\delta} \right).$$

Moreover we have for all $\eta \in L_R$,

$$\|\eta_{0,R}\|_{2;\delta}^2 + \|\eta_{0,R}\|_{2;\delta}^2 = \|\eta\|_{L_R}^2,$$

in which $(\eta_{0,R}, \eta_{1,R}) = \Xi_R(\eta)$.

Proof. We have the same estimates as in the proof of Lemmas 8.3.1 and 8.3.2. Note that we are now in a much simpler situation where the connection is flat and all the parallel transport maps are given by the identity. □

Approximate pseudo-right inverse Fix a finite dimensional subspace $Y \subset L_{01}$ such that D_{01} is transverse to Y, i.e. $\operatorname{im} D_{01} + Y = L_{01}$. Without loss of generality we assume that all functions in Y are compactly supported. Define $X := D_{01}^{-1} Y$.

Lemma 9.4.6. *The linear pregluing operators are injective when restricted to X or Y respectively, for all R sufficiently large.*

Proof. We find s_0 such that functions in Y are supported in $[-s_0, s_0] \times [0,1]$. The fact that $\Omega_R|_Y : Y \to L_R$ is injective for all $R > s_0$ directly follows by its definition. Suppose that $(\xi_0, \xi_1) \in X$. Then $D_0 \xi_0$ is supported inside $[-s_0, s_0] \times [0,1]$ and thus by elliptic regularity ξ_0 must be constant outside. Similarly for ξ_1. Yet if $\Theta_R(\xi_0, \xi_1) = 0$ for $R > s_0$, then ξ_0 and ξ_1 have to vanish. □

Let $X^{\perp} \subset H_{01}$ be some closed complement of X and define $Y^{\perp} := D_{01}(X^{\perp})$. Obviously we have a splitting $L_{01} = Y \oplus Y^{\perp}$ and since D_{01} restricted to $X^{\perp}$ is injective, there exists a unique bounded operator $Q_{01} : L_{01} \to H_{01}$ satisfying

$$\operatorname{im} Q_{01} = X^{\perp}, \qquad \ker Q_{01} = Y, \qquad D_{01} Q_{01} \eta = \eta \quad \forall\, \eta \in Y^{\perp} .$$

We define the *approximate pseudo-right inverse*

$$\widetilde{Q}_R := \Theta_R \circ Q_{01} \circ \Xi_R : L_R \to H_R .$$

Moreover we define subspaces of H_R and L_R by

$$\begin{aligned} X_R &:= \Theta_R(X)\,, & X_R^{\perp} &:= \Theta_R(X^{\perp}), \\ Y_R &:= \Omega_R(Y)\,, & Y_R^{\perp} &:= \Omega_R(Y^{\perp})\,. \end{aligned}$$

By Lemma 9.4.6 and since the linear pregluing maps are surjective we have splittings $H_R = X_R \oplus X_R^{\perp}$ and $L_R = Y_R \oplus Y_R^{\perp}$.

Lemma 9.4.7. *We have* $\operatorname{im} \widetilde{Q}_R = X_R^{\perp}$, $\ker \widetilde{Q}_R = Y_R$ *and there exists constants c and R_0 such that for all $R \geq R_0$ we have*

$$\left\|\widetilde{Q}_R \eta\right\|_{H_R} \leq c \left\|\eta\right\|_{L_R}, \qquad \left\|D_R \widetilde{Q}_R \eta - \eta\right\|_{L_R} \leq c e^{-\delta R} \left\|\eta\right\|_{L_R} .$$

for all $\eta \in Y_R^{\perp}$.

Proof. The statements about the kernel and the image directly follow from the definition. The first estimate follows directly from Lemma 9.4.5. We turn to prove the second estimate. Follow the lines of the proof of Lemma 8.3.4 to show that for all $\xi \in H_{01}$ we have

$$\left\|D_R \Theta_R \xi - \Omega_R D_{01} \xi\right\|_{L_R} \leq O(e^{-\delta R}) \left\|\xi\right\|_{H_{01}} .$$

We abbreviate the norm $\|\cdot\| = \|\cdot\|_{L_R}$. For any $\eta \in Y_R^{\perp}$ we have

$$\left\|D_R \widetilde{Q}_R \eta - \eta\right\| \leq \left\|\Omega_R D_{01} Q_{01} \Xi_R \eta - \eta\right\| + O(e^{-\delta R}) \left\|\eta\right\| .$$

Decompose $\Xi_R \eta = \eta_0 + \eta_1$ along the splitting $Y \oplus Y^{\perp}$ and continue using the fact that Q_{01} is a pseudo-inverse we see

$$\left\|D_R \widetilde{Q}_R \eta - \eta\right\| \leq \left\|\Omega_R \eta_1 - \eta\right\| + O(e^{-\delta R}) \left\|\eta\right\| \leq O(e^{-\delta R}) \left\|\eta\right\| .$$

The term $\Omega_R \eta_1 - \eta$ vanishes because as Ξ_R is a right-inverse to Ω_R we have $\eta = \Omega_R \Xi_R \eta = \Omega_R \eta_0 + \Omega_R \eta_1 = \Omega_R \eta_1$ since by assumption $\eta \in Y_R^{\perp}$ and $\Omega_R \eta_0 \in Y_R$. □

Gluing construction Via (8.3.15) we use the approximate pseudo-inverse $\widetilde{Q}_R$ to define an actual pseudo right-inverse $Q_R : L_R \to H_R$ which is uniformly bounded and satisfies

$$\operatorname{im} Q_R = X_R^{\perp}, \qquad \ker Q_R = Y_R, \qquad D_R Q_R \eta = \eta, \quad \forall\, \eta \in Y_R^{\perp}.$$

Moreover abbreviate $P_R = \mathbb{1} - Q_R D_R$ the projection onto $D_R^{-1} Y_R$ along $\operatorname{im} Q_R$.

Lemma 9.4.8. *For all R sufficiently large, the restriction $P_R|_{X_R} : X_R \to D_R^{-1} Y_R$ is an isomorphism.*

Proof. We write any $\eta \in L_R$ as $\eta = \eta_0 + \eta_1$ along the splitting $L_R = Y_R \oplus Y_R^{\perp}$. Then $\eta = \eta_0 + D_R Q_R \eta_1$. This shows that $\eta \in \operatorname{im} D_R + Y_R$, thus D_R is transverse to Y_R. Hence the space $D_R^{-1} Y_R$ has dimension $\dim Y_R + \operatorname{ind} D_R$. Since pregluing is injective restricted to Y_{01} (cf. Lemma 9.4.6), the dimension of Y_R and Y_{01} are the same. Similarly we conclude that $\dim X_R = \dim X_{01} = \dim Y_{01} + \operatorname{ind} D_{01}$. The index of D_R and D_{01} agree (cf. Lemma 9.4.4). We conclude that the spaces $D_R^{-1} Y_R$ and X_R have the same dimension. It suffices to show that P_R is injective. The kernel of P_R is the intersection $D_R^{-1} Y_R \cap \operatorname{im} Q_R$. Suppose by contradiction that there exists a non-trivial $\xi \in D_R^{-1} Y_R \cap \operatorname{im} Q_R$. Hence $\xi = Q_R \eta$ for some $\eta \in Y_R^{\perp}$ with $D_R Q_R \eta \in Y_R$. By the properties of Q_R we have $\eta = D_R Q_R \eta = 0$. $\square$

Associated to the canonical exact sequences $0 \to \ker D_{01} \to X \xrightarrow{D_{01}} Y \to \operatorname{coker} D_{01} \to 0$ and $0 \to \ker D_R \to D_R^{-1} Y_R \xrightarrow{D_R} Y_R \to \operatorname{coker} D_R \to 0$ are the isomorphisms $|D_{01}| \cong |X| \otimes |Y|^{\vee}$ and $|D_R| \cong |D_R^{-1} Y_R| \otimes |Y_R|^{\vee}$ respectively via (9.1.1). We define the gluing isomorphism as the composition

$$|D_{01}| \longrightarrow |X| \otimes |Y|^{\vee} \xrightarrow{|P_R \Theta_R| \otimes |\Omega_R|^{\vee}} |D_R^{-1} Y_R| \otimes |Y_R|^{\vee} \longrightarrow |D_R| \,. \tag{9.4.8}$$

In order to glue the operators D_0 and D_1 we use the following simple algebraic lemma.

Lemma 9.4.9. *There exists an exact sequence*

$$\begin{aligned} 0 \to \ker D_{01} \to \ker D_0 \oplus \ker D_1 \to \ker A \to \operatorname{coker} D_{01} \to \\ \to \operatorname{coker} D_0 \oplus \operatorname{coker} D_1 \to 0\,, \end{aligned}$$

which together with (9.4.8) *induces an isomorphism $|D_0| \otimes |D_1| \to |\ker A| \otimes |D_R|$.*

Proof. The claim follows directly from the snake lemma on the commutative diagram of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & H_{01} \arrow[r] \arrow[d, "D_{01}"] & H_0 \oplus H_1 \arrow[r] \arrow[d, "D_0 \oplus D_1"] & \ker A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \oplus L \arrow[r, "="] & L \oplus L \arrow[r] & 0\,, & \end{tikzcd}$$

in which the map $H_0 \oplus H_1 \to \ker A$ is given by $(\xi_0, \xi_1) = \xi_0(\infty) - \xi_1(-\infty)$. By equation (9.1.1) we have an isomorphism $|D_0| \otimes |D_1| \cong |\ker A| \otimes |D_{01}|$. $\square$

Naturality We show that the orientation gluing map (9.4.8) is independent of choices and natural with respect to homotopies (cf. Lemmas 9.4.10 and 9.4.11 respectively). Given two linear complements $Y, \widehat{Y} \subset L_{01}$ which are transverse to D_{01} and denote the corresponding maps from (9.4.8) on the level of determinant lines by $\psi_R, \widehat{\psi}_R : \det D_{01} \to \det D_R$ respectively.

Lemma 9.4.10. *For all R sufficiently large, the composition*

$$\psi_R^{-1} \circ \widehat{\psi}_R : \det D_{01} \to \det D_{01}\,,$$

is given by multiplication with a positive number.

Proof. Without loss of generality $Y \subset \widehat{Y}$. We denote the spaces and maps appearing in the construction using $\widehat{Y}$ with $\widehat{X}$, $\widehat{X}^\perp$ etc. To define the maps ψ_R and $\widehat{\psi}_R$ we have the commutative diagram

$$\begin{tikzcd}
 & \det \widehat{X} \otimes \det \widehat{Y}^\vee \arrow[r] & \det D_R^{-1}\widehat{Y}_R \otimes \det \widehat{Y}_R^\vee \arrow[dr] & \\
\det D_{01} \arrow[ur] \arrow[dr] & & & \det D_R \\
 & \det X \otimes \det Y^\vee \arrow[r] \arrow[uu] & \det D_R^{-1} Y_R \otimes \det Y_R^\vee \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

The map ψ_R is obtained by following the diagram along the lower arrows and we get the map $\widehat{\psi}_R$ using the upper path. The two vertical arrows are computed in [2, Lmm. 5.2] and are given as follows: Consider a splitting $\widehat{Y} = Y \oplus Y'$ and set $X' := D_{01}^{-1}(Y')$. We have a splitting $\widehat{X} = X \oplus X'$ and $D_{01}|_{X'} : X' \to Y'$ is an isomorphism. Fix generators $\alpha \in \det X$, $\beta \in \det Y$ and $\gamma \in \det X'$. Then $D_{01}\gamma$ is a generator of $\det Y'$ and the map on the left-hand side is given by $\alpha \otimes \beta^\vee \mapsto (\alpha \wedge \gamma) \otimes (D_{01}\gamma^\vee \wedge \beta^\vee)$. On the other side consider the splitting $\widehat{Y}_R = Y_R \oplus Y'_R$ with $Y_R := \Omega_R(Y)$ and $Y'_R := \Omega_R(Y')$. We have a corresponding splitting $D_R^{-1}\widehat{Y}_R = D_R^{-1}Y_R \oplus D_R^{-1}Y'_R$, generators $\alpha_R \in \det D_R^{-1}Y_R$, $\beta_R \in \det Y_R$ and $\gamma_R \in \det D_R^{-1}Y'_R$ and the map is given by $\alpha_R \otimes \beta_R^\vee \mapsto (\alpha_R \wedge \gamma_R) \otimes (D_R\gamma_R^\vee \wedge \beta_R^\vee)$. We assume that the generators are picked such that $\alpha_R = P_R\Theta_R\alpha$, $\beta_R = \Omega_R(\beta)$ and $D_R\gamma_R = \Omega_R D_{01}\gamma$. We conclude by following the definition of the maps around the square, that $\psi_R^{-1}\psi_R$ is given by multiplication with the determinant of the map

$$\widehat{P}_R\Theta_R : X \to D_R^{-1}\widehat{Y}_R\,,$$

where $\det X$ is oriented by $\alpha \wedge \gamma$ and $\det D_R^{-1}\widehat{Y}_R$ is oriented by $\alpha_R \wedge \gamma_R$.

We claim that $(P_R - \widehat{P}_R)\xi \in D_R^{-1}Y'_R$ for any $\xi \in X_R$. Indeed given $\xi \in X_R$ and split $D_R\xi = \eta_0 + \eta_1 + \eta_2$ along the splitting $Y_R \oplus Y'_R \oplus \widehat{Y}_R^\perp$. Then since they are right-inverses

$$D_R(P_R - \widehat{P}_R)\xi = D_R(\widehat{Q}_R - Q_R)D_R\xi = D_R\widehat{Q}_R\eta_2 - D_RQ_R(\eta_1 + \eta_2) = \\ = \eta_2 - \eta_1 - \eta_2 = \eta_1\,.$$

This shows the claim and we conclude that $\widehat{P}_R\Theta_R\alpha \wedge \widehat{P}_R\Theta_R\gamma = \alpha_R \wedge \widehat{P}_R\Theta_R\gamma$. We are left to compute the number $a \in \mathbb{R}$ which is defined by

$$\pi_{D_R^{-1}Y_R'}\widehat{P}_R\Theta_R\gamma = a \cdot \gamma_R\,,$$

in which $\pi_{D_R^{-1}Y_R'} : D_R^{-1}\widehat{Y}_R \to D_R^{-1}Y_R'$ denotes the projection along $D_R^{-1}Y_R$. We apply D_R on the last equation and obtain

$$\pi_{Y_R'}D_R\widehat{P}_R\Theta_R\gamma = a \cdot D_R\gamma_R = a \cdot \Omega_R D_{01}\gamma\,.$$

Abbreviate the generator $\gamma_R' := \Omega_R D_{01}\gamma \in \det Y_R'$. The inverse of the map $\Omega_R D_{01} : X' \to Y_R'$ is $Q_{01} \circ \Xi_R$ and hence $\gamma = Q_{01}\Xi_R\gamma_R'$. Plugging that back into the last equation we conclude that a is the determinant of the map

$$\pi_{Y_R'}D_R\widehat{P}_R\widetilde{Q}_R : Y_R' \to Y_R'\,.$$

To show that the determinant is positive for all R sufficiently large enough it suffices to show the following: There are constants c and R_0 such that for all $R \geq R_0$ and $\eta \in Y_R'$ we have

$$\left\|D_R\widehat{P}_R\widetilde{Q}_R\eta - \eta\right\|_{L_R} \leq ce^{-\delta R}\left\|\eta\right\|_{L_R}\,.$$

Choose any $\eta \in Y_R'$. Abbreviating $\|\cdot\| = \|\cdot\|_{L_R}$ we compute

$$\begin{aligned}\left\|D_R\widehat{P}_R\widetilde{Q}_R\eta - \eta\right\| &= \left\|D_R(\mathbb{1} - \widehat{Q}_R D_R)\widetilde{Q}_R\eta\right\| \\ &\leq \left\|D_R\widetilde{Q}_R\eta - \eta\right\| + \left\|D_R\widehat{Q}_R D_R\widetilde{Q}_R\eta\right\| \\ &= \left\|D_R\widetilde{Q}_R\eta - \eta\right\| + \left\|D_R\widehat{Q}_R(D_R\widetilde{Q}_R\eta - \eta)\right\| \leq ce^{-\delta R}\left\|\eta\right\|\,.\end{aligned}$$

Using the fact that $\widehat{Q}_R\eta = 0$ and Lemma 9.4.7. $\square$

Lemma 9.4.11. *Given homotopies $(S_0^\tau)_{\tau\in[a,b]}$ and $(S_1^\tau)_{\tau\in[a,b]}$ such that $S_0^\tau \in \mathcal{D}(\sigma_-^\tau, \sigma^\tau)$, $S_1^\tau \in \mathcal{D}(\sigma^\tau, \sigma_+^\tau)$ and $S_0^\tau(s,\cdot) = S_1^\tau(-s,\cdot) = \sigma^\tau$ for all $s \geq 1$ and $\tau \in [a,b]$. For corresponding operators D_{01}^τ and D_R^τ and R sufficiently large we have a commutative diagram*

$$\begin{tikzcd} |D_{01}^a| \arrow[r] \arrow[d] & |D_R^a| \arrow[d] \\ |D_{01}^b| \arrow[r] & |D_R^b| \end{tikzcd}$$

where the horizontal arrows are given by (9.4.8) *and the vertical arrows are induced by the homotopies.*

Proof. Let $\tau \mapsto Y^\tau \subset L$ be a continuous path of subspaces such that for each $\tau \in [a,b]$ the space Y^τ is transverse to D_{01}^τ. For R sufficiently large D_R^τ is transverse to the glued space Y_R^τ for all $\tau \in [a,b]$. By definition of the gluing map we have a commutative diagram

$$\begin{tikzcd}
|D_{01}^\tau| \arrow[r] \arrow[d] & |(D_{01}^\tau)^{-1}Y^\tau| \otimes |Y^\tau| \arrow[d, "|P_R^\tau\Theta_R^\tau|\otimes|\Omega_R^\tau|"] \\
|D_R^\tau| \arrow[r] & |(D_R^\tau)^{-1}Y_R^\tau| \otimes |Y_R^\tau|\,.
\end{tikzcd}$$

The spaces $(D_{01}^\tau)^{-1}(Y^\tau)$, $\widehat{Y}_R^\tau$ etc. are the fibers of vector bundles over $[a,b]$ and the isomorphisms $P_R^\tau\Theta_R^\tau$ and Ω_R^τ are bundle maps. By continuity we obtain a commutative diagram. $\square$

10. Pearl homology

10.1. Overview

Pearl homology is a version of Floer homology of Lagrangian intersection, which has the advantage that if Lagrangians intersect cleanly we do not need to perturb the Lagrangians into transverse position. We call the invariant *pearl homology* because it is a direct generalization of an invariant associated to a single monotone Lagrangian introduced by Biran and Cornea in [12] with that name. The construction was already sketched out by Frauenfelder in [33, Appendix C] under the name of *Floer-Bott homology.* Here we give a detailed account of the theory for the monotone case including orientations.

Pearl trajectories Choose an auxiliary Morse function $f : L_0 \cap L_1 \to \mathbb{R}$ and metric on $L_0 \cap L_1$ and a path of almost complex structures $J : [0, 1] \to \mathrm{End}(TM, \omega)$. Given critical points $p_-, p_+ \in \operatorname{crit} f$, a *pearl trajectory connecting p_- to p_+* is either a negative gradient flow trajectory $u : \mathbb{R} \to L_0 \cap L_1$ with $u(-\infty) = p_-$ and $u(\infty) = p_+$ or a tuple $u = (u_1, \ldots, u_m)$ of non-constant finite energy J-holomorphic strips $\mathbb{R} \times [0, 1] \to M$ with boundary in (L_0, L_1) such that $u_1(-\infty) \in W^u(p_-)$, $u_m(\infty) \in W^s(p_+)$ and for each $j = 1, \ldots, m-1$ there exists a negative gradient flow line from $u_j(\infty)$ to $u_{j+1}(-\infty)$. For reasons of transversality we require that each curve in the tuple is not a reparametrization of another. The number m is called the number of *cascades.* If u is an ordinary Morse flow line, we say that u has zero cascades. We denote by $\mathcal{M}(p_-, p_+)$ the space of all pearl trajectories connecting p_- to p_+ modulo reparametrization with an arbitrary number of cascades. If J is sufficiently generic every connected component of $\mathcal{M}(p_-, p_+)$ is a manifold with corners and for $d \in \mathbb{N}_0$ we denote by $\mathcal{M}(p_-, p_+)_{[d]}$ the union of all d-dimensional components.

Grading Let $\mathcal{P}$ be the space of paths $\gamma : [0, 1] \to M$ such that $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$. We denote by $N \in \mathbb{N}$ the minimal Maslov number of the pair (L_0, L_1) with respect to a fixed element $x_* \in \mathcal{P}$. For every critical point $p \in \operatorname{crit} f$ we choose a map $u_p : [-1, 1] \times [0, 1] \to M$ such that $u_p(s, \cdot) \in \mathcal{P}$ for all $s \in [-1, 1]$, $u_p(-1, \cdot) = x_*$ and $u_p(1, \cdot) \equiv p$. The grading of a critical point $p \in \operatorname{crit} f$ is

$$|p| = \mu(p) - \mu(u_p) - \frac{1}{2} \dim T_p L_0 \cap L_1 \,. \tag{10.1.1}$$

Orientation Let $\mathcal{O} \to L_0 \cap L_1$ be the double cover associated to a fixed relative spin structure (cf. Definition 9.3.4). For any critical point $p \in \operatorname{crit} f$ fix an orientation

$o_p \in |T_p W^u(p)| \otimes \mathcal{O}_p$. In the paragraph before Lemma 10.2.7 we define orientations on $\mathcal{M}(p_-, p_+)$ with these choices. An orientation of $[u] \in \mathcal{M}(p_-, p_+)_{[0]}$ is just a number in $\{\pm 1\}$ which we denote by $\operatorname{sign}(u)$.

Pearl complex Let $\Lambda := A[\lambda^{-1}, \lambda]$ be the ring of Laurent polynomials in one variable of degree $-N$. The *pearl chain complex* $CH_*(L_0, L_1)$ is given as the free Λ-module generated by all critical points of f with grading $|p \otimes \lambda^k| = |p| - kN$ and equipped with the Λ-linear homomorphism

$$\begin{aligned} \partial : CH_*(L_0, L_1) &\to CH_{*-1}(L_0, L_1) \\ p &\mapsto \sum_{q \in \operatorname{crit} f} \sum_{[u] \in \mathcal{M}(p,q)_{[0]}} \operatorname{sign} u \cdot q \otimes \lambda^{(|q|-|p|+1)/N} . \end{aligned} \tag{10.1.2}$$

The next theorem as proven Fukaya et al. in [37] for the very general case of semi-positive symplectic manifolds and unobstructed Lagrangians, which includes the case of monotone Lagrangians.

Theorem 10.1.1. *We have $\partial \circ \partial = 0$. The homology group $QH_*(L_0, L_1) = \ker \partial / \operatorname{im} \partial$ is called* pearl homology *and is independent of choices of J, f, the metric and orientations o_p. Moreover we have a natural isomorphism*

$$QH_*(L_0, L_1) \cong QH_*(L_0, \varphi_H(L_1)) . \tag{10.1.3}$$

for any Hamiltonian H.

By the invariance we conclude that pearl homology is isomorphic to Floer homology. Namely if we choose H such that $\varphi_H(L_1)$ intersects L_0 transversely, then pearl homology for L_0 and $\varphi_H(L_1)$ agrees with Floer homology by definition.

10.2. Pearl trajectories

In the following we abbreviate $\mathcal{I} := L_0 \cap L_1$ and fix an auxiliary Morse function f on $\mathcal{I}$. We denote by $\psi : \mathbb{R} \times \mathcal{I} \to \mathcal{I}$, $\psi^a := \psi(a, \cdot)$ be the negative gradient flow of f with respect to a sufficiently generic metric. Note that in general $\mathcal{I}$ has many connected components with possibly different dimension. Pick a path of almost complex structures $J : [0,1] \to \operatorname{End}(TM, \omega)$. Given an integer $m \in \mathbb{N}$ and submanifolds $W_-, W_+ \subset \mathcal{I}$ we define

$$\widetilde{\mathcal{M}}_m(W_-, W_+; J) := \{(u_1, \ldots, u_m) \subset C^\infty(\Sigma, M) \mid \text{a)} - \text{d)}\} , \tag{10.2.1}$$

to be the space of tuples $(u_1, \ldots, u_m)$ such that

a) for all $j = 1, \ldots, m$ the map u_j is J-holomorphic with boundary in (L_0, L_1),

b) the tuple $(u_1, \ldots, u_m)$ is distinct, i.e. for all $i \neq j$, $a \in \mathbb{R}$ we have $u_i \circ \tau_a \neq u_j$,

c) for all $j = 1, \dots, m-1$ there exists $a_j \geq 0$ such that

$$\psi^{a_j}(u_j(\infty)) = u_{j+1}(-\infty)\,, \tag{10.2.2}$$

d) we have $u_1(-\infty) \in W_-$ and $u_m(\infty) \in W_+$.

For $m = 0$ we define $\widetilde{\mathcal{M}}_0(W_-, W_+; J) := W_- \cap W_+$. The elements of the space $\widetilde{\mathcal{M}}_m(W_-, W_+; J)$ are called *parametrized J-holomorphic pearl trajectory with m cascades connecting W_- to W_+ and boundary in* (L_0, L_1). If in particular $W_- = W^u(p_-)$ and $W_+ = W^s(p_+)$ for some critical points $p_-, p_+ \in \operatorname{crit} f$ we abbreviate furthermore

$$\widetilde{\mathcal{M}}_m(p_-, p_+; J) := \widetilde{\mathcal{M}}_m(W^u(p_-), W^s(p_+); J)\,.$$

The elements of $\widetilde{\mathcal{M}}_m(p_-, p_+; J)$ are simply called *pearl trajectories with m cascades connecting p_- to p_+*. As we shall see in a moment the connected components of these spaces are manifolds with corners if J is chosen sufficiently generic.

Transversality

For the correction description of the space $\widetilde{\mathcal{M}}_m(W_-, W_+; J)$ we need the notion of a *manifold with corners*, which is a mild generalization of the notion of manifolds with boundary. Unfortunately there is no standard concept in the mathematical literature. We stick with the definition of [47]. Briefly a manifold with corners is a topological space $\mathcal{M}$ equipped with an atlas of charts locally modeled on open subsets in $[0, \infty)^k \times \mathbb{R}^{n-k}$ and chart transition maps which extend to smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The *dimension* of the manifold is the number n. The *depth* of a point is the number of zeros among the first k coordinates in a chart. The depth is well-defined independently of the choice of local coordinates and gives rise to a stratification of $\mathcal{M}$. The *top stratum* is given as the space of all points with depth equal to zero. Obviously the each stratum is a manifold in the usual sense. For more details see [47].

Fix an integer $m \in \mathbb{N}$. Let $\widetilde{\mathcal{M}}_m(J)$ denote the space of distinct m-tuples of J-holomorphic curves and consider the evaluation map

$$\begin{aligned} ev : \widetilde{\mathcal{M}}_m(J) &\to \mathcal{I}^{2m} \\ (u_1, \dots, u_m) &\mapsto (u_1(-\infty), u_1(\infty), u_2(-\infty), \cdots, u_m(\infty)) \end{aligned}$$

On the other hand consider the map

$$\begin{aligned} W_- \times \mathcal{I}^{m-1} \times W_+ \times \mathbb{R}^{m-1} &\to \mathcal{I}^{2m}\,, \\ (p_0, \dots, p_m, a_1, \dots, a_{m-1}) &\mapsto \begin{matrix} (p_0, p_1, \psi^{a_1}(p_1), p_2, \psi^{a_2}(p_2), \dots \\ \dots, p_{m-1}, \psi^{a_{m-1}}(p_{m-1}), p_m) \end{matrix} \end{aligned} \tag{10.2.3}$$

We say that J is *regular for W_- and W_+* if J is regular for $X \equiv 0$ and the map (10.2.3) in the sense of Definition 7.3.3, i.e. the operator $D_{u,J}$ is surjective for all J-holomorphic strips u and the evaluation is transverse to (10.2.3).

Lemma 10.2.1. *The subspace of $J \in C^\infty([0,1], \mathrm{End}(TM, \omega))$ which are regular for W_- and W_+ is comeager. If J is regular then each connected component of $\widetilde{\mathcal{M}}_m(W_-, W_+; J)$ is a manifold with corners and the component which contains $u = (u_1, \dots, u_m)$ has dimension*

$$\mu_{\mathrm{Vit}}(u) + \dim W_- - \frac{1}{2}\dim C_- + \dim W_+ - \frac{1}{2}\dim C_+ + m - 1\,,$$

in which $\mu_{\mathrm{Vit}}(u) := \sum_{j=1}^m \mu_{\mathrm{Vit}}(u_j)$ and $C_- \subset \mathcal{I}$, $C_+ \subset \mathcal{I}$ are the connected components containing W_- and W_+ respectively.

Proof. In Theorem 7.3.4 we show that the space of regular structures is comeager. Each connected component of the space $W_- \times \mathcal{I}^{m-1} \times W_+ \times [0,\infty)^{m-1}$ is clearly a manifold with corners. We see that $\widetilde{\mathcal{M}}_m(W_-, W_+; J)$ is the fibre product of the evaluation map with the map (10.2.3) restricted to the subspace $W_- \times \mathcal{I}^{m-1} \times W_+ \times [0,\infty)^{m-1}$. Hence by [47, Thm. 6.4] connected components of $\widetilde{\mathcal{M}}_m(W_-, W_+ J)$ are also manifolds with corners. To compute the dimension, choose some tuple $(u_1, \dots, u_m) \in \widetilde{\mathcal{M}}_m(W_-, W_+; J)$ in the top stratum, which is equivalent to say that the tuple of non-negative numbers $(a_1, \dots, a_{m-1})$ defined by (10.2.2) has no zeros. Let $C_0, C_1, C_2, \dots, C_m \subset \mathcal{I}$ be the connected components of the points $u_1(-\infty), u_1(\infty), u_2(\infty), \dots, u_m(\infty)$ respectively. With the dimension formula from Theorem 7.3.4 we have

$$\dim T_u\widetilde{\mathcal{M}}_m(J) = \sum_{j=1}^m \mu_{\mathrm{Vit}}(u_j) + \frac{1}{2}\dim C_{j-1} + \frac{1}{2}\dim C_j\,.$$

By the exact sequence (9.1.4) we conclude that the dimension d of the fibre product $\widetilde{\mathcal{M}}_m(C_-, C_+; J)$ at u is given by

$$\begin{aligned}
d &= \dim T_u\mathcal{M}_m(J) + \dim W_- \times \prod_{j=1}^{m-1} C_j \times W_+ \times \mathbb{R}^{m-1} - \dim \prod_{j=0}^{m-1} C_j \times C_{j+1} \\
&= \mu(u) + \frac{1}{2}\dim C_0 + \sum_{j=1}^{m-1} \dim C_j + \frac{1}{2}\dim C_m + \dim W_- + \dim W_+ + m - 1 - \\
&\qquad - \sum_{j=0}^{m} \dim C_j \\
&= \mu(u) + \dim W_- - \frac{1}{2}\dim C_0 + \dim W_+ - \frac{1}{2}\dim C_m + m - 1\,.
\end{aligned}$$

This shows the claim. □

From now on we fix an almost complex structure J, which is sufficiently generic in the sense that it is regular with respect to all upcoming pairs of submanifolds and omit the reference to J whenever convenient, eg. we write $\widetilde{\mathcal{M}}(p,q)$ to denote $\widetilde{\mathcal{M}}(p,q;J)$.

Compactness

A *broken J-holomorphic pearl trajectory connecting p_- to p_+* is a tuple of pearl trajectories $v = (v_1, \ldots, v_k)$ such that v_i connects p_i to p_{i+1} for all $i = 1, \ldots, k-1$ and some critical points $p_- = p_1, \ldots, p_k = p_+$. For $i = 1, \ldots, k$ we denote by $m(v_i)$ the number of cascades of v_i.

Definition 10.2.2. We say that a sequence of pearl trajectories

$$(u^\nu)_{\nu\in\mathbb{N}} = (u_1^\nu, \ldots, u_m^\nu)_{\nu\in\mathbb{N}},$$

Floer-Gromov converges to the broken pearl trajectory $v = (v_1, \ldots, v_k)$ if

- for each $j = 1, \ldots, m$ the sequence $(u_j^\nu)_{\nu\in\mathbb{N}}$ Floer-Gromov converges to $w_j = (w_{j,1}, \ldots, w_{j,k_j})$ (cf. Definition 5.1.3)
- for each (i, j) with $1 \le i \le k$ and $1 \le j \le m(v_i)$ there exist a pair (ℓ, κ) such that $v_{i,j} = w_{\ell,\kappa}$,
- the map $\{(i, j) \mid 1 \le i \le k, 1 \le j \le m(v_i)\} \to \{(\ell, \kappa) \mid 1 \le \ell \le m, 1 \le \kappa \le k_\ell\}$, $(i, j) \mapsto (\ell, \kappa)$ mentioned above is surjective and strictly monotone with respect to the lexicographic order.

Lemma 10.2.3. *Assume that L_0 and L_1 are monotone with minimal Maslov number at least three. Let $(u^\nu)_{\nu\in\mathbb{N}} = (u_1^\nu, \ldots, u_m^\nu)_{\nu\in\mathbb{N}}$ be a sequence of pearl trajectories connecting p_- to p_+ such that $\sup_\nu E(u^\nu) < \infty$, then at least one of the following holds:*

(i) a subsequence of (u^ν) Floer-Gromov converges to a broken pearl trajectory connecting p_- to p_+,

(ii) there exists a broken pearl trajectory v connecting p_- to p_+ satisfying $\mu(u^\nu) \ge \mu(v) + 3$ for all ν sufficiently large.

(iii) $p_- = p_+$ and $\mu(u^\nu) \ge 3$ for all ν sufficiently large.

Proof. For each $j = 1, \ldots, m$ we obtain by Theorem 5.1.4 a subsequence of (u_j^ν) still denoted by the same sequence, which Floer-Gromov converges modulo bubbling to the broken strip $w_j = (w_{j,1}, \ldots, w_{j,k_j})$ possibly containing constant components. We also have $u_j^\nu(-\infty) \to w_{j,1}(-\infty)$ and $u_j^\nu(\infty) \to w_{j,k_j}(\infty)$. Let $(b_j^\nu) \subset \mathbb{R}$ be the sequence such that $\psi^{b_j^\nu} u_j^\nu(\infty) = u_{j+1}^\nu(-\infty)$. We distinguish two cases. In the first case $(b_j^\nu)_{\nu\in\mathbb{N}}$ is bounded, then a subsequence converges to b_j and we have

$$\psi^{b_j}(w_{j,k_j}(\infty)) = w_{j+1,1}(-\infty). \tag{10.2.4}$$

In the second case $(b_j^\nu)_{\nu\in\mathbb{N}}$ is unbounded. For each $\nu \in \mathbb{N}$ let γ_j^ν be the Morse trajectory from $u_j^\nu(\infty)$ to $u_{j+1}^\nu(-\infty)$. Then a subsequence of $(\gamma_j^\nu)_{\nu\in\mathbb{N}}$ converges to a broken Morse trajectory $(\gamma_{j,1}, \gamma_{j,2}, \ldots, \gamma_{j,\ell_j})$ with $\ell_j \ge 2$, where $\gamma_{j,1}$ and γ_{j,ℓ_j} are half-trajectories. Set $p_j^- := \gamma_{j,1}(\infty)$ and $p_j^+ := \gamma_{j,\ell_j}(-\infty)$. In particular if $j > 2$ the critical points p_j^-, p_j^+ are joined by the broken Morse trajectory $(\gamma_{j,2},\dots,\gamma_{j,\ell_j-1})$ and if $\ell_j=2$ they are equal $p_j^-=p_j^+$. In any case we have $w_{j,k_j}(\infty)\in W^s(p_j^-)$ and $w_{j+1,1}(-\infty)\in W^u(p_j^+)$. Regrouping the non-constant components of the tuples $(w_{j,i})$ and $(\gamma_{j,2},\dots,\gamma_{j,\ell_j-1})$ using (10.2.4) shows they constitute to a broken pearl trajectory v which connects p_- to p_+. Moreover by Lemma 5.5.4 either each all $w_{j,i}$ are non-constant and (u^ν) Floer-Gromov converges to v or $\mu(u^\nu)\geq\mu(v)+3$ or if all components had been discarded $\mu(u^\nu)\geq 3$ for all ν large enough. □

If $m\geq 1$ the reparametrization group of $\widetilde{\mathcal{M}}_m(p_-,p_+)$ is $\mathbb{R}^m$ which acts freely via $(a_1,\dots,a_m).(u_1,\dots,u_m)=(u_1\circ\tau_{a_1},\dots,u_m\circ\tau_{a_m})$ with $\tau_a:\mathbb{R}\times[0,1]\to\mathbb{R}\times[0,1],(s,t)\mapsto(s-a,t)$ for $a\in\mathbb{R}$. If $m=0$ and $p_-\neq p_+$ there is a free action of $\mathbb{R}$ on $\widetilde{\mathcal{M}}_0(p_-,p_+)$ given by the negative gradient flow. In any case we denote by $\mathcal{M}_m(p_-,p_+)$ the quotient and moreover

$$\mathcal{M}(p_-,p_+):=\bigcup_{m\in\mathbb{N}_0}\mathcal{M}_m(p_-,p_+)\,.$$

For any $d\in\mathbb{N}_0$ we denote by $\mathcal{M}(p_-,p_+)_{[d]}\subset\mathcal{M}(p_-,p_+)$ the union of components with dimension d. Further let

$$\mathcal{M}_m^\ell(p_-,p_+)\subset\mathcal{M}_m(p_-,p_+)\,,$$

be the subspace of points with depth $\ell=0,\dots,m$, i.e. given by equivalence classes of pearl trajectories such that there are exactly ℓ zeros in the tuple $(a_1,\dots,a_{m-1})$ defined by (10.2.2).

Corollary 10.2.4. *For all critical points $p,q\in\operatorname{crit}f$, the space $\mathcal{M}(p,q)_{[0]}$ is finite and the boundary of the Floer-Gromov compactification of $\mathcal{M}_m(p,q)_{[1]}$ is given by the union of*

- $\mathcal{M}_m^1(p,q)_{[0]}$,
- $\mathcal{M}_{m+1}^1(p,q)_{[0]}$,
- $\mathcal{M}_\ell(p,r)_{[0]}\times\mathcal{M}_k(r,q)_{[0]}$ *for all $r\in\operatorname{crit}f$ and $\ell,k\in\mathbb{N}_0$ with $\ell+k=m$.*

Proof. Fix d and let $(u^\nu)=(u_1^\nu,\dots,u_{m^\nu}^\nu)\in\widetilde{\mathcal{M}}_{m^\nu}(p_-,p_+)_{[d+m^\nu]}$ be a sequence of pearl trajectories. Since every non-constant holomorphic strip carries a minimal energy (cf. Proposition 5.4.1) we have an uniform constant $\hbar>0$ such that $E(u_j^\nu)=\int(u_j^\nu)^*\omega>\hbar$. From the dimension formula and the action-index relation (cf. Lemma 5.5.3) we conclude that there exists a constant c such that

$$d=\sum_{j=1}^{m^\nu}\mu(u_j^\nu)+\mu(p_-)-\mu(p_+)-\frac{1}{2}\dim C_-+\frac{1}{2}\dim C_+-1>\tau^{-1}m^\nu\hbar+c\,.$$

Hence the sequence (m^ν) is bounded and after possibly passing to a subsequence we assume without loss of generality that $m^\nu=m$ for all $\nu\geq 1$. By the same token we conclude that $d\geq\tau^{-1}\sum_{j=1}^m\int E(u_j^\nu)+c$ and that the energy of (u_j^ν) is uniformly bounded for all $j=1,\dots,m$. We apply Lemma 10.2.3 and assume by contradiction that the second case holds, i.e. we obtain a broken pearl trajectory $v = (v_1, \ldots, v_k)$ which connects p_- to p_+ such that $\mu(u^\nu) \geq \mu(v) + 3$. Let $p_1, \ldots, p_{k-1}$ (resp. $C_1, \ldots, C_{k-1}$) be critical points (resp. connected components) such that v_ℓ connects $p_{\ell-1}$ to p_ℓ (resp. $C_{\ell-1}$ to C_ℓ) for each $\ell = 1, \ldots, k-1$. The dimension formula for v_ℓ implies

$$\mu(v_\ell) \geq \mu(p_\ell) - \mu(p_{\ell-1}) + \frac{1}{2}\dim C_\ell - \frac{1}{2}\dim C_{\ell-1} + 1\,. \tag{10.2.5}$$

Note that even if v_ℓ is not in the top stratum, the index is still bigger or equal to the number of cascades of v_ℓ. By the dimension formula for u^ν and last estimate

$$\begin{aligned} d &= \sum_{j=1}^{m} \mu(u_j^\nu) + \mu(p_-) - \mu(p_+) - \frac{1}{2}\dim C_- + \frac{1}{2}\dim C_+ - 1 \\ &\geq \sum_{\ell=1}^{k} \mu(v_\ell) + 3 + \mu(p_-) - \mu(p_+) - \frac{1}{2}\dim C_- + \frac{1}{2}\dim C_+ - 1 \\ &\geq k + 3 - 1 \geq 3\,. \end{aligned}$$

Hence if $d = 0, 1$ the second case of Lemma 10.2.3 is impossible. Now assume that the third case holds, i.e. $p_- = p_+$, $C_- = C_+$ and $\mu(u^\nu) \geq 3$. By the same estimate we have $d = \mu(u^\nu) - 1 \geq 2$, which is again impossible for $d = 0, 1$. The first case implies that a subsequence of (u^ν) converges to the broken pearl trajectory v connecting p_- to p_+ with $\mu(u^\nu) = \mu(v)$. By the same estimate as above we have

$$\begin{aligned} d &= \sum_{j=1}^{m} \mu(u_j^\nu) + \mu(p_-) - \mu(p_+) - \frac{1}{2}\dim C_- + \frac{1}{2}\dim C_+ - 1 \\ &= \sum_{\ell=1}^{k} \mu(v_\ell) + \mu(p_-) - \mu(p_+) - \frac{1}{2}\dim C_- + \frac{1}{2}\dim C_+ - 1 \geq k - 1\,. \end{aligned}$$

We conclude that if $d = 0$, then $k = 1$ and the unparametrized curve of v_1 is in fact an element of $\mathcal{M}(p_-, p_+)_{[0]}$. If $d = 1$, then possibly $k = 2$ and hence is an element of the given boundary. To show that every element appears as the boundary consider the proof of Lemma 10.2.7 below. □

Orientations

To define homology with integer coefficients or more generally with coefficients in a ring of characteristic $\neq 2$, we need to orient the moduli spaces. As already mentioned the orientations should be compatible with gluing and breaking (i.e. coherent in the sense of [24]). We now explain that with more detail.

Denote by $\overline{\mathcal{M}}_m(p,q)_{[1]}$ the Floer-Gromov compactification of $\mathcal{M}_m(p,q)_{[1]}$. In view of Corollary 10.2.4 we see that the points $\mathcal{M}^1_m(p,q)_{[0]}$ occur as boundary points of $\overline{\mathcal{M}}_m(p,q)_{[1]}$ and $\overline{\mathcal{M}}_{m-1}(p,q)_{[1]}$. Define the space with common boundary points identified

$$\mathcal{M}_\#(p,q) := \bigcup_{m \in \mathbb{N}_0} \overline{\mathcal{M}}_m(p,q)_{[1]}/\sim\;.$$

Definition 10.2.5. Given orientations of the spaces $\mathcal{M}_m(p,q)_{[0]}$ for all $p,q \in \operatorname{crit} f$ and $m \in \mathbb{N}_0$. We say that the orientations are *coherent*, if there exists an orientation on $\mathcal{M}_\#(p,q)$ such that its oriented boundary is given by

$$\bigcup \mathcal{M}(p,r)_{[0]} \times \mathcal{M}(r,q)_{[0]}\,,$$

with union over all critical points $r \in \operatorname{crit} f$.

Remark 10.2.6. If coherent orientations exists, then pearl homology with integer coefficients is well-defined. Given two sets of orientations which are coherent, there is no reason why the homology should be the same. In particular in [17], Cho found non-isomorphic Floer cohomologies for the same pair of Lagrangians but with different choices of orientations associated to non-equivalent relative spin structures.

Construction of the orientation For all $m \in \mathbb{N}$ and any connected component $C \subset L_0 \cap L_1$ we consider the space $\widetilde{\mathcal{M}}_m(p,C) := \widetilde{\mathcal{M}}_m(W^u(p),C)$ equipped with the evaluation map

$$ev : \widetilde{\mathcal{M}}_m(p,C) \to C, \qquad (u_1,\dots,u_m) \mapsto u_m(\infty).$$

Let $\mathcal{O} \to L_0 \cap L_1$ be the double cover associated to a fixed relative spin structure (cf. Definition 9.3.4). Fix an element in $|T_pW^u(p)| \otimes \mathcal{O}_p^\vee$ for each critical point $p \in \operatorname{crit} f$. We construct recursively an $\mathcal{O}^\vee$-orientation on $ev : \mathcal{M}_m(p,C) \to C$, i.e. a section of $|\mathcal{M}_m(p,C)| \otimes ev^*\mathcal{O}^\vee$. First of all notice that since $W^u(p)$ is contractible to the point p our choices fix an $\mathcal{O}^\vee$-orientation on $W^u(p)$ by parallel transport. Then by Corollary 9.3.7 we obtain an $\mathcal{O}^\vee$-orientation on $\widetilde{\mathcal{M}}_1(p,C) \to C$ and hence an $\mathcal{O}^\vee$-orientation on the quotient $\mathcal{M}_1(p,C) \to C$ by (9.1.8). An $\mathcal{O}^\vee$-orientation on $ev : \mathcal{M}_m(p,C) \to C$ induces and $\mathcal{O}^\vee$-orientation on $ev_\psi : \mathbb{R} \times \mathcal{M}_m(p,C) \to C$, $(a,u) \mapsto \psi^a(ev(u))$ by parallel transport. Suppose for $m \geq 2$ we have an $\mathcal{O}^\vee$-orientation on $ev : \mathcal{M}_{m-1}(p,C') \to C'$ for some connected component C'. There exists an induced $\mathcal{O}^\vee$-orientation on

$$\left[(\mathbb{R} \times \mathcal{M}_{m-1}(p,C'))\ {}_{ev_\psi}\!\times_{ev_-} \widetilde{\mathcal{M}}_1(C',C)\right] / \sim\ \to C\,, \tag{10.2.6}$$

in which the quotient is with respect to the group of reparametrizations acting on the last factor. More precisely the $\mathcal{O}^\vee$-orientation is constructed by the remarks above, Corollary 9.3.7 and the quotient orientation 9.1.8. Here the order of each step matters. In particular we *first* construct an orientation on the fibre product and *then* take the associated orientation on the quotient and not the other way around. Every connected component of $\mathcal{M}_m(p,C)$ is of the form (10.2.6) for some component $C' \subset L_0 \cap L_1$ and thus by induction we obtain an $\mathcal{O}^\vee$-orientation on $\mathcal{M}_m(p,C) \to C$ as promised. We have the isomorphism $|T_qW^u(q)| \otimes \mathcal{O}_q^\vee \cong (\mathcal{O}_q \otimes |T_qC/T_qW^s(q)|)^\vee$. Thus our choices fix an $\mathcal{O}$-coorientation on $W^s(q)$. Finally we obtain an orientation on $\mathcal{M}_m(p,C) \times_C W^s(q) = \mathcal{M}_m(p,q)$ by Lemma 9.1.3.

Lemma 10.2.7. *The orientations are coherent.*

Proof. As mentioned in Section 9.1 the orientation of $\mathcal{M}_m(p,q)_{[1]}$ induces an orientation on its boundary points, denoted $\partial\mathcal{M}_m(p,q)_{[1]}$. We define an orientation on $\mathcal{M}_\#(p,q)$ induced by $(-1)\cdot\mathcal{M}_m(p,q)_{[1]}$. Step 5 and Step 6 show that the orientation is well-defined. Step 1 to Step 4 shows that the orientations are coherent.

Step 1. We show $\mathcal{M}_0(p,r)_{[0]} \times \mathcal{M}_0(r,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_0(p,q)_{[1]}$

Given $(u,v) \in \mathcal{M}_0(p,r)_{[0]} \times \mathcal{M}_0(r,q)_{[0]}$. Pick an orientation of $\mathcal{O}_p$. By parallel transport along u and v we obtain an orientation of $\mathcal{O}_r$ and $\mathcal{O}_q$ respectively. By our choices we obtain orientations of $W^u(p)$, $W^u(r)$ and a coorientation of $W^s(q)$. We identify $\mathcal{M}_0(p,q)_{[1]}$ with the intersection $W^u_a(p) \cap W^s(q)$ where $a \in \mathbb{R}$ is a regular value of f with $f(p) > a > f(q)$. By Lemma 9.1.2 the identification is orientation preserving. By Lemma 8.8.1 we obtain a smooth map $R \mapsto w_R^- \in \mathcal{M}_0(p,q)_{[1]}$ such that orientation of $\partial_R w_R^-$ is $-\operatorname{sign} u \operatorname{sign} v$, $\lim_{R\to\infty} w_R^- = u$ and $\lim_{R\to\infty}\psi^{2R} w_R^- = v$.

Step 2. With $m \geq 1$. We show $\mathcal{M}_m(p,r)_{[0]} \times \mathcal{M}_0(r,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q)_{[1]}$.

Given $(u,v) \in \mathcal{M}_m(p,r)_{[0]} \times \mathcal{M}_0(r,q)_{[0]}$. We have $u \in \mathcal{M}_m(p,C)$ for some component $C \subset L_0 \cap L_1$. Identify an open neighborhood of u in $\mathcal{M}_m(p,C)$ with the image under the evaluation $\mathcal{M}_m(p,C) \to C$. Denote the image by W_-, which is submanifold in a neighborhood of $ev(u)$. The space W_- has an $\mathcal{O}^\vee$-orientation by construction. Pick an orientation of $\mathcal{O}_q$. We obtain a coorientation of $W^s(q)$ and an orientation of W_- for all points in $W_- \cap W^s(q)$ by parallel transport. By Lemma 9.1.2 the identification of an open subset of $\mathcal{M}_m(p,q)$ with $W_- \cap W^s(q)$ is orientation preserving. By Lemma 8.8.1 we obtain a smooth map $R \mapsto w_R^- \in \mathcal{M}_m(p,q)$ with the same properties as in last step.

Step 3. With $m \geq 1$. We show $\mathcal{M}_0(p,r)_{[0]} \times \mathcal{M}_m(r,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q)_{[1]}$

Given $(u,v) \in \mathcal{M}_0(p,r)_{[0]} \times \mathcal{M}_m(r,q)_{[0]}$. Abbreviate $\widetilde{\mathcal{M}}_m(C,q) := \widetilde{\mathcal{M}}_m(C,W^s(q))$ with quotient $\mathcal{M}_m(C,q)$ for some component $C \subset L_0 \cap L_1$ such that $v \in \mathcal{M}_m(C,q)$. We identify an open neighborhood of v in $\mathcal{M}_m(C,q)$ with the image under the evaluation map $\mathcal{M}_m(C,q) \to C$. Denoted the image by $W_+ \subset C$, which is a submanifold in a neighborhood of $ev(v)$. Then an open subset of $\mathcal{M}_m(p,q)$ is identified with $W^u(p) \cap W_+$. Similarly we identify $\mathcal{M}_m(r,q)$ with $W^u(r) \cap W_+$. By Lemma 8.8.1 we obtain $R \mapsto w_R \in W^u(p) \cap W_+$ such that $\lim_{R\to\infty} w_R = v$ and $\lim_{R\to\infty}\psi^{-2R} w_R = u$. It remains to check that the orientation of $\partial_R w_R$ is $-\operatorname{sign}(u)\operatorname{sign}(v)$.

Fix some $w = w_R$ for R sufficiently large. In the following we will often talk a little imprecisely about an orientation of the space $\mathcal{M}_m(p,q)$ or $\widetilde{\mathcal{M}}_m(p,q)$ where in fact we mean an orientation of an open neighborhood of w (or its lift) inside this space. We will not mention that every time.

By Theorem 9.3.6 and Lemma 9.1.3 construct an orientation of $\widetilde{\mathcal{M}}_m(p,q)$ as the fibre product for some connected components $C_0, C_1, \ldots, C_m$ chosen appropriately

$$W^u(p) \times_{C_0} (\widetilde{\mathcal{M}} \times \mathbb{R}) \times_{C_1} (\widetilde{\mathcal{M}} \times \mathbb{R}) \times_{C_2} \cdots \times_{C_{m-2}} (\widetilde{\mathcal{M}} \times \mathbb{R}) \times_{C_{m-1}} \widetilde{\mathcal{M}} \times_{C_m} W^s(q)\,, \quad (10.2.7)$$

in which $\widetilde{\mathcal{M}}$ denotes the space of J-holomorphic strips with obvious evaluation maps. We obtain an alternative orientation on the quotient $\mathcal{M}_m(p,q)$ via (9.1.8) and using the commutation rule (9.1.3) we conclude that this orientation differs from the orientation above by the action with $(-1)^{\Delta(w)}$ where

$$\Delta(w) := \Delta_1 + \Delta_2 + \cdots + \Delta_{m-1}, \qquad \Delta_k = \dim T_{(w_1,\ldots,w_k)}\mathcal{M}_k(p, C_k)\,.$$

By the dimension formula (cf. Lemma 10.2.1)

$$\Delta_k = \mu(p) + \mu(w_1) + \mu(w_2) + \cdots + \mu(w_k) - \frac{1}{2}\dim C_0 + \frac{1}{2}\dim C_k - 1\,.$$

Similarly we obtain another orientation of $\mathcal{M}_m(r, q)$ which agrees with the old orientation up to action with $(-1)^{\Delta(v)}$ where $\Delta(v) = \Delta_1(v) + \Delta_2(v) + \cdots + \Delta_{m-1}(v)$ and for $k = 1, \ldots, m-1$ we have

$$\Delta_k(v) = \mu(r) + \mu(v_1) + \cdots + \mu(v_k) - \frac{1}{2}\dim C_0 + \frac{1}{2}\dim C_k - 1\,.$$

Moreover we obtain an $\mathcal{O}$-coorientation of $\widetilde{\mathcal{M}}_m(C, q) \to C$ by writing $\widetilde{\mathcal{M}}_m(C, q)$ as fibre product (10.2.7) without the first factor. We obtain a canonical $\mathcal{O}$-coorientation on the quotient W_+ by (9.1.8). Let $\widetilde{\mathcal{M}}_m(p, q)'$ and $\mathcal{M}_m(p, q)'$ etc. be the space with orientation induced by (10.2.7). By associativity of the fibre product orientation (cf. Lemma 9.1.1) we conclude

$$\widetilde{\mathcal{M}}_m(p, q)' = W^u(p) \times_C \widetilde{\mathcal{M}}_m(C, q)\,.$$

Let $G = \mathbb{R}^m$ be the group of reparametrizations. By definition we have as oriented spaces $\widetilde{\mathcal{M}}_m(p, q)' = G \times \mathcal{M}_m(p, q)'$ and $\widetilde{\mathcal{M}}_m(C, q) = G \times W_+$. Hence

$$G \times \mathcal{M}_m(p, q)' = W^u(p) \times_C (G \times W_+)\,.$$

By commuting and canceling the factor G we obtain

$$(-1)^g \mathcal{M}_m(p, q)' = (-1)^{gw_+} W^u(p) \times_C W_+ = (-1)^{gw_+} W^u(p) \cap W_+\,,$$

in which we have used small letters to denote the dimensions of the spaces. Similarly we show

$$\mathcal{M}_m(r, q)' = (-1)^{gw_+} W^u(r) \cap W_+\,.$$

Pick an orientation of $\mathcal{O}_p$. By parallel transport along u we obtain an orientation of $\mathcal{O}_r$. By our choices we obtain an orientation on $W^u(p)$ and $W^u(r)$ as well as a coorientation of W_+ such that $\operatorname{sign} u = \varepsilon_0$, where ε_0 is the usual Morse trajectory orientation and $\operatorname{sign} v = (-1)^{\Delta(v)+gw_+}\varepsilon_1$ where ε_1 is the orientation of the point v as an element of the intersection $W^u(r) \cap W_+$. Moreover we conclude with Lemma 8.8.1 that the orientation of $\partial_R w_R$ is

$$(-1)^{\Delta(w)+gw_+ + g}\varepsilon_0\varepsilon_1 = (-1)^{\Delta(w)+2gw_+ + g + \Delta(v)} \operatorname{sign}(u)\operatorname{sign}(v)\,.$$

We have $\Delta_j(w) = \Delta_j(v) + 1$ for all $j = 1, \ldots, m-1$. Hence $\Delta(w) = \Delta(v) + m - 1$. Moreover $g = m$. Hence $\Delta(w) + \Delta(v) + g \equiv 1 \mod 2$. This shows the claim.

Step 4. With $m \geq 2$ and $k + \ell = m$ with $k \neq 0, m$. We show $\mathcal{M}_k(p,r)_{[0]} \times \mathcal{M}_\ell(r,q)_{[0]} \subset (-1) \cdot \partial\mathcal{M}_m(p,q)_{[1]}$.

Given $(u,v) \in \mathcal{M}_k(p,r)_{[0]} \times \mathcal{M}_\ell(r,q)_{[0]}$. We consider u and v as elements in $\mathcal{M}_k(p,C)$ and $\mathcal{M}_\ell(C,q)$ for some component $C \subset L_0 \cap L_1$ respectively. Identify open neighborhoods of u and v in $\mathcal{M}_k(p,C)$ and $\mathcal{M}_\ell(C,q)$ with submanifolds $W_- \subset C$ and $W_+ \subset C$ respectively using the evaluation map. Then an open subset of $\mathcal{M}_m(p,q)$ is identified with $W_- \times_\psi W_+$ (cf. equation (8.8.1)). By Lemma 8.8.1 there exists $R \mapsto w_R \in \mathcal{M}_m(p,q)$ such that $\lim_{R\to\infty} w_R = u$ and $\lim_{R\to\infty} \psi^{2R} w_R = v$. It remains to show that the orientation of $\partial_R w_R$ is $-\operatorname{sign}(u)\operatorname{sign}(v)$.

With alternative orientations as described in the last step we have

$$\widetilde{\mathcal{M}}_m(p,q)' = (\widetilde{\mathcal{M}}_k(p,C)' \times \mathbb{R}) \times_C \widetilde{\mathcal{M}}_\ell(C,q)' .$$

Let $G_- = \mathbb{R}^k$ (resp. $G_+ = \mathbb{R}^\ell$) be the group of reparametrizations of $\widetilde{\mathcal{M}}_k(p,C)$ (resp. $\widetilde{\mathcal{M}}_\ell(C,q)$). Pick an orientation on $\mathcal{O}_r$. We obtain an orientations of $\widetilde{\mathcal{M}}_k(p,C)$ and a coorientation of $\widetilde{\mathcal{M}}_\ell(C,q) \to C$. Hence an orientation on $W_- \subset C$ and a coorientation on $W_+ \subset C$ such that $\widetilde{\mathcal{M}}_k(p,C)' = G_- \times W_-$ and $\widetilde{\mathcal{M}}_\ell(C,q)' = G_+ \times W_+$. Thus (cf. equation (8.8.1))

$$(-1)^{g_+}\mathcal{M}_m(p,q)' = (-1)^{g_+ w_+}(W_- \times \mathbb{R}) \times_C W_+ = (-1)^{g_+ w_+ + w_-} W_- \times_\psi W_+ . \quad (10.2.8)$$

Similarly we conclude that

$$\mathcal{M}(p,r)' = W_- \cap W^s(r), \qquad \mathcal{M}(r,q)' = (-1)^{g_+ w_+} W^u(r) \cap W_+ .$$

Hence we have $\operatorname{sign} u = (-1)^{\Delta(u)}\varepsilon_0$ with ε_0 is the orientation of u as an element in the intersection $W_- \cap W^s(r)$ and $\operatorname{sign} v = (-1)^{\Delta(v)+gw_+}\varepsilon_1$ with ε_1 is the orientation of v as an element in the intersection $W^u(r) \cap W_+$. By Lemma 8.8.1 the orientation of $\partial_R w_R$ is

$$(-1)^{\Delta(w)+g_+ w_+ + w_- + g_+}\varepsilon_0\varepsilon_1 = (-1)^{\Delta(w)+\Delta(u)+\Delta(v)+w_- + g_+} \operatorname{sign} u \operatorname{sign} v .$$

We have $\Delta_j(u) = \Delta_j(w)$ for all $j = 1,\dots,k-1$. Moreover $\mu(r) = w_- = \Delta_k(w)$. By definition we have the recursive formula for all $j = 1,\dots,m-1$

$$\Delta_j(w) = \Delta_{j-1}(w) + \mu(w_j) + \frac{1}{2}\big(\dim T_{w_j(\infty)}L_0 \cap L_1 - \dim T_{w_j(-\infty)}L_0 \cap L_1\big)$$

and similarly for $\Delta_j(v)$. Since for all $j = 1,\dots,\ell-1$ the index of v_j is the same as w_{j+k} and the asymptotics lie on the same connected components we have $\Delta_j(v) = \Delta_{j+k}(w) - 1$. We conclude that $\Delta(w) = \Delta(u) + \Delta(v) + w_- + \ell - 1$ and

$$\Delta(w) + \Delta(u) + \Delta(v) + w_- + g_+ \equiv \ell - 1 + g_+ \equiv 1 \mod 2 .$$

This shows the claim.

Step 5. For $m \geq 2$. We show $\mathcal{M}^1_m(p,q)_{[0]} \subset (-1) \cdot \partial\mathcal{M}_m(p,q)_{[1]}$.

We have not yet constructed an orientation on $\mathcal{M}^1_m(p,q)$ so technically the statement does not make sense on the level of oriented spaces. But our orientation algorithm easily generalizes to give an orientation of the space $\mathcal{M}^1_m(p,q)$, if we leave out the $\mathbb{R}$-factor in (10.2.6) at the appropriate place. So we assume in the following that the space $\mathcal{M}^1_m(p,q)$ is oriented in that way.

Also similarly as above we construct an orientation on $\widetilde{\mathcal{M}}^1_m(p,q)$ via (10.2.7) in which we omit the $\mathbb{R}$-factor at the appropriate place. We obtain an alternative orientation on the quotient and we denote the space equipped with the alternative orientation by $\mathcal{M}^1_m(p,q)'$. Given $u \in \mathcal{M}^1_m(p,q)$. Lets say we have $u_k(\infty) = u_{k+1}(-\infty)$ for some $k = 1,\ldots,m-1$. Then the orientation as an element of $\mathcal{M}^1_m(p,q)'$ is changed with action of $(-1)^{\Delta(u)}$ where $\Delta(u) = \Delta_1(u) + \cdots + \Delta_{m-1}(u)$ and $\Delta_j(u) = \dim T_{(u_1,\ldots,u_j)}\mathcal{M}_j(p,C_j)$ if $j<k$, $\Delta_k(u) = 0$ and $\Delta_j(u) = \dim T_{(u_1,\ldots,u_j)}\mathcal{M}^1_j(p,C_j)$ if $j>k$. We identify $\mathcal{M}_k(p,C)$ as a submanifold $W_- \subset C$ and $\mathcal{M}_\ell(C,q)$ as a submanifold W_+ respectively. Hence by the same computation as (10.2.8) we have

$$\mathcal{M}_m(p,q)' = (-1)^{g_+ + g_+ w_+ + w_-} W_- \times_\psi W_+ \,.$$

Similarly we conclude

$$\mathcal{M}^1_m(p,q)' = (-1)^{g_+ w_+} W_- \cap W_+ \,.$$

We have $\operatorname{sign} u = (-1)^{g_+ w_+ + \Delta(u)}\varepsilon$, where ε is the sign of u seen as an element of $W_- \cap W_+$. A local construction gives a family $R \mapsto w_R = (R, w_R^-, w_R^+) \in W_- \times_\psi W_+$ with $w_0 = (0,u,u)$ (cf. Step 7 in the proof of Lemma 8.8.1). Moreover $\partial_R w_R$ induces an orientation on $W_- \times_\psi W_+$ which is $\varepsilon\, o$ where o is the canonical orientation on $W_- \times_\psi W_+$ (cf. Step 6 of the same proof). For $\mathcal{M}_m(p,q)$ the vector $\partial_R w_R$ gives an orientation that is changed by the action with

$$(-1)^{\Delta(w) + g_+ + w_- + \Delta(u)} \operatorname{sign}(u)\,.$$

We have $\Delta_j(w) = \Delta_j(u)$ for all $j = 1,\ldots,k-1$, $\Delta_k(w) = w_-$, $\Delta_k(u) = 0$ and $\Delta_j(w) = \Delta_j(u)+1$ for all $j = k+1,\ldots,m-1$. We conclude that $\Delta(w) = \Delta(u) + w_- + \ell - 2$. Since $g_+ = \ell$ and with the last equation the vector changes orientation by $\operatorname{sign}(u)$. But since this time the vector $\partial_R w_R$ points inward the induced boundary orientation is $-\operatorname{sign}(u)$.

Step 6. For $m \geq 1$. We have $\mathcal{M}^1_{m+1}(p,q) \subset \partial\mathcal{M}_m(p,q)_{[1]}$.

With oriented spaces as explained in the last step we have

$$\begin{aligned}
\widetilde{\mathcal{M}}^1_{m+1}(p,q)' &= (\widetilde{\mathcal{M}}_k(p,C_-)' \times \mathbb{R}) \times_{C_-} \widetilde{\mathcal{M}} \times_C \widetilde{\mathcal{M}} \times_{C_+} (\mathbb{R} \times \widetilde{\mathcal{M}}_\ell(C_+,q))\,,\\
\widetilde{\mathcal{M}}_m(p,q)' &= (\widetilde{\mathcal{M}}_k(p,C_-)' \times \mathbb{R}) \times_{C_-} \widetilde{\mathcal{M}} \times_{C_+} \times (\mathbb{R} \times \widetilde{\mathcal{M}}_\ell(C_+,q))\,,
\end{aligned}$$

with obvious evaluation maps. Let $G = \mathbb{R}^m$ (resp. $G^1 = \mathbb{R}^{m+1}$, $G_- = \mathbb{R}^k$ and $G_+ = \mathbb{R}^\ell$) be the reparametrization group of $\widetilde{\mathcal{M}}_m(p,q)$ (resp. $\widetilde{\mathcal{M}}^1_{m+1}(p,q)$, $\widetilde{\mathcal{M}}_k(p,C_-)$ and $\widetilde{\mathcal{M}}_\ell(C_+,q)$). Via the evaluation map we identify the quotients $\mathcal{M}_k(p,C_-)$ and $\mathcal{M}_\ell(C_+,q)$ with submanifolds $W'_- \subset C_-$ and $W'_+ \subset C_+$ respectively. Moreover we embed $W_- = \mathbb{R} \times W'_-$ (resp. $W_+ = \mathbb{R} \times W'_+$) into C_- (resp. C_+) via the Morse-flow. We have

$$\begin{aligned}
G^1 \times \mathcal{M}^1_{m+1}(p,q)' &= (-1)^{g_+ w_+} G_- \times (W_- \times_{C_-} \widetilde{\mathcal{M}} \times_C \widetilde{\mathcal{M}} \times_{C_+} W_+) \times G_+\,,\\
G \times \mathcal{M}_m(p,q)' &= (-1)^{g_+ w_+} G_- \times (W_- \times_{C_-} \widetilde{\mathcal{M}} \times_{C_+} W_+) \times G_+\,.
\end{aligned}$$

Denoting the quotients of the terms inside the parentheses on the right-hand side by $\mathcal{M}^1(W_-, W_+)$ and $\mathcal{M}(W_-, W_+)$ respectively, we conclude that

$$\mathcal{M}^1_{m+1}(p,q) = (-1)^{\Delta(u)+g_+w_+}\mathcal{M}^1(W_-, W_+)\,,$$
$$\mathcal{M}_m(p,q) = (-1)^{\Delta(w)+g_+w_++g_+}\mathcal{M}(W_-, W_+)\,.$$

By Theorem 8.1.1 we have $\mathcal{M}^1(W_-, W_+) = (-1)^{g_1} \cdot \partial\mathcal{M}(W_-, W_+)$ in which $g_1 = 1$ is the dimension of the group of reparametrizations acting on the second factor of $\widetilde{\mathcal{M}}^1(W_-, W_+)$. Thus

$$\mathcal{M}^1_{m+1}(p,q) \subset (-1)^{\Delta(u)+\Delta(w)+g_++g_1}\partial\mathcal{M}_m(p,q)\,.$$

Again we have $\Delta_j(u) = \Delta_j(w)$ for all $j = 1, \dots, k$ and $\Delta_k(u) = 0$. By the additivity axiom for the Viterbo index $\mu(u_k) + \mu(u_{k+1}) = \mu(w_k)$ and so $\Delta_j(u) = \Delta_{j-1}(w)$ for all $j = k+1, \dots, m$. So $\Delta(u) + \Delta(w) = m - k - 1 = \ell + 1 \mod 2$. This shows the claim because $g_+ = \ell$. □

10.3. Invariance

In this section we construct a canonical isomorphism

$$QH_*(L_0, L_1) \cong QH_*(L_0, \varphi_H(L_1))\,, \tag{10.3.1}$$

for any clean Hamiltonian function H. The construction of the isomorphism is well-known and goes along the lines of [61], [7], [33, Section 7] and [13]. The novelty we present here is the gluing analysis for cleanly intersecting Lagrangians and the orientations.

10.3.1. Perturbed pearl trajectories

Fix a vector field $X \in C^\infty(\Sigma, \mathrm{Vect}(M))$ such that $X(-s, \cdot) = X_-$ and $X(s, \cdot) = X_+$ for all $s \geq 1$. Moreover let $\mathcal{J}$ denote the space of almost complex structures $J \in C^\infty(\Sigma, \mathrm{End}(TM, \omega))$ such that $J(-s, \cdot) = J_-$ and $J(s, \cdot) = J_+$ for all $s \geq 2$. We abbreviate by $\mathcal{I}_-$ (resp. $\mathcal{I}_+$) the space of perturbed intersection points of H_- (resp. H_+), fix a Morse function f_- (resp. f_+) and denote the negative gradient flow by ψ_- (resp. ψ_+) with respect to some sufficiently generic metric. For any $m \geq 1$, $J \in \mathcal{J}$ and submanifolds $W_- \subset \mathcal{I}_-$, $W_+ \subset \mathcal{I}_+$ we define the space

$$\widetilde{\mathcal{M}}_m(W_-, W_+; J, X) := \{(u_1, \dots, u_m) \in C^\infty(\Sigma, M) \mid \text{a)} - \text{f)}\}\,,$$

as the space of tuples $u = (u_1, \dots, u_m)$ such that there exists $k \in \mathbb{N}$ and

a) the tuple $(u_1, \dots, u_{k-1})$ is a (J_-, X_-)-holomorphic pearl

b) the map u_k is a (J, X)-holomorphic strip,

c) the tuple $(u_{k+1}, \dots, u_m)$ is a (J_+, X_+)-holomorphic pearl

d) all strips have boundary in (L_0, L_1),

e) there exists $a_-, a_+ \geq 0$ such that $\psi_-^{a_-}(u_{k-1}(\infty)) = u_k(-\infty)$ and also $\psi_+^{a_+}(u_k(\infty)) = u_{k+1}(-\infty)$,

f) we have $u_1(-\infty) \in W_-$ and $u_m(\infty) \in W_+$.

The elements of $\widetilde{\mathcal{M}}_m(W_-, W_+; J, X)$ are called (J,X)*-holomorphic pearl trajectories with* m *cascades connecting* W_- *to* W_+. If in particular $W_- = W^u(p)$ and $W_+ = W^s(p)$ for critical points $p \in \operatorname{crit} f_-$ and $q \in \operatorname{crit} f_+$, we abbreviate

$$\widetilde{\mathcal{M}}_m(p, q; J, X) := \widetilde{\mathcal{M}}_m(W^u(p), W^s(q); J, X)\,,$$

with elements called (J,X)*-holomorphic pearl trajectories with* m *cascades connecting* p *to* q.

Transversality

For $m \in \mathbb{N}$ and any connected component $C_- \subset \mathcal{I}_-$ write $\widetilde{\mathcal{M}}_m^- := \widetilde{\mathcal{M}}_m(W_-, C_-; J_-, X_-)$ and denote the evaluation map

$$ev^- : \mathbb{R} \times \widetilde{\mathcal{M}}_m^- \to C_-, \qquad (a, u_1, \ldots, u_m) \mapsto \psi^a(u_m(\infty))\,.$$

Similarly given $C_+ \in \mathcal{I}_+$ let $\widetilde{\mathcal{M}}_m^+ := \widetilde{\mathcal{M}}_m(C_+, W_+; J_+, X_+)$ equipped with the evaluation map

$$ev^+ : \mathbb{R} \times \widetilde{\mathcal{M}}_m^+ \to C_+, \qquad (a, u_1, \ldots, u_m) \mapsto \psi^a(u_1(-\infty))\,.$$

Assume that W_- and W_+ are chosen such that J_- is regular for X_- and W_- and J_+ is regular for X_+ and W_+ (cf. Definition 7.3.3). We call J regular if it is regular for X and the above evaluation maps for any connected components, i.e. the fibre product

$$(\mathbb{R}_+ \times \widetilde{\mathcal{M}}_k^-) \times_{C_-} \widetilde{\mathcal{M}}(C_-, C_+; J, X) \times_{C_+} (\mathbb{R}_- \times \widetilde{\mathcal{M}}_\ell^+)$$

is cut-out transversely with $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$ for any connected components C_- and C_+ and integers k and ℓ. We show in Theorem 7.2.4 that the subspace of regular almost complex structures in $\mathcal{J}$ is comeager and we fix a regular $J \in \mathcal{J}$ for the rest of the section. Obviously the space $\widetilde{\mathcal{M}}_m(W_-, W_+; J, X)$ is the union of the fibre product over all connected components and integers k, ℓ such that $k + \ell = m - 1$ and every connected component of it is a manifold with corners. We conclude with a computation similarly as in the proof of Lemma 10.2.1 the dimension of the component in $\widetilde{\mathcal{M}}_m(W_-, W_+; J, X)$ contain the element u is

$$\dim W_- + \dim W_+ - \frac{1}{2}\dim C_0 - \frac{1}{2}\dim C_m + m - 1 + \mu(u)\,,$$

in which $C_0 \subset \mathcal{I}_-$ is the connected component containing $u_1(-\infty)$ and $C_m \subset \mathcal{I}_+$ is the connected component containing $u_m(\infty)$ respectively. There is a free $\mathbb{R}^{m-1}$-action on the space $\widetilde{\mathcal{M}}_m(p, q; J, X)$ given by reparametrizations and we denote the corresponding quotient by $\mathcal{M}_m(p,q;J,X)$. Since J is fixed, we omit the reference to it and write $\mathcal{M}_m(p,q)$ to denote the space $\mathcal{M}_m(p,q;J,X)$. Moreover for some $\ell, d \in \mathbb{N}_0$ we denote by $\mathcal{M}_m(p,q)_{[d]}$ the union of all d-dimensional components and $\mathcal{M}^\ell_m(p,q)$ the union of all elements of depth ℓ.

Compactness

A *broken (J,X)-holomorphic pearl trajectory connecting p to q of height k* is a tuple of pearl trajectories $(v_1,\dots,v_k)$ such that v_i is a (J_i,X_i)-holomorphic pearl connecting p_i to p_{i+1} for all $i=1,\dots,k-1$ for critical points $p=p_1,\dots,p_\ell \in \operatorname{crit} f_-$, $p_{\ell+1},\dots,p_m = q \in \operatorname{crit} f_+$ for some $\ell \le k$ and we have

$$(J_i,X_i) = \begin{cases} (J_-,X_-) & \text{if } i = 1,\dots,\ell-1 \\ (J,X) & \text{if } i = \ell \\ (J_+,X_+) & \text{if } i = \ell+1,\dots,k\,. \end{cases}$$

Completely analogous to the case without the Hamiltonian perturbations we define the notion of Floer-Gromov convergence, prove that the space $\mathcal{M}_m(p,q)_{[0]}$ is finite and $\mathcal{M}_m(p,q)_{[1]}$ is compact up to breaking of height two, i.e. the Floer-Gromov boundary consists of the union of

- $\mathcal{M}^1_m(p,q)_{[0]}$,
- $\mathcal{M}^1_{m+1}(p,q)_{[0]}$,
- $\mathcal{M}_k(p,r_-)_{[0]} \times \mathcal{M}_\ell(r_-,q)_{[0]}$ for all $r_- \in \operatorname{crit} f_-$ and such that $k+\ell = m$,
- $\mathcal{M}_k(p,r_+)_{[0]} \times \mathcal{M}_\ell(r_+,q)_{[0]}$ for all $r_+ \in \operatorname{crit} f_+$ and such that $k+\ell = m$.

Orientations

Let $\overline{\mathcal{M}}_m(p,q)_{[1]}$ be the Floer-Gromov compactification of $\mathcal{M}_m(p,q)_{[1]}$ and define the space

$$\mathcal{M}_\#(p,q) := \bigcup_{m\in\mathbb{N}} \overline{\mathcal{M}}_m(p,q)_{[1]}/\sim\,,$$

as disjoint union with double boundary points identified.

Definition 10.3.1. Orientations of the spaces $\mathcal{M}(p,q)_{[0]}$ for all $p \in \operatorname{crit} f_-$ and $q \in \operatorname{crit} f_+$ are *coherent*, if there exists an orientation on $\mathcal{M}_\#(p,q)$ such that its oriented boundary is given by the union of

- $(-1)\cdot \mathcal{M}_k(p,r_-)_{[0]} \times \mathcal{M}_\ell(r_-,q)_{[0]}$ for all $r_- \in \operatorname{crit} f_-$ and such that $k+\ell = m$,
- $\mathcal{M}_k(p,r_+)_{[0]} \times \mathcal{M}_\ell(r_+,q)_{[0]}$ for all $r_+ \in \operatorname{crit} f_+$ and such that $k+\ell = m$.

We construct orientations on the spaces $\mathcal{M}_m(p,q)$ completely analogous as described in the paragraph before Lemma 10.2.7. A little more precisely we construct the orientation recursively using the fibre-product (10.2.6) and replace $\widetilde{\mathcal{M}}_1(C',C)$ in the expression with $\widetilde{\mathcal{M}}_1(C',C;J_-,X_-)$, $\widetilde{\mathcal{M}}_1(C',C;J,X)$ or $\widetilde{\mathcal{M}}_1(C',C;J_+,X_+)$ appropriately.

Lemma 10.3.2. *The orientations on $\partial\mathcal{M}_\#(p,q)$ are coherent.*

Proof. The proof follows the steps from the proof of Lemma 10.2.7.

- To show $\mathcal{M}_m(p,r_+)_{[0]} \times \mathcal{M}_0(r_+,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q)_{[1]}$ proceed as Step 3.
- To show $\mathcal{M}_0(p,r_-)_{[0]} \times \mathcal{M}_m(r_-,q)_{[0]} \subset \partial\mathcal{M}_m(p,q)_{[1]}$ proceed as in Step 2 but now the dimension of the group of reparametrizations is $g = m-1$.
- To show $\mathcal{M}_k(p,r_-)_{[0]} \times \mathcal{M}_\ell(r_-,q)_{[0]} \subset \partial\mathcal{M}_m(p,q)_{[1]}$ for $r_- \in \operatorname{crit} f_-$ and $k \neq 0$ proceed as in Step 4 but now the dimension of the group of reparametrization on the right is $g_+ = \ell - 1$.
- To show $\mathcal{M}_k(p,r_+)_{[0]} \times \mathcal{M}_\ell(r_+,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q)_{[1]}$ for $r_+ \in \operatorname{crit} f_+$ and $\ell \neq m$ proceed as in Step 4 but now the dimension of the group of reparametrization on the right is $g_+ = \ell$.
- To show $\mathcal{M}_k(p,C) \times_C \mathcal{M}_\ell(C,q)_{[0]} \subset \partial\mathcal{M}_m(p,q)_{[1]}$ if $C \subset \mathcal{I}_-$ proceed as in Step 5 with $g_+ = \ell - 1$.
- To show $\mathcal{M}_k(p,C) \times_C \mathcal{M}_\ell(C,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q)_{[1]}$ if $C \subset \mathcal{I}_+$ proceed as in Step 5 with $g_+ = \ell$.
- To show $\mathcal{M}_{k+1}(p,C) \times_C \mathcal{M}_{\ell+1}(C,q)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q)_{[1]}$ for $C \subset \mathcal{I}_-$ proceed as in Step 6 with $g_+ + g_1 = \ell$.
- To show $\mathcal{M}_{k+1}(p,C) \times_C \mathcal{M}_{\ell+1}(C,q)_{[0]} \subset \partial\mathcal{M}_m(p,q)_{[1]}$ for $C \subset \mathcal{I}_+$ proceed as Step 6 with $g_+ + g_1 = \ell + 1$.

The above steps show that the orientation on $\mathcal{M}_\#(p,q)$ induced by $(-1)\cdot\mathcal{M}_m(p,q)_{[1]}$ is well-defined and shows that the orientations are coherent. □

10.3.2. Chain map

Define the Λ-linear homomorphism

$$
\begin{aligned}
C\chi_*(J,X) : CH_*(L_0,L_1^-) &\to CH_*(L_0,L_1^+) \\
p &\mapsto \sum\nolimits_{q\in\operatorname{crit} f_+} \sum\nolimits_{[u]\in\mathcal{M}(p,q)_{[0]}} \operatorname{sign}(u)\cdot q \otimes \lambda^{(|q|-|p|)/N}\,.
\end{aligned}
$$

Because the orientations are coherent the homomorphism $C\chi_*(J,X)$ is a chain map. To see that we have to show $C\chi_{*-1}\circ\partial_* = \partial_*\circ C\chi_*$. By definition the coefficient of $C\chi_{*-1}(\partial_* p) - \partial_*(C\chi_* p)$ in front of $q \in \operatorname{crit} f_+$ is given by

$$\sum \operatorname{sign} u_0 \operatorname{sign} u_1 - \operatorname{sign} v_0 \operatorname{sign} v_1\,,$$

with summation over all $(u_0, u_1) \in \mathcal{M}(p, r_-)_{[0]} \times \mathcal{M}(r_-, q)_{[0]}$ and $(v_0, v_1) \in \mathcal{M}(p, r_+)_{[0]} \times \mathcal{M}(r_+, q)_{[0]}$ and all critical points $r_- \in \operatorname{crit} f_-$ and $r_+ \in \operatorname{crit} f_+$. Since the summation agrees with the summation of the signs of the oriented boundary points of $\mathcal{M}_{\#}(p, q)$ it vanishes. Having established that $C\chi_*(J, X)$ is a chain map we denote the induced map on homology by

$$\chi_*(J, X) : QH_*(L_0, L_1^-) \to QH_*(L_0, L_1^+) \,. \tag{10.3.2}$$

Naturality

Lemma 10.3.3. *The homomorphism $\chi_*(J, X)$ does not depend on the choice of X and J.*

Corollary 10.3.4. *The map* (10.3.2) *is the identity if $L_1^- = L_1^+$, $J_- = J_+$ and $f_- = f_+$.*

Proof. By Lemma 10.3.3 we are free to choose X and J. Choose X and J which are $\mathbb{R}$-invariant. Then there is an additional free $\mathbb{R}$-action on the space of tuples $(u_1, \ldots, u_m)$ in $\mathcal{M}_m(p, q)$ with u_k non-constant. Thus such tuples are not counted in the definition of the morphism $\chi(J, X)$. This shows that $\chi(J, X)$ is the identity on chain level. □

Proof of Lemma 10.3.3. Let (X_a, J_a) and (X_b, J_b) be two different choices. Fix a map $X \in C^\infty([a, b] \times \Sigma, \operatorname{Vect}(M))$ such that $X(a, \cdot) = X_a$, $X(b, \cdot) = X_b$ and $X(R, \pm s, \cdot) = X_R(\pm s, \cdot) = X_\pm$ for all $s \geq 1$ and $R \in [a, b]$. We abbreviate by $\mathcal{J}$ the space of $J \in C^\infty([a, b] \times \Sigma, \operatorname{End}(TM, \omega))$ such that $J(a, \cdot) = J_a$, $J(b, \cdot) = J_b$ and $J(R, \pm s, \cdot) = J_R(\pm s, \cdot) = J_\pm$ for all $s \geq 2$ and $R \in [a, b]$. For critical points $p \in \operatorname{crit} f_-$, $q \in \operatorname{crit} f_+$ a natural number $m \in \mathbb{N}$ and $J \in \mathcal{J}$ we define the space

$$\widetilde{\mathcal{M}}_m(p, q; J, X) \,,$$

as the space of pairs (u, R) where $R \in [a, b]$ and u is a (J_R, X_R)-holomorphic pearl trajectory connecting p to q. The space $\widetilde{\mathcal{M}}_m(p, q; J, X)$ is equally defined as the union of the fibre products

$$(\mathbb{R}_+ \times \widetilde{\mathcal{M}}_{k-1}^- \times_{C_-} \widetilde{\mathcal{M}}_1(C_-, C_+; J, X) \times_{C_+} (\mathbb{R}_- \times \widetilde{\mathcal{M}}_\ell^+) \,,$$

over all possible connected components C_-, C_+ and with $k + \ell = m$. We say that J is *regular* if it is regular with respect to X (cf. Definition 7.2.2) and for any connected components and critical points the fibre product is cut-out transversely. We conclude by Theorem 7.2.4 that a generic $J \in \mathcal{J}$ is regular and we fix such a regular J. The components of the the space $\widetilde{\mathcal{M}}_m(p, q; J, X)$ are manifolds with corners. We compute the dimension of a component containing (R, u) to be

$$\mu(u) + \mu(p) - \mu(q) + m - \frac{1}{2} \dim C_0 + \frac{1}{2} \dim C_m \,, \tag{10.3.3}$$

in which again $C_0 \subset \mathcal{I}_-$ (resp. $C_m \subset \mathcal{I}_+$) denotes the connected component containing p (resp. q). The group of reparametrizations is $\mathbb{R}^{m-1}$ and we denote the quotient by $\mathcal{M}_m(p, q; J, X)$. With the same arguments and notations as in the sections above we show that $\mathcal{M}_m(p,q;J,X)_{[0]}$ is finite and the Floer-Gromov boundary of $\mathcal{M}_m(p,q;J,X)_{[1]}$ is given by broken trajectories of height at most two. Moreover we obtain orientations on the spaces constructed recursively using the fibre product (10.2.6) in which $\widetilde{\mathcal{M}}(C',C)$ is given by $\widetilde{\mathcal{M}}(C',C;J_-,X_-)$, $\widetilde{\mathcal{M}}(C',C;J,X)$ or $\widetilde{\mathcal{M}}(C',C';J_+,X_+)$ appropriately. Let $\mathcal{M}_\#(p,q;J,X)$ be the disjoint union of all the Floer-Gromov compactifications of $\mathcal{M}_m(p,q;J,X)_{[1]}$ with double boundary points identified. We say that orientations on the boundary of $\mathcal{M}_\#(p,q;J,X)$ are *coherent*, if there exists an orientation on $\mathcal{M}_\#(p,q;J,X)$ such that its oriented boundary is given by the union of

- $(-1)\cdot\mathcal{M}_m(p,q;J_a,X_a)_{[0]}$,
- $\mathcal{M}_m(p,q;J_b,X_b)_{[0]}$,
- $(-1)\cdot\mathcal{M}_k(p,r;J_-,X_-)_{[0]}\times\mathcal{M}_\ell(r,q;J,X)_{[0]}$ for all $k+\ell=m$ and $r\in\operatorname{crit} f_-$,
- $\mathcal{M}_k(p,r;J,X)_{[0]}\times\mathcal{M}_\ell(r,q;J_+,X_+)_{[0]}$ for all $k+\ell=m$ and $r\in\operatorname{crit} f_+$.

If we show that the orientations are coherent we are done because, then we define the Λ-linear homomorphism

$$\begin{gathered}\Theta_*:CH_*(L_0,L_1^-)\to CH_{*+1}(L_0,L_1^+)\\ p\mapsto\sum_{q\in\operatorname{crit} f_+}\ \sum_{[u]\in\mathcal{M}(p,q;J,X)_{[0]}}\operatorname{sign}(u)\cdot q\otimes\lambda^{(|q|-|p|-1)/N}\,.\end{gathered}$$

Now since the orientations are coherent we conclude that Θ_* is a chain homotopy from $C\chi_*(J_a,X_a)$ to $C\chi_*(J_b,X_b)$, i.e.

$$\partial_*\circ\Theta_*-\Theta_{*-1}\circ\partial_*=C\chi_*(J_a,X_a)-C\chi_*(J_b,X_b)\,,$$

which implies that the induced morphism on homology agree. □

Lemma 10.3.5. *The orientations on $\partial\mathcal{M}_\#(p,q;J,X)$ are coherent.*

Proof. Abbreviate $\partial\mathcal{M}:=\partial\mathcal{M}_m(p,q;J,X)_{[1]}$. The proof of coherence follows the steps from the proof of Lemma 10.2.7.

- To show $\mathcal{M}_m(p,r;J,X)_{[0]}\times\mathcal{M}_0(r,q;J_+,X_+)_{[0]}\subset(-1)\cdot\partial\mathcal{M}$ proceed as in Step 2,
- To show $\mathcal{M}_0(p,r;J_-,X_-)_{[0]}\times\mathcal{M}_m(r,q;J,x)_{[0]}\subset\partial\mathcal{M}$ proceed as in Step 3. We have $g=m-1$.
- To show $\mathcal{M}_k(p,r;J_-,X_-)_{[0]}\times\mathcal{M}_\ell(r,q;J,X)_{[0]}\subset\partial\mathcal{M}$ proceed as in Step 4. We have $g_+=\ell-1$.
- To show $\mathcal{M}_k(p,r;J,X)_{[0]}\times\mathcal{M}_\ell(r,q;J_+,X_+)_{[0]}\subset(-1)\cdot\partial\mathcal{M}$ proceed as in Step 4. We have $g_+=\ell$.

- We show $\mathcal{M}_m(p,q;J_a,X_a)_{[0]} \subset (-1)\cdot\partial\mathcal{M}$. For appropriate submanifolds W_- and W_+ we have $\mathcal{M}_m(p,q;J,X) = \mathcal{M}(W_-,W_+;J,X)$ as well as $\mathcal{M}_m(p,q;J_a,X_a) = \mathcal{M}(W_-,W_+;J_a,X_a)$. Fix some element $(a,u) \in \mathcal{M}(W_-,W_+;J_a,X_a)$. By the implicit function Theorem we obtain $R \mapsto (R, w_R)$ with $w_a = u$. Using notation as the proof of Proposition 8.7.4 in case (C) we conclude that $\operatorname{sign}(u) = \operatorname{sign} D'_R$ which equals the orientation induced by the vector $(1, \partial_R w_R) \in \ker \widehat{D}_R$. Since $(1,\partial_R w_R)$ points inward the orientation of the boundary point is $-\operatorname{sign}(u)$.

- To show $\mathcal{M}_m(p,q;J_b,X_b)_{[0]} \subset \partial\mathcal{M}_m$ proceed as above but this time the vector points outward.

- To show $\mathcal{M}_k(p,C;J_-,X_-) \times_C \mathcal{M}_\ell(C,q;J,X)_{[0]} \subset \partial\mathcal{M}_m$ with $k+\ell = m$ proceed as in Step 5 with $g_+ = \ell - 1$,

- To show $\mathcal{M}_k(p,C;J,X) \times_C \mathcal{M}_\ell(C,q;J_+,X_+)_{[0]} \subset (-1)\cdot\partial\mathcal{M}_m(p,q;J,X)_{[1]}$ proceed as in Step 5 with $g_+ = \ell$,

- To show $\mathcal{M}_{k+1}(p,C;J_-,X_-) \times_C \mathcal{M}_{\ell+1}(C,q;J,X)_{[0]} \subset (-1)\cdot\partial\mathcal{M}$ with $\ell + k = m-1$ proceed as in Step 6 with $g_+ + g_1 = \ell$

- To show $\mathcal{M}_{k+1}(p,C;J,X) \times_C \mathcal{M}_{\ell+1}(C,q;J_+,X_+)_{[0]} \subset \partial\mathcal{M}$ with $\ell + k = m-1$ proceed as in Step 6 with $g_+ + g_1 = \ell + 1$.

This shows the claim by putting the orientation on $\mathcal{M}_\#(p,q;J,X)$ induced by $(-1)\cdot \mathcal{M}_m(p,q;J,X)_{[1]}$ for any $m \in \mathbb{N}$. $\square$

Functoriality

Lemma 10.3.6. *We show that the map* (10.3.2) *is functorial, i.e. the map defined in* (10.3.2) *gives rise to a commutative triangle*

$$\begin{tikzcd} QH(L_0,L_1) \arrow[r] \arrow[rr, bend left] & QH(L_0,L_2) \arrow[r] & QH(L_0,L_3)\,, \end{tikzcd}$$

for any Lagrangians L_0, L_1, L_2 and L_3 such that L_1, L_2 and L_3 are Hamiltonian isotopic.

Corollary 10.3.7. *The map* (10.3.2) *is an isomorphism.*

Proof. The map $QH(L_0,L_1^+) \to QH(L_0,L_1^-)$ is inverse to $QH(L_0,L_1^-) \to QH(L_0,L_1^+)$ because by Lemma 10.3.6 their composition is $QH(L_0,L_1^-) \to QH(L_0,L_1^-)$ which by Corollary 10.3.4 is the identity. $\square$

Proof of Lemma 10.3.6. Suppose that $L_1 = \varphi_{H_-}(L)$, $L_2 = \varphi_H(L)$ and $L_3 = \varphi_{H_+}(L)$ for some Hamiltonians H_-, H and H_+ and a fixed Lagrangian L. We abbreviate the perturbed intersection points $\mathcal{I}_- := \mathcal{I}_{H_-}(L_0,L)$, $\mathcal{I} := \mathcal{I}_H(L_0,L)$ and $\mathcal{I}_+ := \mathcal{I}_{H_+}(L_0,L)$. Fix vector fields $X_0, X_1 \in C^\infty(\Sigma, \operatorname{Vect}(M))$ such that $X_0(-s,\cdot) = X_- := X_{H_-}$, $X_0(s,\cdot) = X_1(-s,\cdot) = X := X_H$ and $X_1(s,\cdot) = X_+ := X_{H_+}$ for all $s \geq 1$. Denote the Morse functions f_-, f and f_+ and paths of almost complex structures J_∞^-, J_∞ and J_∞^+ with respect to which the pearl homology groups are defined. Denote by $\mathcal{J}$ be the space pairs $(J_0, J_1) \subset C^\infty(\Sigma, \mathrm{End}(TM, \omega))$ where $J_0(-s, \cdot) = J_\infty^-$, $J_0(s, \cdot) = J_1(-s, \cdot) = J_\infty$ and $J_1(s, \cdot) = J_\infty^+$ for all $s \geq 2$.

Given some $J = (J_0, J_1) \in \mathcal{J}$ critical points $p \in \mathrm{crit}\, f_-$, $q \in \mathrm{crit}\, f_+$ and a number $m \in \mathbb{N}$ with $m \geq 2$ we denote by

$$\widetilde{\mathcal{N}}_m(p, q; J, X)\,,$$

the space of tuples $u = (u_1, \ldots, u_m)$ such that for some $1 \leq k \leq m-1$

a) the tuple $(u_1, \ldots, u_k)$ is a (J_0, X_0)-holomorphic pearl trajectory,

b) the tuple $(u_{k+1}, \ldots, u_m)$ is a (J_1, X_1)-holomorphic pearl trajectory,

c) all trips have boundary in (L_0, L),

d) there exists $a \geq 0$ such that $\psi^a(u_k(\infty)) = u_{k+1}(-\infty)$,

e) we have $W^u(p) \in u_1(-\infty)$ and $W^s(q) \in u_m(\infty)$.

As usual we conclude that for generic J each component of these spaces are manifolds with corners. The component containing the element $u \in \widetilde{\mathcal{N}}_m(p, q; J, X)$ has dimension

$$\mu(u) + \mu(p) - \mu(q) + \frac{1}{2}\dim C_0 - \frac{1}{2}\dim C_m + m - 1\,.$$

The group of reparametrizations has dimension $m-2$ and acts freely. With usual notations as explained in the last sections we show that the quotient $\mathcal{N}_m(p, q; J, X)_{[0]}$ is finite and the Floer-Gromov boundary of $\mathcal{N}_m(p, q; J, X)_{[1]}$ is given by breaking of height at most two. Also the spaces are oriented using the algorithm given in the paragraph before Lemma 10.2.7. We denote by $\mathcal{N}_\#(p, q; J, X)$ the disjoint union of the Floer-Gromov compactification of $\mathcal{N}_m(p, q; J, X)$ over all $m \in \mathbb{N}_2$ with double boundary points identified. We show as above that there exists an orientation on $\mathcal{N}_\#(p, q; J, X)$ such that its oriented boundary is given by

- $(-1) \cdot \mathcal{K}_m(p, q; J, X)_{[0]}$ (see definition below)
- $\mathcal{N}_k(p, r; J, X)_{[0]} \times \mathcal{M}_\ell(r, q; J_\infty^+, X_+)_{[0]}$ for $r \in \mathrm{crit}\, f_+$ and $k + \ell = m$,
- $(-1) \cdot \mathcal{M}_k(p, r; J_\infty^-, X_-)_{[0]} \times \mathcal{N}_\ell(r, q; J, X)_{[0]}$ for $r \in \mathrm{crit}\, f_-$ and $k + \ell = m$,
- $\mathcal{M}_k(p, r; J_0, X_0)_{[0]} \times \mathcal{M}_\ell(r, q; J_1, X_1)_{[0]}$ for $r \in \mathrm{crit}\, f$ and $k + \ell = m$.

Here $\mathcal{K}_m(p, q; J, X) \subset \mathcal{N}_m^1(p, q; J, X)$ is the subspace of equivalence classes of tuples $(u_1, \ldots, u_m)$ such that u_k is (J_0, X_0)-holomorphic, u_{k+1} is (J_1, X_1)-holomorphic and $u_k(\infty) = u_{k+1}(-\infty)$ for some $k \in \mathbb{N}$.

Remark 10.3.8. The space $\mathcal{K}_m(p, q; J, X)_{[0]}$ appears as a boundary of the glued space $\mathcal{N}_\#(p, q; J, X)$ because if we glue elements in $\mathcal{K}_m(p, q; J, X)_{[0]}$ we do not obtain elements inside the space $\mathcal{N}_{m-1}(p, q; J, X)_{[1]}$.

We need to define another moduli space. For critical points $p \in \operatorname{crit} f_-$, $q \in \operatorname{crit} f_+$ and a number $m \in \mathbb{N}$ we denote by

$$\widetilde{\mathcal{M}}_m(p,q;J,X)\,,$$

the space of pairs (u,R) such that $R \geq 2$ and u is a (J_R, X_R)-holomorphic pearl trajectories connecting p_- to p_+ with glued structures $X_R = X_0 \#_R X_1$ and $J_R = J_0 \#_R J_1$ as defined in (7.2.4) and (7.2.5) respectively. By Theorem 7.2.4 we conclude that for a generic J each connected component of the spaces is a manifold with corners. The dimension of a component containing u is (10.3.3). The group of reparametrizations has dimension $m-1$ and acts freely. The quotient $\mathcal{M}_m(p,q;J,X)_{[0]}$ is finite and the Floer-Gromov boundary of $\mathcal{M}_m(p,q;J,X)_{[1]}$ is give by broken trajectories of height at most two. There also exists an orientation on the spaces as explained in the paragraph before Lemma 10.2.7. Let $\mathcal{M}_\#(p,q;J,X)$ denote the union of all $\mathcal{M}_m(p,q;J,X)_{[1]}$ over $m \in \mathbb{N}$ with double boundary points identified. We show as above there exists an orientation on $\mathcal{M}_\#(p,q;J,X)$ such that its oriented boundary is given by

- $(-1)\cdot \mathcal{M}_k(p,r;J^-_\infty,X_-)_{[0]} \times \mathcal{M}_\ell(r,q;J,X)_{[0]}$ for all $r \in \operatorname{crit} f_-$ and $k+\ell = m$,
- $\mathcal{M}_k(p,r;J,X)_{[0]} \times \mathcal{M}_\ell(r,q;J^+_\infty,X_+)_{[0]}$ for all $r \in \operatorname{crit} f_+$ and $k+\ell = m$,
- $(-1)\cdot \mathcal{M}_m(p,q;J_R,X_R)_{[0]}$ with $R=2$,
- $\mathcal{K}_m(p,q;J,X)_{[0]}$ (which appears considering sequences (u_ν,R_ν) with $R_\nu \to \infty$).

We define the Λ-linear homomorphism

$$\begin{aligned}\Theta_* : CH_*(L_0,L_1) &\to CH_*(L_0,L_3)\\ p &\mapsto \sum_{q\in \operatorname{crit} f_+}\sum_{[u]} \operatorname{sign} u \cdot q \otimes \lambda^{|q|-|p|-1}\,,\end{aligned}$$

in which the second summation is over all elements $[u]$ in the disjoint union

$$\mathcal{N}(p,q;J,X)_{[0]} \sqcup \mathcal{M}(p,q;J,X)_{[0]}\,.$$

If the orientations on $\partial\mathcal{N}_\#(p,q;J,X)$ and $\partial\mathcal{M}_\#(p,q;J,X)$ are coherent, we conclude that Θ_* is a chain-homotopy from the composition $\chi(J_1,X_1)\circ\chi(J_0,X_0)$ to $\chi(J_R,X_R)$ with $R=2$. Note that the boundary components $\mathcal{K}_m(p,q;J,X)$ appears in both spaces but with opposite signs. □

10.4. Spectral sequences

In this section we prove Theorem 2.3.2. Recall that a spectral sequence is a sequence of complexes

$$(E^1_*,\partial),(E^2_*,\partial),\dots$$

such that $E_*^{r+1} \cong \ker \partial^r / \operatorname{im} \partial^r$ for all $r \in \mathbb{N}$. We say that the spectral sequence $(E_*^r, \partial^r)_{r\in\mathbb{N}}$ *collapses (at page r_0)* if there exists $r_0 \in \mathbb{N}$ such that $\partial^r = 0$ for all $r \geq r_0$. In that case we have $E_*^{r+1} \cong E_*^r$ for all $r \geq r_0$ and we define $E_*^\infty := E_*^{r_0}$. We say that a spectral sequence *converges* to the graded module H_* if there exists a filtration $\mathcal{F}$ on H_* such that $E_*^\infty \cong \bigoplus_p \mathcal{F}^p H_* / \mathcal{F}^{p-1} H_*$. If H_* is a vector space this always implies that $E_*^\infty \cong H_*$ although the isomorphism is not canonical. The spectral sequence is *bigraded* if there exists a decomposition $E_*^r = \bigoplus_{k+\ell=*} E_{k,\ell}^r$ for all $r \in \mathbb{N}$ and the boundary operator ∂^r has *degree (i,j)* if $\partial^r(E_{k,\ell}^r) \subset E_{k+i,\ell+j}^r$. We abbreviate a bigraded spectral sequence by E_{**}^*. For more details see [51].

Let N denote the minimal Maslov number of the pair (L_0, L_1) and τ be the monotonicity constant. As explained in Section 2.3 we decompose $L_0 \cap L_1$ into connected components $C_1, \ldots, C_k$ and choose maps $u_j : [-1,1] \times [0,1] \to M$, $u(s,\cdot) \in \mathcal{P}(L_0, L_1)$ such that $u_j(-1) = x_1$ and $u_j(1) = x_j \in C_j$. By concatenating to u_j with path we assume without loss of generality that the caps u_p for all critical points $p \in \operatorname{crit} f$ we have caps u_p that satisfy

$$\mathcal{A}(u_p) = \int u_p^* \omega \in [0, \tau N)\,. \tag{10.4.1}$$

We call a pearl trajectory u connecting p to q *local* if $\mu(u) = \mu(u_p) - \mu(u_q)$. Moreover we define the *local pearl chain complex* $CH_*^{\mathrm{loc}}(L_0, L_1)$ as the free A-module over all critical points $p \in \operatorname{crit} f$ graded by (10.1.1) and equipped with the boundary operator (10.1.2) without the λ factor and summation only over local trajectories. The next lemma shows that local pearl homology, denoted $QH^{\mathrm{loc}}(L_0, L_1)$, is well-defined.

Lemma 10.4.1. *Let (u^ν) be a sequence of local pearl trajectories Floer-Gromov converging to the broken trajectory $v = (v_1, \ldots, v_k)$. Then v_i is local for all $i = 1, \ldots, k$.*

Proof. Let $\bar{v}_i$ be cap of $v_i(\infty)$ with $0 \leq \int \bar{v}_i^* \omega < \tau N$ for all $i = 1, \ldots, k$ and $\bar{v}_0$ be a cap of $v_1(-\infty)$ with $0 \leq \int \bar{v}_0^* \omega < \tau N$. Define $k_i := \mu(v_i) + \mu(\bar{v}_{i-1}) - \mu(\bar{v}_i)$ for all $i = 1, \ldots, k$. We have to show that k_i vanishes for all $i = 1, \ldots, k$. Let m_i denote the number of cascades in v_i and assume without loss of generality that $m_i \geq 1$ since otherwise v_i is local by definition. By the integer axiom we have $k_i \in \mathbb{Z}$ and by monotonicity

$$\tau k_i N = \sum_{j=1}^{m_i} \int v_{i,j}^* \omega + \int \bar{v}_{i-1}^* \omega - \int \bar{v}_i^* \omega\,.$$

Due to the energy condition on the caps and the fact that v_i consists of holomorphic strips we have $\tau k_i N > -\tau N$. Hence $k_i \geq 0$. Again by monotonicity and Floer-Gromov convergence we have

$$\begin{aligned} \tau N \sum_{i=1}^{k} k_i &= E(v) + \int \bar{v}_0^* \omega - \int \bar{v}_k^* \omega \\ &= \lim_{\nu\to\infty} E(u^\nu) + \int \bar{v}_0^* \omega - \int \bar{v}_k^* \omega \\ &= \tau \Big(\lim_{\nu\to\infty} \mu_{\mathrm{Vit}}(u^\nu) + \mu_{\mathrm{Vit}}(\bar{v}_0) - \mu_{\mathrm{Vit}}(\bar{v}_k) \Big) = 0\,. \end{aligned}$$

This shows that $\sum_{i=1}^{k} k_i = 0$. Since all k_i are non-negative, this shows that $k_i = 0$ for all $i = 1, \dots, k$. $\square$

Proof of Theorem 2.3.2. The proof is motivated by [10, proof of Theorem 5.2.A]. Let $(C_*, \partial) = (C_*(f) \otimes \Lambda, \partial)$ denote the pearl-complex. A spectral sequence is canonically determined by an *increasing filtration*, i.e. a sequence of subcomplexes $(\mathcal{F}^k C_*)_{k\in\mathbb{Z}}$ such that

$$\cdots \subset \mathcal{F}^{k-1} C_* \subset \mathcal{F}^k C_* \subset \cdots \subset C_*, \qquad k \in \mathbb{Z} \,. \tag{10.4.2}$$

We construct a filtration by the degree of the Novikov variable. More precisely for every $k \in \mathbb{Z}$ we define the free A-module

$$\mathcal{F}^k C_* := \langle p \otimes \lambda^\ell \mid \ell \geq -k \rangle \,.$$

Clearly the sequence $(\mathcal{F}^k C_*)_{k\in\mathbb{Z}}$ satisfies (10.4.2). To show that $(\mathcal{F}^k C_*)_{k\in\mathbb{Z}}$ is a filtration, it remains to show that the modules are subcomplexes, i.e. $\partial \mathcal{F}^k C_* \subset \mathcal{F}^k C_*$ for all $k \in \mathbb{Z}$. By Λ-linearity of the boundary operator ∂ it suffices to check this for $k = 0$. Moreover it suffices to check this on generators of the form $p \otimes \mathbb{1}$ for some critical point p. Assume that the coefficient for ∂p in front of $q \otimes \lambda^\ell$ is not zero. We need to show that $\ell \geq 0$. By definition there exists a rigid trajectory u connecting the critical points p to q. We have two cases. In the first case u has zero cascades. Then necessarily p and q lie on the same connected component and $\ell N = |p| - |q| + 1 = \mu(p) - \mu(q) + 1 = 0$. Hence $\ell = 0$ and we are finished. In the second case $u = (u_1, \dots, u_m)$ has at least one cascade. Then by the dimension formula of Lemma 10.2.1 and the definition of the grading (10.1.1) we have with connected components C_- and C_+ of p and q respectively

$$\begin{aligned} m &= \sum_{j=1}^{m} \mu(u_j) + \mu(p) - \mu(q) - 1/2 \dim C_- + 1/2 \dim C_+ + m - 1 \\ &= \sum_{j=1}^{m} \mu(u_j) + \mu(u_p) - \mu(u_q) + |p| - |q| - 1 + m \\ &= \sum_{j=1}^{m} \mu(u_j) + \mu(u_p) - \mu(u_q) - \ell N + m \,. \end{aligned}$$

By monotonicity we have

$$\tau \ell N = \sum_{j=1}^{m} \int u_j^* \omega + \int u_p^* \omega - \int u_q^* \omega \,.$$

Since u_j is J-holomorphic $\int u_j^*\omega > 0$ for all j and since by our choice (10.4.1) we have $\int u_p^*\omega \geq 0$ and $\int u_q^*\omega > -\tau N$ we obtain $\tau \ell N > -\tau N$. This shows that $\ell \geq 0$ and that we have truly defined a filtration.

We claim the filtration is *bounded*, i.e. for every $m \in \mathbb{Z}$ there exists $k_-, k_+ \in \mathbb{Z}$ such that

$$0 = \mathcal{F}^{k_-} C_m \subset \mathcal{F}^{k_-+1} C_m \subset \cdots \subset \mathcal{F}^{k_+-1} C_m \subset \mathcal{F}^{k_+} C_m = C_m \,,$$

where $\mathcal{F}^k C_m := \langle p \otimes \lambda^\ell \mid |p| - \ell N = m,\ \ell \geq -k \rangle$. Indeed, define the integers $k_- := \lfloor (m - \overline{m})/N \rfloor - 1$ and $k_+ := \lceil (m - \underline{m})/N \rceil$, where $\underline{m} := \min_{p \in \operatorname{crit} f} |p|$ and $\overline{m} := \max_{p \in \operatorname{crit} f} |p|$. For every $p \otimes \lambda^\ell \in C_m$ we have

$$m = |p| - \ell N \geq \underline{m} - \ell N \Longrightarrow \ell \geq (\underline{m} - m)/N \geq -k_+ \ .$$

Hence $p \otimes \lambda^\ell \in \mathcal{F}^{k_+} C_m$ and this shows $\mathcal{F}^{k_+} C_m = C_m$. On the other hand arguing indirectly assume that there exists $p \otimes \lambda^\ell \in \mathcal{F}^{k_-} C_m$ then

$$m = |p| - \ell N \leq \overline{m} - \ell N \leq \overline{m} + k_- N \ \Longrightarrow \ (m - \overline{m})/N \leq k_- \ .$$

This gives the contradiction $k_- = \lfloor (m - \overline{m})/N \rfloor - 1 < (m - \overline{m})/N \leq k_-$ and shows $\mathcal{F}^{k_-} C_m = \{0\}$. We have deduced that the filtration is bounded.

The rest of the proof follows from standard algebraic arguments. Our main reference here is [51]. In particular the next result is valid for any complex (C_*, ∂) equipped with a bounded increasing filtration $\mathcal{F}$ and a boundary operator of degree -1. For every $p, q \in \mathbb{Z}, r \in \mathbb{N}$ we define

$$\begin{aligned} Z^r_{p,q} &:= \mathcal{F}^p C_{p+q} \cap \partial^{-1} \mathcal{F}^{p-r} C_{p+q-1}, \\ B^r_{p,q} &:= \mathcal{F}^p C_{p+q} \cap \partial \mathcal{F}^{p+r} C_{p+q+1}, \\ E^r_{p,q} &:= Z^r_{p,q} / (Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}) \ . \end{aligned}$$

A simple computation shows $\partial Z^r_{p,q} = B^r_{p-r,q+r-1} \subset Z^r_{p-r,q+r-1}$ and that ∂ induces a morphism

$$\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1} \ . \tag{10.4.3}$$

The proof of [51, Theorem 2.6] adapted to this setting shows that we obtain a spectral sequence and moreover we have the isomorphisms

a) $E^{r+1}_{p,q} \cong \ker \big(\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1} \big) /\ \operatorname{im} \big(\partial^r : E^r_{p+r,q-r+1} \to E^r_{p,q} \big)$,

b) $E^1_{p,q} \cong H_{p+q}(\mathcal{F}^p C_* / F^{p-1} C_*, [\partial])$,

c) $E^\infty_{p,q} \cong \mathcal{F}^p H_{p+q}(C_*) / \mathcal{F}^{p-1} H_{p+q}(C_*)$,

where $\mathcal{F}^p H_{p+q}(C_*) := \operatorname{im} \big(H_{p+q}(\mathcal{F}^p C_*) \to H_{p+q}(C_*) \big)$ and $E^\infty_{p,q}$ denotes $E^r_{p,q}$ with sufficiently large r. Coming back to our specific case, consider the index transformation

$$\tilde{E}^r_{k,\ell} := \begin{cases} E^r_{k/N, \ell + (N-1)k/N} & \text{if } k \in N\mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then by (10.4.3) we have $\partial^r : \tilde{E}^r_{k,\ell} \to \tilde{E}^r_{k-Nr, \ell+Nr-1}$. We interpret that as the rN-th boundary operator setting all other boundary operators to zero. This shows that $\tilde{E}^*_{**}$ is a homological spectral sequence (i.e. the r-th boundary operator has degree $(-r, r-1)$). To obtain the first page of $\tilde{E}^*_{**}$ we use b) and compute

$$\begin{aligned} E^1_{p,q} = H_{p+q}(\mathcal{F}^p C_*/\mathcal{F}^{p-1}C_*, [\partial]) &\cong H_{p+q}(C_*(f) \otimes \langle \lambda^{-p} \rangle, \partial_0 \otimes \mathbb{1}) \\ &\cong H_{p+q-pN}(C_*(f), \partial_0) \otimes \langle \lambda^{-p} \rangle \ , \end{aligned}$$

where $\partial_0 : C_*(f) \to C_{*-1}(f)$ is precisely the boundary operator of the local pearl complex. Hence $E^1_{p,q} \cong QH^{\mathrm{loc}}_{q+(1-N)p} \otimes \langle \lambda^{-p} \rangle \iff \tilde{E}^1_{k,\ell} = QH^{\mathrm{loc}}_{\ell} \otimes \langle \lambda^{-p} \rangle$. This shows the statement (ii) of Theorem 2.3.2.

We show statement (iii) of Theorem 2.3.2. Abbreviate $H_* := QH_*(L_0, L_1)$. By invariance we have $QH_*(L_0, L_1) \cong HF_*(L_0, L_1)$ (cf. equation (10.3.1)). By c) and we obtain an isomorphism of $E^\infty_* \cong \bigoplus_p \mathcal{F}^p H_*/\mathcal{F}^{p-1}H_*$. It remains to show that the graded module is isomorphic to H_* even in the case when the ground ring A is not a field. We define the valuation

$$\nu : \ker \partial \to \mathbb{Z} \cup \{\infty\}, \qquad z \mapsto \begin{cases} \infty & \text{if } z = 0 \\ \min\{k \in \mathbb{Z} \mid x_k \neq 0\} & \text{if } z = \sum_k x_k \otimes \lambda^k \neq 0 \ . \end{cases}$$

Since ∂ is Λ-linear, the module $\ker \partial$ is a Λ-module. In particular there is an automorphism of $\ker \partial$ given by multiplication with λ. It is immediate from the definition that for all $z \in \ker \partial$ and $\ell \in \mathbb{Z}$ we have

$$\nu(\lambda^\ell z) = \ell + \nu(z) \ . \tag{10.4.4}$$

For every $p \in \mathbb{Z}$ we define

$$\begin{aligned} Z^\infty_p &:= \{z \in \ker \partial \mid \nu(z) \geq -p\} = \ker \partial \cap \mathcal{F}^p C_* \ , \\ B^\infty_p &:= \{z \in \operatorname{im} \partial \mid \nu(z) \geq -p\} = \operatorname{im} \partial \cap \mathcal{F}^p C_* \ . \end{aligned}$$

It is easy to see that $\mathcal{F}^p H_* \cong Z^\infty_p / B^\infty_p$. We abbreviate the quotient $\bar{H}_* := \mathcal{F}^0 H_*/\mathcal{F}^{-1}H_*$ and define

$$\phi : \ker \partial \to \bar{H}_* \otimes \Lambda, \qquad z \mapsto [\lambda^{-\nu(z)} z] \otimes \lambda^{\nu(z)} \ .$$

To check that ϕ is well-defined, we need to see that $\lambda^{-\nu(z)} z \in Z^\infty_0$. But with (10.4.4) this is obvious since $\nu(\lambda^{-\nu(z)} z) = -\nu(z) + \nu(z) = 0$. The morphism ϕ is surjective, since every element of $\bar{H}_* \otimes \Lambda$ is a linear combination of elements of the form $[z] \otimes \lambda^\ell$ with $\nu(z) = 0$ and such elements have the preimage $\lambda^\ell z$. The kernel of ϕ is given by $\operatorname{im} \partial$, because $z \in \ker \phi \iff \lambda^{-\nu(z)} z \in \operatorname{im} \partial \iff z \in \operatorname{im} \partial$. Hence ϕ induces an isomorphism

$$H_* \cong \bar{H}_* \otimes \Lambda \ . \tag{10.4.5}$$

Restricting ϕ to Z^∞_p shows that we have the isomorphism $\mathcal{F}^p H_* \cong \bar{H}_* \otimes \mathcal{F}^p \Lambda$, where $\mathcal{F}^p \Lambda := \langle \lambda^\ell \mid \ell \geq -p \rangle$. Since the quotient $\mathcal{F}^p \Lambda / \mathcal{F}^{p-1} \Lambda = \langle \lambda^{-p} \rangle$ is free we have $\mathcal{F}^p H_* / F^{p-1} H_* \cong \bar{H}_* \otimes \langle \lambda^{-p} \rangle$. Together with c) and (10.4.5) we obtain the statement.

We show statement (i). The spectral sequence is constructed again by a filtration. This time we use the local pearl complex. Abbreviate by $C_* := CH_*^{\mathrm{loc}}(L_0, L_1)$ the local pearl complex. Define $\mathcal{F}^j C_* = 0$ if $j \leq 0$ and $\mathcal{F}^j C_* = C_*$ if $j \geq \kappa+1$. If $1 \leq j \leq \kappa$ define $\mathcal{F}^j C_* \subset C_*$ to be the submodule generated by all critical points p with $\mathcal{A}(u_p) \leq a_j$. We need to show that this defines a filtration. By construction $\mathcal{F}^{k-1}C_* \subset \mathcal{F}^k C_*$. Given a critical point p with $\mathcal{A}(u_p) \leq a_k$ and suppose that the coefficient of ∂p in front of q is not zero. We need to show that $\mathcal{A}(u_q) \leq a_k$. There must exists a rigid local pearl u from p to q. If u has zero cascades then p and q are on the same connected component and we are done because then $\mathcal{A}(u_q) = \mathcal{A}(u_p) \leq a_k$. If $u = (u_1, \ldots, u_m)$ has at least one cascade, then since the trajectory is local we have

$$0 = \sum_{j=1}^{m} \mu(u_j) + \mu(u_p) - \mu(u_q) = \tau^{-1} \left(\sum_{j=1}^{m} \int u_j^* \omega - \mathcal{A}(u_p) + \mathcal{A}(u_q) \right) .$$

Since u_j is non-constant and holomorphic we have $\int u_j^* \omega > 0$ for all j and hence

$$a_k \geq \mathcal{A}(u_p) = \sum_{j=1}^{m} \int u_j^* \omega + \mathcal{A}(u_q) > \mathcal{A}(u_q) . \tag{10.4.6}$$

We have deduced that $(\mathcal{F}^k C_*)_{k \in \mathbb{Z}}$ truly defines a filtration, which is evidently bounded by construction. As above we obtain a spectral sequence $E_{**}^{\mathrm{loc},*}$ with first page given by

$$E_{k,\ell}^{\mathrm{loc},1} \cong H_{k+\ell}(\mathcal{F}^k C_* / \mathcal{F}^{k-1} C_*, [\partial]) .$$

The complex $\mathcal{F}^k C_* / \mathcal{F}^{k-1} C_*$ is generated by critical points p such that $\mathcal{A}(u_p) = a_k$ and the boundary operator $[\partial]$ of p is ∂p projected to $\mathcal{F}^k C_* / \mathcal{F}^{k-1} C_*$ (i.e. we forget any critical points of lower action). Suppose that there exists a non-trivial contribution of $[\partial] p$ in front of q. Then by definition there exists a trajectory u connecting p to q. If u has at least one cascade we know by estimate (10.4.6) that $\mathcal{A}(u_q) \neq \mathcal{A}(u_p)$. Hence trajectories connecting critical points with the same action value are only Morse trajectories and hence $[\partial]$ is counting standard Morse trajectories. Taking our orientation algorithm (cf. paragraph before Lemma 10.2.7) and the degree-shift into account shows the claim. □

11. Proofs of the main results

11.1. Abelianization Theorem

In this section we prove theorem 2.1.1. Before we submerge into the details we explain the overall strategy.

Consider the space $V := \mu_G^{-1}(0)/T$, which is called *abelian/non-abelian correspondence*. Abbreviate with $i : V \to M/\!\!/T$ the embedding induced by $\mu_G^{-1}(0) \subset \mu_T^{-1}(0)$ and with $\pi : V \to M/\!\!/G$ the projection $Tx \mapsto Gx$. Written in a diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "\pi"] & M/\!\!/T \\ M/\!\!/G\,, & \end{tikzcd}$$

It is easy to see i is an embedding, π an G/T-fibre bundle and V a Lagrangian submanifold of $M/\!\!/T \times M/\!\!/G$ for the symplectic form $\omega_{M/\!\!/T} \oplus -\omega_{M/\!\!/G}$ embedded via $i \times \pi$ (cf. Proposition 11.1.1 below). Provided that $M/\!\!/G$ are simply connected and $M/\!\!/T$ is monotone, the Lagrangian V is simply connected and monotone with minimal Maslov number equal to $2c_{M/\!\!/T}$ (cf. Lemma 3.1.2). Fix a compatible almost complex structure J. For each non-negative integer k we denote by $\mathcal{M}_k^\Phi$ the space of J-holomorphic discs in $M/\!\!/G \times M/\!\!/T$ with boundary in V, of Maslov index $2kc_{M/\!\!/T}$, one interior marked point and a boundary marked point passing through a cycle representing the class D. Evaluation at the interior marked point gives the diagram

$$\begin{tikzcd} \mathcal{M}_k^\Phi \arrow[r, "i_k"] \arrow[d, "\pi_k"] & M/\!\!/T \\ M/\!\!/G\,, & \end{tikzcd} \tag{11.1.1}$$

where π_k (resp. i_k) is the evaluation on the interior marked point composed with projection to $M/\!\!/G$ (resp. inclusion into $M/\!\!/T$). The push-pull of the diagram (11.1.1) gives maps

$$\Phi_k : H^*(M/\!\!/T;\mathbb{Q}) \to H^{*-2kc_{M/\!\!/T}}(M/\!\!/G;\mathbb{Q}), \qquad a \mapsto \pi_k^! i_k^* a\,.$$

Then define the homomorphism

$$\Phi : QH^*(M/\!\!/T;\Lambda) \to QH^*(M/\!\!/G;\Lambda), \qquad \Phi = \Phi_0 \otimes \mathrm{id} + \Phi_1 \otimes \lambda + \Phi_2 \otimes \lambda^2 + \dots\ ,$$

where by abuse of notation we denote with λ the homomorphism of Λ given by multiplication with λ. The space $\mathcal{M}_0^\Phi$ consists of constant disks and hence $\mathcal{M}_0^\Phi$ is precisely the cycle representing D. This shows that $\Phi_0(a) = \pi^!(i^*a \smile \mathrm{D})$, which is the classical homomorphism considered by Martin. In that sense we think of the maps $\Phi_k \otimes \lambda^k$ of higher order (i.e. $k \geq 1$) as quantum correction terms. To obtain the isomorphism (2.1.2) it suffices to show that Φ is a ring homomorphism when restricted to $QH^*(M/\!\!/T;\Lambda)^W$, that Φ is surjective when restricted to $QH^*(M/\!\!/T;\Lambda)^W$ and to compute its kernel.In the rest of the section abbreviate the symplectic quotients

$$(X,\omega_X) := (M/\!\!/G, \omega_{M/\!\!/G}), \qquad\qquad (Y,\omega_Y) := (M/\!\!/T, \omega_{M/\!\!/T}) \,.$$

We denote by X^- the manifold equipped with the symplectic form $-\omega_X$.

Proposition 11.1.1. *Embedded via $i \times \pi$ the space V is a Lagrangian submanifold of $Y \times X^-$.*

Proof. We show that V is isotropic or equivalently $i^*\omega_Y = \pi^*\omega_X$. Consider the diagram with commuting square

$$\begin{tikzcd}
\mu_G^{-1}(0) \arrow[r, "j"] \arrow[d, "\tau"] & \mu_T^{-1}(0) \arrow[r, "k"] \arrow[d, "\rho"] & M \\
V \arrow[r, "i"] \arrow[d, "\pi"] & Y & \\
X & &
\end{tikzcd}$$

By the defining relation for the symplectic form in symplectic reduction and the commutativity of the diagram we have

$$\tau^* i^* \omega_Y = j^* \rho^* \omega_Y = j^* k^* \omega = (kj)^* \,\omega = (\pi\tau)^* \,\omega_X = \tau^* \pi^* \omega_X \,.$$

Since τ is a submersion, τ^* is injective and thus $i^*\omega_Y = \pi^*\omega_X$ as required.

It remains to show that V has the right dimension. Because the value 0 is regular and the quotients are taken with respect to free group actions we have

$$\begin{aligned}
\dim V &= \dim M - \dim G - \dim T \\
\dim X &= \dim M - 2\,\dim G \\
\dim Y &= \dim M - 2\,\dim T \,.
\end{aligned}$$

Multiply the first equation with 2 and subtract the second and third equation to show $2\dim V = \dim Y + \dim X$. □

Lemma 11.1.2. *Assume that Y is monotone and X is simply connected. Then the Lagrangian $V \subset Y \times X^-$ is monotone and the minimal Maslov number of V given by twice the minimal Chern number of Y.*

Proof. Via the embedding $i : V \hookrightarrow Y$ the space V is a fibred coisotropic submanifold with leaf isomorphic to G/T. It is well-known that the homogeneous space G/T is simply connected (see Proposition D.1.1). This implies by Proposition 3.1.4 that X is monotone with minimal Chern number divisible by c_Y. Hence the product $Y \times X^-$ is monotone with minimal Chern number c_Y. By the exact homotopy sequence shows that V is simply connected. Then by Lemma 3.1.2 we conclude that V is monotone with minimal Maslov number given by $2c_Y$. □

Let $\theta : T \to S^1$ be a root (cf. Section D.1 for more details). We have an associated complex line bundle $L_\theta := \mu_T^{-1}(0) \times_T \mathbb{C}_{(\theta)}$ over Y in which $\mathbb{C}_{(\theta)}$ denotes the complex space equipped with the action $t.z = \theta(t)z$ for all $z \in \mathbb{C}$ and $t \in T$. Let $\Phi^+ = \{\theta_1, \dots, \theta_m\}$ be a set of positive roots we define the *canonical anti-invariant class*

$$\mathrm{D} = \prod_{\theta \in \Phi^+} c_1(L_\theta) \in H^{2m}(Y)\,. \tag{11.1.2}$$

In Lemma D.2.1 we show that the pull-back $i^*\mathrm{D} \in H^{2m}(V)$ agrees with the Euler class of the vertical bundle $\ker \mathrm{d}\pi \subset TV$.

Lemma 11.1.3. *The submanifold $V \subset Y \times X$ is relative spin.*

Proof. First we need to show that V is orientable. This follows because its tangent bundle splits into $\pi^*TX \oplus \ker d\pi$, the bundle π^*TX is orientable because it is a complex bundle and the bundle $\ker \mathrm{d}\pi$ is orientable by Lemma D.2.1. By Proposition 9.2.4 we need to show that there exists a class $w \in H^2(Y \times X, \mathbb{Z}_2)$ which restricted to V is the second Stiefel-Whitney class of the bundle TV. For a root $\theta \in \Phi^+$ let $w_2(L_\theta) \in H^2(Y;\mathbb{Z}_2)$ be the second Stiefel-Whitney class of the associated line bundle L_θ. We claim that the class is given by

$$w := w_2(X) + \sum_{\theta \in \Phi^+} w_2(L_\theta)\,.$$

To see this consider again the splitting $TV \cong \pi^*TX \oplus \ker \mathrm{d}\pi$ and Lemma D.2.1. □

In the following we assume that X and hence V is simply connected and Y is monotone. Let $c_Y \in \mathbb{N}$ be the minimal Chern number of Y. Under these assumptions we work with the simplified Novikov ring, which is the ring of Laurent polynomials

$$\Lambda := \mathbb{Q}[\lambda, \lambda^{-1}], \qquad \deg \lambda = -2c_Y\,. \tag{11.1.3}$$

We denote by $QH^*(X) = H^*(X;\Lambda)$ and $QH^*(Y) = H^*(Y;\Lambda)$ the quantum cohomology rings, which are the cohomology modules equipped with the quantum product. We recapitulate the main steps for the proof

(i) define the homomorphism $\Phi : QH^*(Y) \to QH^*(X)$,

(ii) show that $\Phi|_{QH^*(Y)^W} : QH^*(Y)^W \to QH^*(X)$ is a ring homomorphism,

(iii) show that $\Phi|_{QH^*(Y)^W}$ is surjective and compute the kernel.

The morphism Φ and its properties are deduced via the study of moduli spaces of holomorphic curves and auxiliary Morse functions. For that purpose fix Morse functions f_X, f_Y and f_V on X, Y and V respectively and choose Riemannian metrics which are sufficiently generic. Moreover we fix compatible almost complex structures $J_\infty^- \in \mathrm{End}(TY, \omega_Y)$ and $J_\infty^+ \in \mathrm{End}(TX, \omega_X)$ which are sufficiently generic in the sense that they satisfy certain requirements as explained in the upcoming sections.

11.1.1. Definition of Φ

The map Φ is deduced from a count of holomorphic discs with a boundary marked point passing through a cohomology cycle representing the canonical anti-invariant class.

Moduli space Abbreviate by $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ the closed unit disk with boundary $\partial D = \{z \in \mathbb{C} \mid |z| = 1\}$. We denote by $\mathcal{J}^\Phi$ the space of maps $J : D \to \mathrm{End}(TY, \omega_Y) \oplus \mathrm{End}(TX, -\omega_X)$ such that $J(z) = J_\infty^- \oplus -J_\infty^+$ unless $1/4 \leq |z| \leq 3/4$. Given $J \in \mathcal{J}^\Phi$ let

$$\widetilde{\mathcal{M}}^\Phi(J) := \{u : (D, \partial D) \to (Y \times X, V) \mid \overline{\partial}_J u = 0\},$$

be the space of J-holomorphic disks with boundary on V. For critical points $p \in \mathrm{crit}\, f_Y$, $r \in \mathrm{crit}\, f_V$ and $q \in \mathrm{crit}\, f_X$ we define the subspace

$$\mathcal{M}^\Phi(p, r, q) \subset \widetilde{\mathcal{M}}^\Phi(J),$$

of elements $u = (u^-, u^+) : D \to Y \times X$ satisfying the point constraints $u^-(0) \in W^s(p)$, $u^+(0) \in W^u(q)$ and $u(1) \in W^s(r)$. We call J *regular* if the linearized Cauchy-Riemann operator at every J-holomorphic disc $u \in \widetilde{\mathcal{M}}^\chi(J)$ is surjective and the evaluation at points 0 and 1 is transverse to $W^s(p) \times W^u(q)$ and $W^s(r)$ with respect to all critical points.

Lemma 11.1.4. *The subspace of regular almost complex structures is comeager in $\mathcal{J}^\Phi$.*

Proof. The proof is classical and is a straight forward adaptation of [53, Theorems 3.1.5, 3.4.1]. To obtain the generic almost complex structures of split form in the case where $u = (u^-, u^+)$ and u^+ is constant and u^- is not constant, use [76, Theorem 3.2]. If u^- is constant then this implies that u^+ is constant by the boundary condition. □

Remark 11.1.5. Note that for Lemma 11.1.4 we do not need any monotonicity assumptions on V. Transversality is achieved by allowing domain dependence of J. The analogous result for domain independent J is much harder and needs the monotonicity assumption. For more details see [13, Section 3.3].

Fix a regular structure $J \in \mathcal{J}^\Phi$. With the index formula from [53, Theorem C.1.10] we conclude that the dimension of the component of $\mathcal{M}^\Phi(p, r, q)$ containing u is given by

$$\mu_{\mathrm{Mas}}(u) + \mu_{\mathrm{Mor}}(q) - \mu_{\mathrm{Mor}}(p) - \mu_{\mathrm{Mor}}(r) + \dim G/T. \tag{11.1.4}$$

For any $d \in \mathbb{N}_0$ we denote by $\mathcal{M}^\Phi(p,r,q)_{[d]}$ the union of all connected components of dimension d. We say that a sequence $(u_\nu) \subset \mathcal{M}^\Phi(p,r,q)$ converges to a disk v *up to Morse breaking of index k* if the convergence is uniformly with all derivatives and the Morse half-trajectories from $u_\nu^-(0)$ to p, from q to $u_\nu^+(0)$ and from $u_\nu(1)$ to r converge to broken trajectories of total index k.

Lemma 11.1.6. *Suppose that Y is monotone and X is simply connected. If $\mu(r) \leq 2c_Y - 1$, then $\mathcal{M}^\Phi(p,r,q)_{[0]}$ is finite. If moreover $\mu(r) \leq 2c_Y - 2$, then $\mathcal{M}^\Phi(p,r,q)_{[1]}$ is compact up to Morse breaking of index one, i.e. the boundary of the compactification is given by*

- $\mathcal{M}_0(p',p) \times \mathcal{M}^\Phi(p',r,q)_{[0]}$ *for all critical points p' with $\mu(p') = \mu(p)+1$,*
- $\mathcal{M}_0(r',r) \times \mathcal{M}^\Phi(p,r',q)_{[0]}$ *for all critical points r' with $\mu(r') = \mu(r)+1$,*
- $\mathcal{M}^\Phi(p,r,q')_{[0]} \times \mathcal{M}_0(q,q')$ *for all critical points q' with $\mu(q') = \mu(q)-1$.*

Proof. By Lemma 11.1.2 we conclude that V is monotone with minimal Maslov number given by $2c_Y$. Since bubbling has codimension $2c_Y$ the markings which evaluate into V converge under the limit. For details see [13, Section 3.4]. □

By Lemma 11.1.3 the submanifold V is relative spin. We fix a relative spin structure. With the choice of a relative spin structure we obtain a class of stable trivializations of $(\partial u)^*TV$ which gives a canonical orientation on the Cauchy-Riemann operator D_u hence an orientation on $\mathcal{M}^\Phi(J)$ (cf. [74, Prop. 4.1.1]). We also obtain by Lemma 9.1.3 a canonical orientation on the fibre product $\mathcal{M}^\Phi(p,r,q)$ from choices of coorientations of $W^s(p)$, $W^s(r)$ and $W^u(q)$.

Chain map For some $d \in \mathbb{N}_0$ let $C^{*\leq d}(f_V;\Lambda) \subset C^*(f_Y;\Lambda)$ denote the submodule of the Morse complex which is generated by critical points of index $\leq d$. Similarly we abbreviate $H^{*\leq d}(V;\Lambda)$ the submodule of the cohomology of degree $\leq d$. With orientation on the spaces $\mathcal{M}^\Phi(p,r,q)$ as explained above we define the Λ-linear map of degree $-2m$ with $2m := \dim G/T$,

$$\begin{aligned} \widehat{\Phi} : C^*(f_Y;\Lambda) \otimes C^{*\leq 2m+1}(f_V;\Lambda) &\to C^{*-2m}(f_X;\Lambda)\,, \\ p \otimes r &\mapsto \sum_{q \in \operatorname{crit} f_X} \; \sum_{[u] \in \mathcal{M}^\Phi(p,r,q)_{[0]}} \operatorname{sign} u \cdot q \otimes \lambda^{\mu_{\mathrm{Mas}}(u)/2c_Y}\,. \end{aligned}$$

By the compactness Lemma 11.1.6 the homomorphism $\widehat{\Phi}$ is well-defined and moreover similarly to Lemma 10.2.7 we show the identity

$$\widehat{\Phi}(\mathrm{d}p \otimes r) + (-1)^{\mu(p)}\widehat{\Phi}(p \otimes \mathrm{d}r) = \mathrm{d}\widehat{\Phi}(p \otimes r)\,,$$

where d denotes the Morse differential (cf. equation (3.3.3)). The last equation shows that $\widehat{\Phi}$ induces an homomorphism $H^*(Y;\Lambda) \otimes H^{*\leq 2m}(V;\Lambda) \to H^*(X;\Lambda)$ still denoted by the same symbol and we finally define using the canonical anti-invariant class (cf. equation (11.1.2))

$$\Phi : H^*(Y;\Lambda) \to H^*(X;\Lambda), \qquad a \mapsto \frac{1}{|W|}\widehat{\Phi}(a \otimes \mathrm{D})\,.$$

This finishes the construction of the homomorphism Φ.

11.1.2. The ring homomorphism property

In this subsection we show that Φ is a ring homomorphism for the quantum cup product when restricted to the subring of elements which are invariant under the action of the Weyl group W. For details on the Weyl group see Section D.1.

Lemma 11.1.7. *For all $a \in QH^*(Y)^W$ and $b \in QH^*(Y)$ we have*

$$\Phi(a) * \Phi(b) = \Phi(a * b)\,. \tag{11.1.5}$$

Proof. Here we give the principal steps, which are each proven in separate lemmas later on. First we define a homomorphism

$$\chi : H^*(Y;\Lambda) \otimes H^*(Y;\Lambda) \otimes H^*(V \times_\pi V;\Lambda) \to H^*(X;\Lambda)\,,$$

in which $V \times_\pi V := \{(p,p') \in V \times V \mid \pi(p) = \pi(p')\}$ denotes the fibre square. In Lemma 11.1.14 we show that for all $a, b \in QH^*(Y)$ we have

$$\Phi(a * b) = \chi(a \otimes b \otimes d_V)\,, \tag{11.1.6}$$

in which $d_V \in H^*(V \times_\pi V;\Lambda)$ denotes the class which is Poincare dual to the embedded submanifold $V \subset V \times_\pi V$ given by the diagonal embedding. In Lemma 11.1.16 we show that for all $a, b \in QH^*(Y)$ we have we have

$$\Phi(a) * \Phi(b) = \frac{1}{|W|}\,\chi(a \otimes b \otimes \mathrm{pr}_2^*\mathrm{D})\,, \tag{11.1.7}$$

where $\mathrm{pr}_2 : V \times_\pi V \to V$ denotes the restriction of the projection to the second factor. Since the action of W on Y is vertical, the group $W \times W$ acts on $V \times_\pi V$ and hence on $H^*(V \times_\pi V;\Lambda)$. In Lemma 11.1.17 we show that for any element of the Weyl group $w \in W$ and for all $a, b \in H^*(Y;\Lambda)$, $c \in H^*(V \times_\pi V;\Lambda)$ we have

$$\chi(a \otimes b \otimes c) = \chi(w.a \otimes b \otimes (w,1).c)\,. \tag{11.1.8}$$

In Proposition D.2.2 we show that

$$\sum_{w \in W} (w,1).d_V = \mathrm{pr}_2^*\mathrm{D}\,.$$

Given an invariant element $a \in QH^*(Y)^W$ and $b \in QH^*(Y)$. We conclude

$$\begin{aligned}\Phi(a * b) &= \frac{1}{|W|} \sum_{w \in W} \Phi(w.a * b) \\ &= \frac{1}{|W|} \sum_{w \in W} \chi(w.a \otimes b \otimes d_V) \\ &= \frac{1}{|W|} \sum_{w \in W} \chi(a \otimes b \otimes (w,1).d_V) \\ &= \frac{1}{|W|} \chi(a \otimes b \otimes \mathrm{pr}_2^* \mathrm{D}) \\ &= \Phi(a) * \Phi(b) \ .\end{aligned}$$

This shows (11.1.5). □

Remark 11.1.8. Identities similar to (11.1.6) and (11.1.7) where already observed in [75, Theorem 6.2] and are deduced by geometric homotopies of moduli spaces. Our homotopy is slightly different because we keep track of a marked point.

Definition of χ

In order to show the ring homomorphism property we express the left and the right-hand side as counts of certain holomorphic curves which we describe now.

Moduli space We abbreviate by $\mathcal{J}^\chi$ the space of maps $J = (J_0^-, J_1^-, J^+) : D \to \mathrm{End}(TY, \omega_Y) \oplus \mathrm{End}(TY, \omega_Y) \oplus \mathrm{End}(TX, -\omega_X)$ such that such that $J(z) = J_\infty^- \oplus J_\infty^- \oplus - J_\infty^+$ unless $1/4 \leq |z| \leq 3/4$. Given $J \in \mathcal{J}^\chi$ we define the space

$$\mathcal{M}^\chi(J) := \{u = (u_0^-, u_1^-, u^+) : D \to Y \times Y \times X \mid \partial_s u + J \partial_t u = 0,\ (11.1.9)\} \,,$$

with boundary condition

$$(u_0^-(e^{2i\theta}), u^+(e^{i\theta})) \in V, \qquad (u_1^-(e^{2i\theta}), u^+(-e^{i\theta})) \in V, \qquad \forall\, \theta \in [0, \pi] \,. \tag{11.1.9}$$

It is easy to convince oneself that for all elements $(u_0^-, u_1^-, u^+) \in \mathcal{M}^\chi(J)$ we have $u^+(1) = u^+(-1)$ and thus $(u_0^-(1), u_1^-(1)) \in V \times_\pi V := \{(p, p') \in V \times V \mid \pi(p) = \pi(p')\}$.

Remark 11.1.9. With terminology from Wehrheim&Woodward the elements of the space $\mathcal{M}^\chi(J)$ are pseudo-holomorphic quilts. In particular see [75, $S_{3\mathrm{comp}}$ in Figure 15]. Pseudo-holomorphic quilts are holomorphic maps with certain non-local Lagrangian boundary conditions (i.e. Lagrangian seams) and are studied by the mentioned authors in great detail in [75] and [74]. The statements we use here are almost already proven in their work (cf. [75, Theorem 3.9]). We have written "almost" because the mentioned authors assume that at boundary punctures the Lagrangians seams intersect transversely in an appropriate sense. We would need an extension of their results to include clean intersections. Although such an extension is conceivable using the results which we have developed in this work, we choose to give an ad-hoc treatment of the space $\mathcal{M}^\chi(J)$ to be concrete.

We put a smooth structure on the moduli space $\mathcal{M}^\chi(J)$ for a generic J. For that we follow the same general steps as given for the space of holomorphic strips, viz. construct an appropriate Banach manifold, show that the non-linear Cauchy-Riemann operator is Fredholm, show that for generic choice of almost complex structure it intersects the zero-section transversely,

Banach manifold Let $\Sigma_+ := \{z = s + it \in \mathbb{C} \mid s \geq 0, t \in [0,1]\}$ be the half-strip and consider the holomorphic function $\epsilon : \Sigma_+ \to D\backslash\{1\}$, $z \mapsto (e^{\pi z}-i)/(e^{\pi z}+i)$. The function is called *strip-like end* and satisfies $\lim_{s\to\infty} \epsilon(s + it) = 1$ for all $t \in [0,1]$, $\epsilon(s) \in \partial D_-$ and $\epsilon(s+i) \in \partial D_+$ for all $s \geq 0$, where ∂D_- (resp. ∂D_+) is the boundary with negative (resp. positive) imaginary part. To a map $(u_0^-, u_1^-, u^+) : D \to Y \times Y \times X$ we associate the half-strip $u_\epsilon : \Sigma_+ \to Z := Y \times X \times Y \times X$ given by

$$u_\epsilon(z) = (u_0^-(w^2), u^+(w), u_1^-(w^2), u^+(-w)), \qquad w = \epsilon(z)\,. \tag{11.1.10}$$

By construction we have

$$u_\epsilon(\infty) := \lim_{s\to\infty} u_\epsilon(s,\cdot) = (u_0^-(1), u^+(1), u_1^-(1), u^+(-1))\,,$$

and if u satisfies the boundary condition (11.1.9) then u_ϵ is a half-strip with boundary condition

$$\begin{aligned} u_\epsilon(\cdot,0) \subset L_0 &:= \{(y,x',y',x) \in Z \mid (x,y),(x',y') \in V\}, \\ u_\epsilon(\cdot,1) \subset L_1 &:= \{(y,x,y',x') \in Z \mid (x,y),(x',y') \in V\}\,. \end{aligned}$$

The spaces L_0 and L_1 are Lagrangian submanifolds of the symplectic manifold $Z = Y \times X^- \times Y \times X^-$ which intersect cleanly and their intersection $L_0 \cap L_1$ is diffeomorphic to $V \times_\pi V$. Moreover if $u \in \mathcal{M}^\chi(J)$, then u_ϵ is J_Z-holomorphic where $J_Z := J_\infty^- \oplus -J_\infty^+ \oplus J_\infty^- \oplus -J_\infty^+$.

Fix $p > 2$ and $\delta > 0$ sufficiently small and define $\mathcal{B}$ to be the space of continuous maps $u = (u_0^-, u_1^-, u^+) : D \to Y \times Y \times X$ such that

- $(u_0^-, u_1^-) \in H^{1,p}_{\mathrm{loc}}(D \setminus \{1\})$ and $u^+ \in H^{1,p}_{\mathrm{loc}}(D \setminus \{-1,1\})$,
- u satisfies the boundary condition (11.1.9),
- the integral $\int_{\Sigma_+} (\mathrm{dist}\,(u_\epsilon, u_\epsilon(\infty))^p + |\mathrm{d}u_\epsilon|^p) e^{\delta sp} \mathrm{d}s\mathrm{d}t$ is finite.

The tangent space of $\mathcal{B}$ at u, denoted $T_u\mathcal{B}$, is given by sections $\xi = (\xi_0^-, \xi_1^-, \xi^+) \in C^0(u^*TM)$ such that

- $(\xi_0^-, \xi_1^-) \in H^{1,p}_{\mathrm{loc}}(D \setminus \{1\})$ and $\xi^+ \in H^{1,p}_{\mathrm{loc}}(D \setminus \{\pm 1\})$,
- ξ satisfies the linearized version of condition (11.1.9),

- the following norm is finite,

$$\|\xi\|_{1,p;\delta} := \Big(\|\xi\|^p_{H^{1,p}(D\setminus\epsilon(\Sigma_+))} + \\ + \int_{\Sigma_+} |\xi_\epsilon - \widehat{\Pi}^u_x \xi(\infty)|^p + |\nabla(\xi_\epsilon - \widehat{\Pi}^u_x\xi(\infty))|^p) e^{\delta ps} \mathrm{d}s\mathrm{d}t\Big)^{1/p},$$

  with $x = u(\infty)$, $\xi(\infty) = (\xi_0^-(1), \xi^+(1), \xi_1^-(1), \xi^+(-1))$ and ξ_ϵ defined by (11.1.10) using ξ instead of u.

We also define the the Banach bundle $\mathcal{E}$ over $\mathcal{B}$ with fibre over $u \in \mathcal{B}$ given the space of sections $\eta = (\eta_0^-, \eta_1^-, \eta^+)$ of the bundle u^*TM such that $(\eta_0^-, \eta_1^-) \in L^p_{\mathrm{loc}}(D \setminus \{1\})$, $\eta^+ \in L^p_{\mathrm{loc}}(D \setminus \{\pm 1\})$ and the norm is finite

$$\|\eta\|^p_{p;\delta} := \|\eta\|^p_{L^p(D\setminus\epsilon(\Sigma_+))} + \int_{\Sigma_+} |\eta_\epsilon|^p e^{(\delta-\pi)ps} \mathrm{d}s\mathrm{d}t\,.$$

where again we define η_ϵ via (11.1.10) using η in place of u. Note that we have changed the exponential weight! This is necessary so that the linearized Cauchy-Riemann operator

$$D_u := \nabla_s + J(u)\nabla_t + (\nabla J(u))\partial_t u : T_u\mathcal{B} \to \mathcal{E}_u\,,$$

is a bounded operator (cf. Lemma 11.1.10 below). To relate the index of D_u to topological data we define the Viterbo index of an element $u = (u_0^-, u_1^-, u^+) \in \mathcal{M}^\chi(J)$ as follows. Let $U \subset Z$ be a small neighborhood of $x = u(\infty)$ and choose an unitary trivialization $\Phi : U \times \mathbb{R}^{2n} \cong TZ|_U, (p,\xi) \mapsto \Phi(p)\xi \in T_pZ$ such that there exists linear Lagrangians Λ_0, Λ_1 with $\Phi(p)\Lambda_0 = T_pL_0$ for all $p \in U \cap L_0$ and $\Phi(p)\Lambda_1 = T_pL_1$ for all $p \in U \cap L_1$ (cf. Lemmas 3.2.10 and 3.2.11). Then choose an unitary trivialization $\Phi_u = (\Phi_0^-, \Phi_1^-, \Phi^+)$ of $u^*T(Y \times Y \times X^-)$ such that $\Phi_u(\epsilon(s,t)) = \Phi(u_\epsilon(s,t))$ for all $s \geq s_0$, $t \in [0,1]$ and some s_0 large enough. Define paths of linear Lagrangians

$$\begin{aligned} F_0(\theta) &= \big(\Phi_0^-(e^{2i\theta}) \oplus \Phi^+(e^{i\theta})\big) T_{(u_0^-(e^{2i\theta}), u^+(e^{i\theta}))} V\,, \\ F_1(\theta) &= \big(\Phi_1^-(e^{2i\theta}) \oplus \Phi^+(-e^{i\theta})\big) T_{(u_1^-(e^{2i\theta}), u^+(e^{i\theta}))} V\,. \end{aligned} \tag{11.1.11}$$

Define the Viterbo index using the Robbin Salamon index

$$\mu(u) := \mu_{\mathrm{RS}}(F_0, F_0(0)) + \mu_{\mathrm{RS}}(F_1, F_1(0))\,. \tag{11.1.12}$$

The index does not the depend on the choice of the trivialization, because another choice will lead paths which are homotopic to F_0 and F_1 with fixed endpoints.

Lemma 11.1.10. *The operator $D_u : T_u\mathcal{B} \to \mathcal{E}_u$ is Fredholm of index $\mu(u) + \dim V \times_\pi V$.*

Proof. We follow the proof of Theorem 6.1.10. Using the trivialization Φ_u we conclude that the differential operator D_u is conjugated to $D_S = \partial_s + J_{\mathrm{std}}\partial_t + S$, with lower order term $S : D \to \mathbb{R}^{2m\times 2m}$ where $2m = 2\dim Y + \dim X$. The domain of the operator is the Banach space, denoted H_F, given by maps $\xi = (\xi_0^-, \xi_1^-, \xi^+)$ satisfying

- $\xi_0^-, \xi_1^- \in H^{1,p}_{\mathrm{loc}}(D \setminus \{1\})$ and $\xi^+ \in H^{1,p}_{\mathrm{loc}}(D \setminus \{\pm 1\})$,
- with boundary condition for all $\theta \in [0, \pi]$,
$$(\xi_0^-(e^{2i\theta}), \xi^+(e^{i\theta})) \in F_0(\theta), \qquad (\xi_1^-(e^{2i\theta}), \xi^+(-e^{i\theta})) \in F_1(\theta)\,,$$
- there exists $\xi(\infty) \in \Lambda_0 \cap \Lambda_1$ such that the norm is finite
$$\|\xi\|_{1,p;\delta} := \|\xi\|_{H^{1,p}(D\setminus\epsilon(\Sigma_+))} + \|(\xi_\epsilon - \xi(\infty))\kappa_\delta\|_{H^{1,p}(\Sigma_+)}\ ,$$
with weight-function $\kappa_\delta(s) = e^{\delta s}$ and ξ_ϵ defined via (11.1.10) using ξ in place of u.

Using the trivialization $\Phi_\epsilon : \Sigma_+ \times \mathbb{R}^{2n} \to u_\epsilon^* TZ$, $\Phi_\epsilon(s,t) = \Phi(u_\epsilon(s,t))$ we see that $D_{u_\epsilon} = \nabla_s + J_Z(u_\epsilon)\nabla_t + (\nabla J_Z(u_\epsilon))\partial_t u_\epsilon$ is conjugated to $\partial_s + J_{\mathrm{std}}\partial_t + S_\epsilon$, with lower order term $S_\epsilon : \Sigma_+ \to \mathbb{R}^{2n\times 2n}$. By Lemma 4.4.1 we conclude that $S_\epsilon(s,\cdot)$ converges to 0 and that S_ϵ has μ-decay. We conclude via (6.2.9) that for any smooth $\xi_\epsilon : \Sigma_+ \to \mathbb{R}^{2n}$ and $\xi(\infty) \in \Lambda_0 \cap \Lambda_1$ we have
$$|(\partial_s + J_{\mathrm{std}}\partial_t + S_\epsilon)\xi_\epsilon| \leq |d(\xi_\epsilon - \xi(\infty)| + O(1)|\xi_\epsilon - \xi(\infty)| + O(e^{-\mu s})|\xi(\infty)|\,.$$
Given any $\xi : D \to \mathbb{R}^{2m}$ set $\eta := D_S\xi = (\partial_s + J_{\mathrm{std}}\partial_t + S)\,\xi$ and define η_ϵ and ξ_ϵ via (11.1.10) using η and ξ respectively in place of u. A simple computation using the chain-rule and that u, ϵ are holomorphic as well as $|d\epsilon| \sim e^{-\pi s}$ as $s \to \infty$ we conclude that
$$|\eta_\epsilon| = O(e^{\pi s})\,|(\partial_s + J_{\mathrm{std}}\partial_t + S_\epsilon)\xi_\epsilon|\,. \tag{11.1.13}$$
With this and the last estimate we conclude that the operator D_S is bounded. Also by that estimate and the isomorphism $\partial_s + J_{\mathrm{std}}\partial_t : H^{1,p;\delta}_\Lambda(\Sigma, \mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma, \mathbb{R}^{2n})$ with $\Lambda(s) = \Lambda_0 \times \Lambda_1$ for all $s \in \mathbb{R}$, we conclude that exists a constant $s_0 \geq 0$ such that
$$\|\xi\|_{1,p;\delta} \leq O(1)\,\|D_S\xi\|_{p;\delta}\ , \tag{11.1.14}$$
for all $\xi \in H_F$ supported in $\epsilon(\Sigma^\infty_{s_0})$ and with $\xi(\infty) = 0$. Using standard elliptic estimates and cut-off functions we conclude that
$$\|\xi\|_{1,p;\delta} \leq c(\|D_S\xi\|_{p;\delta} + \|\xi\|_{L^p(D\setminus\epsilon(\Sigma^\infty_{s_0}))})\,,$$
for all $\xi \in H_F$ with $\xi(\infty) = 0$. This implies that D_S is semi-Fredholm because the restriction $H_F \to L^p(D \setminus \epsilon(\Sigma^\infty_{s_0}))$ is compact. By considering the formal adjoint we conclude that D_S is Fredholm. To compute the index see that D_S is homotop to the glued operator $D_0 \# D_1$, of the operators $D_0 = \partial_s + J_{\mathrm{std}}\partial_t : H_{F'} \to L^{p;\delta}(D, \mathbb{R}^{2m})$ with constant boundary conditions $F' = (F_0', F_1')$ such that $F_0'(\theta) = F_0(0)$, $F_1'(\theta) = F_1(0)$ for all $\theta \in [0,\pi]$ and $D_1 = \partial_s + J_{\mathrm{std}}\partial_t + S' : H^{1,p;\delta}_{F,W}(\Sigma, \mathbb{R}^{2n}) \to L^{p;\delta}(\Sigma, \mathbb{R}^{2n})$ with $F = (F_0, F_1^\vee)$, $W = (\Lambda_0 \cap \Lambda_1, \Lambda_0 \cap \Lambda_1)$ and $S' : \Sigma \to \mathbb{R}^{2n\times 2n}$ has the asymptotics $S'(\pm\infty) = 0$. By linear gluing we have
$$\operatorname{ind} D_0 + \operatorname{ind} D_1 = \operatorname{ind} D_u + \dim \Lambda_0 \cap \Lambda_1\,.$$
We also by Corollary 6.2.7 we know that $\operatorname{ind} D_1 = \mu(F) + \dim \Lambda_0 \cap \Lambda_1$. The operator D_0 is surjective an has kernel consisting of constants, hence $\operatorname{ind} D_0 = \dim \Lambda_0 \cap \Lambda_1$. This shows the claim since $\mu(F) = \mu(u)$ and $\dim \Lambda_0 \cap \Lambda_1 = \dim V \times_\pi V$. □

Choose a Morse function $f_{V\times_\pi V} : V\times_\pi V \to \mathbb{R}$ and a sufficiently generic metric. Given critical points $p_0, p_1 \in \operatorname{crit} f_Y$, $r \in \operatorname{crit} f_V$, $s \in \operatorname{crit} f_{V\times_\pi V}$ and $q \in \operatorname{crit} f_X$ we define the subspace

$$\mathcal{M}^\chi(p_0, p_1, r, s, q) \subset \mathcal{M}^\chi(J)\,,$$

to be the space of $u = (u_0^-, u_1^-, u^+)$ satisfying the point-wise constraints

$$\begin{aligned} &u_0^-(0) \in W^s(p_0)\,, \qquad &&u_0^-(-1) \in W^s(r)\,, \quad u^+(0) \in W^u(q)\,,\\ &u_1^-(0) \in W^s(p_1)\,, \quad &&(u_0^-(1), u_1^-(1)) \in W^s(s)\,. \end{aligned} \tag{11.1.15}$$

We call $J \in \mathcal{J}^\chi$ *regular* if the operator D_u is surjective for all $u \in \mathcal{M}^\chi(J)$ and the space $\mathcal{M}^\chi(p_0, p_1, r, s, q)$ is cut-out transversely for all possible critical points. We show similarly to Theorem 7.2.1 that the space of regular J is comeager and fix J.

Lemma 11.1.11 (Energy-Index relation)**.** *Suppose that X is simply connected and that Y is τ-monotone, then for every $u = (u_0^-, u_1^-, u^+) \in \mathcal{M}^\chi(J)$ we have $\tau\mu(u) = E(u) = \int (u_0^-)^*\omega_Y + \int (u_1^-)^*\omega_Y - \int (u^+)^*\omega_X$.*

Proof. By Lemma 11.1.2 we see that V is a τ-monotone Lagrangian. Similar to [75, Remark 2.2(2)] we show that there exists an uniform constant c such that $\tau\mu(u) = E(u) + c$. However the constant map u has vanishing energy and vanishing index, thus $c = 0$. $\square$

Lemma 11.1.12. *Suppose that Y is monotone and X is simply connected. If $\mu(r) \leq 2c_Y - 1$ and $\mu(s) \leq 2c_Y - 1$ then $\mathcal{M}^\chi(p_0, p_1, r, s, q)_{[0]}$ is finite and if moreover $\mu(r) \leq 2c_Y - 2$ and $\mu(s) \leq 2c_Y - 2$ then $\mathcal{M}^\chi(p_0, p_1, r, s, q)_{[1]}$ is compact up adding broken Morse trajectories of index one.*

Proof. The principal strategy of the proof is similar to Lemma 11.1.4. We show that bubbling has codimension $2c_Y$, hence the evaluation into V and $V \times_\pi V$ converges. We give some more details.

Fix $d = 0, 1$. Given a sequence $(u^\nu) = (u_0^{-,\nu}, u_1^{-,\nu}, u^{+,\nu}) \subset \mathcal{M}^\chi(p_0, p_1, r, s, q)_{[d]}$. It suffices to show that $|\partial_s u^\nu|$ is uniformly bounded. Assume by contradiction that it is not. Provided with the index formula we have

$$d = \mu(u^\nu) + \dim Y - \dim X - \mu(p_0) - \mu(p_1) - \mu(r) - \mu(s) + \mu(q)\ . \tag{11.1.16}$$

By the energy-index relation we conclude that the energy $E(u^\nu)$ is uniformly bounded. By Gromov compactness (u^ν) is compact up to bubbling, i.e. a subsequence of u^ν converges to a stable map $v = (v_i)_{i\in T}$ modeled on a rooted tree T with root $i_0 \in T$ such that v_{i_0} is an element in $\mathcal{M}^\chi(J)$ and for each $i \in T \setminus \{i_0\}$ the map v_i is either

- a J_∞^--holomorphic sphere in Y,
- a $-J_\infty^+$-holomorphic sphere in X,
- a $(J_\infty^- \oplus -J_\infty^+)$-holomorphic disk in $Y \times X$ with boundary on V,

- a J_Z-holomorphic strip in Z with boundary on (L_0, L_1).

Also the subsequence of the energies $E(u^\nu)$ converges to $E(v) = \sum_{i\in T} E(v_i)$. Consider the bubble energies

$$\begin{aligned} m_s &:= \lim_{s\to\infty} \lim_{\nu\to\infty} E(u_\epsilon^\nu; [s,\infty)\times[0,1]),\\ m_r &:= \lim_{\varepsilon\to 0} \lim_{\nu\to\infty} E(z \mapsto (u_0^{-,\nu}(z^2), u^{+,\nu}(z)); D \cap B_\varepsilon(i))\,. \end{aligned}$$

We distinguish two cases

(i) we have $m_s = m_r = 0$,

(ii) we have $m_s > 0$ or $m_r > 0$.

For case (i): In this case there is no $i \in T$ such that v_i is a strip in Z and we have convergence of the points $(u_0^{-,\nu}(1), u_1^{-,\nu}(1))$ and $(u_0^{-,\nu}(-1), u^{+,\nu}(i))$ evaluating into $V \times_\pi V$ and V respectively. Therefore the root $v_{i_0} = (v_{0,i_0}^-, v_{1,i_0}^-, v_{i_0}^+) \in \mathcal{M}^\chi(J)$ satisfies

$$(v_{0,i_0}^-(1), v_{1,i_0}^-(1)) \in W^s(s') \quad \text{and} \quad v_{i_0}^+(-1) \in W^s(r')\,, \tag{11.1.17}$$

with $\mu(s') \geq \mu(s)$ and $\mu(r') \geq \mu(r)$. Moreover we have three leaves $i_0^-, i_1^-, i^+ \in T$ such that $v_{i_0^-}$, $v_{i_1^-}$ are spheres in Y and v_{i^+} is a sphere in X with $v_{i_0^-}(0) \in W^s(p_0')$ and $v_{i_1^-}(0) \in W^s(p_1')$ and $v_{i^+}(0) \in W^u(q')$ in which $p_0', p_1' \in \operatorname{crit} f_Y$ and $q' \in \operatorname{crit} f_X$ are critical points satisfying $\mu(p_0') \geq \mu(p_0)$, $\mu(p_1') \geq \mu(p_1)$ and $\mu(q') \leq \mu(q)$. Consider the subtree $T' \subset T$ where we have discarded all discs and spheres which are not between any of those leaves and the root. Then we reduce all remaining spheres to simple spheres without increasing the index. Assuming that J, J_∞^- and J_∞^+ are sufficiently generic the space of such configurations is a manifold of dimension

$$d_v := \sum_{i\in T'} \mu(v_i) - \mu(p_0') - \mu(p_1') - \mu(r') - \mu(s') + \mu(q') + \dim Y - \dim X\,.$$

We have $\lim_{\nu\to\infty} E(u^\nu) \geq \sum_{i\in T'} E(v_i)$. By monotonicity and the energy index relation $\tau\mu(v_{i_0}) = E(v_{i_0})$ we conclude $\mu(u^\nu) \geq \sum_{i\in T'} \mu(v_i)$. Thus

$$d_v \leq \mu(u^\nu) - \mu(p_0) - \mu(p_1) - \mu(r) - \mu(s) + \mu(q) + \dim Y - \dim X = d \leq 1\,.$$

But if there is a non-constant sphere v_i for some $i \in T' \setminus \{i_0\}$ then d_v is a least two since the space admits a free action of a 2-dimensional subgroup of $PSL(2,\mathbb{C})$. If instead $T' = \{i_0\}$ but $|\partial_s u^\nu|$ is unbounded, we must have discarded a non-constant bubble and continue as in case (ii).

For case (ii): Assume that $m_r > 0$ and $m_s > 0$. The other subcases are similar. By the same reasoning as above we obtain a stable map $v = (v_i)_{i\in T'}$ indexed on a rooted tree T' with exactly three leaves and with root $v_{i_0} \in \mathcal{M}^\chi(J)$ and all other components are simple holomorphic spheres in X or Y and such that for the leaves i_0^-, i_1^- and i^+ we have the point condition $v_{i_0^-}(0) \in W^s(p_0')$, $v_{i_1^-}(0) \in W^s(p_1')$ and $v_{i^+}(0) \in W^u(q')$. If J_∞^-, J_∞^+ and J are sufficiently generic the space of such configuration is of dimension

$$\sum_{i \in T'} \mu(v_i) - \mu(p_0') - \mu(p_1') + \mu(q') + \dim Y - \dim X\,.$$

There possibly exists $i \in T \setminus T'$ such that v_i is a non-constant strip. The intersections $L_0 \cap L_1$ is connected. We find a strip $\bar{v}$ lying completely in $L_0 \cap L_1$ such that $\bar{v}(\infty) = v_i(-\infty)$ and $\bar{v}(-\infty) = v_i(\infty)$. By the concatenation and zero axiom of the Viterbo index we have $\mu(v \# v_i) = \mu(v) + \mu(v_i) = \mu(v_i)$. The connected sum $w := v \# v_i$ is a annulus with boundary in (L_0, L_1). Because L_0 is simply connected we find a disk $\bar{w}$ lying completely in L_0 such that $\bar{w} \# w$ is a disk on L_1. Because L_1 is monotone with minimal Maslov index $2c_Y$ we have $\mu(\bar{w} \# w) = \mu(\bar{w}) + \mu(w) = \mu(w) = \mu(v_i) \in 2c_Y\mathbb{Z}$. Altogether we conclude that since v_i is non-constant $\mu(v_i) \geq 2c_Y$. Similar we have $\mu(v_i) \geq 2c_Y$ if v_i is a holomorphic disk or sphere. Thus $\mu(u^\nu) = \sum_{i \in T} \mu(v_i) \geq \sum_{i \in T'} \mu(v_i) + 4c_Y$ for some $\nu \in \mathbb{N}$ large enough and thus

$$\begin{aligned} 0 &\leq \sum_{i \in T'} \mu(v_i) - \mu(p_0') - \mu(p_1') + \mu(q') + \dim Y - \dim X \\ &\leq \mu(u^\nu) - 4c_Y - \mu(p_0) - \mu(p_1) + \mu(q) + \dim Y - \dim X \\ &\leq d + \mu(r) + \mu(s) - 4c_Y \leq d - 2\,. \end{aligned}$$

Which implies that $d \geq 2$ and contradicts the assumption. $\square$

The pair (L_0, L_1) is relative spin with background class $\mathrm{pr}_1^* w_2(X) \oplus \mathrm{pr}_2^* \mathrm{D} \oplus \mathrm{pr}_3^* w_2(X) \oplus \mathrm{pr}_4^* \mathrm{D}$ where pr_i denotes the projection to the i-th factor. Let $\mathcal{O} \to L_0 \cap L_1$ be corresponding double cover given in Definition 9.3.4. A slight generalization of Theorem 9.3.6 shows that we have a canonical isomorphism $|\mathcal{M}^\chi(J)|_u \cong \mathcal{O}_x^\vee \otimes |C|_x$ for every $u \in \mathcal{M}^\chi(J)$ where $x = u(\infty)$ and $C = L_0 \cap L_1$. Hence by an $\mathcal{O}$-coorientation of $W^u(s)$ and coorientations of $W^s(p_0)$, $W^s(p_1)$, $W^s(r)$ and $W^u(q)$ we obtain an orientation on $\mathcal{M}^\chi(p_0, p_1, r, s, q)$ via the Lemma 9.1.3.

Chain map We define the Λ-linear homomorphism of degree $-4m$

$$\begin{aligned} \widehat{\chi} : C^*(f_Y) \otimes C^*(f_Y) \otimes C^{*\leq 2m+1}(f_V) \otimes C^{*\leq 2m+1}(f_{V \times_\pi V}) \otimes \Lambda &\to C^{*-4m}(f_X) \otimes \Lambda \\ p_0 \otimes p_1 \otimes r \otimes s \mapsto \sum_{q \in \mathrm{crit}\, f_X} \sum_{[u] \in \mathcal{M}^\chi(p_0, p_1, r, s, q)_{[0]}} &\mathrm{sign}\, u \cdot q \otimes \lambda^{\mu_{\mathrm{Vit}}(u)/2c_Y}\,. \end{aligned}$$

Lemma 11.1.12 shows that $\widehat{\chi}$ is well-defined and induces a homomorphism on the homology groups which we still denote by the same symbol. Finally we define

$$\begin{aligned} \chi : H^*(Y; \Lambda) \otimes H^*(Y; \Lambda) \otimes H^{*\leq 2m}(V \times_\pi V; \Lambda) &\to H^{*-d}(X; \Lambda), \\ a \otimes b \otimes c &\mapsto \frac{1}{|W|} \widehat{\chi}(a \otimes b \otimes \mathrm{v} \otimes c)\,. \end{aligned} \tag{11.1.18}$$

This finishes the section.

Left homotopy

In this subsection we show equation (11.1.6).

Moduli space Using an idea of [3] we write down a concrete model of the Riemannian surface which moderates the homotopy. Define the half-disks D^-, D^+ via $D^\pm = \{z = s+it \in \mathbb{C} \mid |z| \leq 1, \pm t \geq 0\}$. We establish a chain-homotopy using the count of pseudo-holomorphic curves defined on a family of Riemann surfaces D_ρ with $\rho \in [1/2, 1]$ defined by $D_\rho = D^+ \sqcup D^- / \sim$ with identifications

$$(s, 0^+) \sim (s, 0^-) \text{ if } |s| \geq \rho \quad \text{and} \quad \begin{matrix} (s,0^-) \sim (-s, 0^-) \\ (s, 0^+) \sim (-s, 0^+) \end{matrix} \text{ if } |s| \leq \rho \,.$$

For $1/2 \leq \rho < 1$ the surface D_ρ is homeomorphic to a disk and for $\rho = 1$ the surface D_ρ is homeomorphic to a wedge of two disks connected in one boundary point. Equivalently one can think of the family $(D_\rho)_{1/2 \leq \rho < 1}$ as a fixed disk D equipped with a family of varying almost complex structures. Technically speaking the almost complex structure is singular at the points $-1/2, 0$ and $1/2$, but these singularities are mild in the sense that a punctured neighborhood is bi-holomorphic to an annulus. So in that sense we think of the holomorphic curves which we are about to define, as finite energy holomorphic curves on the punctured surfaces which are continuously extended over the punctures.

Let $\mathcal{J}^\Theta \subset C^\infty\big(D, \mathrm{End}(TY, \omega_Y) \oplus \mathrm{End}(TX, -\omega_X)\big)$ the space of maps $J = (J^-, J^+)$ such that $J^-(z) = J^-_\infty$ unless $1/8 \leq |z \pm i/2| \leq 1/4$ and $J^+(z) = -J^+_\infty$ unless $1/4 \leq |z| \leq 3/4$. Note that J^- defines a map $C^\infty\big(D_\rho, \mathrm{End}(TY, \omega_Y)\big)$ for each ρ since it is constant at points which are identified. Given $J = (J^-, J^+) \in \mathcal{J}^\Theta$ we define the space

$$\mathcal{M}^\Theta_\rho(J) := \left\{ \begin{pmatrix} u^- : D_\rho \to Y \\ u^+ : D \to X \end{pmatrix} \,\middle|\, \begin{matrix} \partial_s u^\pm + J^\pm(u^\pm)\partial_t u^\pm = 0\,, \\ \forall\, \theta : (u^-(e^{i\theta}), u^+(e^{i\theta})) \in V \end{matrix} \right\} .$$

Also define $\mathcal{M}^\Theta(J) := \{(u, \rho) \mid u \in \mathcal{M}^\Theta_\rho(J)\}$ and the subspace

$$\mathcal{M}^\Theta(p_0, p_1, r, q) \subset \mathcal{M}^\Theta(J)$$

consisting of elements with pointwise constraints

$$u^-(i/2) \in W^s(p_0), \quad u^-(-i/2) \in W^s(p_1), \quad u^-(i) \in W^s(r), \quad u^+(0) \in W^u(q)\,.$$

Topologically every element $u \in \mathcal{M}^\Theta_\rho(J)$ is a disk in $Y \times X$ with boundary on V and we denote by $\mu_{\mathrm{Mas}}(u) \in \mathbb{Z}$ its Maslov index. We show similarly to Lemma 11.1.4 that there exists a comeager subset of almost complex structures $\mathcal{J}^\Theta_{\mathrm{reg}} \subset \mathcal{J}^\Theta$ such that for all $J \in \mathcal{J}^\Theta_{\mathrm{reg}}$ the space $\mathcal{M}^\Theta(J)$ is component-wise a smooth manifold of dimension $\mu_{\mathrm{Mas}}(u) + \dim V + 1$ at $(u, \rho) \in \mathcal{M}^\Theta(J)$ and that $\mathcal{M}^\Theta(p_0, p_1, r, q)$ is cut-out transversely. Abbreviate again $\mathcal{M}^\Theta(p_0, p_1, r, q)_{[d]}$ the union of the d-dimensional components for any $d \in \mathbb{N}_0$. Not so obvious is the following compactness lemma. We assume without loss of generality that the Morse function $f_{V \times_\pi V}$ on the fibre square $V \times_\pi V$ is chosen such that in a tubular neighborhood of the submanifold $\Delta_V \subset V \times_\pi V$ it equals $f_V +$(neg. quadratic form) and that f_V has a unique minimum. We denote the minimum considered as a critical point of the function $f_{V\times_\pi V}$ by $s_{\min}$. The Morse index of $s_{\min}$ is $\dim G/T$. By these assumption the Poincaré dual to the diagonal $[\Delta_V]$ is represented by $s_{\min}$ (see [21, Section 3]).

Lemma 11.1.13. *Suppose that Y is monotone and X simply connected. Suppose that $\dim G/T \leq 2c_Y - 2$. If $\mu(r) \leq 2c_Y - 1$ then $\mathcal{M}^\Theta(p_0, p_1, r, q)_{[0]}$ is finite and if moreover $\mu(r) \leq 2c_Y - 2$ then $\mathcal{M}^\Theta(p_0, p_1, r, q)_{[1]}$ is compact up to adding the boundary given by*

- *broken Morse trajectories of index one,*
- *$\mathcal{M}^\chi(p_0, p_1, r, s_{\min}, q)_{[0]}$ as $\rho \to 1$,*
- *$\mathcal{M}^\Theta_{1/2}(p_0, p_1, r, q)_{[0]}$ as $\rho \to 1/2$.*

Proof. Fix $d = 0, 1$ and let $(u^\nu, \rho^\nu) \in \mathcal{M}^\Theta(p_0, p_1, r, q)_{[d]}$ be given. If ρ^ν is bounded away from 1 we conclude similar as in the proof of Lemma 11.1.6 that $|\partial_s u^\nu|$ is uniformly bounded, which is sufficient to show the claim. In the case when ρ^ν is not bounded away from 1 we need a little different argument, which we summarize at first: We identify u^ν with holomorphic strips in $Z := X^4 \times Y^4$ with boundary on four Lagrangians. As ρ^ν converges to 1 we show that the strip breaks into two strips, one of which is constant in a generic situation and the other is identified with an element in $\mathcal{M}^\chi(p_0, p_1, r, s_{\min}, q)_{[0]}$. The reverse process is then given by gluing a constant strip to an element in $\mathcal{M}^\chi(p_0, p_1, r, s_{\min}, q)_{[0]}$. We give details.

Define the quater disk $D^{++} := \{z = s + it \in \mathbb{C} \mid |z| \leq 1, s, t \geq 0\}$ the strip $\Sigma := \{z = s + it \in \mathbb{C} \mid t \in [0, 1]\}$. There exists a homeomorphism $\varphi : \Sigma \to D^{++} \setminus \{0, 1\}$ which fixes i and is holomorphic in the interior. Given $(u^-, u^+) \in \mathcal{M}^\Theta_\rho(J)$ we define a map $\widetilde{u} = (\widetilde{u}^-, \widetilde{u}^+) : \Sigma \to Z := Y^4 \times X^4$ by

$$
\begin{aligned}
&\widetilde{u}_0^\pm(z) = u^\pm(w), \qquad && \widetilde{u}_1^\pm(z) = u^\pm(-\bar{w}), \qquad && w = \varphi(z)\,,\\
&\widetilde{u}_2^\pm(z) = u^\pm(-w), && \widetilde{u}_3^\pm(z) = u^\pm(\bar{w})\,.
\end{aligned}
$$

Denote the spaces $\Delta_Y^{1122} = \{(y, y, y', y') \mid y, y' \in Y\} \subset Y^4$, $\Delta_Y^{1221} = \{(y, y', y', y) \mid y, y' \in Y\}$ etc. and $R^\nu := \varphi^{-1}(\rho^\nu)$. We obtain a sequence of strips $(\widetilde{u}^\nu)$ satisfies the boundary conditions

$$
\begin{aligned}
\tilde{u}^\nu\big|_{s\leq 0, t=1} &\in \Delta_Y^{1122} \times \Delta_X^{1122}, \quad & \tilde{u}^\nu\big|_{s\geq 0, t=1} &\in V \times V \times V \times V\,,\\
\tilde{u}^\nu\big|_{s\leq R^\nu, t=0} &\in \Delta_Y^{1122} \times \Delta_X^{1221}, & \tilde{u}^\nu\big|_{s\geq R^\nu, t=0} &\in \Delta_Y^{1221} \times \Delta_X^{1221}\,.
\end{aligned}
\tag{11.1.19}
$$

By Gromov-compactness a subsequence converges to a stable map $(v_i)_{i\in T}$ such that $E(u^\nu) \to \sum_{i\in T} E(v_i)$, the tree T has two distinguished vertices $i_-, i_+ \in T$ such that the map $v_- := v_{i_-} : \Sigma \to Z$ satisfies the boundary condition

$$
v_-\big|_{s\leq 0, t=1} \in \Delta_Y^{1122} \times \Delta_X^{1122}, \qquad v_-\big|_{s\geq 0, t=1} \in V^4, \qquad v_-\big|_{t=0} \in \Delta_Y^{1122} \times \Delta_X^{1221}\,, \tag{11.1.20}
$$

the map $v_+ := v_{i_+} : \Sigma \to Z$ satisfies the boundary condition

$$
v_+\big|_{t=1} \in V^4, \qquad v_+\big|_{s\leq 0, t=0} \in \Delta_Y^{1122} \times \Delta_X^{1221}, \qquad v_+\big|_{s\geq 0, t=0} \in \Delta_Y^{1221} \times \Delta_X^{1221}\,,
$$

for all vertices $i \in T$ between i_- and i_+ the map v_i is a J_Z-holomorphic strip with boundary on $(V^4, \Lambda_Y^{1122} \times \Lambda_X^{1221})$ and for all other indices $i \in T$ the map is either a J_Z-holomorphic sphere, disk with boundary on any of the Lagrangians or strip with boundary on $(\Lambda_Y^{1122} \times \Lambda_X^{1122}, \Lambda_Y^{1122} \times \Lambda_X^{1221})$ or $(V^4, \Lambda_Y^{1221} \times \Lambda_X^{1221})$.

We have $V^4 \cap \Delta_Y^{1122} \times \Delta_X^{1221} = V^4 \cap \Delta_Y^{1122} \times \Delta_X^{1111} \cong \Delta_{V \times_\pi V}$. The constant map $\bar{v} : \Sigma^+ \to Z$, $(s,t) \mapsto v_+(-\infty)$ is a point in $\bar{v} \in V^4 \cap \Delta_Y^{1122} \times \Delta_X^{1221} \cap \Delta_Y^{1122} \times \Delta_X^{1122}$, hence satisfies (11.1.20). Thus the glued map $\bar{v} \# v_+$ satisfies the boundary condition (11.1.19). Reversing the identification $\bar{v} \# v_+$ is a disk in $Y \times X$ with boundary on V. Since V is τ-monotone with minimal Maslov number $2c_Y$, we conclude that if v_+ is non-constant, then

$$2k\tau c_Y = \int (\bar{v} \# v_+)^* \omega_Z = \int v_+^* \omega_Z + \int \bar{v}^* \omega_Z = \int v_+^* \omega_Z > 0 \,.$$

This shows that $E(v_+) \geq 2\tau c_Y$. The strip v_- gives an element $(v^-, v^+) \in \mathcal{M}_1^\Theta(J)$ which is identified with an element $(u_0^-, u_1^-, u^+) \in \mathcal{M}^\chi(J)$ satisfying $u_0^-(1) = u_1^-(1)$ via $u_0^-(z^2) = v^-(z)$ for all $z \in D_+$, $u_1^-(z^2) = v^-(z)$ for all $z \in D_-$ and $u^+(z) = v^+(z)$. Using that $E(v_i) \geq 2\tau c_Y$ for all $i \in T \setminus \{i_-\}$, we show as Lemma 11.1.12 that v_+ must be constant and we have no bubbling. Generically since $u_0^-(1) = u_1^-(1)$ we have$(u_0^-(1), u_1^-(1)) \in W^s(s_{\min})$. This shows the claim. □

Lemma 11.1.14. *For all $a, b \in QH^*(Y;\Lambda)$*

$$\Phi(a * b) = \chi(a \otimes b \otimes d_V) \,,$$

in which $d_V \in H^{2m}(V \times_\pi V;\Lambda)$ is the Poincare dual of the diagonal $\Delta_V \subset V \times_\pi V$.

Proof. The elements of $\mathcal{M}^\Theta(J)$ are holomorphic disk on V, hence the space is canonically oriented by the relative spin structure and we obtain corresponding orientations on $\mathcal{M}^\Theta(p_0, p_1, r, q)$ using Lemma 9.1.3. We define the Λ-linear map

$$\begin{aligned} \Theta : C^*(f_Y;\Lambda) \otimes C^*(f_Y;\Lambda) \otimes C^{*\leq 2m+1}(f_V;\Lambda) &\to C^{*-2m-1}(f_X;\Lambda)\,, \\ p_0 \otimes p_1 \otimes r &\mapsto \sum\nolimits_{q \in \operatorname{crit} f_X} \sum\nolimits_{u \in \mathcal{M}^\Theta(p_0,p_1,r,q)_{[0]}} \operatorname{sign} u \cdot q \otimes \lambda^{\mu_{\mathrm{Vit}}(u)/2c_Y} \,. \end{aligned}$$

From Lemma 11.1.13 we conclude that Θ is well-defined and establishes a cochain homotopy between $\widehat{\chi}(\cdot \otimes s_{\min})$ and the cochain homomorphism θ obtained by counting the elements of $\mathcal{M}^\Theta_{1/2}(p_0, p_1, r, q)_{[0]}$. Another homotopy argument shows that θ is cochain homotopic to $\widehat{\Phi} \circ (\mu \otimes \mathrm{id})$, where μ denotes the cochain homomorphism of the quantum cup product. Since $s_{\min}$ is a cycle generating the Poincaré dual of the diagonal class d_V we conclude

$$\Phi(a * b) = \frac{1}{|W|} \widehat{\Phi}(a * b \otimes \mathrm{D}) = \frac{1}{|W|} \widehat{\chi}(a \otimes b \otimes \mathrm{D} \otimes d_V) = \chi(a \otimes b \otimes d_V) \,.$$

This proves the lemma. □

Right homotopy

In this subsection we deduce equation (11.1.7). We mirror the construction from the previous step to obtain a cochain homotopy relating the quantum product of two Φ with χ. This time the branch point is approaching from the right-hand side.

Moduli space We study maps defined for every $\rho \in [1/2, 1]$ on $S_\rho := D^+ \sqcup D^- / \sim$ with identifications

$$(s, 0^+) \sim (s, 0^-) \text{ if } |s| \leq \rho \quad \text{and} \quad \begin{matrix} (s, 0^-) \sim (-s, 0^-) \\ (s, 0^+) \sim (-s, 0^+) \end{matrix} \text{ if } |s| \geq \rho\,.$$

For all $1/2 \leq \rho < 1$ the surface S_ρ is homeomorphic to the cylinder $S^1 \times [0, 1]$. If $\rho = 1$ it is homeomorphic to a disk with two boundary points identified. Define $\mathcal{J}^\Xi$ as the space of maps $J = (J_0^-, J_1^-, J^+) \in C^\infty\big(D, \mathrm{End}(TY, \omega_Y) \oplus \mathrm{End}(TY, \omega_Y) \oplus \mathrm{End}(TX, -\omega_X)\big)$ such that where $J_0^-(z) = J_1^-(z) = J_\infty^-$ unless $1/4 \leq |z| \leq 3/4$ and $J^+(z) = -J_\infty^-$ unless $1/8 \leq |z \pm i/2| \leq 1/4$. Given a tuple $J = (J_0^-, J_1^-, J^+) \in \mathcal{J}^\Xi$ and $\rho \in [1/2, 1)$ we define

$$\mathcal{M}^\Xi_\rho(J) = \left\{ \begin{pmatrix} u_0^- : D \to Y \\ u_1^- : D \to Y \\ u^+ : S_\rho \to X \end{pmatrix} \middle| \begin{matrix} \partial_s u + J(u)\partial_t u = 0\,, \\ \forall\, \theta \in [0, \pi] : (u_0^-(e^{2i\theta}), u^+(e^{i\theta})) \in V\,, \\ \forall\, \theta \in [0, \pi] : (u_1^-(e^{2i\theta}), u^+(-e^{i\theta})) \in V\,. \end{matrix} \right\}.$$

Moreover define $\mathcal{M}^\Xi(J) = \{(u, \rho) \mid u \in \mathcal{M}^\Xi_\rho(J), \rho \in [1/2, 1)\}$ and the subspace

$$\mathcal{M}^\Xi(p_0, p_1, r_0, r_1, q) \subset \mathcal{M}^\Xi(J)$$

of all elements satisfying the pointwise constraints

$$\begin{matrix} u_0^-(0) \in W^s(p_0)\,, & u_0^-(-1) \in W^s(r_0)\,, & u^+(0) \in W^s(q)\,, \\ u_1^-(0) \in W^s(p_1)\,, & u_1^-(1) \in W^s(r_1)\,. & \end{matrix}$$

The elements of the space $\mathcal{M}^\Xi(J)$ are holomorphic quilt without punctures hence by [75, Theorem 3.9] and index formula in [53, Appendix C] we conclude that $\mathcal{M}^\Xi(J)$ is a manifold of dimension $\mu_{\mathrm{Mas}}(u) + \dim Y + 1$ at $u \in \mathcal{M}^\Xi(J)$. Similarly we show that $\mathcal{M}^\Xi(p_0, p_1, r_0, r_1, q)$ is cut-out transversely. Let $\mathrm{pr}_2 : V \times_\pi V \to V$ denote the restriction of the projection to the second factor. For the next lemma we assume that the Morse function $f_{V \times_\pi V}$ is a small perturbation of the function $\mathrm{pr}_2^* f$ such that

- the critical points of $f_{V \times_\pi V}$ project to critical points of f_V via pr_2,
- gradient flow lines of $f_{V \times_\pi V}$ project to gradient flow lines of f_V via pr_2,
- for every $r \in \mathrm{crit}\, f_V$ the function $f_{V \times_\pi V}$ restricted to each fibre $\mathrm{pr}_2^{-1}(r) \subset V \times_\pi V$ is a Morse function with only one minimum, denoted $\hat{r}$.

If we assume this special form of $f_{V \times_\pi V}$ the pull-back $\mathrm{pr}_2^* : H^*(Y) \to H^*(Y \times_\pi Y)$ is defined on cochain level by sending r to $\hat{r}$ (for details see [57])

Lemma 11.1.15. *If $\mu(r_0), \mu(r_1) \leq 2c_Y - 1$, then $\mathcal{M}^{\Xi}(p_0, p_1, r_0, r_1, q)_{[0]}$ is finite and if moreover $\mu(r_0), \mu(r_1) \leq 2c_Y - 2$, then the space $\mathcal{M}^{\Xi}(p_0, p_1, r_0, r_1, q)_{[1]}$ is compact up to adding*

- *broken Morse trajectories of index one,*
- *$\mathcal{M}^{\Xi}_{1/2}(p_0, p_1, r_0, r_1, q)_{[0]}$ as $\rho \to 1/2$,*
- *$\mathcal{M}^{\chi}(p_0, p_1, r_0, \hat{r}_1, q)_{[0]}$ as $\rho \to 1$.*

Proof. The new part is when ρ approaches 1. Let $(u^\nu, \rho^\nu) \in \mathcal{M}^{\Xi}(J)$ be a sequence with $\rho^\nu \to 1$. We use notations from the proof of Lemma 11.1.13. Given $(u, \rho) \in \mathcal{M}^{\Xi}(J)$ we define the strip $\tilde{u} = (\tilde{u}^-, \tilde{u}^+) : \Sigma \to Z := Y^4 \times X^4$ via where $w = \varphi(z)$

$$\begin{aligned} \tilde{u}_0^-(z) &= u_0^-(w^2)\,, & \tilde{u}_1^-(z) &= u_0^-(\bar{w}^2)\,, & \tilde{u}_0^+(z) &= u^+(w)\,, & \tilde{u}_1^+(z) &= u^+(-\bar{w})\,, \\ \tilde{u}_2^-(z) &= u_1^-(w^2)\,, & \tilde{u}_3^-(z) &= u_1^-(\bar{w}^2)\,, & \tilde{u}_2^+(z) &= u^+(-w)\,, & \tilde{u}_3^+(z) &= u^+(\bar{w})\,. \end{aligned}$$

Set $R^\nu := \varphi^{-1}(\rho^\nu) \to \infty$. We have the boundary condition

$$\begin{aligned} \tilde{u}^\nu\big|_{s\leq 0, t=1} &\in \Delta_Y^{1122} \times \Delta_X^{1122}, & \tilde{u}^\nu\big|_{s\geq 0, t=1} &\in V \times V \times V \times V\,, \\ \tilde{u}^\nu\big|_{s\leq R^\nu, t=0} &\in \Delta_Y^{1122} \times \Delta_X^{1221}, & \tilde{u}^\nu\big|_{s\geq R^\nu, t=0} &\in \Delta_Y^{1122} \times \Delta_X^{1122}\,. \end{aligned}$$

As in the proof of Lemma 11.1.13 we obtain a stable map $(v_i)_{i\in T}$ with two distinguished vertices i_-, i_+ such that $v_- := v_{i_-}$ is identified with an element in $\mathcal{M}^{\Xi}_1(J) \cong \mathcal{M}^{\chi}(J)$ and $v_+ := v_{i_+}$ satisfies the boundary condition

$$v_+\big|_{t=1} \subset V^4, \qquad v_+\big|_{s\leq 0, t=0} \subset \Delta_Y^{1122} \times \Delta_X^{1221}, \qquad v_+\big|_{s\geq 0, t=0} \subset \Delta_Y^{1122} \times \Delta_X^{1122}\,.$$

It suffices to show either v_+ is constant or $E(v_+) \geq 2\tau c_Y$. We have $v_+(-\infty) \in V^4 \cap \Delta_Y^{1122} \times \Delta_X^{1221} = V^4 \cap \Delta_Y^{1221} \times \Delta_X^{1111} \cong \Delta_{V\times_\pi V}$, i.e.

$$v_+(-\infty) = (y, y, y', y', x, x, x, x)\,,$$

for some $(y, y') \in V \times_\pi V$ with $\pi(y) = \pi(y') = x$. The fibre $\pi^{-1}(x) \subset V$ is isomorphic to G/T thus connected. We find a path $\gamma : \mathbb{R} \to \pi^{-1}(x)$ such that $y = \gamma(-s)$ and $y' = \gamma(s)$ for all $s \geq 1$. Define the map $\bar{v} : \Sigma \to Z$, $(s,t) \mapsto \bar{v}(s,t)$ by

$$\bar{v}(s,t) = (y, y, \gamma(s), \gamma(s), x, x, x, x)\,.$$

For some $R \geq 0$ the connected sum $w := \bar{v} \# v'$ satisfies the boundary conditions

$$\begin{aligned} w\big|_{s\leq 0, t=1} &\subset \Delta_Y^{1221} \times \Delta_X^{1221}, & w\big|_{s\geq 0, t=1} &\subset V^4\,, \\ w\big|_{s\leq R, t=0} &\subset \Delta_Y^{1122} \times \Delta_X^{1122} & w\big|_{s\geq R, t=0} &\subset \Delta_Y^{1122} \times \Delta_X^{1122}\,. \end{aligned}$$

A similar unfolding process as in the proof of Lemma 11.1.13 shows that w defines a disk with boundary on V. Hence $E(w) = 2k\tau c_Y$ for some $k \in \mathbb{Z}$. Moreover $E(w) = E(v_+) > 0$. This shows that $E(v_+) \geq 2\tau c_Y$ for a non-constant v_+. $\square$

Lemma 11.1.16. *For all $a, b \in QH^*(Y;\Lambda)$ we have*

$$\Phi(a) * \Phi(b) = \frac{1}{|W|}\chi(a \otimes b \otimes \mathrm{pr}_2^*\mathrm{D}) \ .$$

Proof. The relative spin structure defines an orientation on $\mathcal{M}^{\Xi}(J)$ and we obtain an orientation on $\mathcal{M}^{\Xi}(p_0, p_1, r_0, r_1, q)$ via Lemma 9.1.3. We define the homomorphism

$$\begin{aligned} \Xi : C^*(f_Y) \otimes C^*(f_Y) \otimes C^{*\leq 2m+1}(f_V) \otimes C^{*\leq 2m+1}(f_V) \otimes \Lambda \to C^{*-4m-1}(f_X) \otimes \Lambda \, , \\ p_0 \otimes p_1 \otimes r_0 \otimes r_1 \mapsto \sum\nolimits_{q \in \mathrm{crit}\, f_X} \sum\nolimits_{u \in \mathcal{M}^{\Xi}(p_0,p_1,r_0,r_1,q)_{[0]}} \mathrm{sign}\, u \cdot q \otimes \lambda^{\mu_{\mathrm{Mas}}(u)/2c_Y} \ . \end{aligned}$$

Using Lemma 11.1.15, we see that the homomorphism Ξ is well-defined and establishes a cochain homotopy between $\widehat{\chi} \circ (\mathrm{id} \otimes \mathrm{pr}_2^*)$ and the homomorphism ξ obtained by counting elements in the moduli space $\mathcal{M}^{\Xi}_{1/2}(p_0, p_1, r_0, r_1, q)_{[0]}$. Another homotopy argument shows that ξ is cochain homotopic to $\mu \circ (\widehat{\Phi} \otimes \widehat{\Phi})$, where μ denotes the cochain homomorphism of the quantum cup product. We conclude for $a, b \in QH^*(Y)$

$$\begin{aligned} \Phi(a) * \Phi(b) = \frac{1}{2|W|}\widehat{\Phi}(a \otimes \mathrm{D}) * \widehat{\Phi}(b \otimes \mathrm{D}) = \frac{1}{2|W|}\widehat{\chi}(a \otimes b \otimes \mathrm{D} \otimes \mathrm{pr}_2^*\mathrm{D}) = \\ = \frac{1}{|W|}\chi(a \otimes b \otimes \mathrm{pr}_2^*\mathrm{D}) \ . \end{aligned}$$

This shows the lemma. □

Weyl group action

In this subsection deduce the identity (11.1.8).

Lemma 11.1.17. *For all $w \in W$, $a, b \in H^*(Y;\Lambda)$ and $c \in H^*(V \times_\pi V;\Lambda)$ we have*

$$\chi(a \otimes b \otimes c) = \chi(w.a \otimes b \otimes (w,1).c) \ .$$

Proof. As before we construct a cochain homotopy between the morphisms on cochain level. Actually we will use two cochain homotopies. The first comes from a cobordism obtained by a homotopy of almost complex structures and the second by varying the length of the Morse trajectories. In the process we possibly pick up a minus sign in the first homotopy which is then canceled by the second homotopy. We give details.

Fix an arbitrary element $w \in W$ and denote by $\psi : Y \to Y$ the symplectomorphism given by the action with w. Recall that the sign $(-1)^w \in \{\pm 1\}$ is given as the determinant of the linear action of w on the Lie algebra of the maximal torus (cf. equation (D.1.5)). The map ψ preserves the submanifold $V \subset Y$ and $\psi\big|_V : V \to V$ is orientation preserving (resp. reversing) if the sign of w is positive (resp. negative). Given $u = (u_0^-, u_1^-, u^+) \in \mathcal{M}^{\chi}(p_0, p_1, r, s, q)$ then $(v_0^-, v_1^-, v^+) := (\psi \circ u_0^-, u_1^-, u^+)$ is $(\psi_* J_0^-, J_1^-, J^+)$-holomorphic, where $\psi_* J_0^- := \mathrm{d}\psi \circ J_0^- \circ \mathrm{d}\psi^{-1}$, and satisfies the point constraints

$$\begin{aligned} &v_0^-(0) \in \psi(W^s(p_0)), \qquad v_0^-(-1) \in \psi\big(W^s(r)\big), \qquad v^+(0) \in W^v(q) \\ &v_1^-(0) \in W^s(p_1), \qquad (v_0^-(1), v_1^-(1)) \in (\psi \times \mathrm{id})\,(W^s(s)) \ . \end{aligned} \tag{11.1.21}$$

Let $\mathcal{M}^{\Omega}_0(p_0, p_1, r, s, q)$ be the space of elements in $\mathcal{M}^{\chi}(J)$ with these constraints. We define a the Λ-linear homomorphism Ω similar to $\widehat{\chi}$ by replacing $\mathcal{M}^{\chi}(p_0, p_1, r, s, q)_{[0]}$ with $\mathcal{M}^{\Omega}_0(p_0, p_1, r, s, q)_{[0]}$ in the formula (11.1.18). Using a homotopy of $\psi_* J_0^-$ to J_0^- we define a cochain homotopy between $\widehat{\chi}$ and the homomorphism $(-1)^w \cdot \Omega$. The sign $(-1)^w$ comes up because the identification with $\mathcal{M}^{\chi}(p_0, p_1, r, s, q)_{[0]}$ changes sign depending on the orientation of ψ restricted to V.

We come to the second homotopy. Denote by φ_{f_Y}, φ_{f_V} and $\varphi_{f_{V\times_\pi V}}$ the the negative gradient flow of f_Y, f_V and $f_{V\times_\pi V}$ respectively. We abbreviate the compositions $\psi_Y^a := \varphi^a_{f_Y} \circ \psi$, $\psi_V^a := \varphi^a_{f_V} \circ \psi$ and $(\psi \times \mathrm{id})^a := \varphi^a_{f_{V\times_\pi V}} \circ (\psi \times \mathrm{id})$ for all $a \geq 0$. Let $\mathcal{M}^{\Omega}(p_0, p_1, r, s, q)$ be the space of tuples (u, a) where $u \in \mathcal{M}^{\chi}(J)$ and $a \geq 0$ such that the following pointwise constraints hold

$$\begin{aligned} &v_0^-(0) \in \psi_Y^a(W^s(p_0)), && v_0^-(-1) \in \psi_V^a\big(W^s(r)\big), && v^+(0) \in W^v(q), \\ &v_1^-(0) \in W^s(p_1), && (v_0^-(1), v_1^-(1)) \in (\psi \times \mathrm{id})^a\,(W^s(s)) \,. \end{aligned}$$

The compactification of the space $\mathcal{M}^{\Omega}(p_0, p_1, r, s, q)_{[1]}$ has the boundary (the spaces $\mathcal{M}^{\psi}(p_0, p_0')$ etc. are defined in (3.3.2))

- Morse breaking of index one,
- $\mathcal{M}^{\Omega}_0(p_0, p_1, r, s, q)_{[0]}$ as $a \to 0$,
- $\mathcal{M}^{\chi}(p_0', p_1, r', s', q)_{[0]} \times \mathcal{M}^{\psi}(p_0, p_0') \times \mathcal{M}^{\psi}(r, r') \times \mathcal{M}^{\psi\times\mathrm{id}}(s, s')$ for all critical points p_0', s' and r' with $\mu(s') = \mu(s)$, $\mu(r') = \mu(r)$ and $\mu(p_0') = \mu(p_0)$ as $a \to \infty$.

We conclude that $\mathcal{M}^{\Omega}(p_0, p_1, r, s, q)_{[1]}$ defines a cobordism showing that Ω is cochain homotopic to $\widehat{\chi} \circ (\psi^* \otimes \mathrm{id} \otimes (\psi|_V)^* \otimes (\psi \times \mathrm{id})^*)$. The class D is anti-invariant (see equation (D.2.5)). Using all above homotopies we conclude that on cohomology level

$$\begin{aligned} \chi(a \otimes b \otimes c) &= \frac{1}{|W|}\,\widehat{\chi}(a \otimes b \otimes \mathrm{D} \otimes c) \\ &= \frac{(-1)^w}{|W|}\,\Omega(a \otimes b \otimes \mathrm{D} \otimes c) \\ &= \frac{(-1)^w}{|W|}\,\widehat{\chi}(w.a \otimes b \otimes w.\mathrm{D} \otimes (w,1)c) \\ &= \frac{(-1)^w}{|W|}\,(-1)^w\widehat{\chi}(w.a \otimes b \otimes \mathrm{D} \otimes (w,1)c) \\ &= \frac{1}{|W|}\,\widehat{\chi}(w.a \otimes b \otimes \mathrm{D} \otimes (w,1)c) \\ &= \chi(w.a \otimes b \otimes (w,1)c) \,. \end{aligned}$$

This shows the lemma. □

11.1.3. Surjectivity and the kernel of Φ

We are left to show that Φ restricted to $QH^*(Y;\Lambda)^W$ is surjective and compute the kernel.

Lemma 11.1.18. *The homomorphism $\Phi|_{QH^*(Y;\Lambda)^W}$ is surjective.*

Proof. Recall that $QH^*(Y;\Lambda) = H^*(Y;\mathbb{Q}) \otimes \Lambda$. By monotonicity we have $\mu_{\mathrm{Mas}}(u) \geq 0$ for all $u \in \mathcal{M}^\Phi(J)$ which implies that Φ respects the filtration induced by the powers of λ and we have the expansion

$$\Phi = \Phi_0 \otimes \mathrm{id} + \Phi_1 \otimes \lambda + \Phi_2 \otimes \lambda^2 + \dots$$

with $\Phi_k : H^*(Y;\mathbb{Q}) \to H^{*-2kc_Y}(X;\mathbb{Q})$ for all $k \in \mathbb{N}_0$. The homomorphism Φ_0 is defined by purely Morse-theoretical means and given by $\pi^! \circ (\mathrm{D} \smile \cdot) \circ i^* : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$. In [50] it is shown that Φ_0 is surjective when restricted to $H^*(Y;\mathbb{Q})^W$ (in particular see [50, equation (3.2)]). Let $\Psi : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})^W$ be a right-inverse. With $\Theta_k := \Phi_k \circ \Psi : H^*(X;\mathbb{Q}) \to H^{*-2kc_Y}(X;\mathbb{Q})$ and $\Theta := \sum_{k\geq 1} \Theta_k \otimes \lambda^k$ we have

$$\Phi \circ (\Psi \otimes \mathrm{id}) = \mathrm{id} + \Theta_1 \otimes \lambda + \Theta_2 \otimes \lambda^2 + \dots = \mathrm{id} + \Theta\ .$$

The homomorphism Θ_k is nil-potent for all $k \geq 1$, thus the homomorphism Θ is nil-potent and $(\mathrm{id} + \Theta)$ is invertible. Hence $(\Psi \otimes \mathrm{id}) \circ (\mathrm{id} + \Theta)^{-1}$ defines a right-inverse of Φ. $\square$

Lemma 11.1.19. *The kernel of $\Phi|_{QH^*(Y;\Lambda)^W}$ is given by $\{a \in QH^*(Y;\Lambda)^W \mid a * \mathrm{D} = 0\}$.*

Proof. A simple homotopy argument, which turns a boundary marked point into an interior marked point, shows that $\widehat{\Phi}(a \otimes i^*\mathrm{D}) = \widehat{\Phi}(a * \mathrm{D} \otimes 1)$ for all $a \in QH^*(Y;\Lambda)$. To define $\widehat{\Phi}(a * \mathrm{D} \otimes 1)$ we only need to prescribe the interior marked point of the disk, which without loss of generality is 0. After possibly another homotopy, we assume that the almost complex structure J used to define $\mathcal{M}^\Phi(J)$ is invariant under the S^1-rotation of the disk. We conclude that the subspace of non-constant maps in $\mathcal{M}^\Phi(J)$ admits a free S^1-action and will not show-up in the count of $\widehat{\Phi}(a * \mathrm{D} \otimes 1)$. Hence only the constant maps are counted and $\widehat{\Phi}(a * \mathrm{D} \otimes 1) = \pi^!(i^*(a * \mathrm{D}))$. Gathering all, we conclude similarly to the proof of [50, Thm. A] for some $a \in H^*(Y;\mathbb{Q})^W$

$$\begin{aligned}
a \in \ker\Phi &\iff \Phi(a) * \Phi(b) = 0,\ \forall\, b \in QH^*(Y;\Lambda) && \text{surjectivity of } \Phi\\
&\iff \Phi(a * b) = 0,\ \forall\, b \in QH^*(Y;\Lambda) && \text{equation (11.1.5)}\\
&\iff \widehat{\Phi}(a * b \otimes i^*\mathrm{D}) = 0,\ \forall\, b \in QH^*(Y;\Lambda) && \text{Definition of } \Phi\\
&\iff \widehat{\Phi}(a * b * \mathrm{D} \otimes 1) = 0,\ \forall\, b \in QH^*(Y;\Lambda) && \text{see above}\\
&\iff \pi^!(i^*(a * b * \mathrm{D})) = 0,\ \forall b \in QH^*(Y;\Lambda) && \text{see above}\\
&\iff a * b * \mathrm{D} = 0,\ \forall\, b \in QH^*(Y;\Lambda)\\
&\iff a * \mathrm{D} = 0\ .
\end{aligned}$$

This shows the claim. $\square$

11.2. Quantum Leray-Hirsch Theorem

Fix a fibre $F := \pi^{-1}(\mathrm{pt}) \subset L^V$ and denote the embedding $j : F \subset L^V$. Assume that there exists classes $a_1, \ldots, a_n \in H^*(L^V)$ which pull-back to a basis of $H^*(F)$. The classical Leray-Hirsch isomorphism is given as the linear extension of

$$H^*(F) \otimes H^*(L) \to H^*(L^V), \qquad j^* a_i \otimes b \mapsto a_i \smile \pi^*(b)\,. \tag{11.2.1}$$

We prove the isomorphism (2.2.2) by defining a map with “quantum correction terms”.

Special Morse functions Choose Morse functions $f : L \to \mathbb{R}$ and $f_V : L^V \to \mathbb{R}$ with sufficiently generic Riemannian metrics on L and L^V respectively. We assume without loss of generality that f_V is chosen such that in a tubular neighborhood of $F \subset L^V$ such that it is given by $h +$ (neg. quadratic form) where h is a Morse function on F. Under these assumptions every critical point for h of index k is identified with a critical point for f_V with index $k + \dim L$ and we have a similar identification for negative gradient flow lines between critical points of h. In this form a right-inverse of the umkehr map map associated to the embedding $F \subset L^V$ on Morse chain level is given by the inclusion $C^*(h) \subset C^*(f^V)$ (see [21, Section 3]). The subspaces $L \times L^V \subset X \times Y^-$ and $V \subset X \times Y^-$ intersect cleanly in a manifold which is isomorphic to L^V. We identify the intersection manifold with L^V and assume it is equipped with the Morse function f_V.

Moduli space Fix almost complex structures $J_X \in C^\infty([0,1], \mathrm{End}(TX, \omega_X)$ and $J_Y \in C^\infty([0,1], \mathrm{End}(TY, \omega_Y)$. Let $\mathcal{J}$ be the space of almost complex structures $J \in C^\infty(\mathbb{R} \times [0,1], \mathrm{End}(TX, \omega_X) \oplus \mathrm{End}(TY, -\omega_Y))$ such that $J(\pm s, \cdot) = J_X \oplus -J_Y$ for all $s \geq 1$. Given $J \in \mathcal{J}$, $m \in \mathbb{N}$ and critical points $p \in \mathrm{crit}\, f$ and $r, q \in \mathrm{crit}\, f_V$ we define

$$\widetilde{\mathcal{M}}^{\phi}_m(p, r, q) := \{u = (u_1, \ldots, u_m) \mid \text{a)} - \text{e)}\}$$

be space of tuples $u = (u_1, \ldots, u_m)$ such that there exists $\ell \in \{1, \ldots, m\}$ satisfying

a) $(u_1, \ldots, u_{\ell-1})$ is an J_X-holomorphic pearl trajectory with boundary on (L, L),

b) $u_\ell = (u^X_\ell, u^Y_\ell) : (-\infty, 0] \times [0,1] \to X \times Y^-$ satisfies $\partial_s u_\ell + J(u_\ell)\partial_t u_\ell = 0$, $E(u_\ell) < \infty$ and
$$u_\ell|_{t=0} \subset L \times L^V, \qquad u_\ell|_{t=1} \subset L \times L^V, \qquad u_\ell|_{s=0} \subset V\,, \tag{11.2.2}$$

c) $(u_{\ell+1}, \ldots, u_m)$ is a J_Y-holomorphic pearl trajectory with boundary on (L^V, L^V),

d) there exists numbers $a_-, a_+ \geq 0$ such that
$$\psi^{a_-}(u_{\ell-1}(\infty)) = u^X_\ell(-\infty), \qquad \psi^{a_+}(u^Y_\ell(-\infty)) = u_{\ell+1}(-\infty)\,,$$

e) we have the point constraints
$$u_1(-\infty) \in W^u(p), \quad u_\ell(0) \in W^u(r), \qquad u_m(\infty) \in W^s(q)\,.$$

We denote by $\mathcal{M}_m^\phi(p,r,q)$ the space of $\widetilde{\mathcal{M}}_m^\phi(p,r,q)$ modulo reparametrizations and $\mathcal{M}^\phi(p,r,q) = \bigcup_{m\in\mathbb{N}} \mathcal{M}_m^\phi(p,r,q)$. Provided that the almost complex structures are sufficiently generic each connected component of the space $\mathcal{M}^\phi(p,r,q)$ is a manifold with corners and the dimension of the component containing u is given by

$$\mu(u) + \mu(p) + \mu(r) - \mu(q) - \dim L\,,$$

in which $\mu(u)$ denotes the sum of the Viterbo indices of all u_i. Let N denote the minimal Maslov number of $(L \times L^V, V)$. Assume that $\mu(r) \geq \dim L^V - N + 1$, then standard compactness arguments similarly to Lemma 11.1.6 show that the union of the zero dimensional components $\mathcal{M}^\phi(p,r,q)_{[0]}$ is finite and if moreover $\mu(r) \geq \dim L^V - N + 2$ then compactness and gluing shows that the union of the one-dimensional components $\mathcal{M}^\phi(p,r,q)_{[1]}$ has a compactification up to breaking of height one, i.e. is given by

- $\mathcal{M}(p,p')_{[0]} \times \mathcal{M}^\phi(p',r,q)_{[0]}$ for all critical points $p' \in \operatorname{crit} f$,
- $\mathcal{M}^\phi(p,r,q')_{[0]} \times \mathcal{M}(q',q)_{[0]}$ for all critical points $q' \in \operatorname{crit} g$,
- $\mathcal{M}_0(r,r')_{[0]} \times \mathcal{M}^\phi(p,r',q)_{[0]}$ for all critical points $r' \in \operatorname{crit} g$.

We point out that the last line states that only honest Morse trajectories break off at the corner, which holds due to the bound of the index of r.

Chain map Let $\Lambda := \mathbb{Z}_2[\lambda,\lambda^{-1}]$ denote the ring of Laurent polynomials with $\deg \lambda = -N$. We define the Λ-linear homomorphism

$$\begin{aligned} C\phi : C_*(f;\Lambda) \otimes C_*(h;\Lambda) &\mapsto C_*(f_V;\Lambda)\,, \\ p \otimes r &\mapsto \sum\nolimits_{q\in\operatorname{crit} f} \sum\nolimits_{u\in\mathcal{M}^\phi(p,r,q)_{[0]}} q \otimes \lambda^{\mu(u)/N}\,. \end{aligned} \tag{11.2.3}$$

Here we have used the identification of critical points of h with critical points of f_V which have index at least $\dim L$. Note that since $N \geq \dim F + 2$ we have $\mu(r) \geq \dim L \geq \dim L^V - N + 2$, which was necessary for compactness. We conclude that $C\phi$ is a chain map with respect to the pearl differential and thus induces a map on homology, denoted

$$\phi : QH_*(L,L;\Lambda) \otimes H_*(F;\Lambda) \to QH_*(L^V,L^V;\Lambda)\,. \tag{11.2.4}$$

We want to show that ϕ is an isomorphism. The intersection of $L \times L^V$ with V is connected and using the energy-index relation we conclude that $\tau\mu(u) = E(u)$. In particular $\mu(u) \geq 0$ for all $u \in \mathcal{M}^\phi(p,r,q)$ and $\mu(u) = 0$ if and only if u is constant. The subspace of all $u \in \mathcal{M}^\Phi(p,r,q)$ which are constant is given by the triple intersection $\pi^{-1}(W^u(p)) \cap W^u(r) \cap W^s(q)$, which is transverse after a suitable choice of Morse functions. We conclude that $C\phi$ respects the filtration given by powers of λ and the induced morphism on the first page of the associated spectral sequence is (11.2.3) with $\mathcal{M}^\phi(p,r,q)$ replaced by $\pi^{-1}(W^u(p)) \cap W^u(r) \cap W^s(q)$. This is precisely the Morse-theoretic description of the classical Leray-Hirsch morphism (11.2.1) (up to Poincaré duality). Hence by the Leray-Hirsch theorem $C\phi$ induces an isomorphism on the second page of the spectral sequence. Then it follows by standard algebraic arguments that $C\phi$ also induces an isomorphism on the final page of the spectral sequence (see for example [51, Theorem 3.5]) or in other words ϕ is an isomorphism.

12. Applications

12.1. Quantum cohomology of the complex Grassmannian

Choose integers $k < n$ and define the complex Grassmannian as the moduli space of all complex k-dimensional subspaces in $\mathbb{C}^n$, i.e.

$$\mathrm{Gr}_{\mathbb{C}}(n,k) := \{V \subset \mathbb{C}^n \mid \dim_{\mathbb{C}} V = k\} .$$

equipped with the Fubini-Studi metric the complex Grassmannian is a Kähler manifold and in partiuclar a monotone compact symplectic manifold. The quantum cohomology of $\mathrm{Gr}_{\mathbb{C}}(n,k)$ was first computed by Witten. We illustrate how Theorem 2.1.1 is used to compute it. For simplicity we set the Novikov variable to one.

Proposition 12.1.1. *The rational quantum cohomology of* $\mathrm{Gr}_{\mathbb{C}}(n,k)$ *is given by*

$$\mathbb{Q}[\sigma_1, \sigma_2, \ldots, \sigma_k]/\langle h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n - 1\rangle .$$

in which σ_j *(resp.* h_j*) are the elementary (resp. complete) symmetric polynomials.*

The space $\mathrm{Gr}_{\mathbb{C}}(n,k)$ is also obtained via symplectic reduction of an $U(k)$-action on $\mathbb{C}^{n\times k}$ via $g.A = Ag^*$ where $g \in U(k)$ and $A \in \mathbb{C}^{n\times k}$ thought of as a $n \times k$-matrix (cf. [50, Section 7]). The abelian quotient Y is the k-fold product of the complex projective space $\mathbb{C}\mathbf{P}^{n-1}$ and the abelian/non-abelian correspondence V is given by the complex Stiefel manifold

$$V = \{(\ell_1, \ell_2, \ldots, \ell_k) \in Y \mid \ell_i \perp_{\mathbb{C}} \ell_j \ \forall i \neq j\} ,$$

where by $\ell_i \perp_{\mathbb{C}} \ell_j$ we mean that the complex lines ℓ_i and ℓ_j are perpendicular with respect to the standard Hermitian product on $\mathbb{C}^n$. The projection $\pi : V \to \mathrm{Gr}_{\mathbb{C}}(n,k)$ sends the tuple $(\ell_1, \ell_2, \ldots, \ell_k)$ the the complex k-plane spanned by the lines. The minimal Chern number of Y is n and the dimension of $U(k)/T$ is $k^2 - k$. Hence the theorem applies as long as $2n \geq k^2 - k + 2$.

It is well-known that the rational quantum cohomology ring of $\mathbb{C}\mathbf{P}^{n-1}$ is $\mathbb{Q}[x]/\langle x^n - 1\rangle$. By the Künneth formula the quantum cohomology of the product Y is

$$\mathbb{Q}[x_1, \ldots, x_k]/\langle x_1^n - 1, \ldots, x_k^n - 1\rangle .$$

The Weyl group W of $U(k)$ is the symmetric group of k letters. The group W is acting on $S := \mathbb{Q}[x_1, \ldots, x_k]$ by exchanging the arguments of a polynomial, i.e. for $w \in W$ and $p \in S$ we define

$$(w.p)\,(x_1, x_2, \ldots, x_k) \stackrel{\text{def}}{=} p(x_{w(1)}, x_{w(2)}, \ldots, x_{w(k)}) .$$

The ring of such invariant polynomials is well studied. In particular we know that S^W is a polynomial ring generated by the *elementary symmetric polynomials* $\sigma_1, \dots, \sigma_k \in S$ given by

$$\sigma_d(x_1, \dots, x_k) := \sum_{1 \le j_1 < j_2 < \dots < j_d \le k} x_{j_1} x_{j_2} \dots x_{j_d}\,, \tag{12.1.1}$$

or equivalently generated by the *complete symmetric polynomials* $h_1, \dots, h_k \in S$

$$h_d(x_1, \dots, x_k) := \sum_{1 \le j_1 \le j_2 \le \dots \le j_d \le k} x_{j_1} x_{j_2} \dots x_{j_d}\,. \tag{12.1.2}$$

Denote the ideal $I := \langle x_1^n - 1, \dots, x_k^n - 1 \rangle$. In view of Theorem 2.1.1 we have to compute the ring $(S/I)^W / \operatorname{ann} \mathrm{D}$. We claim that there is an isomorphism

$$(S/I)^W / \operatorname{ann} \mathrm{D} \cong S^W / (I : \mathrm{D} \cap S^W)\,,$$

in which $I : \mathrm{D}$ is the ideal quotient given by $\{p \in S \mid \mathrm{D}p \in I\}$. Indeed, consider the map $\varphi : S \to S^W$, $p \mapsto |W|^{-1} \sum_{w \in W} w.p$. The map φ descends to $\bar{\varphi} : S/I \to S^W/(I : \mathrm{D} \cap S^W)$. It is easy to see that $\bar{\varphi}$ is surjective restricted to the subring $(S/I)^W$ (a preimage of $p + (I : \mathrm{D}) \cap S^W$ being $p + I$) and that the kernel of $\bar{\varphi}$ is the annihilator of D.

The rank of $U(k)$ is k and we identify the symmetric $\mathbb{Q}$-algebra associated to the weight space of $U(k)$ with the polynomial ring S (cf. Section D.1). It is well-known that a set of positive roots for $U(k)$ is given by $(x_i - x_j)_{1 \le i < j \le k} \subset S$ (cf. [55, p. 285]). We conclude that the canonical anti-invariant class is given by

$$\mathrm{D} = \prod_{i<j} x_i - x_j\,.$$

To compute the ideal quotient $I : \mathrm{D}$ we use classical results about the ring of invariant polynomials. We quote from [72, Chapter 7]. A *composition α of length k* is a tuple of k non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$. To such a composition α we associate the an anti-symmetric polynomial, called *alternant*, given by

$$a_\alpha(x_1, x_2, \dots, x_k) := \det \left(x_j^{\alpha_i} \right)_{1 \le i,j \le k}\,.$$

A well-known computation shows that

$$\mathrm{D} = a_\delta, \qquad \delta := (k-1, k-2, \dots, 1, 0)\,. \tag{12.1.3}$$

A *partition λ of length k* is an tuple of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_k)$ which is ordered, i.e.

$$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k\,.$$

The degree of a partition λ is defined by $\deg \lambda := \lambda_1 + \lambda_2 + \dots + \lambda_k$. We denote by Par_k the space of partitions of length k. To a partition λ we associate the *complete symmetric polynomial* given by (12.1.2) and

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}\,,$$

and the *Schur polynomial*

$$s_\lambda := \frac{a_{\lambda+\delta}}{a_\delta} . \tag{12.1.4}$$

In [72, Prop. 7.8.9 and Cor 7.10.6] it is proved that

Proposition 12.1.2. *Each set of polynomials* $\{h_\lambda \mid \lambda \in \mathrm{Par}_k\}$ *and* $\{s_\lambda \mid \lambda \in \mathrm{Par}_k\}$ *constitutes to a* $\mathbb{Q}$*-basis of* S^W*.*

On the space of partitions with the same degree there is a partial order, called *dominance order.* This is defined by $\mu \preceq \lambda$ iff $\deg \mu = \deg \lambda$ and for all $i \leq k$,

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i .$$

Moreover, write $\mu \prec \lambda$ if $\mu \preceq \lambda$ and $\mu \neq \lambda$.

Lemma 12.1.3. *There are numbers* $K_{\lambda\mu} \in \mathbb{N}$ *such that*

$$a_\delta h_\mu = a_{\mu+\delta} + \sum_{\mu \prec \lambda} K_{\lambda\mu} a_{\delta+\lambda} . \tag{12.1.5}$$

These numbers are called the Kostka numbers*.*

Proof. By [72, Cor. 7.12.4] we have $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$ where s_λ have a combinatorial definition. In [72, Thm. 7.15.1] it is shown that this agrees with (12.1.4) and finally in [72, Prop. 7.10.5] that $K_{\lambda\mu}$ has the required properties, that is $K_{\mu\mu} = 1$ and $K_{\lambda\mu} = 0$ unless $\mu \preceq \lambda$. □

Lemma 12.1.4. *Let* $n-k+1 \leq d \leq n-1$ *we have* $\mathrm{D}h_d \in I$*. Moreover we have for all* $d \geq 0$ *we have* $\mathrm{D}(h_{n+d} - h_d) \in I$*.*

Proof. By (12.1.3) we have $\mathrm{D} = a_\delta$. Set $\ell = d - n + k - 1$. The element $\mu = (d, 0, \ldots, 0)$ is maximal for the dominance order. Thus (12.1.5) has a particular simple form,

$$a_\delta h_d = a_{\delta+(d,0,\ldots,0)} = a_{(d+k-1,k-2,\ldots,1,0)} .$$

We have $0 \leq \ell \leq k-2$ and thus $a_{(\ell,k-2,k-3,\ldots,1,0)} = 0$. Therefore after the last equation we have

$$\begin{aligned}
a_\delta h_d &= a_{(k-1+d,k-2,\ldots,1,0)} - a_{(\ell,k-2,k-3,\ldots,1,0)} \\[1em]
&= \det \begin{pmatrix} x_1^{k-1+d} - x_1^\ell & x_1^{k-2} & x_1^{k-3} & \ldots & x_1 & 1 \\ x_2^{k-1+d} - x_2^\ell & x_2^{k-2} & x_2^{k-3} & \ldots & x_2 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ x_k^{k-1+d} - x_k^\ell & x_k^{k-2} & x_k^{k-3} & \ldots & x_k & 1 \end{pmatrix} \\[1em]
&= \det \begin{pmatrix} x_1^\ell(x_1^n - 1) & x_1^{k-2} & x_1^{k-3} & \ldots & x_1 & 1 \\ x_2^\ell(x_2^n - 1) & x_2^{k-2} & x_2^{k-3} & \ldots & x_2 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ x_k^\ell(x_n - 1) & x_k^{k-2} & x_k^{k-3} & \ldots & x_k & 1 \end{pmatrix} .
\end{aligned}$$

Expanding the determinant in the first column shows that $a_\delta h_d$ is of the form

$$(x_1^n - 1)q_1 + (x_2^n - 1)q_2 + \cdots + (x_k^n - 1)q_k \ ,$$

for some polynomials $q_1, q_2, \ldots, q_k \in S$ or equivalently an element of I. This shows the first claim. The second follows from a similar computation. □

Corollary 12.1.5. *For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $n-k+1 \le \lambda_1 \le n-1$ we have $a_\delta h_\lambda \in I$. Moreover for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_j \ge n$ for some j we have $a_\delta(h_\lambda - h_{\lambda'}) \in I$ where λ' is a reordering of $(\lambda_1, \lambda_2, \ldots, \lambda_j - n, \ldots, \lambda_k)$.*

Given $\ell \in \mathbb{N}$ we denote by $\mathrm{Par}_{k,\ell} \subset \mathrm{Par}_k$ the subset of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \le \ell$. By subtracting a multiple of n from the components we see that for any partition $\lambda \in \mathrm{Par}_k$ there exists a unique $\nu \in \mathrm{Par}_{k,n-1}$ such that ν is a reordering of $\lambda - n\alpha$ for some composition α. We call ν the *base* of λ and write $\lambda \sim \nu$. More generally given two partitions $\lambda, \mu \in \mathrm{Par}_k$ we define $\lambda \sim \mu$ if there exists $\nu \in \mathrm{Par}_{k,n-1}$ such that $\lambda \sim \nu$ and $\mu \sim \nu$ and say that λ and μ are *base-equivalent*.

Lemma 12.1.6. *For every $\lambda \in \mathrm{Par}_k$ there is an alternative: Either $a_{\lambda+\delta} \in I$ or there exists $\sigma = \sigma(\lambda) \in \{-1, 1\}$ and a unique $\nu \in \mathrm{Par}_{k,n-k}$ with $\lambda + \delta \sim \nu + \delta$ and $a_{\lambda+\delta} - \sigma a_{\nu+\delta} \in I$.*

Proof. Let $\nu' \in \mathrm{Par}_{k,n-1}$ be the base of $\lambda + \delta$. If $\nu' = (\nu'_1, \nu'_2, \ldots, \nu'_k)$ is not a strict partition (meaning that not necessarily $\nu'_j > \nu'_{j+1}$ for all $j = 1, \ldots, k-1$) then the associated alternant $a_{\nu'}$ vanishes. By a similar computation as in the proof of Lemma 12.1.4 we conclude that $a_{\lambda+\delta} \in I$. If instead ν' has only distinct entries then there exists $\nu \in \mathrm{Par}_{k,n-k}$ such that $\nu' = \nu + \delta$. Then we have $a_{\lambda+\delta} - \sigma a_{\nu+\delta} \in I$. The sign σ is the sign of the permutation of $\lambda + \delta - n\alpha$ to $\nu + \delta$. □

Lemma 12.1.7. *The two ideals are the same*

$$\overline{J} = \langle h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n - 1, h_{n+1} - h_1, h_{n+2} - h_2, \ldots \rangle$$
$$J = \langle h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n - 1 \rangle \ .$$

Proof. Clearly $J \subset \overline{J}$. We show the other inclusion. By [72, Equation (7.13)] and (12.1.1) we have for all $r \ge 0$

$$\sum_{d=0}^{r} (-1)^d \sigma_d h_{r-d} = 0 \ .$$

Note that $\sigma_d = 0$ for $d > k$. We distinguish two cases. First case $d < k$, then

$$h_{d+n} = h_{d+n-1}\sigma_1 - h_{d+n-2}\sigma_2 + \cdots \pm h_{d+n-k}\sigma_k,$$
$$h_d = h_{d-1}\sigma_1 - h_{d-2}\sigma_2 + \cdots \pm \sigma_d \ .$$

Subtracting

$$h_{d+n} - h_d = (h_{d+n-1} - h_{d-1})\sigma_1 - (h_{d+n-2} - h_{d-2})\sigma_2 + \ldots$$
$$\pm (h_n - 1)\sigma_d \mp h_{n-1}\sigma_{d+1} \pm \cdots \pm h_{n+d-k}\sigma_k \ .$$

Thus by induction on d we find that $h_{d+n} - h_d \in J$ for all $d < n$. Suppose now that $d \geq n$, then

$$\begin{aligned} h_{d+n} &= h_{d+n-1}\sigma_1 - h_{d+n-2}\sigma_2 + \cdots \pm h_{d+n-k}\sigma_k \\ h_d &= h_{d-1}\sigma_1 - h_{d-2}\sigma_2 + \cdots \pm h_{d-k}\sigma_k \ . \end{aligned}$$

Subtracting

$$h_{d+n} - h_d = (h_{d+n-1} - h_{d-1})\sigma_1 - (h_{d+n-2} - h_{d-2})\sigma_2 + \cdots \pm (h_{d+n-k} - h_{d-k})\sigma_k \ .$$

Again an induction over d shows that $h_{d+n} - h_d \in J$ for all $d \geq k$. Thus $\overline{J} \subset J$. □

Lemma 12.1.8. *For any $d \geq 0$, the set $\{\mathrm{D}h_\mu \mid \mu \in \mathrm{Par}_{k,n-k},\ \deg\mu = d\}$ is linearly independent in S/I over $\mathbb{Q}$.*

Proof. Suppose there is a dependence relation $\sum_\mu u_\mu \mathrm{D}h_\mu \in I$ with some numbers $u_\mu \in \mathbb{Q}$ where the sum runs over all $\mu \in \mathrm{Par}_{k,n-k}$ with $\deg\mu = d$. By (12.1.5), $\mathrm{D} = a_\delta$ and our definition of the Schur polynomials (12.1.4) as the quotient of two alternates we obtain

$$\sum_\mu \sum_{\{\lambda|\mu\preceq\lambda\}} u_\mu K_{\lambda\mu} a_{\lambda+\delta} \in I \ .$$

We use Lemma 12.1.6 to reduce the expression modulo I. More precisely the alternant $a_{\lambda+\delta}$ is either in I or equivalent to $\sigma(\lambda)a_{\nu+\delta}$ modulo I where $\nu+\delta$ is the base of $\lambda+\delta$ and $\sigma(\lambda) \in \{-1,1\}$ is the sign of the permutation. Using this fact we regroup the previous sum modulo I as

$$\sum_{\nu\in\mathrm{Par}_{k,n-k}} \Big(\sum_{\left\{\lambda \,\middle|\, \substack{\lambda+\delta\sim\nu+\delta \\ \deg\lambda = d}\right\}} \sigma(\lambda) \sum_{\{\mu|\mu\preceq\lambda\}} u_\mu K_{\lambda\mu} \Big) a_{\nu+\delta} \ .$$

Let $w_\nu \in \mathbb{Q}$ denote the term in the parenthesis. By assumption we have $\sum_\nu w_\nu a_{\nu+\delta} \in I$. Since $\nu \in \mathrm{Par}_{k,n-k}$, in the alternant $a_{\nu+\delta}$ no variable x_j appears with power greater or equal to n. Hence the statement $\sum_\nu w_\nu a_{\nu+\delta} \in I$ is equivalent to $\sum_\nu w_\nu a_{\nu+\delta} = 0$. Dividing by a_δ we obtain $\sum_\nu w_\nu s_\nu = 0$ and using the fact from Proposition 12.1.2, that the Schur polynomials are linearly independent, we conclude that for all $\nu \in \mathrm{Par}_{k,n-k}$ we have $w_\nu = 0$. Obviously if $\lambda+\delta \sim \nu+\delta$ and $\deg\lambda = \deg\nu$, then $\lambda = \nu$ and $\sigma(\lambda) = 1$. Thus for all $\nu \in \mathrm{Par}_{k,n-k}$ with $\deg\nu = d$ we have

$$w_\nu = \sum_{\mu\preceq\nu} u_\mu K_{\nu\mu} = 0 \ .$$

Finally the upper triangular form of $K_{\nu\mu}$ (see [72, Prop. 7.10.5]) implies that $u_\mu = 0$. □

Proof of Proposition 12.1.1. In view of Theorem 2.1.1 it suffices to show $(I : \mathrm{D}) \cap S^W = J$. By Lemma 12.1.7 it is even enough to show that $(I : \mathrm{D}) \cap S^W = \overline{J}$. Suppose that $\sum_\lambda u_\lambda h_\lambda \in (I : \mathrm{D}) \cap S^W$ for some $u_\lambda \in \mathbb{Q}$ with $\lambda \in \mathrm{Par}_k$ such that $\deg\lambda = d$. By Corollary 12.1.5 we have:

- if $\lambda \sim \lambda'$ then $\mathrm{D}h_\lambda = \mathrm{D}h_{\lambda'} \mod I$,
- $\mathrm{D}h_\lambda \in I$ whenever $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $n-k+1 \leq \lambda_1 \leq n-1$.

Hence

$$\begin{aligned}\sum_{\deg\lambda=d} u_\lambda h_\lambda \in (I:\mathrm{D}) &\iff \sum_{\deg\lambda=d} u_\lambda \mathrm{D}h_\lambda \in I \\ &\iff \sum_{\mu\in\mathrm{Par}_{k,n-k}} \sum_{\left\{\lambda \;\middle|\; \substack{\deg\lambda=d\\ \lambda\sim\mu}\right\}} u_\lambda \mathrm{D}h_\mu \in I \\ &\iff \sum_{\left\{\lambda \;\middle|\; \substack{\deg\lambda=d\\ \lambda\sim\mu}\right\}} u_\lambda = 0 \qquad \forall \mu \in \mathrm{Par}_{k,n-k}\ .\end{aligned}$$

The last equivalence follows by Lemma 12.1.8. Whenever $\deg\mu = d$ we have $\{\lambda \mid \deg\lambda = d,\ \lambda\sim\mu\} = \{\mu\}$ and hence $u_\mu = 0$ in that case. This shows that $\sum_\lambda u_\lambda h_\lambda \in \overline{J}$. The converse just follow the equivalence upwards. □

Remark 12.1.9. The previous considerations generalizes to other examples. For example yield a computation of the quantum cohomology of the partial or incomplete flag manifold. Given a tuple of natural numbers $(k_1, k_2, \dots, k_r) \in \mathbb{N}^r$ define $n = \sum_j k_j$. The partial flag manifold $\mathrm{Fl}(k_1, \dots, k_r)$ is defined in one of the following equivalent ways:

- the space of filtration $F = (V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset \mathbb{C}^n)$ such that $\dim V_j - \dim V_{j-1} = k_j$ for all $j = 1, \dots, r-1$ (where $\dim V_0 = 0$),
- the homogeneous quotient $\mathrm{Fl}(k_1, \dots, k_r) \cong U(n)/U(k_1) \times \cdots \times U(k_r)$,
- a coadjoint orbit of the action of $U(n)$ on the its dual Lie algebra $\mathfrak{u}(n)^\vee$, where, after an identification of the latter with the space of Hermitian matrices, the orbit is taken at an Hermitian matrix with r eigenvalues of geometric multiplicities $(k_1, \dots, k_r)$ (see [77, Example 5.1.1]).
- the Hamiltonian quotient given as follows. Define the tuple $(\ell_1, \dots, \ell_r) := (k_1, k_1 + k_2, k_1+k_2+k_3, \dots, k_1+\cdots+k_r)$. The group $G := U(\ell_1)\times U(\ell_2)\times\cdots\times U(\ell_{r-1})$ acts on the vector space $V = \mathrm{Hom}(\mathbb{C}^{\ell_1}, \mathbb{C}^{\ell_2}) \oplus \mathrm{Hom}(\mathbb{C}^{\ell_2}, \mathbb{C}^{\ell_3}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^{\ell_{r-1}}, \mathbb{C}^{\ell_r})$ via

  $$(g_1, g_2, \dots, g_{r-1}).(A_1, A_2, \dots, A_{r-1}) = (g_1 A_1 g_2^{-1}, g_2 A_2 g_3^{-1}, \dots, g_{r-1} A_{r-1})\ .$$

  The action is Hamiltonian with moment map $\mu : V \to \mathfrak{u}(\ell_1) \oplus \mathfrak{u}(\ell_2) \oplus \cdots \oplus \mathfrak{u}(\ell_{r-1})$ given by

  $$\mu(A_1, \dots, A_{r-1}) = \frac{i}{2}(A_1^* A_1, A_1 A_1^* - A_2^* A_2, \dots, A_{r-1}^* A_{r-1} - A_r A_r^* - \mathbb{1})\ .$$

  However the associated abelian quotient is not as simple anymore.

12.2. Lagrangian spheres in symplectic quotients

In this section we give the proof of Corollary 2.2.2. We quickly repeat the statement. Let G be a compact group which acts on a complex vector space M via linear maps with moment map μ_G. Let $T \subset G$ be a maximal torus which acts on M with moment map $\mu_T : M \to \mathfrak{t}^\vee$. Let $w \in \mathfrak{t}^\vee$ be the unique value such that the symplectic quotients $Y := \mu_T^{-1}(w)/T$ is a monotone symplectic manifold and set $X := \mu_G^{-1}(w)/G$. We denote the minimal Chern numbers by c_Y and c_X respectively. Assume that $2c_Y \geq \dim G/T + 2$. Since c_X divides c_Y we also have $2c_X \geq \dim G/T + 2$. Suppose that X contains a closed Lagrangian submanifold L which has the same homology as a sphere. The claim of Corollary 2.2.2 is that one of the following holds

- $2c_Y$ divides $n+1$
- $\dim G/T \leq 2$ and $n \leq 4$,

where $\dim L = n$. Without loss of generality we assume that $n \geq 2$ and $\dim G/T \geq 2$. Since the Lagrangian L satisfies $H_1(L) = 0$ we conclude that the minimal Maslov number is divisable by $2c_X$, which is larger than 3 because $2c_X \geq \dim G/T + 2 \geq 4$. Hence Floer homology of L is well-defined. Assume by contradiction that $2c_Y$ does not divide $n+1$. This implies that $2c_X$ does not divide $n+1$. Thus Oh's spectral sequence collapses at the first page and we have

$$HF_*(L, L) \cong H_*(L) \otimes \Lambda, \qquad \Lambda = \mathbb{Z}_2[\lambda, \lambda^{-1}] \text{ with } \deg \lambda = -2c_Y\,.$$

Let $V \subset Y \times X^-$ be the abelian/non-abelian correspondence, which is a fibered correspondence with fibre G/T (cf. Prop. 11.1.1). It is a classical fact that G/T is simply connected. According to the exact homotopy sequence we conclude that L^V is simply connected. This implies that L^V is monotone with minimal Maslov number divisible by $2c_Y$ (cf. Lemma 3.1.2). Since V arises as a T-quotient of the G-principle bundle $\mu_G^{-1}(0)$ the pull-back to the fibre G/T is surjective (cf. equation (D.2.1)) By Theorem 2.2.1 we conclude that

$$HF_*(L^V, L^V) \cong HF_*(L, L) \otimes H_*(G/T; \mathbb{Z}_2) \cong H_*(L; \mathbb{Z}_2) \otimes H_*(G/T; \mathbb{Z}_2) \otimes \Lambda\,.$$

By definition Y is a symplectic toric manifold. Using the Seidel element, there exists an invertible element of degree two in the quantum cohomology of Y (cf. [53, p. 441]) which by the quantum action induces an isomorphism of degree two of $HF_*(L^V, L^V)$. In other words $HF_k(L^V, L^V) \cong HF_{k+2}(L^V, L^V)$ for all $k \in \mathbb{Z}$. In particular

$$\mathbb{Z}_2 \cong HF_0(L^V, L^V) \cong HF_2(L^V, L^V) \cong H_2(G/T)\,.$$

But if $\dim G/T > 2$ then the dimension of $H_2(G/T)$ is at least two (cf. Corollary D.1.8) and we obtain a contradiction. If on the other hand $\dim G/T = 2$ but $n > 4$ then

$$\begin{aligned}\mathbb{Z}_2 \cong HF_0(L^V, L^V) \cong HF_4(L^V, L^V) \cong& \\ &\cong (H_4(L) \otimes H_0(G/T)) \oplus (H_2(L) \otimes H_2(G/T))\,.\end{aligned}$$

Since by assumption $\dim L = n > 4$ the right-hand side vanishes. This leads to a contradiction and hence $n \leq 4$ as claimed.

A. Estimates

A.1. Derivative of the exponential map

In the section we have collected estimates for the derivative of the exponential map of the Levi-Civita connection. These results are well-known, yet we have always included the proofs, since we have not found a good reference. Let M be a compact Riemannian manifold with Levi-Civita connection ∇. The connection induces a splitting of the tangent space $T_\xi(TM)$ at $\xi \in T_pM$ into horizontal and vertical space and we define the horizontal and vertical lift

$$L^{\mathrm{hor}}(\xi) : T_pM \to T_\xi^{\mathrm{hor}}(TM), \qquad L^{\mathrm{ver}}(\xi) : T_pM \to T_\xi^{\mathrm{ver}}(TM) .$$

Associated to the connection is an exponential map $\exp : TM \to M$. Using the horizontal and vertical lifts we define the horizontal and vertical differential of the exponential map at some $\xi \in T_pM$

$$\begin{aligned} E_p^{\mathrm{hor}}(\xi) &:= \mathrm{d}_\xi \exp \circ L^h(\xi) : T_pM \longrightarrow T_{\exp(\xi)}M , \\ E_p^{\mathrm{ver}}(\xi) &:= \mathrm{d}_\xi \exp \circ L^v(\xi) : T_pM \longrightarrow T_{\exp(\xi)}M . \end{aligned}$$

Given a smooth curve $u : (a,b) \to M$ and a smooth vector field $\xi \in \Gamma(u^*TM)$ along u, we write $u_\xi = \exp_u \xi : (a,b) \to M$ where $u_\xi(x) = \exp_{u(x)} \xi(x)$. With the above definitions we have

$$\partial_x u_\xi = E_u^{\mathrm{hor}}(\xi)\partial_x u + E_u^{\mathrm{ver}}(\xi)\nabla_x \xi \; . \tag{A.1.1}$$

Proposition A.1.1. *For all $\varepsilon > 0$ there exists an universal constant c with the following significance:*

- *Given vectors $\xi, \xi' \in T_pM$ with $|\xi| < \varepsilon$, we have the estimates*

$$\left|E_p(\xi)\xi'\right| \le c\left|\xi'\right| , \qquad \left|E_p(\xi)\xi' - \Pi_p(\xi)\xi'\right| \le c\,|\xi|\left|\xi'\right| ,$$

- *Let $u : (a,b) \to M$ be a smooth curve and given vector fields $\xi, \xi' \in \Gamma(u^*TM)$ such that $\|\xi\|_{L^\infty} < \varepsilon$ then we have the estimates*

$$\left|\nabla_x E_u(\xi)\xi' - E_u(\xi)\nabla_x \xi'\right| \le c\left|\xi'\right| |\xi| \left(|\partial_x u| + |\nabla_x \xi|\right) , \tag{A.1.2}$$

where $\Pi_p(\xi) : T_pM \to T_{\exp(\xi)}M$ is the parallel transport along the geodesic curve $y \mapsto \exp_p(y\xi)$ and $E_p(\xi)$ denotes either $E_p^{\mathrm{ver}}(\xi)$ or $E_p^{\mathrm{hor}}(\xi)$.

Proof. By (A.1.1) the vector field $Y(y) := E_p(y\xi)\xi'$ is a Jacobi vector field along the geodesic $c : [0,1] \to M$, $y \mapsto \exp_p(y\xi)$, i.e. solves the equation $\nabla_y\nabla_y Y = R(\dot{c}, Y)\dot{c}$ where R denotes the curvature tensor. Given any Jacobi field Y, we define the function $f : [0,1] \to \mathbb{R}$, $y \mapsto |Y(y)| + |\nabla_y Y(y)|$. We have

$$f'(y) \le |\nabla_y Y| + |\nabla_y\nabla_y Y| \le |\nabla_y Y| + \|R\|_\infty |\xi|^2 |Y| \le (1 + \|R\|_\infty \varepsilon^2) f\,.$$

Hence $f(y) \le c_1 f(0)$ with constant $c_1 := e^{(1+\|R\|\varepsilon^2)}$ and so

$$|Y(1)| + |\nabla_y Y(1)| \le c_1(|Y(0)| + |\nabla_y Y(0)|)\,. \tag{A.1.3}$$

Since the estimate holds for any Jacobi field Y we have in particular the estimates $\left|E^{\mathrm{hor}}(\xi)\xi'\right| \le c_1 |\xi'|$ and $|E^{\mathrm{ver}}(\xi)\xi'| \le c_1 |\xi'|$ as required.

We show the second inequality. We define the vector field $X \in \Gamma(c^*TM)$ via

$$X(y) := \Pi_p(y\xi)Y(0) + y\Pi_p(y\xi)\nabla_y Y(0)\,.$$

Consider the function

$$f : [0,1] \to \mathbb{R},\ f(y) := |Y(y) - X(y)| + |\nabla_y Y(y) - \nabla_y X(y)| + c_1\varepsilon \|R\| |\xi| \left|\xi'\right|\,.$$

We derive

$$f'(y) \le |\nabla_y Y - \nabla_y X| + |\nabla_y\nabla_y Y| \le |\nabla_y Y - \nabla_y X| + |R(\dot{c}, Y)\dot{c}| \le f(y)\,.$$

This shows that $|E_p(\xi)\xi' - \Pi_p(\xi)\xi'| \le f(1) \le ef(0) = ec_1\varepsilon \|R\| |\xi| |\xi'|$.

We come to the third inequality. Define the map $w(x,y) := \exp_{u(x)} y\xi(x)$ and the family of geodesics $c_x := w(x,\cdot)$. The vector fields $Y(x,y) = E(y\xi(x))\xi'(x)$ and $Z(x,y) := E(y\xi(x))\nabla_x\xi'(x)$ are vector fields along w which are Jacobi fields when restricted to c_x. We claim that there exists a uniform constant c_2 such that

$$|\nabla_y\nabla_x\nabla_y Y - \nabla_y\nabla_y Z| \le c_2 |\xi| \left|\xi'\right| (|\partial_x u| + |\nabla_x \xi|) + \varepsilon^2 \|R\| |\nabla_x Y - Z|\,. \tag{A.1.4}$$

Indeed use (A.1.3) to show in particular that $|\partial_x w| + |\nabla_y\partial_x w| \le c_1(|\partial_x u| + |\nabla_x\xi|)$ and $|Y| + |\nabla_y Y| \le c_1 |\xi'|$. Abbreviate $R(\partial_x, \partial_y) = R(\partial_x w, \partial_y w)$ etc. and estimate

$$\begin{aligned}
&|\nabla_y\nabla_x\nabla_y Y - \nabla_y\nabla_y Z| \\
&\le |R(\partial_y, \partial_x)\nabla_y Y| + |\nabla_x R(\partial_y, Y)\partial_y w - R(\partial_y, Z)\partial_y w| \\
&\le |R(\partial_y, \partial_x)\nabla_y Y| + |\nabla_x R(\partial_y, Y)\partial_y w - R(\partial_y, \nabla_x Y)\partial_y w| + \\
&\qquad\qquad + |R(\partial_y, \nabla_x Y - Z)\partial_y w| \\
&\le \|R\| |\xi| |\partial_x w| |\nabla_y Y| + \|\nabla R\| |\xi|^2 |\partial_x w| |Y| + 2\|R\| |\xi| |\nabla_y \partial_x w| |Y| + \\
&\qquad\qquad + \|R\| |\xi|^2 |\nabla_x Y - Z| \\
&\le c_2 |\xi| \left|\xi'\right| (|\partial_x u| + |\nabla_x\xi|) + \varepsilon^2 \|R\| |\nabla_x Y - Z|\,.
\end{aligned}$$

Define the function $f : [0,1] \to M$, where $c_3 := c_2 + c_1^2\varepsilon \|R\|$,

$$f(y) := |\nabla_x Y - Z| + |\nabla_x\nabla_y Y - \nabla_y Z| + c_3 |\xi| \left|\xi'\right| (|\partial_x u| + |\nabla_x\xi|)\,.$$

We derive using (A.1.4)

$$\begin{aligned} f'(y) &\leq |\nabla_y \nabla_x Y - \nabla_y Z| + |\nabla_y \nabla_x \nabla_y Y - \nabla_y \nabla_y Z| \\ &\leq |R(\partial_y, \partial_x) Y| + |\nabla_x \nabla_y Y - \nabla_y Z| + c_2 |\xi| \left|\xi'\right| (|\partial_x u| + |\nabla_x \xi|) + \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \|R\| \varepsilon^2 |\nabla_x Y - Z| \\ &\leq (1 + \|R\| \varepsilon^2) f(y) . \end{aligned}$$

This shows that $|\nabla_x Y - Z| \leq f(1) \leq c_1 f(0) = c_1 c_3 |\xi| |\xi'| (|\partial_x u| + |\nabla_x \xi|)$. □

Corollary A.1.2. *There exists universal constants c and ε with the following significance. Given a smooth curve $u : (a,b) \to M$ and a vector field $\xi \in \Gamma(u^*TM)$ such that $\|\xi\|_{L^\infty} \leq \varepsilon$ we have*

$$|\partial_x u_\xi| \leq c \left(|\partial_x u| + |\nabla_x \xi|\right) , \tag{A.1.5}$$
$$|\nabla_x \xi| \leq c \left(|\partial_x u| + |\partial_x u_\xi|\right) , \tag{A.1.6}$$
$$\left|\partial_x u_\xi - \Pi_u^{u_\xi} \partial_x u\right| \leq c \left(|\partial_x u| |\xi| + |\nabla_x \xi|\right) . \tag{A.1.7}$$

Moreover for all vector fields $\xi, \xi' \in \Gamma(u^*TM)$ with $\|\xi\|_\infty + \|\xi'\|_\infty < \varepsilon$ we have

$$\left| \exp_{u_\xi}^{-1} \exp_u \xi' - \Pi_u^{u_\xi} \xi' \right| \leq c |\xi| , \tag{A.1.8}$$
$$\left|\nabla_x \exp_{u_\xi}^{-1} \exp_u \xi' - \Pi_u^{u_\xi} \nabla_x \xi'\right| \leq c \left(|\partial_x u| |\xi| + |\nabla_x \xi|\right) . \tag{A.1.9}$$

Proof. Estimate (A.1.5) follows by the first inequality of Proposition A.1.1 and (A.1.1). We show the estimate (A.1.6). By the second inequality of Proposition A.1.1 we have an universal constant c_1 such that

$$\left|\mathbb{1} - \Pi_{u_\xi}^u E_u^{\text{ver}}(\xi)\right| = \left|\Pi_u^{u_\xi} - E_u^{\text{ver}}(\xi)\right| \leq c_1 |\xi| \ .$$

If $|\xi| < 1/2c_1$ then the operator $E_u^{\text{ver}}(\xi)$ is invertible with inverse $\sum_{k\geq 0}(1 - \Pi_{u_\xi}^u E_u^{\text{ver}})^k \circ \Pi_{u_\xi}^u$ which is bounded by 2. By (A.1.1) we have

$$|\nabla_x \xi| = \left|(E_u^{\text{ver}})^{-1} E_u^{\text{hor}} \partial_s u - (E_u^{\text{ver}})^{-1} \partial_s u_\xi\right| \leq 2c_1 |\partial_s u| + 2 |\partial_s u_\xi| \ .$$

This shows the claimed bound.

Estimate (A.1.7) follows because after Proposition A.1.1 we estimate the norm of $\partial_x u_\xi - \Pi_u^{u_\xi} \partial_x u$ by

$$\left|E_u^{\text{hor}}(\xi) \partial_x u - \Pi_u^{u_\xi} \partial_x u\right| + |E_u^{\text{ver}}(\xi) \nabla_x \xi| \ ,$$

which is bounded by $O(1) |\partial_s u| |\xi| + O(1) |\nabla_x \xi|$.

For estimate (A.1.8) we define the curve $w : (a,b) \times [0,1] \to M$, via $w(x,y) := \exp_{u(x)} y\xi(x)$. If ξ', ξ are sufficiently small we define implicitly a vector field ζ along w via

$$\exp_{u(x)} \xi'(x) = \exp_{w(x,y)} \zeta(x,y) \,,$$

for all $x \in (a,b)$ and $y \in [0,1]$. Deriving the last equation by ∂_y using (A.1.1) gives

$$E_w^{\mathrm{hor}}(\zeta)\partial_y w + E_w^{\mathrm{ver}}(\zeta)\nabla_y \zeta = 0\,. \tag{A.1.10}$$

Fix $x \in (a,b)$ and define the function

$$f : [0,1] \to \mathbb{R}, \qquad y \mapsto \left|\zeta(x,y) - \Pi_{u(x)}^{w(x,y)} \xi'(x)\right|.$$

By construction we have $w(x,0) = u(x)$, $w(x,1) = u_\xi(x)$, $\zeta(x,0) = \xi'(x)$ and $\zeta(x,1) = \exp_{u_\xi(x)}^{-1} \exp_{u(x)} \xi'(x)$ for all $x \in (a,b)$, which implies that $f(0) = 0$ and that we have to estimate $f(1)$. We compute using (A.1.10) and the mean-value theorem omitting the arguments x, y whenever convenient

$$\left|\exp_{u_\xi}^{-1} \exp_u \xi' - \Pi_u^{u_\xi} \xi'\right| = f(1) = \partial_y f(y) \le |\nabla_y \zeta| = \left|E_w^{\mathrm{ver}}(\zeta)^{-1} E_w^{\mathrm{hor}}(\zeta)\partial_y w\right|,$$

which is in particular bounded by $O(1)\,|\xi|$.

We show (A.1.9). By definition we have $\zeta(1) = \exp_{u_\xi}^{-1} \exp_u \xi'$ and after the mean-value theorem

$$\left|\nabla_x \zeta - \Pi_u^w \nabla_x \xi'\right| = \partial_y \left|\nabla_x \zeta - \Pi_u^w \nabla_x \xi'\right| \le |\nabla_y \nabla_x \zeta| \le |R(\partial_y w, \partial_x w)\zeta| + |\nabla_x \nabla_y \zeta|\ .$$

By (A.1.5) the first term on the right-hand side is in $O(|\partial_x u|\,|\xi| + |\nabla_x \xi|)$. Hence it suffices to estimate $|\nabla_x \nabla_y \zeta|$. Abbreviate $E_w^{\mathrm{ver}} := E_w^{\mathrm{ver}}(\zeta)$ and estimate

$$\begin{aligned} |\nabla_x \nabla_y \zeta| \le O(1)\,|E_w^{\mathrm{ver}} \nabla_x \nabla_y \zeta| &\le \\ &\le O(1)\,|\nabla_x E_w^{\mathrm{ver}} \nabla_y \zeta| + O(1)\,|(\nabla_x E_w^{\mathrm{ver}} - E_w^{\mathrm{ver}} \nabla_x)\nabla_y \zeta| \end{aligned}$$

With (A.1.10) we have $|\nabla_y \zeta| \le O(1)$. By (A.1.2) the second term on the right-hand side of the last estimate is in $O(|\partial_x u|\,|\xi| + |\nabla_x \xi|)$. We continue to estimate the first. Using (A.1.10) again we have

$$|\nabla_x E_w^{\mathrm{ver}} \nabla_y \zeta| = \left|\nabla_x E_w^{\mathrm{hor}} \partial_y w\right| \le \left|(\nabla_x E_w^{\mathrm{hor}} - E_w^{\mathrm{hor}} \nabla_x)\partial_y w\right| + O(1)\,|\nabla_x \partial_y w|\ .$$

Again by (A.1.2) the first term is in $O(|\partial_x u|\,|\xi| + |\nabla_x \xi|)$ and it suffices to bound $|\nabla_x \partial_y w| = |\nabla_y \partial_x w|$.

$$\begin{aligned} |\nabla_y \partial_x w| &\le \left|\nabla_y E_w^{\mathrm{hor}}(y\xi)\partial_x u\right| + |\nabla_y E_w^{\mathrm{ver}}(y\xi)(y\nabla_x \xi)| \\ &\le O(1)\,|\xi|\,|\partial_x u| + O(1)\,|\xi|\,|\nabla_x \xi| + \left|E_w^{\mathrm{ver}}(y\xi)\nabla_x \xi\right| \\ &\le O(1)(|\xi|\,|\partial_x u| + |\nabla_x \xi|)\,. \end{aligned}$$

This shows the claim using the last four estimates. □

Corollary A.1.3. *There exists constants c and ε with the following significance. Given a point $p \in M$ and two vectors $\xi_0, \xi_1 \in T_pM$ such that $|\xi_0| + |\xi_1| < \varepsilon$, then we have*

$$1/c\,|\xi_0 - \xi_1| \le \operatorname{dist}\left(\exp_p(\xi_0), \exp_p(\xi_1)\right) \le c\,|\xi_0 - \xi_1|\ . \tag{A.1.11}$$

Moreover let $u : [a,b] \to M$ be a curve and $\xi, \xi' \in \Gamma(u^*TM)$ be a vector field along u with $\|\xi\|_\infty + \|\xi'\|_\infty < \varepsilon$, then

$$\left|\nabla_x \exp_{u_{\xi'}}^{-1} u - \nabla_x \exp_{u_{\xi'}}^{-1} u_\xi\right| \le c\big(\,|\xi|\ |\partial_x u| + |\xi|\ \left|\nabla_x \xi'\right| + |\nabla_x \xi|\,\big)\,. \tag{A.1.12}$$

Proof. To show the estimate on the right-hand side of (A.1.11) set $\xi(x) = (1-x)\xi_0 + x\xi_1$ and $u(x) = p$ for all $x \in [0,1]$. Then the inequality follows after integrating the norm of $\partial_x u_\xi$ over $[0,1]$ and using estimate (A.1.5) of Corollary A.1.2.

To show the estimate on the left-hand side of (A.1.11) let $c : [0,1] \to M$ be the unique shortest geodesic from $\exp_p \xi_0$ to $\exp_p \xi_1$. We define the path of vectors $\xi : [0,1] \to T_pM$ via $\xi(y) := \exp_p^{-1} c(y)$. By the mean-value theorem we have a value $y \in [0,1]$ such that $\xi_1 - \xi_0 = \nabla_y \xi(y)$ and hence after (A.1.6) there exists a uniform constant c_1 such that $|\xi_1 - \xi_0| = |\nabla_y \xi(y)| \leq c_1 \left|\partial_y c\right| = c_1 \mathrm{dist}\left(\exp_p \xi_0, \exp_p \xi_1\right)$.

We show (A.1.12). Abbreviate the curve $v := \exp_u \xi'$ and define implicitly the vector field $\zeta : [a,b] \times [0,1] \to v^*TM$ via

$$\exp_{v(x)} \zeta(x,y) = \exp_{u(x)} y\xi(y)\,.$$

Deriving the equation by ∇_x and by ∇_y we get

$$\begin{aligned} E_v^{\mathrm{ver}}(\zeta)\nabla_x\zeta &= E_v^{\mathrm{hor}}(y\xi)\partial_x u + yE_u^{\mathrm{ver}}(y\xi)\nabla_x\xi - E_v^{\mathrm{hor}}(\zeta)\partial_x v \\ E_v^{\mathrm{ver}}(\zeta)\nabla_y\zeta &= E_u^{\mathrm{ver}}(y\xi)\xi\,. \end{aligned} \tag{A.1.13}$$

By construction we have $\zeta_0 := \zeta(\cdot,0) = \exp_v^{-1} u$ and $\zeta_1 := \zeta(\cdot,1) = \exp_v^{-1} u_\xi$. Hence after the mean-value theorem

$$\left|\nabla_x \exp_v^{-1} u - \nabla_x \exp_v^{-1} u_\xi\right| = \left|\nabla_y \nabla_x \zeta\right| \leq \left|\nabla_x \nabla_y \zeta\right| + \left|R(\partial_y u, \partial_x u)\zeta\right| \,.$$

Since u does not depend on y the last term vanishes and we are left to estimate the norm of $\nabla_x \nabla_y \zeta$. Abbreviate $E_v = E_v^{\mathrm{vert}}(\zeta)$, then

$$\left|\nabla_x \nabla_y \zeta\right| \leq O(1) \left|E_v \nabla_x \nabla_y \zeta\right| \leq O(1) \left|(E_v \nabla_x - \nabla_x E_v)\nabla_y \zeta\right| + O(1) \left|\nabla_x E_v \nabla_y \zeta\right| \,.$$

Via (A.1.2) the first term on the right-hand side is bounded by

$$\begin{aligned} \left|(E_v \nabla_x - \nabla_x E_v)\nabla_y \zeta\right| &\leq O(1) \left|\nabla_y \zeta\right| \left|\zeta\right| \left(\left|\partial_x v\right| + \left|\nabla_x \zeta\right|\right) \leq \\ &\leq O(1) \left|\xi\right| \left(\left|\partial_x u\right| + \left|\nabla_x \xi'\right| + \left|\nabla_x \xi\right|\right). \end{aligned}$$

For the last estimate we have used (A.1.13). To show the claim it suffices to estimate the norm of $\nabla_x E_v \nabla_y \zeta = \nabla_x E_u^{\mathrm{ver}}(y\xi)\xi$. Abbreviate $E_u^{\mathrm{ver}}(y\xi) = E_u$ and estimate

$$\left|\nabla_x E_u \xi\right| \leq \left|(\nabla_x E_u - E_u \nabla_x)\xi\right| + \left|E_u \nabla_x \xi\right| \leq O(1) \left|\xi\right| \left(\left|\partial_x u\right| + \left|\nabla_x \xi\right|\right) + O(1) \left|\nabla_x \xi\right| \,.$$

This shows the claim using the last three estimates. □

The next corollary states that the distance between parallel geodesics is uniformly bounded by the distance of their starting point.

Corollary A.1.4. *There exists positive constants ε and c such that given points $p, q \in M$ and a vector $\xi \in T_pM$ satisfying $\mathrm{dist}\,(p,q) + |\xi| \leq \varepsilon$ then we have*

$$\mathrm{dist}\left(\exp_p \xi, \exp_q \Pi_p^q \xi\right) \leq c \,\mathrm{dist}\,(p,q) \,.$$

Proof. Let $u : [0,1] \to M$ be the unique shortest geodesic from p to q. Extend ξ to a parallel vector field along u. Integrate the estimate (A.1.5) of Corollary A.1.2 over $[0,1]$, using $\nabla_x \xi = 0$ and that $|\partial_x u| = \mathrm{dist}\,(p,q)$. □

A.2. Parallel Transport

Let M be a compact Riemannian manifold equipped with a metric connection ∇. For any curve $\gamma : [a, b] \to M$, let

$$\Pi(\gamma) : T_{\gamma(a)}M \to T_{\gamma(b)}M \,,$$

denote the parallel transport along γ with respect to the connection ∇.

Lemma A.2.1. *There exists a constant c such that for any map $w : [0, 1]^2 \to M$ we have*

$$|\Pi(\gamma_1)\Pi(u_0) - \Pi(u_1)\Pi(\gamma_0)| \leq c \int_{[0,1]^2} |\partial_x w(x, y)|\, |\partial_y w(x, y)| \,\mathrm{d}x\mathrm{d}y \,.$$

with curves $u_\tau = w(\tau, \cdot)$ and $\gamma_\tau = w(\cdot, \tau)$ for $\tau = 0, 1$.

Proof. Fix $\xi_0 \in T_{w(0,0)}M$ and define vector fields $\xi, \eta \in \Gamma(w^*TM)$ along w such that

$$\begin{aligned} \nabla_x \xi(x, y) &= 0 \,, & \nabla_x \eta(x, 0) &= 0 \,, \\ \nabla_y \xi(0, y) &= 0 \,, & \nabla_y \eta(x, y) &= 0 \,, \\ \xi(0, 0) &= \xi_0 \,, & \eta(0, 0) &= \xi_0 \,. \end{aligned}$$

We have to estimate the norm of $\xi(1, 1) - \eta(1, 1)$. Let R be the curvature tensor. We have

$$\partial_y |\nabla_x \eta(x, y)| \leq |\nabla_y \nabla_x \eta| = |R(\partial_x w, \partial_y w)\eta| \leq \|R\|_\infty |\partial_x w|\, |\partial_y w|\, |\xi_0| \,.$$

Then $\nabla_x \eta(x, 0) = 0$ and by integrating the last inequality we obtain

$$|\nabla_x \eta(x, y)| \leq \|R\|_\infty |\xi_0| \int_0^1 |\partial_x w(x, y)|\, |\partial_y w(x, y)| \,\mathrm{d}y \,.$$

We have $\partial_x |\xi - \eta| \leq |\nabla_x \eta|$ and integrate again using the last estimate and $\xi(0, y) - \eta(0, y) = 0$ we show the claim. □

Lemma A.2.2. *There exists a constant c such that for any curve $w : [0, 1]^2 \to M$ and section $\xi \in \Gamma(w(\cdot, 0)^*TM)$ we have*

$$|\nabla_x \Pi(\gamma_x)\xi - \Pi(\gamma_x)\nabla_x \xi| \leq c\,|\xi| \int_0^1 |\partial_x w(x, y)|\, |\partial_y w(x, y)| \,\mathrm{d}y \,,$$

with curves $\gamma_x = w(x, \cdot)$ for $x \in [0, 1]$.

Proof. Define $\xi, \eta \in \Gamma(w^*TM)$ via

$$\begin{aligned} \nabla_y \xi(x, y) &= 0, & \nabla_y \eta(x, y) &= 0 \,, \\ \xi(x, 0) &= \xi(x), & \eta(x, 0) &= \nabla_x \xi(x) \,. \end{aligned}$$

We compute

$$\partial_y |\nabla_x \xi - \eta| \leq |\nabla_y \nabla_x \xi| = |R(\partial_y w, \partial_x w)\xi| \leq \|R\|_\infty |\partial_x w|\, |\partial_y w|\, |\xi| \,.$$

Since $\nabla_x \xi(x, 0) = \eta(x, 0)$ the result follows by integration. □

Corollary A.2.3. *There exists uniform constants ε and c such that for all curves $u : [0,1] \to M$ and vector fields $\xi \in \Gamma(u^*TM)$ with $\|\xi\|_\infty < \varepsilon$ we have*

$$|\Pi(\gamma_1)\Pi(u_0) - \Pi(u_1)\Pi(\gamma_0)| \leq c \int_0^1 |\xi| \left(|\partial_x u| + |\nabla_x \xi|\right) \mathrm{d}x\,.$$

with curves $\gamma_x, u_y : [0,1] \to M$ given by $\gamma_x(y) = u_y(x) := \exp_{u(x)} y\xi(x)$ for all $x, y \in [0,1]$. In particular if u_0 and u_1 are short geodesics we have

$$|\Pi(\gamma_1)\Pi(u_0) - \Pi(u_1)\Pi(\gamma_0)| \leq c \left(\operatorname{dist}\left(u_0(0), u_0(1)\right) + \operatorname{dist}\left(u_1(0), u_1(1)\right)\right)\,.$$

Proof. Define $w(x,y) := \gamma_x(y)$. We have $|\partial_y w| = |\xi| < \varepsilon$. By Corollary A.1.2 there exists constants c and ε such if $|\xi| < \varepsilon$ we have $|\partial_x w| \leq c(|\partial_x u| + |\nabla_x \xi|)$ and $|\partial_x w| \leq c(|\partial_x u_0| + |\partial_x u_1|)$. Then conclude by Lemma A.2.1. □

Corollary A.2.4. *There exists constants ε and c such that for all curves $u : [0,1] \to M$ and vector fields $\xi, \xi' \in \Gamma(u^*TM)$ with $\|\xi\|_\infty < \varepsilon$ we have*

$$\left|\nabla_x \Pi(\gamma_x)\xi' - \Pi(\gamma_x)\nabla_x \xi'\right| \leq c\,|\xi|\,\left|\xi'\right| \left(|\partial_x u| + |\nabla_x \xi|\right),$$

with curves $\gamma_x : [0,1] \to M$ given by $\gamma_x(y) := \exp_{u(x)} y\xi(x)$ for $x, y \in [0,1]$.

Proof. Define $w(x,y) := \gamma_x(y)$. Conclude by Lemma A.2.2, $|\partial_y w| = |\xi|$ and Corollary A.1.2. □

A.3. Estimates for strips

For a smooth map $u : \mathbb{R} \times [0,1] \to M$ consider the differential operator $\mathcal{F}_u$ and its linearization D_u given by equation (6.1.5) and (6.1.6) respectively. In this section we establish auxiliary estimates for these operators. We assume for simplicity that $X \equiv 0$.

Pointwise estimates

For the following estimate $\|J\|_{C^2}$ denotes the C^2-norm of the tensor J using the induced norm on $\mathrm{End}(TM)$ coming from the fixed Riemannian metric on M. All universal constants are independent of J.

Lemma A.3.1. *There exists universal constants c and ε such that for all smooth maps $u : \mathbb{R} \times [0,1] \to M$ and vector fields $\xi \in \Gamma(u^*TM)$ with $\|\xi\|_\infty < \varepsilon$ we have*

$$\left|D_{u_\xi}\Pi_u^{u_\xi}\xi' - \Pi_u^{u_\xi} D_u \xi'\right| \leq c(1 + \|J\|_{C^2}) \left(|\xi|\,\left|\xi'\right|\,|\mathrm{d}u| + |\nabla\xi|\,\left|\xi'\right| + |\xi|\,\left|\nabla\xi'\right|\right)\,. \qquad \text{(A.3.1)}$$

with $u_\xi : \mathbb{R} \times [0,1] \to M$ defined by $u_\xi(s,t) = \exp_{u(s,t)} \xi(s,t)$.

Proof. We have to estimate the norm of

$$D_{u_\xi}\Pi_u^{u_\xi}\xi' - \Pi_u^{u_\xi}D_u\xi' = \nabla_s\Pi_u^{u_\xi}\xi' - \Pi_u^{u_\xi}\nabla_s\xi' + J(u_\xi)\nabla_t\Pi_u^{u_\xi}\xi' - \Pi_u^{u_\xi}J(u)\nabla_t\xi'$$
$$+ \left(\nabla_{\Pi_u^{u_\xi}\xi'}J(u_\xi)\right)\partial_t u_\xi - \Pi_u^{u_\xi}\left(\nabla_{\xi'}J(u)\right)\partial_t u$$

Denote the norm of the successive differences of the right-hand side by T_1, T_2 and T_3. By Corollary A.2.4 we have a constant ε such that if $|\xi| < \varepsilon$ then

$$T_1 \leq O(1)\,|\xi|\,\left|\xi'\right|\left(|\partial_s u| + |\nabla_s\xi|\right)\,.$$

Similarly

$$\begin{aligned} T_2 &\leq \left|J(u_\xi)\nabla_t\Pi_u^{u_\xi}\xi' - J(u_\xi)\Pi_u^{u_\xi}\nabla_t\xi'\right| + + \left|J(u_\xi)\Pi_u^{u_\xi}\nabla_t\xi' - \Pi_u^{u_\xi}J(u)\nabla_t\xi'\right| \\ &\leq \|J\|_\infty\left|\nabla_t\Pi_u^{u_\xi}\xi' - \Pi_u^{u_\xi}\nabla_t\xi'\right| + \|\nabla J\|_\infty\left|\nabla_t\xi'\right||\xi| \\ &\leq O(1)\,\|J\|_{C^2}\,|\xi|\,\left|\xi'\right|\left(|\partial_t u| + |\nabla_t\xi|\right) + \|J\|_{C^2}\,|\xi|\left|\nabla_t\xi'\right|\,, \end{aligned}$$

and

$$\begin{aligned} T_3 &\leq \left|\left(\nabla_{\Pi_u^{u_\xi}\xi'}J(u_\xi)\right)\left(\partial_t u_\xi - \Pi_u^{u_\xi}\partial_t u\right)\right| + \\ &\qquad + \left|\,\left(\nabla_{\Pi_u^{u_\xi}\xi'}J(u_\xi)\right)\Pi_u^{u_\xi}\partial_t u - \Pi_u^{u_\xi}\left(\nabla_{\xi'}J(u)\right)\partial_t u\right| \\ &\leq \|\nabla J\|_\infty\left|\xi'\right|\left|\partial_t u_\xi - \Pi_u^{u_\xi}\partial_t u\right| + \|\nabla(\nabla J)\|_\infty\left|\xi'\right||\xi|\,|\partial_t u| \\ &\leq O(1)\,\|J\|_{C^2}\left(|\partial_t u|\,|\xi|\left|\xi'\right| + \left|\xi'\right||\nabla_t\xi|\right)\,. \end{aligned}$$

This shows the claim. □

Lemma A.3.2. *There exists universal constants c and ε such that for all smooth $u: \mathbb{R}\times[0,1]\to M$ we have and vector fields $\xi,\xi'\in\Gamma(u^*TM)$ with $\|\xi\|_\infty < \varepsilon$ we have*

$$\left|\mathrm{d}\mathcal{F}_u(\xi)\xi' - D_u\xi'\right| \leq c(1 + \|J\|_{C^2})\left(|\mathrm{d}u|\,|\xi|\left|\xi'\right| + |\nabla\xi|\left|\xi'\right| + |\xi|\left|\nabla\xi'\right|\right)\,.$$

Proof. For $\tau\in\mathbb{R}$ small enough denote $u_{\xi_\tau} := \exp_u(\xi + \tau\xi')$, $u_\xi := \exp_u\xi$ and the vector field $\eta_\tau := \mathcal{F}_u(\xi + \tau\xi')$. By definition we have $\Pi_u^{u_{\xi_\tau}}\eta_\tau = \Pi_u^{u_{\xi_\tau}}\mathcal{F}_u(\xi+\tau\xi') = \overline{\partial}_J u_{\xi_\tau}$ and after deriving that equation covariantly for τ and restricting to $\tau = 0$ we obtain

$$D_{u_\xi}\left(E_u^{\mathrm{ver}}(\xi)\xi'\right) = \nabla_\tau\Pi_u^{u_{\xi_\tau}}\eta_\tau\big|_{\tau=0} = \nabla_\tau\Pi_u^{u_{\xi_\tau}}\eta_\tau - \Pi_u^{u_{\xi_\tau}}\partial_\tau\eta_\tau\big|_{\tau=0} + \Pi_u^{u_\xi}\mathrm{d}\mathcal{F}_u(\xi)\xi'\,.$$

For the second identity we just added zero and used that by definition $\partial_\tau\eta_\tau|_{\tau=0} = \mathrm{d}\mathcal{F}_u(\xi)\xi'$. Hence

$$\begin{aligned} \left|\mathrm{d}\mathcal{F}_u(\xi)\xi' - D_u\xi'\right| &= \left|\Pi_u^{u_\xi}\mathrm{d}\mathcal{F}_u(\xi)\xi' - \Pi_u^{u_\xi}D_u\xi'\right| \\ &\leq \left|\nabla_\tau\Pi_u^{u_{\xi_\tau}}\eta_\tau - \Pi_u^{u_{\xi_\tau}}\partial_\tau\eta_\tau|_{\tau=0}\right| + \left|D_{u_\xi}E_u^{\mathrm{ver}}(\xi)\xi' - \Pi_u^{u_\xi}D_u\xi'\right|\,. \end{aligned} \tag{A.3.2}$$

To estimate the first term of the right-hand side, we use corollaries A.2.4 and A.1.2

$$\begin{aligned} \left|\nabla_\tau\Pi_u^{u_{\xi_\tau}}\eta_\tau - \Pi_u^{u_{\xi_\tau}}\partial_\tau\eta_\tau|_{\tau=0}\right| &\leq O(1)\,|\partial_\tau u_{\xi_\tau}|\,|\xi_\tau|\,|\eta_\tau|\,\big|_{\tau=0} \leq O(1)\left|\xi'\right||\xi|\left|\overline{\partial}_J u\right| \\ &\leq O(1)\left|\xi'\right||\xi|\left(1 + \|J\|_\infty\right)|\mathrm{d}u| \end{aligned} \tag{A.3.3}$$

We now focus on the second summand of the right-hand side of (A.3.2). The differene $D_{u_\xi} E_u^{\mathrm{ver}}(\xi)\xi' - \Pi_u^{u_\xi} D_u \xi'$ equals

$$\nabla_s E_u^{\mathrm{ver}}(\xi)\xi' - \Pi_u^{u_\xi}\nabla_s \xi' + J(u_\xi)\nabla_t E_u^{\mathrm{ver}}(\xi)\xi' - \Pi_u^{u_\xi} J(u)\nabla_t \xi' \\ + \left(\nabla_{E_u^{\mathrm{ver}}(\xi)\xi'} J(u_\xi)\right)\partial_t u_\xi - \Pi_u^{u_\xi}\left(\nabla_{\xi'} J(u)\right)\partial_t u\,.$$

Denote the norm of the successive differences of the right-hand side by T_1, T_2 and T_3. By Proposition A.1.1 and Corollary A.1.2 we have a constant $\varepsilon > 0$ such that if $|\xi| < \varepsilon$

$$\begin{aligned} T_1 &\le \left|\nabla_s E_u^{\mathrm{ver}}(\xi)\xi' - E_u^{\mathrm{ver}}(\xi)\nabla_s\xi'\right| + \left|E_u^{\mathrm{ver}}(\xi)\nabla_s\xi' - \Pi_u^{u_\xi}\nabla_s\xi'\right| \\ &\le O(1)\left|\xi'\right|(|\partial_s u|\,|\xi| + |\nabla_s\xi|) + O(1)\,|\xi|\left|\nabla_s\xi'\right| \ , \end{aligned}$$

Abbreviate $E(\xi)\xi' = E_u^{\mathrm{ver}}(\xi)\xi')$ and estimate using Proposition A.1.1 and Corollary A.1.2

$$\begin{aligned} T_2 &\le \left|J(u_\xi)\nabla_t E(\xi)\xi' - J(u_\xi)\Pi_u^{u_\xi}\nabla_t\xi'\right| + \left|J(u_\xi)\Pi_u^{u_\xi}\nabla_t\xi' - \Pi_u^{u_\xi} J(u)\nabla_t\xi'\right| \\ &\le \|J\|_\infty\left|\nabla_t E(\xi)\xi' - \Pi_u^{u_\xi}\nabla_t\xi'\right| + \|\nabla J\|_\infty\,|\xi|\left|\nabla_t\xi'\right| \\ &\le O(1)\,\|J\|_\infty\left|\xi'\right|(|\partial_t u|\,|\xi| + |\nabla_t\xi|) + \|J\|_{C^1}\,|\xi|\left|\nabla_t\xi'\right| \ . \end{aligned}$$

and

$$\begin{aligned} T_3 \le\ & \left|\left(\nabla_{E(\xi)\xi'} J(u_\xi)\right)\partial_t u_\xi - \left(\nabla_{E(\xi)\xi'} J(u_\xi)\right)\Pi_u^{u_\xi}\partial_t u\right| + \\ & + \left|\left(\nabla_{E(\xi)\xi'} J(u_\xi)\right)\Pi_u^{u_\xi}\partial_t u - \left(\nabla_{\Pi_u^{u_\xi}\xi'} J(u_\xi)\right)\Pi_u^{u_\xi}\partial_t u\right| + \\ & + \left|\left(\nabla_{\Pi_u^{u_\xi}\xi'} J(u_\xi)\right)\Pi_u^{u_\xi}\partial_t u - \Pi_u^{u_\xi}\left(\nabla_{\xi'} J(u)\right)\partial_t u\right| \\ \le\ & \|\nabla J\|_\infty\left|E(\xi)\xi'\right|\left|\partial_t u_\xi - \Pi_u^{u_\xi}\partial_t u\right| + \|\nabla J\|_\infty\left|E(\xi)\xi' - \Pi_u^{u_\xi}\xi'\right||\partial_t u| + \\ & \qquad + \left\|\nabla^2 J\right\|_\infty\left|\xi'\right||\xi|\,|\partial_t u| \\ \le\ & O(1)\,\|\nabla J\|_\infty\left|\xi'\right|(|\partial_t u|\,|\xi| + |\nabla_t\xi|) + \|\nabla J\|_\infty\,|\xi|\left|\xi'\right||\partial_t u| + \\ & \qquad + \left\|\nabla^2 J\right\|_\infty\left|\xi'\right||\xi|\,|\partial_t u| \\ \le\ & O(1)(1 + \|J\|_{C^2})\left(|\xi|\left|\xi'\right||\partial_t u| + \left|\xi'\right||\nabla_t\xi|\right) \ . \end{aligned}$$

Putting everything together we obtain the result by the last three estimates and estimate (A.3.3) plugged into the identity (A.3.2). □

Sobolev estimates

Here we have collected estimates for the weighted Sobolev norms. For any $a < b$ consider the domain $\Sigma_a^b := [a,b] \times [0,1]$. We also abbreviate the half-open strips $\Sigma_a^\infty = [a,\infty) \times [0,1]$ and $\Sigma_{-\infty}^b = (-\infty,b] \times [0,1]$. The next lemma states that functions on Σ_a^b satisfy a Sobolev estimate with a constant independent of a and b as long as the strip is long enough.

Lemma A.3.3. *For all constants $p > 2$, a and b with possibly $a = -\infty$ or $b = \infty$ satisfying $b - a \geq 1$ and all functions $f \in H^{1,p}(\Sigma_a^b, \mathbb{R})$ we have*

$$\|f\|_{C^0(\Sigma_a^b)} \leq \frac{4p}{p-2} \left(\int_{\Sigma_a^b} |f|^p + |\mathrm{d}f|^p \,\mathrm{d}s\mathrm{d}t \right)^{1/p} .$$

Proof. It suffices to show the estimate for any given smooth function $f : \Sigma_a^b \to \mathbb{R}$. Fix an arbitrary $z = s + it \in \Sigma_a^b$. An easy geometric observation shows there exists $s_0 \in \mathbb{R}$ such that $|s - s_0| < 1$ and $\Sigma_{s_0}^{s_0+1} \subset \Sigma_a^b$. We have for every $z' \in [s_0, s_0 + 1] \times [0, 1]$

$$f(z) = f(z') + \int_0^1 df(\theta z + (1-\theta)z')[z - z']\mathrm{d}\theta .$$

Integrate z' over $\Sigma_{s_0}^{s_0+1} := [s_0, s_0 + 1] \times [0, 1]$ and estimate using $|z - z'| < 2$ and the Hölder inequality

$$\begin{aligned}
|f(z)| &\leq \int_{\Sigma_{s_0}^{s_0+1}} \left|f(z')\right| \mathrm{d}z' + 2 \int_0^1 \int_{\Sigma_{s_0}^{s_0+1}} \left|\mathrm{d}f(\theta z + (1-\theta)z')\right| \mathrm{d}z'\mathrm{d}\theta \\
&\leq \|f\|_{L^p} + 2 \int_0^1 \left(\int_{\Sigma_{s_0}^{s_0+1}} \left|\mathrm{d}f(\theta z + (1-\theta)z')\right|^p \mathrm{d}z' \right)^{1/p} \mathrm{d}\theta \\
&= \|f\|_{L^p} + 2 \int_0^1 \left(\int_{(1-\theta)\Sigma_{s_0}^{s_0+1} + \theta z} \frac{|\mathrm{d}f(\hat{z})|^p}{(1-\theta)^2} \mathrm{d}\hat{z} \right)^{1/p} \mathrm{d}\theta \\
&\leq \|f\|_{L^p} + 2 \|\mathrm{d}f\|_{L^p} \int_0^1 (1-\theta)^{-2/p} \mathrm{d}\theta \\
&\leq \frac{2p}{p-2} \left(\|f\|_{L^p} + \|\mathrm{d}f\|_{L^p} \right) \leq \frac{4p}{p-2} \|f\|_{H^{1,p}} .
\end{aligned}$$

This shows the claim by taking the supremum over all z □

Lemma A.3.4. *For all $p > 2$, there exists a constant c such that for all strips $u \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$, vector fields $\xi \in T_u\mathcal{B}^{1,p;\delta}$ and $R \geq 1$ we have*

$$\begin{aligned}
&\|\xi\|_\infty \leq c \|\xi\|_{1,p;\delta} , && \|\nabla\xi\|_{p;\delta} \leq c(1 + \|\mathrm{d}u\|_{p;\delta}) \|\xi\|_{1,p;\delta} \\
&\|\xi\|_\infty \leq c \|\xi\|_{1,p;\delta,R} , && \|\nabla\xi\|_{p;\delta,R} \leq c(1 + \|\mathrm{d}u\|_{p;\delta,R}) \|\xi\|_{1,p;\delta,R} ,
\end{aligned} \tag{A.3.4}$$

in which the norm $\|\cdot\|_{1,p;\delta,R}$ is defined in (8.2.5).

Proof. The proof is given in [5, Lemma 10.8] and [5, Lemma 10.9]. For some vector field $\xi \in \Gamma(u^*TM)$ we have the inequality

$$|\mathrm{d}\,|\xi|| = |\langle\nabla\xi, \xi\rangle| \,/\, |\xi| \leq |\nabla\xi| . \tag{A.3.5}$$

For any $(s, t) \in \Sigma_0^\infty$ we have by Lemma A.3.3

$$|\xi|^p \leq 2^p \left(\left|\xi - \widehat{\Pi}_{u(\infty)}^u \xi(\infty)\right|^p + |\xi(\infty)|^p \right)$$

Then by estimate (A.3.5) and since $e^{\delta|s|} \geq 1$ we have

$$|\xi|^p \leq O(1) \int_{\Sigma_0^\infty} \left|\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\right|^p + \left|\mathrm{d}\left|\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\right|\right|^p \mathrm{d}s\mathrm{d}t + 2^p\,|\xi(\infty)|^p$$

which is easily bounded by $O(1)\,\|\xi\|_{1,p;\delta}$. Similarily we proceed for the negative end and for Σ^{2R}_{-2R} appearing in norm $\|\cdot\|_{1,p;\delta,R}$. Note that the Sobolev constant of Σ^{2R}_{-2R} is independent of R by Lemma A.3.3. This shows the two inequalities on the left-hand side of (A.3.4).

For the two inequalities on the right-hand side we use Corollary A.2.4 to see that the norm of $\nabla\xi$ is bounded by

$$\left|\nabla\big(\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\big)\right| + \left|\nabla\widehat{\Pi}^u_{u(\infty)}\xi(\infty)\right| \leq \left|\nabla\big(\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\big)\right| + c_3\,|\mathrm{d}u|\,|\xi(\infty)|\ .$$

Multiply the estimate with $e^{\delta|s|}$, use the inequality $(a+b)^p \leq 2^p(a^p+b^p)$ for all positive a, b and integrate over Σ_0^∞ to conclude that there exists a constant c_4 such that

$$\begin{aligned}\int_{\Sigma_0^\infty} &|\nabla\xi|^p\,e^{\delta|s|p}\mathrm{d}s\mathrm{d}t\\ &\leq c_4\int_{\Sigma_0^\infty}\left|\nabla\big(\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\big)\right|^p e^{\delta|s|p}\mathrm{d}s\mathrm{d}t + c_4\,|\xi(\infty)|^p\int_{\Sigma_0^\infty}|\mathrm{d}u|^p\,e^{\delta|s|p}\mathrm{d}s\mathrm{d}t\,.\end{aligned}$$

Similar we proceed with the negative end and Σ^{2R}_{-2R}. This shows the claim. □

Lemma A.3.5. *For all $p > 2$, there exists a constant c such that for all $\delta \geq 0$, strips $u \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$, vector fields $\xi \in T_u\mathcal{B}^{1,p;\delta}$ and $s \geq 0$ we have*

$$\left\|\xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)\right\|_{C^0([s,\infty]\times[0,1])} \leq ce^{-\delta s}\,\|\xi\|_{1,p;\delta}\ .$$

A similar estimate holds for the negative end.

Proof. A proof is given in [5, Lemma 4.4]. Abbreviate $\xi^+ := \xi - \widehat{\Pi}^u_{u(\infty)}\xi(\infty)$. By Lemma A.2.2, the Sobolev estimate A.3.3 and estimate (A.3.5) we have uniform constants c_1 and c_2 such that

$$\left|e^{\delta s}\xi^+\right|^p \leq c_1\int_{\Sigma_s^\infty}\left(\left|\xi^+\right|^p + \left|\mathrm{d}\left|\xi^+\right|\right|^p\right)e^{\delta\sigma}\mathrm{d}\sigma \leq c_2\,\|\xi\|^p_{1,p;\delta}\ .$$

This shows the claim after multiplying with $e^{-p\delta s}$ on both sides and taking the p-th root. □

Corollary A.3.6. *There exists constant ε and c such that for all $u \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$ and smooth vector fields $\xi, \xi' \in \Gamma(u^*TM)$ satisfying $\|\xi\|_\infty < \varepsilon$ we have*

$$\left\|D_{u_\xi}\Pi^{u_\xi}_u\xi' - \Pi^{u_\xi}_u D_u\xi'\right\|_{p;\delta} \leq c(1 + \|J\|_{C^2})(1 + \|\mathrm{d}u\|_{p;\delta})\,\|\xi\|_{1,p;\delta}\left\|\xi'\right\|_{1,p;\delta}\ .$$

Proof. Integrate the pointwise estimate from Lemma A.3.1 and then use the Sobolev estimates from Lemma A.3.4. A completely similar argument appears in the proof of Lemma 8.4.1. □

Corollary A.3.7. *There exists a constant c such that for all $u \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$ and $\xi \in T_u\mathcal{B}^{1,p;\delta}$ we have*

$$\|D_u\xi\|_{p;\delta} \leq c(1 + \|J\|_{C^2})(1 + \|\mathrm{d}u\|_{p;\delta}) \|\xi\|_{1,p;\delta} \, .$$

Proof. By definition of the operator D_u we have the point-wise estimate

$$|D_u\xi| \leq (1 + \|J\|_\infty) |\nabla\xi| + \|J\|_{C^1} |\xi| |\mathrm{d}u| \, .$$

Now integrate the estimate and use the estimates given in Lemma A.3.4. □

Lemma A.3.8. *There exists a constant $\varepsilon > 0$ such that for all $u \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$ and $\xi \in \Gamma(u^*TM)$ with $u_\xi = \exp_u \xi \in \mathcal{B}^{1,p;\delta}(C_-, C_+)$ and $\|\xi\|_\infty < \varepsilon$ we have $\xi \in T_uB^{1,p;\delta}$.*

Proof. Abbreviate $p = u(\infty)$ and $q = u_\xi(\infty)$. Define the vector $\xi(\infty) \in T_pM$ via $q = \exp_p \xi(\infty)$. Since the distance between parallel geodesics is uniformly bounded (cf. Corollary A.1.4) we have for all $(s,t) \in \mathbb{R} \times [0,1]$ with s large enough

$$\begin{aligned}
\left|\xi - \Pi_p^u\xi(\infty)\right| &\leq O\big(\mathrm{dist}\left(u_\xi, \exp_u \Pi_p^u\xi(\infty)\right)\big) \\
&\leq O\big(\mathrm{dist}\,(u_\xi, q) + \mathrm{dist}\left(\exp_p \xi(\infty), \exp_u \Pi_p^u\xi(\infty)\right)\big) \\
&\leq O\big(\mathrm{dist}\,(u_\xi, q) + \mathrm{dist}\,(u,p)\big)\, ,
\end{aligned}$$

and using bounds on the derivative of the exponential map (cf. Corollary A.1.2) as well as the bound for the commutator of Π_p^u with ∇ (cf. Corollary A.2.4) we obtain

$$\left|\nabla\big(\xi - \Pi_p^u\xi(\infty)\big)\right| \leq O\left(|\mathrm{d}u| + |\mathrm{d}u_\xi| + \mathrm{dist}\,(u,p)\right) \, .$$

Since u and u_ξ are elements of $\mathcal{B}^{1,p;\delta}(C_-, C_+)$ the integral is finite

$$\begin{aligned}
&\int_{\Sigma_0^\infty} \Big(\left|\xi - \widehat{\Pi}_p^u\xi(\infty)\right|^p + \left|\nabla\big(\xi - \widehat{\Pi}_p^u\xi(\infty)\big)\right|^p\Big) e^{\delta ps}\mathrm{d}s\mathrm{d}t \leq \\
&\qquad O(1) \int_{\Sigma_0^\infty} \left(|\mathrm{d}u|^p + |\mathrm{d}v|^p + \mathrm{dist}\,(u,p)^p + \mathrm{dist}\,(v,q)^p\right) e^{\delta ps}\mathrm{d}s\mathrm{d}t < \infty \, .
\end{aligned}$$

Similar we proceed on the negative end. This shows the claim. □

Lemma A.3.9. *Gromov topology is finer than the topology of $\mathcal{B}^{1,p;\delta}(C_-, C_+)$ if $\delta > 0$ is sufficiently small, i.e. given a sequence $(u_\nu)_{\nu\in\mathbb{N}}$ of (J_ν, X_ν)-holomorphic curves which Floer-Gromov converges to the (J, X)-holomorphic strip u, then for all $\delta > 0$ small enough and $\nu \in \mathbb{N}$ large enough we have $u_\nu = \exp_u \xi_\nu$ for some vector field $\xi_\nu \in T_u\mathcal{B}^{1,p;\delta}$ and moreover $\|\xi_\nu\|_{1,p;\delta}$ converges to zero.*

Proof. By Floer-Gromov convergence we have in particular that u_ν converges to u uniformly on $\mathbb{R} \times [0,1]$ (cf. Lemma 5.3.1). Hence there exists $\xi_\nu \in \Gamma(u^*TM)$ such that $u_\nu = \exp_u \xi_\nu$ for all ν large enough. Lemma A.3.8 shows that the norm $\|\xi_\nu\|_{1,p;\delta}$ is finite for all ν large enough. It remains to show that $\|\xi_\nu\|_{1,p;\delta}$ converges to zero.

By Lemma 4.3.2 we see that there exists a constant μ such that for all $s > 0$ and $\nu \in \mathbb{N}$ large enough

$$\operatorname{dist}(u_\nu, u_\nu(\infty)) + |\mathrm{d}u_\nu| \leq c_1 e^{-\mu s}\,.$$

The same holds with u_ν replaced by u. Abbreviate $p_+ := u(\infty)$, $p_+^\nu := u_\nu(\infty)$, $\xi_\nu(\infty) := \exp_{p_+}^{-1} p_+^\nu$ and $\xi_\nu^+ := \xi_\nu - \widehat{\Pi}_{p_+}^u \xi_\nu(\infty)$. We use Corollary A.2.4 to get

$$\left|\nabla \xi_\nu^+\right| \leq O(|\mathrm{d}u| + |\mathrm{d}u_\nu| + \operatorname{dist}(u,p)) \leq O(e^{-\mu s})\,.$$

By Lemma A.3.8 we have $\lim_{s\to\infty} |\xi_\nu^+(s,t)| = 0$ and hence

$$\begin{aligned}\left|\xi_\nu^+(s,t)\right| = \int_s^\infty -\partial_\sigma \left|\xi_\nu^+(\sigma,t)\right| \mathrm{d}\sigma \leq \int_s^\infty \left|\nabla\xi_\nu^+(\sigma,t)\right| \mathrm{d}\sigma \leq \\ \leq O(1)\int_s^\infty e^{-\mu\sigma}\mathrm{d}\sigma \leq O(e^{-\mu s})\,.\end{aligned}$$

For $\delta < \mu$ and $s > s_0$ with s_0 large enough we conclude

$$\begin{aligned}\int_{\Sigma_0^\infty} \left(\left|\xi_\nu^+\right|^p + \left|\nabla\xi_\nu^+\right|^p\right) e^{\delta p s}\mathrm{d}s\mathrm{d}t \leq \|\xi_\nu\|^p_{C^1(\Sigma_0^{s_0})} \int_0^s e^{\delta\sigma}\mathrm{d}\sigma + O(1)\int_s^\infty e^{-(\mu-\delta)\sigma}\mathrm{d}\sigma \\ \leq O(e^{\delta s})\,\|\xi_\nu\|_{C^1(\Sigma_0^s)} + O(e^{-(\mu-\delta)s})\,.\end{aligned}$$

Similar we proceed with the negative end to show that for all $s \geq s_0$ we have

$$\|\xi_\nu\|_{1,p;\delta} \leq O(e^{\delta s})\,\|\xi_\nu\|_{C^1(\Sigma_{-s}^s)} + O(e^{-(\mu-\delta)s}) \leq o(1) + O(e^{-(\mu-\delta)s})\,.$$

Because s was chosen freely the left-hand side converges to zero. □

Lemma A.3.10. *With the same assumptions as Lemma A.3.9. There exists a constant c such that for all $\xi \in T_u\mathcal{B}^{1,p;\delta}(C_-,C_+)$ and $\nu \in \mathbb{N}$ large enough we have $u_\nu = \exp_u \xi_\nu$ for some $\xi_\nu \in T_u\mathcal{B}^{1,p;\delta}$ and*

$$\left\|\left(D_{u_\nu,J_\nu}\Pi_u^{u_\nu} - \Pi_u^{u_\nu} D_{u,J}\right)\xi\right\|_{p;\delta} = c(\|\xi_\nu\|_{1,p;\delta} + \|J_\nu - J\|_{C^1})\,\|\xi\|_{1,p;\delta}\;.$$

In particular the operator $D_{u_\nu,J_\nu}\Pi_u^{u_\nu} - \Pi_u^{u_\nu} D_{u,J}$ converges to zero in operator norm.

Proof. By Lemma A.3.9 the vector field ξ_ν exists. Corollary A.3.6 implies that there exists a constant c_1 possibly depending on u and J but independent of ν such that for all sections $\xi \in \Gamma(u^*TM)$ and ν large enough we have

$$\|(D_{u_\nu,J_\nu}\Pi_u^{u_\nu} - \Pi_u^{u_\nu} D_{u,J_\nu})\,\xi\|_{p;\delta} \leq c_1\,\|\xi_\nu\|_{1,p;\delta}\,\|\xi\|_{1,p;\delta}\;.$$

Directly from the Definition we have

$$\left\|\left(D_{u,J_\nu} - D_{u,J}\right)\xi\right\|_{p;\delta} \leq 2^{p+1}\,\|J - J_\nu\|_{C^1}\,(\|\xi\|_\infty\,\|\partial_t u\|_{p;\delta} + \|\nabla\xi\|_{p;\delta})\,.$$

This shows the estimate using Lemma A.3.4. With the estimate we conclude convergence of the operator since by Lemma A.3.9 the norm $\|\xi_\nu\|_{1,p;\delta}$ converges to zero. □

B. Operators on Hilbert spaces

Let H be a separable real Hilbert space. We will write $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for the inner product and respectively the norm of H. In this section we consider unbounded self-adjoint operators A and with dense $\operatorname{dom} A \subset H$. Let $\mathcal{L}(H)$ denote the space of bounded linear operators of H and for $B \in \mathcal{L}(H)$ we denote by $\|B\|$ the operator norm.

B.1. Spectral gap

Given a self-adjoint operator $A : \operatorname{dom} A \to H$ we denote by $\sigma(A) \subset \mathbb{R}$ its spectrum and by

$$\iota(A) := \inf\{|\lambda| \mid \lambda \in \sigma(A) \setminus \{0\}\} \tag{B.1.1}$$

the *spectral gap of A*.

Lemma B.1.1. *Let A be self-adjoint operator with domain $\operatorname{dom} A \subset H$. Assume that the spectrum $\sigma(A) \subset \mathbb{R}$ is bounded from below, then for all $\xi \in \operatorname{dom} A$ we have*

$$\langle A\xi, \xi \rangle \geq \inf \sigma(A) \|\xi\|^2 \, .$$

If additionally the range of A is closed then the spectral gap of A is positive and satisfies

$$\|A\xi\| \geq \iota(A) \|\xi\| \, ,$$

for all $\xi \in \operatorname{dom} A$ with $\xi \perp \ker A$. Both inequalities are sharp.

Proof. The first part is proven in [48, Section 10]. To show the second inequality use the first inequality with A^2. It remains to show that $\iota(A)$ is positive. Assume without loss of generality that A is injective. Since A is self-adjoint with closed range it is invertible. Since A is a closed operator, the closed graph theorem implies that A^{-1} is bounded. Hence for some constant $c > 0$ we have $\left\|A^{-1}\eta\right\| \leq c \|\eta\|$ for all $\eta \in H$ which implies $\|\xi\| \leq c \|A\xi\|$ for all $\xi \in \operatorname{dom} A$. This shows that $\iota(A) \geq 1/c$ since the inequalities are sharp. □

Corollary B.1.2. *Let A be a self-adjoint operator with domain $\operatorname{dom} A$ and closed range, then we have for all $\xi \in \operatorname{dom} A$*

$$\|A\xi\|^2 \geq \iota(A) \langle A\xi, \xi \rangle \, .$$

Proof. Let $P : H \to \ker A$ denote the orthogonal projector to $\ker A$ as subset of H. With Lemma B.1.1 we have

$$\langle A\xi, \xi \rangle = \langle A(1-P)\xi, \xi \rangle = \langle (1-P)\xi, A\xi \rangle \leq \|(1-P)\xi\| \, \|A\xi\| \leq \iota(A)^{-1} \|A\xi\|^2 \, .$$

This shows the claim. □

The next lemma states that the spectral gap is lower semi-continuous for bounded perturbations of A which preserve the dimension of the kernel.

Lemma B.1.3. *Let A be a self-adjoint operator with closed range and finite dimensional kernel. For all $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for any bounded symmetric operator B with $\|B\| < \delta$ and* $\dim \ker A + B = \dim \ker A$ *we have $\iota(A+B) \geq \iota(A) - \epsilon$.*

Proof. Write $A' = A + B$ and denote by P, P' the orthogonal projection to the kernel of A, A' respectively. We claim that for any $\varepsilon > 0$ there exists δ such that

$$\left\|A - A'\right\| < \delta \quad \Rightarrow \quad \left\|P - P'\right\| < \frac{\varepsilon}{2\iota(A)}\,. \tag{B.1.2}$$

Let $\{E(\lambda)\}, \{E'(\lambda)\}$ be the spectral families associated to A, A' respectively. The spectrum of A has a gap at $\pm\iota(A)/2$. By [48, Thm. 5.10] we have for any $\varepsilon > 0$ a constant δ such that ($\iota := \iota(A)$)

$$\left\|A - A'\right\| < \delta \quad \Rightarrow \quad \left\|E(\iota/2) - E(-\iota/2) - (E'(\iota/2) - E'(-\iota/2))\right\| < \frac{\varepsilon}{2\iota}\,. \tag{B.1.3}$$

Note that in our situation the quantity $\hat{\delta}(A, A')$ as defined in [48, p. 197] reduces to $\|A - A'\|$. Since zero is the only spectral value in the interval $[-\iota/2, \iota/2]$ we have $E(\iota/2) - E(-\iota/2) = P$. To show (B.1.2) it remains to show $E'(\iota/2) - E'(-\iota/2) = P'$. By monotonicity of the spectral family we have

$$\operatorname{im} P' = \operatorname{im}\left(E'(0) - \lim_{\lambda\uparrow 0} E'(\lambda)\right) \subset \operatorname{im}\left(E'(\iota/2) - E'(-\iota/2)\right), \tag{B.1.4}$$

By (B.1.3) the projection $E'(\iota/2) - E'(-\iota/2)$ converges to P, in particular their images have the same dimension. Hence

$$\dim \operatorname{im} P' \leq \dim \operatorname{im}\left(E'(\iota/2) - E'(-\iota/2)\right) = \dim \operatorname{im} P\,.$$

By assumption we have $\dim \operatorname{im} P' = \dim \operatorname{im} P$, hence we have equality in the last estimate, which shows that we have equality in (B.1.4) and thus $P' = E'(\iota/2) - E'(-\iota/2)$. Hence (B.1.2) follows from (B.1.3).

We now proof the lemma. Since A' is a bounded perturbation we have $\operatorname{dom} A' = \operatorname{dom} A$. By possibly decreasing δ we assume that $\delta < \frac{\varepsilon}{2}$ and estimate using (B.1.2) for any $\xi \in \operatorname{dom} A$ with $\|\xi\| = 1$ and $\xi \perp \ker A'$

$$\begin{aligned} 1 = \|\xi\| &\leq \|(\mathbb{1} - P)\xi\| + \left\|(P - P')\xi\right\| \\ &\leq \frac{1}{\iota}\,\|A\xi\| + \left\|P - P'\right\| \\ &\leq \frac{1}{\iota}\left\|A'\xi\right\| + \frac{1}{\iota}\left\|A - A'\right\| + \left\|P - P'\right\| \\ &\leq \frac{1}{\iota}\left\|A'\xi\right\| + \frac{\varepsilon}{2\iota} + \frac{\varepsilon}{2\iota}\,. \end{aligned}$$

Hence $\iota - \varepsilon \leq \|A'\xi\|$ and the lemma follows by taking the infimum over all $\xi \in \operatorname{dom} A$ with $\|\xi\| = 1$ and $\xi \perp \ker A'$. $\square$

B.2. Flow operator

Given a Banach space V such that there exists a compact and dense inclusion $V \subset H$. Let $\mathcal{L}(V,H)$ denote the space of bounded operators from V to H. In this section we analyze the asymptotic properties of bounded functions $\xi : [0,\infty) \to V$ which solve the differential equation

$$\partial_s \xi(s) + A(s)\xi(s) + B(s)\xi(s) = \eta(s)\,, \tag{B.2.1}$$

where $\eta : [0,\infty) \to H$ and $A : [0,\infty) \to \mathcal{L}(V,H)$, $B : [0,\infty) \to \mathcal{L}(H)$ are continuously differentiable functions satisfying the assumptions:

(i) The operator $A(s)$ is symmetric for every s. There exists an operator $A_\infty \in \mathcal{L}(V,H)$ such that $A(s) - A_\infty$ and $\partial_s A(s)$ extend to bounded linear operators on H and we have

$$\lim_{s\to\infty} \|A(s) - A_\infty\| = \lim_{s\to\infty} \|\partial_s A(s)\| = 0\,. \tag{B.2.2}$$

(ii) The operator A_∞ is Fredholm but not necessarily injective.

(iii) The operator $B(s)$ is skew-symmetric for every $s \geq 0$ and

$$\lim_{s\to\infty} \|B(s)\| = 0\,. \tag{B.2.3}$$

Remark B.2.1. These assumptions are almost identical to the assumptions in [65, Section 3] except that we do not suppose that A_∞ is injective.

Lemma B.2.2. *Let $P : H \to \ker A_\infty$ denote the orthogonal projection. Assume that $\lim_{s\to\infty} \|\xi(s)\| = 0$ and for every constant ε there exists s_0 such that for all $s \geq s_0$ we have*

$$\|P\xi(s)\| \leq \varepsilon \|\xi(s)\|\,. \tag{B.2.4}$$

Further suppose that there exists positive constants δ and c such that for all $s \geq 0$ we have

$$\|\eta(s)\| + \|\partial_s \eta(s)\| \leq c e^{-\delta s}\,.$$

Then for any $\mu < \min\{\iota(A_\infty), \delta\}$ there exists a constant $s_0 = s_0(\mu)$ such that $\|\xi(s)\| \leq e^{-\mu s}$ for all $s \geq s_0$. Moreover if $\eta = 0$ then we have

$$\langle A(s)\xi(s), \xi(s)\rangle \geq \mu \|\xi(s)\|^2\,,$$

for all $s \geq s_0$.

Proof. We follow closely the lines of the proof of [65, Lemma 3.1]. We just need to insert assumption (B.2.4) at the right place. Consider the function $g : [0,\infty) \to \mathbb{R}$ given by

$$g(s) := \frac{1}{2}\|\xi(s)\|^2\,.$$

We suppress the argument s whenever convenient and write $\dot{g}$ etc. to denote the derivative by ∂_s. Since $B(s)$ is skew-symmetric we have

$$\dot{g}(s) = \langle \xi, \dot{\xi} \rangle = \langle \xi, \eta - A\xi \rangle \,. \tag{B.2.5}$$

Differentiating again we have with assumptions (B.2.2) and (B.2.3) for any $\varepsilon > 0$

$$\begin{aligned}
\ddot{g} &= \langle \dot{\xi}, \eta - 2A\xi \rangle + \langle \xi, \dot{\eta} - \dot{A}\xi \rangle \\
&= 2\left\| A\xi \right\|^2 + \left\|\eta\right\|^2 - \langle A\xi, 3\eta \rangle - \langle B\xi, \eta \rangle + \langle 2B\xi, A\xi \rangle + \langle \xi, \dot{\eta} - \dot{A}\xi \rangle \\
&\geq (2-\varepsilon)\left\| A\xi \right\|^2 - \Big((1+4\varepsilon^{-1})\left\| B \right\|^2 + \left\| \dot{A} \right\| + \varepsilon \Big) \left\| \xi \right\|^2 - \left(1 + 9\varepsilon^{-1}\right)\left\| \eta \right\|^2 - \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \varepsilon^{-1}\left\| \dot{\eta} \right\|^2 \\
&\geq (2-\varepsilon)\left\| A\xi \right\|^2 - (o(1) + \varepsilon)\left\| \xi \right\|^2 - c^2(1 + 10\varepsilon^{-1})e^{-2\delta s} \,,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the estimate $-ab \geq -\varepsilon a^2 - b^2/\varepsilon$ for all $a, b > 0$. Similarly we have

$$\begin{aligned}
\left\| A_\infty \xi \right\|^2 &= \left\| (A - A_\infty)\xi \right\|^2 + 2\langle (A - A_\infty)\xi, A\xi \rangle + \left\| A\xi \right\|^2 \\
&\leq (1 + \varepsilon^{-1})\left\| A - A_\infty \right\|^2 \left\| \xi \right\|^2 + (1+\varepsilon)\left\| A\xi \right\|^2 \\
&\leq o(1)\left\| \xi \right\|^2 + (1+\varepsilon)\left\| A\xi \right\|^2 \,.
\end{aligned}$$

Combining the last two estimates we get a constant $c_1 = c_1(\varepsilon)$ such that

$$\begin{aligned}
\ddot{g} &\geq \frac{2-\varepsilon}{1+\varepsilon}\left\| A_\infty \xi \right\|^2 - (o(1) + \varepsilon)\left\| \xi \right\|^2 - c_1 e^{-2\delta s} \\
&\geq (2 - 4\varepsilon)\left\| A_\infty \xi \right\|^2 - (o(1) + \varepsilon)\left\| \xi \right\|^2 - c_1 e^{-2\delta s}
\end{aligned}$$

Let $\iota := \iota(A_\infty)$ denote the spectral gap of A_∞. With Lemma B.1.1 we have

$$\left\| A_\infty \xi \right\|^2 \geq \iota^2 \left\| (1-P)\xi \right\|^2 = \iota^2 \left\| \xi \right\|^2 - \iota^2 \left\| P\xi \right\|^2 \geq \iota^2 \left\| \xi \right\|^2 - o(1)\left\| \xi \right\|^2 \,,$$

in which we have used the assumption (B.2.4). Combining the last two estimates shows

$$\begin{aligned}
\ddot{g}(s) &\geq \iota^2(2 - 4\varepsilon)\left\| \xi \right\|^2 - (o(1) + \varepsilon)\left\| \xi \right\|^2 - c_1 e^{-2\delta s} \\
&\geq (2\iota^2 - 4\iota^2\varepsilon - \varepsilon - o(1))\left\| \xi \right\|^2 - c_1 e^{-2\delta s} \,.
\end{aligned}$$

In particular there exists a constant $s_0 = s_0(\varepsilon)$ such that for all $s \geq s_0$ we have

$$\ddot{g}(s) \geq 2(2\iota^2 - 4\iota^2\varepsilon - 4\varepsilon)g(s) - c_1 e^{-2\delta s} \,.$$

The previous computation holds with any ε. Now choose $\varepsilon < (\iota^2 - \mu^2)/(2\iota^2 + 2)$ to conclude

$$\ddot{g}(s) \geq 4\mu^2 g(s) - c_1 e^{-2\delta s} \,.$$

By assumption we also have $\lim_{s\to\infty} g(s) = 0$. Provided with the last estimate the rest of the proof is completely analogous to the proof of [65, Lemma 3.1]. $\square$

Lemma B.2.3. *Assume that $\eta = 0$ and the integral is finite*

$$\int_0^\infty \|A(s) - A_\infty\| + \|B(s)\| \, \mathrm{d}s \,,$$

then there exists an element $\zeta \in \ker A_\infty$ such that $\lim_{s\to\infty} \xi(s) = \zeta$. Moreover assume that there exist constants $\varepsilon > 0$ and c_1 such that for all $s \geq 0$

$$\|A(s) - A_\infty\| + \|B(s)\| \leq c_1 e^{-\varepsilon s} \,,$$

then for all $\mu < \min\{\varepsilon, \iota(A_\infty)\}$ we have a constant $s_0 = s_0(\mu)$ such that $\|\xi(s) - \zeta\| \leq e^{-\mu s}$ for all $s \geq s_0$.

Proof. Let $P : H \to \ker A_\infty$ be the orthogonal projection. Apply P on (B.2.1) to show that

$$\partial_s P\xi = -PA\xi - PB\xi = P(A_\infty - A)\xi - PB\xi \,.$$

Since ξ is bounded we conclude that there exists a constant c such that

$$\|\partial_s P\xi\| \leq c(\|A_\infty - A\| + \|B\|) \,.$$

Since $\partial_s P\xi$ is integrable and $\ker A_\infty$ is finite dimensional the path $s \mapsto P\xi(s)$ converges to an element $\zeta \in \ker A_\infty$. The difference $s \mapsto \xi(s) - \zeta$ solves the equation

$$\partial_s(\xi - \zeta) + A(\xi - \zeta) + B(\xi - \zeta) = \eta \,,$$

with $\eta(s) = (A_\infty - A(s))\zeta - B(s)\zeta$. We conclude using Lemma B.2.2. □

For the rest of the section we assume that $\eta = 0$ in (B.2.1). The proof of Agmon-Nirenberg Lemma (cf. [65, Lemma 3.3]) goes through without any change. Thus $\xi(s) \neq 0$ for all $s \geq 0$ and we define

$$v(s) = \frac{\xi(s)}{\|\xi(s)\|}, \qquad \lambda(s) = \langle v(s), A(s)v(s) \rangle \,.$$

The proof of [65, Lemma 3.4] requires an adjustment.

Lemma B.2.4. *With the assumptions of Lemma B.2.2 and $\eta = 0$. Suppose that the two integrals are finite*

$$\int_0^\infty \|A(s) - A_\infty\| + \|B(s)\| \, \mathrm{d}s, \quad N := \lambda(0) + \int_0^\infty \|B(s)\|^2 + \left\|\dot{A}(s)\right\| \mathrm{d}s \,.$$

Let μ denote the constant from Lemma B.2.2. Then the limits

$$N \geq \lim_{s\to\infty} \lambda(s) = \lambda_\infty \geq \mu, \qquad \lim_{s\to\infty} v(s) = v_\infty \,,$$

exist, where the latter convergence is in H and we have $A_\infty v_\infty = \lambda_\infty v_\infty$.

Proof. We differentiate

$$\dot{v} = \lambda v - Av - Bv \;, \tag{B.2.6}$$

and

$$\begin{aligned}\dot{\lambda} &= 2\langle \dot{v}, Av\rangle + \langle v, \dot{A}v\rangle \\ &= 2\langle \lambda v - Av - Bv, Av\rangle + \langle v, \dot{A}v\rangle \\ &= -2\left\|\lambda v - Av\right\|^2 + 2\langle Bv, \lambda v - Av\rangle + \langle v, \dot{A}v\rangle \\ &\leq \|B\|^2 + \|\dot{A}\| - \|\lambda v - Av\|^2 \;.\end{aligned} \tag{B.2.7}$$

Consider the function $\gamma : [0,\infty) \to \mathbb{R}$ defined by

$$\gamma(s) := \lambda(s) + \int_s^\infty \|B(\sigma)\|^2 + \left\|\dot{A}(\sigma)\right\| \mathrm{d}\sigma \;.$$

Since $\dot{\gamma} = \dot{\lambda} - \|B\|^2 - \left\|\dot{A}\right\|$ it follows from (B.2.7) that for every $s \geq 0$

$$\dot{\gamma} + \|Av - \lambda v\|^2 \leq 0 \,. \tag{B.2.8}$$

We see that γ is decreasing. Moreover γ is bounded from below since with Lemma B.2.2 we have $\gamma(s) \geq \lambda(s) \geq \mu$. Hence $\gamma(s)$ converges to a positive real number

$$\lambda_\infty := \lim_{s\to\infty} \gamma(s) = \lim_{s\to\infty} \lambda(s) \,.$$

Since $\gamma(0) = N$ and $\gamma(s) \geq \mu$ for all $s \geq 0$ we have that

$$\mu \leq \lambda_\infty \leq N \;.$$

We claim that λ_∞ is an eigenvalue of A_∞. By contradiction we assume that $A_\infty - \lambda_\infty$ is injective. Since $V \hookrightarrow H$ is compact and A_∞ is a Fredholm operator, $A_\infty - \lambda_\infty$ is a Fredholm operator as well. In particular $A_\infty - \lambda_\infty$ is closed and there exists a constant $c_1 > 0$ such that

$$1 = \|v\| \leq c_1 \left\|A_\infty v - \lambda_\infty v\right\| \;.$$

We estimate

$$\|A_\infty v - \lambda_\infty v\| \leq \|A - A_\infty\| + |\lambda - \lambda_\infty| + \|Av - \lambda v\| \;.$$

Hence by assumption and definition of λ_∞ for any $\varepsilon > 0$ we find s_0 such that for all $s \geq s_0$ we have

$$\|A_\infty v - \lambda_\infty v\| \leq \varepsilon + \|Av - \lambda v\| \;. \tag{B.2.9}$$

Suppose $\varepsilon < 1/2c_1$ then the last estimates show that $\|Av - \lambda v\| \geq 1/2c_1$ and by (B.2.8) also $\dot{\gamma}(s) \leq -1/4c_1^2 < 0$ for all $s \geq s_0$. This contradicts the fact that $\gamma(s)$ converges. Thus we have proved that λ_∞ is an eigenvalue.

Consider the eigenspace $E = \ker(A_\infty - \lambda_\infty)$ and the orthogonal projection $P : H \to E$. We want to show that

$$\lim_{s\to\infty} \|v(s) - Pv(s)\|^2 = 0 \;. \tag{B.2.10}$$

Set $\sigma(s) := 1/2 \left\| v(s) - Pv(s) \right\|^2$ and compute with $\langle \dot{v}, v \rangle = 0$ and (B.2.6)

$$\begin{aligned}\dot{\sigma} = \langle \dot{v}, (1-P)v \rangle = -\langle \dot{v}, Pv \rangle = \langle Bv, Pv \rangle + \langle Av - \lambda v, Pv \rangle = \\ = \langle Bv, Pv \rangle + \langle Av - A_\infty v, Pv \rangle + (\lambda_\infty - \lambda)\langle v, Pv \rangle + \langle A_\infty v - \lambda_\infty v, Pv \rangle \,.\end{aligned}$$

The last term on the right-hand side vanishes because P projects to the kernel of $A_\infty - \lambda_\infty$ and we conclude that the derivative of σ converges to zero. Now suppose by contradiction that (B.2.10) does not hold. Then we find a constant $\varepsilon > 0$ and a sequence $s_\nu \to \infty$ such that $\sigma(s_\nu) \geq \varepsilon$ for all $\nu \in \mathbb{N}$. But since the derivative of σ converges to zero we have for all $\nu \in \mathbb{N}$ sufficiently large

$$\left| s_\nu - s \right| \leq 1 \Longrightarrow \sigma(s) \geq \varepsilon/2 \,.$$

Since $A_\infty - \lambda_\infty$ is closed there exists a constant c_2 such that

$$\left\| v - Pv \right\| \leq c_2 \left\| A_\infty v - \lambda_\infty v \right\| \,.$$

By (B.2.9) we conclude that for all ν sufficiently large and $s \in [s_\nu - 1, s_\nu + 1]$ we have

$$\frac{\varepsilon}{2} \leq \sigma(s) \leq c_2^2 \left\| A(s)v(s) - \lambda(s)v(s) \right\|^2 + \frac{\varepsilon}{4} \,.$$

Again by (B.2.8) it follows $\dot{\gamma}(s) \leq -\varepsilon/4c_2^2 < 0$ for all $s \in [s_\nu - 1, s_\nu + 1]$, which contradicts that $\gamma(s)$ converges. Thus we have proved that

$$\lim_{s\to\infty} \left\| v(s) - Pv(s) \right\| = 0, \qquad \lim_{s\to\infty} \left\| Pv(s) \right\| = \lim_{s\to\infty} \langle v(s), Pv(s) \rangle = 1 \,. \tag{B.2.11}$$

Hence for s large enough we have $\left\| Pv(s) \right\| \neq 0$ and define

$$w(s) = \frac{P\xi(s)}{\left\| P\xi(s) \right\|} = \frac{Pv(s)}{\left\| Pv(s) \right\|} \,.$$

By assumption ξ solves $\partial_s \xi + A\xi + B\xi = 0$ and $\lambda_\infty P\xi = PA_\infty \xi$. We conclude

$$P\dot{\xi} = P(A_\infty - A)\xi - PB\xi - \lambda_\infty P\xi \,.$$

Abbreviate $\zeta := (A_\infty - A)\xi - B\xi$ and plug this into the computation of the derivative of w to obtain

$$\dot{w} = \frac{P\dot{\xi}}{\left\| P\xi \right\|} - \frac{\langle P\dot{\xi}, \xi \rangle}{\left\| P\xi \right\|^2} w = \frac{P\zeta}{\left\| P\xi \right\|} - \lambda_\infty w - \frac{\langle \zeta, P\xi \rangle}{\left\| P\xi \right\|^2} w + \lambda_\infty w = \frac{P\zeta}{\left\| P\xi \right\|} - \frac{\langle \zeta, P\xi \rangle}{\left\| P\xi \right\|^2} w \,.$$

According to (B.2.11) we have for all s large enough

$$\left\| \xi(s) \right\| \leq 2 \left\| P\xi(s) \right\|, \qquad \left\| \zeta(s) \right\| \leq 2 \left(\left\| A(s) - A_\infty \right\| + \left\| B(s) \right\| \right) \left\| P\xi(s) \right\| \,,$$

for hence

$$\left\| \dot{w} \right\| \leq 4 \left\| A - A_\infty \right\| + 4 \left\| B \right\| \,, \tag{B.2.12}$$

Thus $\left\| \dot{w} \right\|$ is integrable after the assumptions and $w(s)$ converges to some element $v_\infty \in E$. By (B.2.11) we have furthermore

$$v_\infty = \lim_{s\to\infty} w(s) = \lim_{s\to\infty} Pv(s) = \lim_{s\to\infty} v(s) \,.$$

This proves the lemma. □

Provided Lemma B.2.4 and equations (B.2.11), (B.2.12) the proof of [65, Lemma 3.5] and [65, Lemma 3.6] goes through up to very small changes. We state it here.

Lemma B.2.5. *In the situation of Lemma B.2.4. Assume that $s \mapsto B(s)$ is continuously differentiable and that there exists positive constants c and ε such that*

$$\|A(s) - A_\infty\| + \|B(s)\| + \|\dot{A}(s)\| + \|\dot{B}(s)\| \leq ce^{-\varepsilon s} ,$$

then there exists a non-zero eigenvalue λ_∞ of A_∞ with corresponding eigenvector v_∞ and for every $\mu < \varepsilon$ there exists a constant c such that for all $s \geq 0$ we have

$$\|\xi(s) - e^{-\lambda_\infty s} v_\infty\| \leq ce^{-(\lambda_\infty + \mu)s} .$$

C. Viterbo index

In order to relate the Fredholm index of the Cauchy-Riemann-Floer operator to topological data we generalize the index defined by Viterbo [73] and Floer [26] to maps with boundary on not necessarily transversely intersecting Lagrangians in terms of they Robbin-Salamon index for paths μ_{RS} given in [63].

Let $L_0, L_1 \subset M$ be any two Lagrangian submanifolds and $H_-, H_+ : M \times [0,1] \to \mathbb{R}$ be any two Hamiltonian functions. Consider the perturbed intersection points $\mathcal{I}_{H_-}(L_0, L_1)$ and $\mathcal{I}_{H_+}(L_0, L_1)$ as defined in (3.2.6). To a continuous map $u : [-1,1] \times [0,1] \to M$ satisfying

$$u(\pm 1, \cdot) \in \mathcal{I}_{H_\pm}(L_0, L_1), \qquad u(\cdot, 0) \subset L_0, \qquad u(\cdot, 1) \subset L_1 \ , \tag{C.0.1}$$

we assign an half-integer $\mu_{\mathrm{Vit}}(u)$. Let us explain the construction. Since the base $[-1,1] \times [0,1]$ is contractible the symplectic bundle u^*TM is trivial and any two trivializations are homotopic. Choose a symplectic trivialization

$$\Phi_u : [-1,1] \times [0,1] \times \mathbb{R}^{2n} \to u^*TM, \qquad (s,t,\xi) \mapsto \Phi_u(s,t)\xi \in T_{u(s,t)}M \ ,$$

that is a bundle isomorphism Φ_u such that for all $(s,t) \in [-1,1] \times [0,1]$ and $\xi, \xi' \in \mathbb{R}^{2n}$

$$\omega_{u(s,t)}(\Phi_u(s,t)\xi, \Phi_u(s,t)\xi') = \omega_{\mathrm{std}}(\xi, \xi') \ .$$

Denote by $\mathcal{L}(n)$ the space of linear Lagrangian subspaces in $\mathbb{R}^{2n}$ and by $\varphi_{H_\pm} : [0,1] \times M \to M$, $\varphi^t_{H_\pm} = \varphi_{H_\pm}(t, \cdot)$ the Hamiltonian flow associated to $H_\pm$. For $s \in [-1,1]$ and $t \in [0,1]$ define the Lagrangian spaces

$$\begin{aligned} F_0(s) &= \Phi_u(s,0)^{-1} T_{u(s,0)} L_0, \quad & F_-(t) &= \Phi_u(-1,t)^{-1} \mathrm{d}\varphi^t_{H_-} T_{u(-1,0)} L_0 \\ F_1(s) &= \Phi_u(s,1)^{-1} T_{u(s,1)} L_1, \quad & F_+(t) &= \Phi_u(+1,t)^{-1} \mathrm{d}\varphi^t_{H_+} T_{u(1,0)} L_1 \ . \end{aligned} \tag{C.0.2}$$

We denote by F_0, F_1, F_- and F_+ the continuous paths of Lagrangian spaces defined by $s \mapsto F_k(s)$ for $k = 0, 1$ and $t \mapsto F_\pm(t)$ respectively. Finally the index $\mu_{\mathrm{Vit}}(u)$ is defined by

$$\mu_{\mathrm{Vit}}(u) := \mu_{\mathrm{RS}}(F_0, F_1) + \mu_{\mathrm{RS}}(F_+, F_1(1)) - \mu_{\mathrm{RS}}(F_-, F_1(-1)) \ . \tag{C.0.3}$$

Lemma C.0.1. *The index $\mu_{\mathrm{Vit}}(u)$ is defined independently of the choice of Φ_u. Moreover given two Hamiltonian functions $H_-, H_+ : M \times [0,1] \to \mathbb{R}$ and let $u : [0,1] \times [-1,1] \times [0,1] \to M$ be such that $u_\tau := u(\tau, \cdot)$ satisfies* (C.0.1) *for all $\tau \in [0,1]$, then $\mu_{\mathrm{Vit}}(u_\tau) = \mu_{\mathrm{Vit}}(u_0)$ for all $\tau \in [0,1]$.*

Proof. Given two trivializations Φ_0 and Φ_1. Because $[-1,1] \times [0,1]$ is contractible there exists a homotopy Φ_τ between Φ_0 and Φ_1. Using Φ_τ we define via (C.0.2) homotopies of paths. Note that throughout the homotopy the dimension of the intersection of the two spaces at the endpoints is constant. A homotopy satisfying that property is called stratum homotopy and leaves the Robbin-Salamon index invariant (see [63, Theorem 2.4]). Similar we proceed for the family u_τ. □

Proposition C.0.2. *The index μ_{Vit} has the following properties*

(i) *If $H_+ = H_- = 0$ vanishes and L_0, L_1 intersect transversely, then the index $\mu_{\mathrm{Vit}}(u)$ agrees with the one given in [73].*

(ii) *Given $u : [-1,1] \times [0,1] \to M$ satisfying* (C.0.1). *Let $v : S^2 \to M$ be a sphere (resp. $v : (D, \partial D) \to (M, L_k)$ be a disk with $k \in \{0,1\}$), such that v has an point (resp. a boundary point) in common with u, then we have*

$$\mu_{\mathrm{Vit}}(u \# v) = \mu_{\mathrm{Vit}}(u) + 2\langle c_1(TM), [v]\rangle,$$
$$(\textit{resp. } \mu_{\mathrm{Vit}}(u \# v) = \mu_{\mathrm{Vit}}(u) + \mu_{\mathrm{Mas}}(v))\ ,$$

where by $\#$ we denote the connected sum at the common point (see the proof for the concrete definition).

(iii) *Given $u : [-1,1] \times [0,1] \to M$ satisfying* (C.0.1) *and $v : [0,1]^2 \to M$ satisfying $v(0,\cdot) = v(1,\cdot)$, $v(\cdot,0) \subset L_0$ and $v(\cdot,1) \subset L_1$. Assume that there exists $s_0 \in [-1,1]$ such that $u(s_0,\cdot) = v(0,\cdot)$ then it holds*

$$\mu_{\mathrm{Vit}}(u \# v) = \mu_{\mathrm{Vit}}(u) + \mu_{\mathrm{Mas}}(v)\ ,$$

here $u \# v$ denotes the connected sum along the path $u(s_0,\cdot)$.

(iv) *Given three Hamiltonian functions $H_0, H_1, H_2 : [0,1] \times M \to \mathbb{R}$ and maps u_0, u_1 satisfying the boundary condition $u_k|_{t=0} \subset L_0$, $u_k|_{t=1} \subset L_1$ for $k \in \{0,1\}$ and*

$$\begin{aligned} u_0(-1,\cdot) &\in I_{H_0}(L_0, L_1), \\ u_0(1,\cdot) = u_1(-1,\cdot) &\in I_{H_1}(L_0, L_1), \\ u_1(1,\cdot) &\in I_{H_2}(L_0, L_1)\ . \end{aligned}$$

Then we have

$$\mu_{\mathrm{Vit}}(u_0 \# u_1) = \mu_{\mathrm{Vit}}(u_0) + \mu_{\mathrm{Vit}}(u_1)\ .$$

Where $u_0 \# u_1$ denotes the connected sum.

(v) *Assume that $H = H_- = H_+$ is clean for the pair (L_0, L_1) and given $u : [-1,1] \times [0,1] \to M$ such that $u(s,\cdot) \in \mathcal{I}_H(L_0, L_1)$ for all $s \in [-1,1]$ then $\mu_{\mathrm{Vit}}(u) = 0$.*

(vi) *Let H_-, H_+ be clean. Given two connected components $C_- \subset \mathcal{I}_{H_-}(L_0, L_1)$ and $C_+ \subset \mathcal{I}_{H_+}(L_0, L_1)$. Suppose $u : [-1,1] \times [0,1] \to M$ satisfies* (C.0.1) *and $u(\pm 1,\cdot) = x_\pm \in C_\pm$, then*

$$\mu_{\mathrm{Vit}}(u) + \frac{1}{2}(\dim C_+ + \dim C_-) \in \mathbb{Z}\ .$$

Proof. We will deduce these properties from the axioms of the Robbin-Salamon index.

Step 1. We show (i)

Since L_0, L_1 intersect transversely, the intersection $L_0 \cap L_1$ is a discrete set of points. Necessarily the map $t \mapsto u(\pm 1, t) = p_\pm$ is constant. We choose a trivialization Φ satisfying for all $t \in [0,1]$

$$\Phi(\pm 1, t) T_{p_\pm} L_0 = \mathbb{R}^n \oplus \{0\}, \qquad \Phi(\pm 1, t) T_{p_\pm} L_1 = \{0\} \oplus \mathbb{R}^n \ .$$

Since $H_- = H_+ = 0$ the paths $F_\pm$ defined in (C.0.2) are constant. Define the path $\overline{F} : [0,1] \to \mathcal{L}(n)$ via

$$t \mapsto \overline{F}(t) = \left(\cos(\pi t/2)\mathbb{1} + \sin(\pi t/2) J_{\mathrm{std}}\right)(\mathbb{R}^n \oplus \{0\}) \ .$$

Denote by $\overline{F}^\vee$, $F_1^\vee$ etc. the path $\overline{F}$, F_1 run with reverse orientation. We define a loop $F_{\mathrm{loop}} : S^1 \to \mathcal{L}(n)$ via the concatenation of the paths

$$F_{\mathrm{loop}} = F_0 \# \overline{F} \# F_1^\vee \# \overline{F}^\vee \ .$$

In [73] the index is defined as the Maslov index of the loop F_{loop}. Using [63, Remark 2.6] we have fix

$$\mu_{\mathrm{Mas}}(F_{\mathrm{loop}}) = \mu_{\mathrm{RS}}(F_{\mathrm{loop}}, F_1(0)) \ . \tag{C.0.4}$$

By the concatenation axiom, the homotopy axiom and the zero axiom we have

$$\mu_{\mathrm{RS}}(\overline{F}, F_1(1)) + \mu_{\mathrm{RS}}(\overline{F}^\vee, F_1(1)) = \mu_{\mathrm{RS}}(\overline{F} \# \overline{F}^\vee, F_1(1)) = \mu_{\mathrm{RS}}(\overline{F}(0), F_1(1)) = 0.$$

Similarly $\mu_{\mathrm{RS}}(F_1, F_1(1)) + \mu_{\mathrm{RS}}(F_1^\vee, F_1(1)) = 0$ and

$$\mu_{\mathrm{RS}}(F_1(0), F_1) + \mu_{\mathrm{RS}}(F_1, F_1(1)) = 0 \,,$$

which shows $\mu_{\mathrm{RS}}(F_1^\vee, F_1(1)) = \mu_{\mathrm{RS}}(F_1(0), F_1)$. Since F_1 is a loop we have

$$\mu_{\mathrm{RS}}(F_1(0), F_1) = \mu_{\mathrm{RS}}(F_0(1), F_1) \,.$$

Using the last identities we continue with (C.0.4)

$$\begin{aligned}
\mu_{\mathrm{Mas}}(F_{\mathrm{loop}}) &= \mu_{\mathrm{RS}}(F_0 \# \overline{F} \# F_1^\vee \# \overline{F}^\vee, F_1(1)) \\
&= \mu_{\mathrm{RS}}(F_0, F_1(1)) + \mu_{\mathrm{RS}}(\overline{F}, F_1(1)) - \mu_{\mathrm{RS}}(F_1, F_1(1)) - \mu_{\mathrm{RS}}(\overline{F}, F_1(1)) \\
&= \mu_{\mathrm{RS}}(F_0, F_1(1)) + \mu_{\mathrm{RS}}(F_1(0), F_1) \\
&= \mu_{\mathrm{RS}}(F_0, F_1) \\
&= \mu_{\mathrm{Vit}}(u) \ .
\end{aligned}$$

Step 2. We show (ii)

Let $v : (D, \partial D) \to (M, p)$ be a sphere and assume for simplicity that the sphere has an interior point in common with u, say $u(1/2, 0) = p$. Without loss of generality we assume that $D := \{(s,t) \in \mathbb{R}^2 \mid s^2 + (t-1/2)^2 \le 1/4\} \subset [-1,1] \times [0,1]$. Let $\varphi : [-1,1] \times [0,1] \to [-1,1] \times [0,1]$ be a continuous map, which maps D to the point $\{(1/2, 0)\}$, is a homeomorphism on the complement and fixes each arc of the boundary. We define the connected sum

$$(u\#v)(s,t) := \begin{cases} v(s,t) & \text{if } (s,t) \in D \\ u(\varphi(s,t)) & \text{if } (s,t) \notin D \,. \end{cases}$$

Choose symplectic trivializations $\Phi_u : u^*TM \to \underline{\mathbb{R}}^{2n}$ and $\Phi_v : v^*TM \to \underline{\mathbb{R}}^{2n}$. The change of trivialization defines a loop $\psi : \partial D \to Sp(2n)$, $(s,t) \mapsto \Phi_v(s,t) \circ \Phi_u(1/2, 0)^{-1}$. We assume with loss of generality that ψ is unitary. Let $\Psi : [-1,1] \times [0,1] \setminus D \to U(n)$ be an extension of ψ such that $\Psi(s,t) = \psi(s,t)$ for all $(s,t) \in \partial D$ and $\Psi(s,t) = \mathbb{1}$ for $(s,t) \in \{-1\} \times [0,1] \cup [-1,1] \times \{1\} \cup \{1\} \times [0,1]$. Abbreviate $w = u\#v$. We define the symplectic trivialization $\Phi_w : w^*TM \to \underline{\mathbb{R}}^{2n}$ via

$$\Phi_w(s,t) = \begin{cases} \Phi_v(s,t) & \text{if } (s,t) \in D \\ \Psi(s,t) \circ \Phi_u(\varphi(s,t)) & \text{if } (s,t) \notin D \,. \end{cases}$$

Define F_0^u, F_1^u, $F_\pm^u$ (resp. F_0^w, F_1^w, $F_\pm^w$) via (C.0.2) using Φ_u (resp. Φ_w). By construction we have $F_0^w(s) = \Psi(s,0) F_0^u(\varphi(s,0))$ for all $s \in [-1,1]$, $F_\pm^u = F_\pm^w$ and $F_1^u = F_1^w$. We have after a homotopy in the domain $\psi' F_0^u = \psi' F_0(-1) \# F_0$. Hence by the concatenation axiom

$$\begin{aligned} \mu_{\mathrm{Vit}}(w) &= \mu_{\mathrm{RS}}(F_0^w, F_1^w) + \mu_{\mathrm{RS}}(F_+^w, F_1^w(1)) - \mu_{\mathrm{RS}}(F_-^w, F_1^w(-1)) \\ &= \mu_{\mathrm{RS}}(\psi' F_0^u, F_1^u) + \mu_{\mathrm{RS}}(F_+^u, F_1^u(1)) - \mu_{\mathrm{RS}}(F_-^u, F_1^u(-1)) \\ &= \mu_{\mathrm{RS}}(\psi' F_0^u(-1), F_1^u(-1)) + \mu_{\mathrm{RS}}(F_0^u, F_1^u) + \mu_{\mathrm{RS}}(F_+^u, F_1^u(1)) - \\ &\qquad\qquad - \mu_{\mathrm{RS}}(F_-^u, F_1^u(-1)) \\ &= \mu_{\mathrm{Mas}}(\psi' F_0^u(-1)) + \mu_{\mathrm{Vit}}(u) \,. \end{aligned} \tag{C.0.5}$$

Abbreviate $F := F_0^u(-1)$. Since ψ and ψ' are homotop within $U(n)$ (via Ψ), the Maslov index of the loop $\psi' F$ is the same as ψF, which by definition is given as the degree of the map $\det(\psi \circ \psi) : S^1 \to S^1$ (see [53, C.3.1]). On the other hand it is a classical fact that $\deg \det \psi = \langle c_1(TM), [v] \rangle$. We summarize the argument

$$\mu_{\mathrm{Mas}}(\psi' F) = \mu_{\mathrm{Mas}}(\psi F) = \deg \det(\psi \circ \psi) = 2 \deg \det \psi = 2 \langle c_1(TM), [v] \rangle \,.$$

This shows the identity for spheres using equation (C.0.5).

Now let $v : (D, \partial D) \to (M, L_0)$ be a disk. Denote by $z_0 \in \partial D$ and $s_0 \in [-1,1]$ the points such that $v(z_0) = u(s_0, 0)$. We give the definition of the connected sum: Define $\Omega_0 := \{(s,t) \in [-1,1] \times [0,1] \mid s^2 + t^2 \le 1/2\}$ and let $\varphi_0 : \Omega_0 \to D$ be a continuous map which maps the arc $\gamma := \{(s,t) \in [-1,1] \times [0,1] \mid s^2 + t^2 = 1/2\}$ to the point $\{z\}$ and which is a homeomorphism on the complement. Secondly define

$\Omega_1 := \{(s,t) \in [-1,1] \times [0,1] \mid s^2 + t^2 \geq 1/2\}$ and let $\varphi_1 : \Omega_1 \to [-1,1] \times [0,1]$ be a continuous map such that $\varphi(\gamma) = \{(s_0, 0)\}$, which is a homeomorphism on the complement and which fixes the arcs $\{-1\} \times [0,1]$, $[-1,1] \times \{1\}$ and $\{1\} \times [-1,1]$. With these preparations we define the *connected sum*

$$(u\#v)(s,t) := \begin{cases} v(\varphi_0(s,t)) & \text{if } (s,t) \in \Omega_0 \\ u(\varphi_1(s,t)) & \text{if } (s,t) \in \Omega_1 \ . \end{cases}$$

We deduce the equation. Choose symplectic trivializations $\Phi_u : u^*TM \to \underline{\mathbb{R}}^{2n}$ and $\Phi_v : v^*TM \to \underline{\mathbb{R}}^{2n}$ that agree on $u(s_0, 0) = v(z_0)$. Using Φ_u and Φ_v we obtain a trivialization $\Phi_w : w^*TM \to \underline{\mathbb{R}}^{2n}$ where $w = u\#v$. Define F_0^u, F_1^u, $F_\pm^u$ (resp. F_0^w, F_1^w, $F_\pm^w$) via (C.0.2) using Φ_u (resp. Φ_w). Secondly define $F^v(s) := \Phi_v(\varphi_0(s,0))T_{v(\varphi_0(s,0))}L_0$ for all $s \in [-1/2, 1/2]$. By construction we have $F^v(-1/2) = F^v(1/2)$, $F_1^w = F_1^u$, $F_\pm^w = F_\pm^u$ and $F_0^w = F_0^u|_{[-1,s_0]}\#F^v\#F_0^u|_{[s_0,1]}$. Then using the concatenation axiom

$$\begin{aligned} \mu_{\text{Vit}}(w) &= \mu_{\text{RS}}(F_0^w, F_0^w) + \mu_{\text{RS}}(F_+^w, F_1^w(1)) - \mu_{\text{RS}}(F_-^w, F_1^w(-1)) \\ &= \mu_{\text{RS}}(F_0^u|_{[-1,s_0]}, F_1^u|_{[-1,s_0]}) + \mu_{\text{RS}}(F^v, F_1^u(s_0)) + \mu_{\text{RS}}(F_0^u|_{[s_0,1]}, F_1^u|_{[s_0,1]}) \\ &\qquad \mu_{\text{RS}}(F_+^u, F_1^u(1)) - \mu_{\text{RS}}(F_-^u, F_1^u(-1)) \\ &= \mu_{\text{RS}}(F_0^u, F_1^u) + \mu_{\text{RS}}(F_+^u, F_1^u(1)) - \mu_{\text{RS}}(F_-^u, F_1^u(-1)) + \mu_{\text{RS}}(F^v, F_1^u(s_0)) \\ &= \mu_{\text{Vit}}(u) + \mu_{\text{Mas}}(v) \ . \end{aligned}$$

Step 3. We show (iii)

Choose a symplectic trivialization $\Phi_u : u^*TM \to \underline{\mathbb{R}}^{2n}$ and $\Phi_v : v^*TM \to \underline{\mathbb{R}}^{2n}$ that agree over $v(0,\cdot) = u(s,\cdot)$. We obtain a symplectic trivialization Φ_w of w^*TM where w is the connected sum $u\#v$. Define paths of Lagrangians F_0^u, F_1^u, F_-^u and F_+^u via (C.0.2) using Φ_u. Similarly define

$$F_0^v(s) := \Phi_v(s,0)T_{v(s,t)}L_0, \quad F_1^v(s) := \Phi_v(s,1)T_{v(s,1)}L_1 \,,$$

for all $s \in [0,1]$. Further define F_0^w, F_1^w, F_-^w and F_+^w via (C.0.2) using Φ_w. By construction we have $F_k^v(0) = F_k^v(1) = F_k^u(s_0)$, $F_k^w = F_k^u|_{[-1,s_0]}\#F_k^v\#F_k^u|_{[s_0,1]}$ for $k = 0,1$ and $F_\pm^w = F_\pm^u$. Using the concatenation axiom we have

$$\begin{aligned} \mu_{\text{Vit}}(w) &= \mu_{\text{RS}}(F_0^w, F_1^w) + \mu_{\text{RS}}(F_+^w, F_1^w(1)) - \mu_{\text{RS}}(F_-^w, F_1^w(-1)) \\ &= \mu_{\text{RS}}(F_0^u|_{[-1,s_0]}, F_1^u|_{[-1,s_0]}) + \mu_{\text{RS}}(F_0^v, F_1^v) + \mu_{\text{RS}}(F_0^u|_{[s_0,1]}, F_0^u|_{[s_0,1]}) \\ &\qquad + \mu_{\text{RS}}(F_+^u, F_1^u(1)) - \mu_{\text{RS}}(F_-^u, F_1^u(-1)) \\ &= \mu_{\text{RS}}(F_0^u, F_1^u) + \mu_{\text{RS}}(F_+^u, F_1^u(1)) - \mu_{\text{RS}}(F_-^u, F_1^u(1)) + \mu_{\text{RS}}(F_0^v, F_1^v) \\ &= \mu_{\text{Vit}}(u) + \mu_{\text{Mas}}(v) \ . \end{aligned}$$

Step 4. We show (iv)

Choose symplectic trivialization $\Phi_k : u_k^*TM \to \underline{\mathbb{R}}^{2n}$ for $k = 0,1$ that agree over $u_0(1,\cdot)$ and $u_1(-1,\cdot)$. We obtain a symplectic trivialization of the connected sum $u_2 =$

$u_0\#u_1$ denoted Φ_2. For $k=0,1,2$ denote the paths $F_0^{\Phi_k}$,$F_1^{\Phi_k}$ and $F_\pm^{\Phi_k}$ associated to Φ_k by (C.0.2). We have $F_+^{\Phi_0}=F_-^{\Phi_1}$ and $F_k^{\Phi_0}(1)=F_k^{\Phi_1}(-1)$ for $k=0,1$. By the concatenation axiom

$$\mu(F_0^{\Phi_0},F_1^{\Phi_0})+\mu(F_0^{\Phi_1},F_1^{\Phi_1})=\mu(F_0^{\Phi_2},F_1^{\Phi_2})\,.$$

The shows $\mu(u_0)+\mu(u_1)=\mu(u_2)$ inserting the definitions.

Step 5. We show (v).

Choose a symplectic trivialization Φ_0 of $u(\cdot,0)^*TM$ and define the trivialization Φ of u^*TM by

$$\Phi(s,t)=\mathrm{d}\varphi_H^t\Phi_0(s)\ ,$$

where φ_H^t denotes the Hamiltonian flow of H. By the property of the Hamiltonian flow Φ is a symplectic trivialization. By definition

$$\begin{aligned}F_\pm(t)=\Phi(\pm1,t)^{-1}\mathrm{d}\varphi_H^tT_{u(\pm1,0)}L_0=\Phi_0(\pm1)^{-1}\left(\mathrm{d}\varphi_H^t\right)^{-1}\mathrm{d}\varphi_H^tT_{u(\pm1,0)}L_0=\\ =F_0(\pm1)\ .\end{aligned}$$

Thus $\mu_{\mathrm{RS}}(F_\pm,F_1(\pm1))$ vanishes after the zero axiom. Since $\varphi_H(L_0)$ intersects L_1 cleanly we have for all $s\in[0,1]$

$$\begin{aligned}F_0(s)\cap F_1(s)&=\Phi(s,0)^{-1}T_{u(s,0)}L_0\cap\Phi(s,1)^{-1}T_{u(s,1)}L_1\\&=\Phi(s,1)^{-1}\left(\mathrm{d}\varphi_H^1T_{u(s,0)}L_0\cap T_{u(s,1)}L_1\right)\\&=\Phi(s,1)^{-1}\left(T_{u(s,1)}\varphi_H^1(L_0)\cap T_{u(s,1)}L_1\right)\\&=\Phi(s,1)^{-1}T_{u(s,1)}\left(\varphi_H^1\left(L_0\right)\cap L_1\right)\ .\end{aligned}$$

We see that the dimension of $F_0(s)\cap F_1(s)$ is constant for all $s\in[-1,1]$. This shows that $\mu_{\mathrm{RS}}(F_0,F_1)=0$ by the zero axiom.

Step 6. We show (vi)

Choose a symplectic trivialization Φ. Define the Lagrangian paths F_0,F_1,F_- and F_+ by (C.0.2). We have by definition $F_0(-1)=F_-(0)$ and $F_0(1)=F_+(0)$. As a result the concatenation $F_{-,0,+}=F_-^{-1}\#F_0\#F_+$ is a well-defined continuous path of Lagrangian subspaces starting from $F_-(1)$ and ending at $F_+(1)$. By the concatenation axiom and [63, Theorem 2.4] we have

$$\mu_{\mathrm{Vit}}(u)=\mu_{\mathrm{RS}}(F_{-,0,+},F_1)=\frac{1}{2}\left(\dim F_+(1)\cap F_1(1)-\dim F_-(1)\cap F_1(-1)\right)+\mathbb{Z}\ .$$

This shows the claim since

$$T_{x_+}C_+=\Phi(1,1)\left(F_+(1)\cap F_1(1)\right),\qquad T_{x_-}C_-=\Phi(-1,1)\left(F_-(1)\cap F_1(-1)\right)\ .$$

□

D. Quotients of principal bundles by maximal tori

Let $(G, P, P/G, \pi_G)$ be a principle G-fibre bundle where G is a compact Lie group. Fix a closed subgroup $T \subset G$. The restriction of the G-action to the subgroup T turns P into a principal T-bundle with base P/T and moreover these spaces fit into a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "\pi_T"'] \arrow[dd, "\pi_G"] \\ P/T \arrow[dr, "\pi"] & \\ & P/G\,, \end{tikzcd} \tag{D.0.1}$$

where $\pi : P/T \to G/T$, $Tx \mapsto Gx$ is a fibre bundle with fibre G/T. In this section we study the cohomology of P/T in case when T is a maximal torus. As a precursor we review the classical facts of compact Lie groups and their quotients by maximal tori. The main reference for this section is the article by Borel [15] and the introduction of the article by Borel and Hirzebruch [14]. For a more modern treatment we also refer the reader to book by Mimura and Toda [55].

D.1. Compact Lie groups

Let G be a compact connected Lie group. A closed, connected and abelian subgroup of G which is not contained in a strictly larger suchlike subgroup is called a *maximal torus*. Two maximal tori are conjugated by inner automorphism of G and their common dimension ℓ defines the *rank of* G. A well-known theorem of Hopf states that the cohomology ring $H^*(G, \mathbb{Q})$ is isomorphic to the exterior algebra of a vector space generated by ℓ homogeneous elements of odd degree (see [15, Prp. 7.2] or [55, VI.5.3] for a proof). The tuple of the degrees of the generators in ascending order is the *(rational) type* of G.

Fix a maximal torus $T \subset G$. The *Weyl group* W is defined as the group of automorphisms of T induced by inner automorphisms of G leaving the subgroup T as a set invariant. The group W is isomorphic to $N(T)/T$, where $N(T) := \{g \in G \mid gTg^{-1} = T\}$ is the *normalizer of* T *in* G. It is a finite group and its order $|W|$ is given by the number of connected components of $N(T)$ in G. Since any two maximal tori are conjugated by inner automorphisms the corresponding Weyl groups are isomorphic. Thus it makes sense to talk about the Weyl group of G without giving a reference to the maximal torus used to define it.

Proposition D.1.1. *Given a compact Lie group G of type* $(2m_1{+}1, 2m_2{+}1, \ldots, 2m_\ell{+}1)$. *The Betti numbers of G/T vanish for odd degrees. The Euler characteristic of G/T satisfies*

$$\chi(G/T) = |W| \ ,$$

where $|W|$ denotes the order of the Weyl group. The Poincaré polynomial of G/T is given by

$$P(G/T, x) = (1 - x^{2m_1+2})(1 - x^{2m_2+2}) \ldots (1 - x^{2m_\ell+2})/(1 - x^2)^\ell \ . \tag{D.1.1}$$

Proof. The first statement is [15, Lmm. 26.1], the second is [55, V.3.14] and the third is [15, Prp. 26.1]. □

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebra of G and T respectively. The adjoint representation $t \mapsto \mathrm{Ad}(t)$ of T in $\mathfrak{g}$ is fully reducible and there exists a direct sum decomposition

$$\mathfrak{g} \cong \mathfrak{t} \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_m, \tag{D.1.2}$$

into invariant subspaces where $\mathfrak{t}$ is the largest subspace on which T operates trivially. Each $\mathfrak{n}_j$ $(j = 1, \ldots, m)$ is two-dimensional and the adjoint action restricted to $\mathfrak{n}_j$ has the matrix form

$$\mathrm{Ad}(t)\big|_{\mathfrak{n}_j} = \begin{pmatrix} \cos 2\pi\bar{\theta}_j(t) & -\sin 2\pi\bar{\theta}_j(t) \\ \sin 2\pi\bar{\theta}_j(t) & \cos 2\pi\bar{\theta}_j(t) \end{pmatrix}, \qquad \forall\, t \in T\,,$$

for some non-trivial homomorphism $\bar{\theta}_j : T \to \mathbb{R}/\mathbb{Z}$ uniquely determined up to sign. The homomorphism $\bar{\theta}_j$ is a *root of* G and we denote by

$$\Phi := \{\bar{\theta}_1, -\bar{\theta}_1, \bar{\theta}_2, -\bar{\theta}_2, \ldots, \bar{\theta}_m, -\bar{\theta}_m\}\,,$$

the set of all roots.

A *weight* is a homomorphism $\bar{\theta} : T \to \mathbb{R}/\mathbb{Z}$. In particular any root is a weight. Let $\exp : \mathfrak{t} \to T$ denote the exponential map and $\Gamma := \exp^{-1}(e)$ the *unit lattice*, where e is the unit element of the group T. Every weight $\bar{\theta}$ uniquely lifts to a linear form $\theta : \mathfrak{t} \to \mathbb{R}$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "\theta"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d] \\ T \arrow[r, "\bar{\theta}"] & \mathbb{R}/\mathbb{Z}\,. \end{tikzcd} \tag{D.1.3}$$

Conversely every linear form on $\mathfrak{t}$ that maps the unit lattice Γ to $\mathbb{Z}$ defines a homomorphism $T \to \mathbb{R}/\mathbb{Z}$ via the above diagram. From now on we take the liberty to identify weights and in particular roots with linear forms on $\mathfrak{t}$ and make no notational distinction between $\bar{\theta}$ and θ anymore.

Choose a basis $x_1, \dots, x_\ell \in \mathfrak{t}^\vee$ of vectors that map the unit lattice Γ to $\mathbb{Z}$, hence constitute to an integer basis of the space of all weights. Abbreviate by M the space of weights and S its associated symmetric algebra, i.e.

$$M := \operatorname{Hom}(T, \mathbb{R}/\mathbb{Z}), \qquad S := \operatorname{Sym}_{\mathbb{Z}}(M) \cong \mathbb{Z}[x_1, \dots, x_\ell] \,.$$

We equip S with grading by $\deg \theta = 2$ for all $\theta \in M$ and abbreviate $S_{\mathbb{Q}} := S \otimes \mathbb{Q}$. Given a weight $\theta \in M$, we denote by $\mathbb{C}_{(\theta)} \cong \mathbb{C}$ the representation space on which T acts via $t.z = e^{2\pi i \theta(t)} z$. We define the *characteristic homomorphism*

$$c : S \to H^*(G/T, \mathbb{Z}) \,, \tag{D.1.4}$$

as the ring homomorphism induced by sending $\theta \in M$ to $c_1(L_\theta)$, the first Chern class of the line bundle $L_\theta := G \times_T \mathbb{C}_{(\theta)}$.

Proposition D.1.2. *The characteristic homomorphism $c \otimes \mathbb{Q}$ is surjective.*

Proof. This is a classical fact. We give references for the cornerstones of the proof. Let μ be the fundamental class of $\mathbb{R}/\mathbb{Z}$. The map $M \to H^1(\mathbb{R}/\mathbb{Z})$, $\theta \mapsto \theta^*\mu$ defines an isomorphism. Thus we have a degree preserving ring isomorphism $S_{\mathbb{Q}} \cong H^*(BT, \mathbb{Q})$, induced by $M \ni \theta \mapsto \tau(\theta^*\mu)$, where τ is the transgression for the fibration $T \hookrightarrow ET \to BT$ and BT is the classifying space (see [15, Thm. 19.1] or [55, III.5.4]). The pull-back for the rational cohomology of the classifying map $i : G/T \to BT$ is surjective (see [15, Prp. 29.1] or [55, VII.3.29]). To finish the proof it remains to show that $c_1(L_\theta) = i^*\tau(\theta^*\mu)$ for all $\theta \in M$. This follows from the naturality of the transgression (see [55, III.(6.4)]). $\square$

By the very definition the Weyl group acts on G with action descending to the quotient G/T inducing an action on $H^*(G/T)$. Moreover the linearization of the action at the unit element gives an action of W on the Lie algebra $\mathfrak{g}$ that leaves $\mathfrak{t}$ invariant. We obtain an action on $\mathfrak{t}$ and by duality on $\mathfrak{t}^\vee$. A simple observation shows that the action on $\mathfrak{t}^\vee$ maps weights to weights and even roots to roots (see [55, V.4.21]). Finally we note that the action on M induces an action on S. It is obvious from the naturality of the first Chern class, that the characteristic homomorphism is W-equivariant. By a Lemma of Leray the action of W on $H^*(G/T, \mathbb{Q})$ is equivalent to the regular representation (see [15, Lmm. 27.1] or [55, VII.3.26])

We define $\theta = \sum_{i=1}^{\ell} \alpha_i x_i \in \mathfrak{t}^\vee$ as *positive*, if the first non-vanishing coefficient is positive. Of course the definition depends on the choice of the basis. Up to changing the signs we assume without loss of generality that $\Phi^+ := \{\theta_1, \theta_2, \dots, \theta_m\}$ is the set of *positive roots*, i.e. the set of roots which are positive. A root is *simple* if it is positive and not the sum of two positive roots. For every root $\theta \in \Phi$, there exists an element $w_\theta \in W$ such that $w_\theta.\theta = -\theta$ and w_θ fixes a linear complement of $\mathbb{R} \cdot \theta$ in $\mathfrak{t}^\vee$ (see [55, V.4.27]). Such an element w_θ is called a *reflection*.

Proposition D.1.3. *Every reflection w_θ associated to a simple root θ permutes the set $\Phi^+ \setminus \{\theta\}$. Moreover W is generated by reflections associated to simple roots.*

Proof. The first statement is [55, V.6.15] the second statement is [55, V.6.12]. □

Let $g \in N(T)$ belong to the coset of $w \in W$. Then the adjoint action $\mathrm{Ad}(g)$ leaves the subspace $\mathfrak{t}$ invariant and we define *sign of* w as the determinant

$$(-1)^w := \det(\mathrm{Ad}(g) : \mathfrak{t} \to \mathfrak{t}) . \tag{D.1.5}$$

For instance any reflection has a negative sign. Let H be a module equipped with an action of W. An element $a \in H$ is (anti-)invariant if $w.a = a$ (resp. $w.a = (-1)^w a$) for all $w \in W$. We denote by H^W (resp. H^a) the invariant (resp. anti-invariant) elements. It is a deep fact that the subring $S_{\mathbb{Q}}^W \subset S_{\mathbb{Q}}$ is generated by ℓ homogeneous polynomials of degrees $(2m_1 + 2, \dots, 2m_\ell + 2)$, where $(2m_1 + 1, 2m_2 + 1, \dots, 2m_\ell + 1)$ denotes the rational type of G (see [15, Thm. 19.1, Prp. 27.1] or [55, VII.2.12, VII.3.29]).

Proposition D.1.4. *The kernel of the characteristic map $c \otimes \mathbb{Q}$ is given by $S^W_{\mathbb{Q},+}$, the subspace generated by W-invariant elements of positive degree.*

Proof. See [55, VII.3.29] or [15, Prp. 29.2, Prp. 27.1]. □

An important anti-invariant element of S is the *discriminant*

$$\Delta := \prod_{\theta \in \Phi^+} \theta .$$

Lemma D.1.5. *We have $c(\Delta) = |W|\mu$, where μ is the fundamental class of G/T.*

Proof. Choose and Ad-invariant metric on $\mathfrak{g}$. We have induced splittings $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ and $TG \cong (G \times \mathfrak{t}) \oplus (G \times \mathfrak{n})$. The first bundle is spanned by the fundamental vector fields of the T-action on G, hence $T(G/T) \cong G \times_T \mathfrak{n}$, where the action on $\mathfrak{n}$ of T is via the adjoint representation. By (D.1.2) we obtain a further splitting $\mathfrak{n} = \bigoplus_{j=1}^m \mathfrak{n}_j$ and by our choice of positive roots we have the T-equivariant isomorphism $\mathfrak{n}_j \cong \mathbb{C}_{(\theta_j)}$. Now with Proposition D.1.1 we have (recall $L_\theta = G \times_T \mathbb{C}_{(\theta)}$)

$$|W|\mu = \chi(G/T)\mu = \mathrm{eu}(TG/T) = \mathrm{eu}\big(\bigoplus\nolimits_{\theta \in \Phi^+} L_\theta\big) = \prod\nolimits_{\theta \in \Phi^+} c_1(L_\theta) = c(\Delta) .$$

This shows the claim. □

Lemma D.1.6. *The submodule $S_{\mathbb{Q}}^a$ is a free $S_{\mathbb{Q}}^W$-module of rank one generated by Δ.*

Proof. That Δ is anti-invariant follows by Proposition D.1.3. The rest is proven in [23, Lmm. 1.2]. □

The *root lattice* M^Φ is the submodule of M generated by Φ. It is freely generated by the simple roots, which implies that if G is semi-simple, then $M^\Phi = M$.

Corollary D.1.7. *The characteristic homomorphism induces the isomorphism*

$$H^*(G/T, \mathbb{Q}) \cong S_{\mathbb{Q}}/S^W_{\mathbb{Q},+} \tag{D.1.6}$$
$$H^2(G/T, \mathbb{Q}) \cong M^\Phi \otimes \mathbb{Q} \,, \tag{D.1.7}$$

further we have

$$H^*(G/T, \mathbb{Q})^W = H^0(G/T, \mathbb{Q}) \tag{D.1.8}$$
$$H^*(G/T, \mathbb{Q})^a = H^{2m}(G/T, \mathbb{Q}) \,. \tag{D.1.9}$$

Proof. Equation (D.1.6) directly follows from propositions D.1.2 and D.1.4. We further deduce equation (D.1.8) from (D.1.6) using equivariance of the characteristic homomorphism. Equation (D.1.9) holds since by Lemma D.1.6 and D.1.5 we have

$$H^*(G/T, \mathbb{Q})^a \cong S^a_{\mathbb{Q}}/S^W_{\mathbb{Q},+} \cong \Delta S^W_{\mathbb{Q}}/S^W_{\mathbb{Q},+} \cong \Delta \cdot \mathbb{Q} \cong H^{2m}(G/T, \mathbb{Q}) \,.$$

For (D.1.7) it suffices to show the W-equivariant splitting $M = M^W \oplus M^\Phi$. Choose an Ad-invariant inner product. A reflection w_θ fixes all points of $(\mathbb{R} \cdot \theta)^\perp$. Thus by Proposition D.1.3 the subspace of $\mathfrak{t}^\vee$ which is fixed by W is given by $\bigcap_\theta (\theta \cdot \mathbb{R})^\perp = \left(M^\Phi\right)^\perp$. Hence elements of M which are orthogonal to M^Φ are exactly all invariant elements. □

Corollary D.1.8. *Assume that* $\dim G/T > 2$, *then* $\dim H^2(G/T, \mathbb{Q}) \geq 2$.

Proof. If $\dim G/T > 2$, then $m \geq 2$ and there are at least two different positive roots $\theta_1, \theta_2 \in \Phi^+$. By standard properties of roots these are linear independent (see [14, 2.2] or [55, V.4.25]). Hence by (D.1.7) we have $2 \leq \dim M^\Phi \otimes \mathbb{Q} = \dim H^2(G/T, \mathbb{Q})$. □

D.2. The cohomology of the quotient of principle bundles by maximal tori

The quotient $V := P/T$ is naturally a G/T-bundle over $X := P/G$ with projection

$$\pi : V \to X, \qquad Tx \mapsto Gx \,.$$

The Weyl group W naturally acts on the T-orbits V via $w.Tx = Tg.x$ if $g \in N(T)$ belongs to the coset of $w \in W$. If $\theta \in M$ is a weight we have the associated line bundle

$$L_\theta := P \times_T \mathbb{C}_{(\theta)} \to V \,.$$

Take a basis $\theta_1, \ldots, \theta_\ell$ of the weight space M, which we identify with $H^2(G/T, \mathbb{Q})$ using (D.1.7) assuming that G is semi-simple without loss of generality. Since by (D.1.6) these elements generate $H^*(G/T; \mathbb{Q})$ as a ring the following W-equivariant homomorphism is well defined on the generators

$$H^*(G/T; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) \to H^*(V; \mathbb{Q}), \qquad \theta_i \otimes a \mapsto c_1(L_{\theta_i})\pi^* a \,. \tag{D.2.1}$$

We conclude by the Leray-Hirsch theorem (see [15, Prop. 4.1] or [55, III.4.4]) that this is an isomorphism of $H^*(X;\mathbb{Q})$-modules. If the restrict the map to the invariant elements we obtain by (D.1.8) the well-known isomorphism

$$H^*(X,\mathbb{Q}) \to H^*(V,\mathbb{Q})^W, \qquad a \mapsto \pi^* a\,. \tag{D.2.2}$$

If we restrict to the anti-invariant elements we obtain by (D.1.9) with $2m = \dim G/T$ and $\mathrm{D} = \prod_{\theta\in\Phi^+} c_1(L_\theta)$ the isomorphism

$$H^*(X,\mathbb{Q}) \to H^{*+2m}(V,\mathbb{Q})^a, \qquad a \mapsto \mathrm{D}\pi^* a\,. \tag{D.2.3}$$

Lemma D.2.1. *We have a splitting* $\ker \mathrm{d}\pi \cong \bigoplus_{\theta\in\Phi^+} L_\theta$ *into complex line bundles.*

Proof. Choose an Ad-invariant metric. We obtain a splitting $\mathfrak{g} \cong \mathfrak{t} \oplus \mathfrak{n}$. Using Killing fields and a G-connection we get an G-equivariant isomorphism

$$TP \cong P \times \mathfrak{t} \oplus P \times \mathfrak{n} \oplus \pi_G^* TX\,,$$

where on the right-hand side G acts on $P\times\mathfrak{t}$ and $P\times\mathfrak{n}$ with the adjoint action. Under this identification the bundle $P\times\mathfrak{t}$ is spanned by the fundamental vector fields that generate the T action. We obtain

$$TV \cong P \times_T \mathfrak{n} \oplus \pi^* TX\,.$$

Again under the identification $P\times_T\mathfrak{n}$ is spanned by the vector fields which generate the G-action, hence $\ker \mathrm{d}\pi \cong P\times_T \mathfrak{n}$. Now consider the splitting (D.1.2). $\square$

Fibre square The *fibre square* $V\times_\pi V := \{(p,p') \in V\times V \mid \pi(p) = \pi(p')\}$ fits into the commutative diagram of G/T-fibre bundles

$$\begin{tikzcd} V\times_\pi V \arrow[r, "pr_1"] \arrow[d, "pr_2"] & V \arrow[d, "\pi"] \\ V \arrow[r, "\pi"] & X\,, \end{tikzcd} \tag{D.2.4}$$

here pr_1 (resp. pr_2) denotes the projection to the first (resp. second) factor. On the other hand the fibre square is a the quotient of the $(G\times G)$-principle bundle $P\times_{\pi_G} P$ by the maximal torus $T\times T$. Its Weyl group $W\times W$ acts naturally on $V\times_\pi V$. The diagonal

$$\Delta_V := \{(p,p)\in V\times_\pi V \mid p\in V\}\,,$$

embeds into the fibre square as a submanifold of codimension $\dim G/T$.

Proposition D.2.2. *A choice of positive roots induces an orientation of* $\ker \mathrm{d}\pi$ *and a coorientation of* $\Delta_V \subset V\times_\pi V$ *and with respect to these orientations we have*

$$\sum_{w\in W} (1,w).d_V = \mathrm{pr}_1^*\mathrm{D}, \qquad \sum_{w\in W} (w,1).d_V = \mathrm{pr}_2^*\mathrm{D}\,,$$

where $d_V \in H^*(V\times_\pi V,\mathbb{Q})$ *is the Poincaré dual of the diagonal* $\Delta_V \subset V\times_\pi V$ *and* $\mathrm{D}\in H^*(V,\mathbb{Q})$ *is the Euler class of the bundle* $\ker d\pi \to V$.

Proof. By analogy we only prove the first identity. By Lemma D.2.1 the bundle $\ker d\pi$ is oriented by a choice of positive roots. Obviously the pull back $\mathrm{pr}_1^* \ker d\pi$ is point-wise a linear complement to the tangent bundle of Δ_V in $V \times_\pi V$. Thus $\mathrm{pr}_1^* \ker d\pi$ gives a normal bundle. Via this identification and again Lemma D.2.1 we have for any element $w \in W$

$$w.\mathrm{D} = (-1)^w \mathrm{D}, \qquad (w,w).d_V = (-1)^w d_V\,. \tag{D.2.5}$$

The class $\sum_{w\in W}(1,w).d_V$ is invariant with respect to the $\{1\}\times W$ action. By (D.2.2) there exists $a \in H^*(V,\mathbb{Q})$ such that

$$\sum_{w\in W}(1,w).d_V = \mathrm{pr}_1^* a\,.$$

For any $w' \in W$ we compute with (D.2.5)

$$\begin{aligned}\mathrm{pr}_1^* w'a = (w',1)\mathrm{pr}_1^* a = (w',1)\sum_{w\in W}(1,w)d_V = (-1)^{w'}\sum_{w''\in W}(1,w'')d_V = \\ = (-1)^{w'}\mathrm{pr}_1^* a\,.\end{aligned}$$

Since pr_1^* is injective we conclude that a is anti-invariant. By (D.2.5) the class D is also anti-invariant. Since both classes are anti-invariant and of degree $2m = \dim G/T$ we have with (D.2.3) $\mathrm{D} = ra$ for some $r \in \mathbb{Q}$. It remains to show that $r = 1$.

Consider the commutative diagram

$$\begin{tikzcd} G/T \times G/T \arrow[r, "i"] \arrow[d, "\tau_1"] & V \times_\pi V \arrow[d, "pr_1"] \\ G/T \arrow[r, "j"] & V\,, \end{tikzcd}$$

where τ_1 denotes the projection to the second coordinate and i the inclusion of the fibre. Let $d_{G/T}$ denote the Poincaré dual of the diagonal in $G/T \times G/T$. We compute via functoriality of the Poincaré dual (see [16, p.69])

$$\tau_1^* j^* a = i^* pr_1^* a = \sum_{w\in W}(1,w)i^* d_V = \sum_{w\in W}(1,w)d_{G/T}\,.$$

Consider a homogeneous basis $e_1, e_2, \ldots, e_k$ of $H^*(G/T;\mathbb{Q})$ over $\mathbb{Q}$ and assume that e_k is the generator of $H^{2m}(G/T;\mathbb{Q})$ with $2m = \dim G/T$. Consider the dual basis $e_1^\vee, e_2^\vee, \ldots, e_k^\vee$, i.e. we have $\deg e_i + \deg e_i^\vee = 2m$ and if $\deg e_i + \deg e_j = 2m$ we have $e_i e_j^\vee = \delta_{ij} e_k$ (Kronecker delta). Since the cohomology is supported only in even degrees a classical computation (cf. [54, Thm. 11.11]) shows that

$$d_{G/T} = \sum_{i=1}^{k}(-1)^{\deg e_i} e_i \otimes e_i^\vee = \sum_{i=1}^{k} e_i \otimes e_i^\vee\,.$$

Plugging this into the last equation shows

$$\tau_1^* j^* a = \sum_{w \in W} (1, w) \sum_{i=1}^{k} e_i \otimes e_i^\vee = \sum_{i=1}^{k} e_i \otimes \bar{e}_i^\vee ,$$

where $\bar{e}_i^\vee := \sum_{w \in W} w.e_i^\vee \in H^*(G/T)^W$. By (D.1.8) the only invariant class in $H^*(G/T)$ is $e_k^\vee$ which generates $H^0(G/T)$ and we conclude that

$$\tau_1^* j^* a = |W| \tau_1^* e_k$$

since τ_1^* is an isomorphism onto the invariant elements,

$$j^* a = |W| \, e_k .$$

On the other hand by naturality of the Euler class and Proposition D.1.1 we have

$$j^* \mathrm{D} = \mathrm{eu}(TG/T) = \chi(G/T) e_k = |W| \, e_k .$$

With the last two equations we see that $r = 1$. This shows the claim. □

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