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On geodesic flows with symmetries and closed magnetic geodesics on orbifolds

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Abstract. Let Q be a closed manifold admitting a locally free action of a compact Lie group G. In this paper, we study the properties of geodesic flows on Q given by suitable G-invariant Riemannian metrics. In particular, we will be interested in the existence of geodesics that are closed up to the action of some element in the group G, since they project to closed magnetic geodesics on the quotient orbifold Q/G.

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Contents

1. *Introduction*

Let Q be a *closed locally free principal* G*-bundle*, that is, a smooth closed manifold equipped with a smooth, effective and locally free action of a compact Lie group G.

On Q we consider a Riemannian metric g_Q that is G-invariant and restricts to a fixed Ad-invariant metric on fundamental vectors. The study of the geodesic flow of (Q, g_Q) is made particularly interesting by the fact that from the existence of geodesics that are closed up to G-action and satisfy some additional constraint (roughly speaking, the angle that such geodesics form with fundamental vector fields is equal to some *a priori* fixed angle), *constrained G-closed geodesics* for short, one obtains the existence of closed magnetic geodesics on the quotient orbifold Q/G.

More precisely, let (M, g_M) be a closed (throughout this paper, always effective) Riemannian orbifold and let σ be a closed two-form on M. A *closed magnetic geodesic for the pair* (g_M, σ) is a loop $\mu : [0, T] \to M$ that locally lifts to a classical magnetic geodesic, meaning that, for every $t \in [0, T]$, there exists $\epsilon > 0$ small enough such that the restriction of μ to $(t - \epsilon, t + \epsilon)$ is entirely contained in an orbifold chart (U, Γ, φ) and any lift $\tilde{\mu} : (t - \epsilon, t + \epsilon) \to U$ of μ to U is a magnetic geodesic (in the classical sense) for the Riemannian metric $\tilde{g}_M$ and the two-form $\tilde{\sigma}$ obtained, respectively, by lifting the Riemannian metric g_M and the two-form σ to U. We say that the closed magnetic geodesic μ has *energy* k if every local lift $\tilde{\mu}$ has energy k in the classical sense. We readily see that this definition naturally extends the usual definition of closed geodesics for Riemannian orbifolds (see, e.g., [**26**] or [**30**]).

To the authors' best knowledge, the magnetic problem on Riemannian orbifolds has not been studied yet, whereas the corresponding problem for manifolds has received the attention of numerous outstanding mathematicians over the past decades (e.g. Contreras, Ginzburg, Novikov, and Taimanov, to mention just some of them). Nowadays, a rich literature about magnetic flows on manifolds is available (see, e.g., [**3**, **10**, **12**, **13**, **17**, **23**, **24**, **33–35**, **37**, **38**]; for generalities we refer to [**8**]). As orbifolds are perhaps the simplest generalization of manifolds to singular spaces, and since they arise naturally in several different fields of mathematics—for instance in dynamical systems presenting symmetries, but also in representation theory, algebraic geometry, topology and so on— it is interesting to determine to what extent known results for magnetic flows on manifolds extend to this more general setting.

Closed magnetic geodesics on orbifolds are related with constrained G-closed geodesics by projection, meaning that every closed magnetic geodesic on M for the pair (g_M, σ) lifts to a constrained G-closed geodesic on a suitable Riemannian locally free principal G-bundle (Q, g_Q) over M and, conversely, every constrained G-closed geodesic on (Q, g_Q) projects to a closed magnetic geodesic on M for the pair (g_M, σ). The latter turn out to correspond to critical points of the functional

$$\mathbb{S}_k : \mathcal{M} := W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}) \times (0, +\infty) \to \mathbb{R}$$

given by

$$\mathbb{S}_k(\gamma, \phi, T) = \frac{1}{2T} \int_0^1 |\dot{\gamma}(t) + \underline{\phi(t)}(\gamma(t))|^2 \, dt - \int_0^1 \langle \phi(t), Z \rangle \, dt + kT. \tag{1.1}$$

Here $\mathfrak{g}$ denotes the Lie algebra of G, $Z \in \mathfrak{g}$ is a suitable central vector of unit length with respect to some fixed Ad-invariant metric on $\mathfrak{g}$ and $\underline{\phi(t)}$ denotes the fundamental vector field on Q associated with the Lie algebra element $\phi(t)$. All definitions will be given rigorously in §§2 and 3.

Remark 1.1. This correspondence also shows that magnetic flows on orbifolds are intimately related to the field of dynamics of mechanical systems with non-holonomic constraints (see, e.g., [**14**, **20**]), i.e., systems given by a Lagrangian L defined on TQ in which the equations of motion are derived by the Lagrange d'Alembert principle on curves $\gamma(t)$ that satisfy the constraint $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for some non-integrable distribution $\mathcal{D} \subset TQ$. When Q is the total space of a principal bundle, $\mathcal{D}$ is the horizontal distribution of a principal connection and the Lagrangian L is invariant under the group action, one speaks of a *(generalized) Chaplygin system*.

The first result of the present paper is the following generalization of the main theorem in [**8**]. In the statement below, $\pi_\ell^{\mathrm{orb}}(M)$ denotes the orbifold-theoretic homotopy groups as defined in [**5**, Definition 1.50].

THEOREM 1.2. *Let (M, g_M) be a closed non-rationally aspherical Riemannian orbifold, i.e., such that $\pi_\ell^{\mathrm{orb}}(M) \otimes \mathbb{Q} \neq 0$ for some $\ell \geq 2$, and let σ be a closed 2-form on M. Then, for almost every $k > 0$, there exists a closed magnetic geodesic for the pair (g_M, σ) with energy k.*

COROLLARY 1.3. *Suppose that (M, g_M) is a closed Riemannian orbifold and σ is a closed 2-form on M such that one of the following conditions is satisfied.*

(i) $\pi_1^{\mathrm{orb}}(M)$ *is finite.*

(ii) σ *is not weakly-exact (i.e., its lift to any cover is not exact).*

Then, for almost every $k > 0$, there exists a closed magnetic geodesic for the pair (g_M, σ) with energy k.

An orbifold M is called *developable* if it is isomorphic to a quotient $\tilde{M}/\Lambda$, where $\tilde{M}$ is a manifold and Λ is a discrete (not necessarily finite) group acting properly on $\tilde{M}$, and it is *non-developable* otherwise. In this latter case, adapting the argument in [**26**] to our setting yields the following theorem.

THEOREM 1.4. *Let (M, g_M) be a non-developable Riemmanian orbifold and let $\sigma \in \Omega^2(M)$ be a closed two-form on M. Then, for almost every $k > 0$, there exists a closed magnetic geodesic for the pair (g_M, σ) with energy k.*

The easiest example of non-developable orbifolds is given by the so-called *weighted projective spaces* (cf. [**5**, Example 1.15]), whose definition we now recall. For two coprime integers $k, \ell \in \mathbb{Z}$, the weighted projective space $W\mathbb{P}^1(k, \ell)$ is defined as the quotient of the standard unit sphere $S^3 \subset \mathbb{C}^2$ by the S^1-action given by

$$u.(z_1, z_2) := (u^k z_1, u^\ell z_2). \tag{1.2}$$

Topologically, $W\mathbb{P}^1(k, \ell)$ is a two-sphere, and as the quotient of a manifold by a locally free action of a compact group it carries a canonical orbifold structure, namely, with two singular points with isotropy groups $\mathbb{Z}_k$ and $\mathbb{Z}_\ell$, respectively. One easily sees that $W\mathbb{P}^1(k, \ell)$ is non-developable, for instance since its orbifold fundamental group is trivial. Hence, Theorem 1.4 immediately applies to $W\mathbb{P}^1(k, \ell)$ and yields closed magnetic geodesics for almost every energy for any choice of Riemannian metric and closed two-form. Observe that Theorem 1.2 applies as well, for $\pi_2^{\mathrm{orb}}(W\mathbb{P}^1(k, \ell)) \otimes \mathbb{Q} = \mathbb{Q}$. As a concrete example, we consider the metric g_M and magnetic form σ constructed as follows. The S^1-action (1.2) preserves the ellipsoid

$$E = \{(z_1, z_2) \in \mathbb{C}^2 \mid k^2|z_1|^2 + \ell^2|z_2|^2 = 1\}$$

and the quotient $E/\sim$ is an orbifold isomorphic to $W\mathbb{P}^1(k, \ell)$. The metric g given by restricting the Euclidean metric to E is S^1-invariant and has the property that the norm of the fundamental vector field $Z_{(z_1,z_2)} = i(kz_1, \ell z_2)$ is constant along E. In particular, we obtain a connection form and a metric g_M on the quotient (see §3.1). Furthermore, the connection form induces a curvature form and a closed two-form $\sigma = \sigma_Z$ on the quotient via (3.4). Using Lemma 3.3, we see that any geodesic $\dot{\gamma}$ on E with $\langle \dot{\gamma}, Z\rangle = 1$ projects to a magnetic geodesic for the pair (g_M, σ). We shall notice that, in general, although integrable, the dynamics on E is quite complicated (cf. [**18**]), and hence it is not straightforward to determine the energy levels for which there are closed magnetic geodesics by explictly looking at the geodesic flow on E, except for the case $k = \ell = 1$, in which we retrieve the standard geodesic flow on S^3, and on the quotient $\mathbb{CP}^1$, the magnetic flow defined by the round metric and the induced area form.

We now briefly give an account of the methods used to prove Theorems 1.2, 1.4 and Corollary 1.3: the idea is to use critical point theory for the functional $\mathbb{S}_k$ and consists roughly speaking on building, under the given assumptions, a non-trivial minimax class for $\mathbb{S}_k$ to which (an infinite dimensional version of) Morse theory will be applied. This approach has been already implemented in [**11**] in the particular case of free principal S^1-bundles and yielded an alternative proof of the existence of one closed magnetic geodesic for almost every energy level on closed non-aspherical Riemannian manifolds equipped with a closed two-form representing an integer cohomology class. However, extending this approach to general Lie group actions is by no means straightforward. Also, the orbifold structure enters in an essential way into several steps of this process.

- It forces G to be non-abelian (for manifolds we can always assume that $G = \mathbb{T}^N$ for some $N \in \mathbb{N}$). This is the reason why the functional $\mathbb{S}_k$ takes the form (1.1) rather than the simpler one suggested by [**11**] (see also Remark 5.3). Indeed, non-abelianity obligates us to allow also time-dependent fundamental vector fields, as constant Lie algebra elements do not give enough symmetry to infer that projected curves are magnetic geodesics. Moreover, since the infinite dimensional group $W^{1,2}(S^1, G)$ acts on $\mathcal{M}$ leaving the critical set of $\mathbb{S}_k$ invariant, one has to formulate a version of the Palais–Smale condition modulo this gauge group action.
- In the study of the geometry of $\mathbb{S}_k$ close to the set of *vertical loops* (that is, loops which are almost everywhere tangential to fundamental vector fields) it is important to exclude Palais–Smale sequences that have T-variable going to zero. In doing this, we have to take into account that there are exceptional 'shorter' fibers, as the G-action is only locally free.
- In all arguments involving exact homotopy sequences, one has to replace the orbifold M by its classifying space. As a consequence, the construction of a non-trivial minimax class for $\mathbb{S}_k$ requires the use of rational homotopy theory techniques. Speaking of which, an analogy with the manifold case would hint that Theorem 1.2 holds true also if M is merely *non-aspherical*, i.e., $\pi_\ell^{\mathrm{orb}}(M) \neq 0$ for some $\ell \geq 2$.

However, in this case, we are not able to prove an analogue of Lemma 6.1, which is the key tool for building a non-trivial minimax class for $\mathbb{S}_k$.

The approach introduced in this paper can be used also to study the existence of closed geodesics on Riemannian orbifolds. As an application, in §7, we give an alternative proof of [**26**, Theorem 5.1.1].

THEOREM 1.5. *A closed Riemannian orbifold* (M, g_M) *carries a closed geodesic, provided one of the following conditions is satisfied.*

(1) *M is not developable.*

(2) *$\pi_1^{\mathrm{orb}}(M)$ is either finite or contains an element of infinite order.*

To the author's best knowledge, all known methods to produce closed geodesics with positive length fail for developable orbifolds whose orbifold-theoretic fundamental group is a so-called *monster group*, that is, an infinite finitely presented group whose elements are all torsion (see [**26**]), even though recent developments provided positive answers for suitable subclasses of such orbifolds (see [**19**] for more details)†. The methods developed in this paper could be used to treat such orbifolds and potentially yield new results; this is subject of ongoing research.

We finish this introduction with a brief summary of the contents of the paper.

- In §2, we recall the definition and basic properties of orbifolds.
- In §3, we construct the locally free principal G-bundle Q starting from a closed Riemannian orbifold (M, g_M) and a closed two-form σ on M.
- In §4, we explain the relation between constrained G-closed geodesics and critical points of a suitable Rabinowitz-type action functional $\mathbb{A}_k$.
- In §5, we introduce the functional $\mathbb{S}_k$ and study its properties.
- In §6, we prove Theorem 1.2 and Corollary 1.3.
- Finally, in §7, we prove Theorem 1.5.

2. *Orbifolds*

Orbifolds appear naturally in several different fields of mathematics, including representation theory, algebraic geometry, physics and topology. Roughly speaking, they are generalizations of manifolds by allowing certain singularities. As such, they have many properties in common with manifolds and share known constructions, such as tangent bundles, differential forms, vector fields etc. We quickly recall these definitions and highlight basic properties, accounting to [**5**].

Let M be a paracompact Hausdorff space and fix $n \in \mathbb{N}$. An *orbifold chart* is a triple (U, Γ, φ) such that $U \subset \mathbb{R}^n$ is a connected open subset, Γ is a finite group acting effectively on U by smooth automorphisms and $\varphi : U \to M$ is a Γ-invariant map inducing a homeomorphism of U/Γ onto its image. An *embedding* $(U, \Gamma, \varphi) \hookrightarrow (U', \Gamma', \varphi')$ of orbifold charts is an embedding $\lambda : U \hookrightarrow U'$ such that $\varphi \circ \lambda = \varphi'$. An *orbifold atlas* $\mathcal{U} = \{(U, \Gamma, \varphi)\}$ is a set of *compatible* orbifold charts such that $\bigcup_{\mathcal{U}} \varphi(U) = M$: for all $p \in M$ and all orbifold charts $(U_1, \Gamma_1, \varphi_1), (U_2, \Gamma_2, \varphi_2) \in \mathcal{U}$ with $p \in \varphi_1(U_1) \cap \varphi_2(U_2)$ and there

† It is actually not known whether such finitely presented monster groups exist, even though they are generally believed to (cf. [**32**]). Also, many such examples are known under the assumption that the group is only finitely generated.

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exist $(U, \Gamma, \varphi) \in \mathcal{U}$ such that $p \in \varphi(U)$ and embeddings $(U, \Gamma, \varphi) \hookrightarrow (U_i, \Gamma_i, \varphi_i)$ for $i = 1, 2$. We define the *transition function* near p as the diffeomorphism $\phi_{12} = \lambda_2 \circ \lambda_1^{-1} : \lambda_1(U) \to \lambda_2(U)$. We say that an atlas $\mathcal{U}$ is a *refinement* of $\mathcal{U}'$ if, for every chart of $\mathcal{U}$, there exists an embedding into some chart of $\mathcal{U}'$. Two atlases are *equivalent* if they have a common refinement. Every orbifold atlas has a unique maximal refinement and two orbifold atlases are equivalent if and only if they have the same maximal refinement.

Definition 2.1. An *effective orbifold of dimension n* is a paracompact Hausdorff space M equipped with an equivalence class of n-dimensional orbifold atlases.

As the definition above suggests, there is a further generalization of the notion of an orbifold by allowing non-effective local actions. However, we will not consider these objects in this paper and refer to effective orbifolds simply as orbifolds.

Definition 2.2. A map $f : M \to N$ between two orbifolds is *smooth* if, for every $p \in M$, there exist charts (U, Γ, φ) and (U', Γ', φ') with $p \in \varphi(U)$ and $f(p) \in \varphi'(U')$ together with a smooth map $\tilde{f} : U \to U'$ satisfying $f \circ \varphi = \varphi' \circ \tilde{f}$.

Exactly as for manifolds, we define the tangent bundle of an orbifold M by gluing together tangent bundles of the local charts using transition functions.

Definition 2.3. We define the *tangent bundle* as the space

$$TM = \bigsqcup_{(U,\Gamma,\varphi)\in\mathcal{U}} (U \times_\Gamma \mathbb{R}^n) \Big/ \sim,$$

where $U \times_\Gamma \mathbb{R}^n$ denotes the quotient space of $U \times \mathbb{R}^n$ by the diagonal action of Γ and $\sim$ is defined via $[p, v] \sim [p', v']$ if and only if $\phi_{12}(p) = p'$ and $d_p\phi_{12}(v) = v'$ for all charts $(U, \Gamma, \varphi), (U', \Gamma', \varphi') \in \mathcal{U}$ such that $p \in U$ and $p' \in U'$.

The tangent bundle of an orbifold carries a natural orbifold structure and there is a canonical foot-point projection map $TM \to M$. It is worth mentioning that fibers are no longer vector spaces but rather quotients $\mathbb{R}^n/\Gamma_p$, where the finite group Γ_p varies with $p \in M$. By a similar gluing construction, we define the cotangent bundle T^*M and its exterior as well as symmetric tensor powers. Further, we define *vector fields*, *differential forms* and *Riemannian metrics* as smooth sections of such bundles. One shows that by virtue of the definitions we find local representatives of such sections which, in addition to satisfying the usual transformation rules, are equivariant with respect to the local group action. In particular, integral curves to a local representative of a vector field depend equivariantly on the starting point, which implies that vector fields on orbifolds induce flows as usual. We define metric connections verbatim as for the manifold definition and, by the previous observation, we see that the geodesic flow and the magnetic geodesic flow on an orbifold are well defined. In this sense, a magnetic geodesic on M is a curve which locally (i.e., in every orbifold chart) lifts to a classical magnetic geodesic. More precisely, given a Riemannian metric g_M and a closed two-form σ on M, a path $\mu : (a, b) \to M$ is said to be a *magnetic geodesic* if, for each $t_0 \in (a, b)$, there exists $\varepsilon > 0$, an orbifold chart (U, Γ, φ) such that $\mu(t) \in \varphi(U)$ for all $t \in (t_0 - \varepsilon, t_0 + t)$ and a smooth map $\tilde{\mu} : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ with $\mu|_{(t_0-\varepsilon,t_0+\varepsilon)} = \varphi \circ \tilde{\mu}$ such that

$$\langle \nabla_t \dot{\tilde{\mu}}, u \rangle = \tilde{\sigma}_{\tilde{\mu}}(u, \dot{\tilde{\mu}}) \quad \text{for all } u \in T_q U. \tag{2.1}$$

Here ∇, $\langle \cdot\,,\,\cdot\rangle$ are $\tilde\sigma$, respectively, the Levi-Civita connection and the local representatives of g_M and σ. We say that the magnetic geodesic μ has *energy* k if every lift $\tilde\mu$ has energy k in the classical sense.

Example 2.4. The $\mathbb{Z}_n$*-football orbifold* F_n is defined as the quotient of S^2 by the $\mathbb{Z}_n$-action generated by the rotation of angle $2\pi/n$ around the z-axis. A magnetic pair on F_n corresponds to a magnetic pair on S^2 given by a Riemannian metric and a closed two-form which are invariant under the $\mathbb{Z}_n$-action. In particular, magnetic geodesics on F_n are obtained by projecting down the magnetic geodesics of S^2.

More generally, magnetic geodesics on developable orbifolds are always the projection of magnetic geodesics on the (not necessarily compact) manifold cover.

Example 2.5. A manifold N with non-empty boundary ∂N can be seen as an orbifold in which the boundary points have $\mathbb{Z}_2$-charts with the $\mathbb{Z}_2$ acting by reflection about a hyperplane. An orbifold magnetic field in this setting then corresponds to a two-form on N that is invariant under reflection along ∂N. For instance, in the case $n = 2$, this means that σ vanishes on the boundary, while in the case $n = 3$ (where we can identify magnetic fields with vector fields), the magnetic field is orthogonal to the boundary. Furthermore, orbifold magnetic geodesics correspond to magnetic billiard trajectories in N, that is, curves in N that satisfy the Lorenz equation in the interior of N and Snell's law of reflection when hitting ∂N.

Another important source of examples of orbifolds are quotients of smooth manifolds by compact Lie groups which act *locally freely*, i.e., with all stabilizer groups finite. Orbifold charts for such quotients are provided by the slice theorem. As it turns out, every orbifold is of this form. We cite [**5**, Corollary 1.24].

PROPOSITION 2.6. *Every n-orbifold is diffeomorphic to a quotient orbifold for a smooth, effective and locally free* $\mathrm{O}(n)$*-action on a smooth manifold.*

3. *Locally free actions*

Let Q be a smooth manifold equipped with a smooth action $G \times Q \to Q$, $(g, q) \mapsto g \cdot q$ of a compact Lie group G. Throughout the paper, we assume that the action is *effective* and *locally free*, i.e., with all stabilizer groups finite. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ denote the adjoint action map for $g \in G$. For any $X \in \mathfrak{g}$, we denote by $\underline{X} : Q \to TQ$ the *fundamental vector field* on Q defined by

$$\underline{X}_q = \frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot q \in T_q Q \quad \text{for all } q \in Q. \tag{3.1}$$

By assumption, the linear map $j_q : \mathfrak{g} \to T_q Q$, $X \mapsto \underline{X}_q$ is injective for all $q \in Q$. Moreover, since locally there is no difference between locally free and free actions, the concept of a principal connection form literally carries over.

Definition 3.1. A *principal connection form* $\theta \in \Omega^1(Q; \mathfrak{g})$ is a $\mathfrak{g}$-valued differential one-form such that:

(i) $\theta(\underline{X}) = X$ for all $X \in \mathfrak{g}$; and

(ii) $\varphi_g^*\theta = \mathrm{Ad}_g \cdot \theta$ for all $g \in G$, where $\varphi_g : Q \to Q, q \mapsto g \cdot q$.

One easily checks that the standard results, as given, e.g., in [**29**, Ch. II], hold true in the locally free setting: the space of principal connection forms is a non-empty affine space. Also, any given θ induces a projection $j_q \circ \theta_q : T_q Q \to T_q Q$ for all $q \in Q$ as well as a splitting of the tangent space into subspaces

$$T_q Q = \ker \theta_q \oplus \operatorname{im} j_q \quad \text{for all } q \in Q, \tag{3.2}$$

which are called *horizontal* and *vertical*, respectively, and we have a corresponding *curvature form* $\Omega \in \Omega^2(Q; \mathfrak{g})$ defined by

$$\Omega = d\theta + [\theta, \theta], \tag{3.3}$$

which satisfies $\varphi_g^* \Omega = \mathrm{Ad}_g \cdot \Omega$ for all $g \in G$ and is *horizontal*, meaning that $\Omega(u, v) = 0$ whenever at least one of $u, v \in T_q Q$ is vertical (cf. [**29**, Theorem 5.2]).

A *central* covector, that is, any $Z \in \mathfrak{g}^\vee$ such that $\mathrm{Ad}_g^* Z = Z$ for all $g \in G$, defines a G-invariant two-form $\Omega_Z := \langle Z, \Omega \rangle$, where $\langle \cdot\, , \cdot \rangle$ denotes the duality pairing. As Ω_Z is also horizontal, it descends to the quotient orbifold $M = Q/G$. More precisely, we denote by $\tau : Q \to M$ the quotient map, which is smooth in the sense of Definition 2.2, and we define a two-form $\sigma_Z \in \Omega^2(M)$ by

$$\tau^* \sigma_Z := \Omega_Z = \langle Z, \Omega \rangle. \tag{3.4}$$

This means that if $\tilde{\Omega} \in \Omega^2(U, \mathfrak{g})$ and $\tilde{\tau} : U \to U'$ are local representatives of Ω and τ, respectively, then the local representative $\tilde{\sigma}_Z$ of σ_Z is given by

$$\tilde{\sigma}_Z(d_q \tilde{\tau} u, d_q \tilde{\tau} v) = \langle Z, \tilde{\Omega}(u, v) \rangle \quad \text{for all } q \in U, \; u, v \in T_q U.$$

It is straightforward to check using Bianchi's identity (cf. [**29**, Theorem 5.4]) that σ_Z is a well-defined closed two-form.

3.1. *Horizontally lifted metric.* Suppose that the quotient $M = Q/G$ is equipped with a metric g_M and that Q carries a fixed connection form θ. The key observation of this section is that for a particular metric on Q, which we call *horizontally lifted metric*, the geodesic flow on Q restricted to a certain invariant sub-bundle projects to the magnetic geodesic flow on M, where the magnetic form is determined by the curvature. To this end we fix an Ad-invariant positive definite bilinear form B on $\mathfrak{g}$, which is possible since G is assumed to be compact, and we define a Riemannian metric g_Q on Q by requiring that:

- the splitting (3.2) is orthogonal;
- $\tau : Q \to M$ is a Riemannian submersion, i.e., $d_q \tilde{\tau} : \ker \tilde{\theta}_q \to T_{\tilde{\tau}(q)} U'$ is an isometry for all $q \in U$, where $\tilde{\tau}$ and $\tilde{\theta}$ are local representatives; and
- the metric restricted to the kernel of τ is given by B, i.e., $j_q : \mathfrak{g} \to T_q Q$ is an isometry onto its image for all $q \in Q$.

In other words, the local representative of g_Q is given by

$$B(\tilde{\theta}_q(v), \tilde{\theta}_q(w)) + \tilde{g}_{\tilde{\tau}(q)}(d_q \tilde{\tau}(v), d_q \tilde{\tau}(v)) \quad \text{for all } q \in U \text{ and for all } v, w \in T_q U,$$

where $\tilde{g}$ denotes the local representative of g_M. It is easy to see that the metric g_Q is G-invariant and restricts to the fixed Ad-invariant bilinear form on the fundamental vectors. Conversely, given a metric g_Q on Q satisfying these properties, we obtain a connection form θ and a metric g_M on M by setting $\langle \theta_q(v), X\rangle = (g_Q)_q(v, \underline{X}_q)$ for all $q \in Q$, $v \in T_qQ$, $X \in \mathfrak{g}$ and by requiring that τ be a Riemannian submersion.

In what follows, we denote the metrics g_Q, g_M and the bilinear form B on $\mathfrak{g}$ all with $\langle \cdot\,, \cdot\rangle$ if they cannot be confused. Also, we denote by ∇ the Levi-Civita connection of g_Q or g_M and we identify $\mathfrak{g}^\vee$ with $\mathfrak{g}$ using B.

LEMMA 3.2. *For any $X \in \mathfrak{g}$, consider the two-form $\Omega_X := \langle X, \Omega\rangle \in \Omega^2(Q)$. We have*

$$\Omega_X(u, v) = -2\langle u, \nabla_v \underline{X}\rangle + \langle X, [\theta(u), \theta(v)]\rangle \tag{3.5}$$

for all $p \in Q$ and all vectors $u, v \in T_pQ$.

Proof. We first observe that, for every $X \in \mathfrak{g}$ and all $p \in Q$,

$$T_pQ \times T_pQ \ni (u, v) \mapsto \langle u, \nabla_v \underline{X}\rangle$$

is an antisymmetric bilinear form. Indeed, after extending u and v along the flow-line $t \mapsto g_t p$, $g_t := \exp(tX)$ by $u(t) = d_p\varphi_{g_t} u$ and $v(t) = d_p\varphi_{g_t} v$, respectively, we have

$$0 = \underline{X}\langle u, v\rangle = \langle \nabla_{\underline{X}} u, v\rangle + \langle u, \nabla_{\underline{X}} v\rangle = \langle \nabla_u \underline{X}, v\rangle + \langle u, \nabla_v \underline{X}\rangle,$$

where we have $0 = [\underline{X}, u] = [\underline{X}, v]$.

Now let $p \in Q$ and $u, v \in T_pQ$ be fixed and extend u and v to vector fields, which we still denote by u and v, respectively. By (3.3),

$$\Omega_X(u, v) = d\theta_X(u, v) + \langle X, [\theta(u), \theta(v)]\rangle, \tag{3.6}$$

where $\theta_X := \langle X, \theta\rangle \in \Omega^1(Q)$. We assume, for the moment, that u and v are both horizontal and compute

$$\begin{aligned}\Omega_X(u, v) &= u(\theta_X(v)) - v(\theta_X(u)) - \theta_X([u, v]).\\ &= -\langle [u, v], \underline{X}\rangle\\ &= -\langle \nabla_u v, \underline{X}\rangle + \langle \nabla_v u, \underline{X}\rangle\\ &= \langle v, \nabla_u \underline{X}\rangle - \langle u, \nabla_v \underline{X}\rangle\\ &= -2\langle u, \nabla_v \underline{X}\rangle,\end{aligned}$$

where in the last step we used antisymmetry and in the second-from-last we used the identity

$$0 = u\langle v, \underline{X}\rangle = \langle \nabla_u v, \underline{X}\rangle + \langle v, \nabla_u \underline{X}\rangle.$$

Since Ω is horizontal, it suffices to check that the right-hand side of (3.5) is horizontal as well. For that, we assume, without loss of generality, that $v = \underline{Y}$ for some $Y \in \mathfrak{g}$ and compute using antisymmetry and the fact that $\langle \underline{Y}, \underline{X}\rangle = \langle Y, X\rangle$ is constant on Q

$$\langle u, \nabla_{\underline{Y}} \underline{X}\rangle = -\langle \underline{Y}, \nabla_u \underline{X}\rangle = \langle \nabla_u \underline{Y}, \underline{X}\rangle = -\langle \nabla_{\underline{X}} \underline{Y}, u\rangle = -\langle \nabla_{\underline{Y}} \underline{X}, u\rangle - \langle [\underline{X}, \underline{Y}], u\rangle.$$

We conclude that

$$2\langle u, \nabla_v \underline{X}\rangle = -\langle [\underline{X}, \underline{Y}], u\rangle.$$

If u is horizontal, then the right-hand side vanishes as $[\underline{X}, \underline{Y}]$ is vertical. Finally, if u is vertical and given by $u = \underline{Z}$ with $Z \in \mathfrak{g}$, then

$$2\langle u, \nabla_v \underline{X}\rangle = -\langle [\underline{X}, \underline{Y}], \underline{Z}\rangle = -\langle [X, Y], Z\rangle = \langle X, [Z, Y]\rangle = \langle X, [\theta(u), \theta(v)]\rangle.$$

Using antisymmetry we conclude that the right-hand side of (3.5) vanishes whenever at least one of u and v is vertical. This completes the proof. □

LEMMA 3.3. *For any geodesic $\gamma : (a, b) \to Q$ for g_Q we have that $X := \theta(\dot{\gamma}) \in \mathfrak{g}$ is constant along γ. Moreover, if $X \in \mathfrak{g}$ is central, then $\mu = \tau \circ \gamma : (a, b) \to M$ is a magnetic geodesic for the pair (g_M, σ_X).*

Proof. The first statement follows directly from antisymmetry, i.e.,

$$\partial_t \langle \theta(\dot{\gamma}), Y \rangle = \partial_t \langle \dot{\gamma}, \underline{Y} \rangle = \langle \dot{\gamma}, \nabla_t \underline{Y} \rangle = 0 \quad \text{for all } Y \in \mathfrak{g}.$$

For $t \in (a, b)$, we set $p := \mu(t)$ and extend the vector $\dot{\mu}(t)$ to a vector field $\bar{v}$. Fix any other vector $\bar{u}$ at p and extend it to a vector field, again denoted by $\bar{u}$, such that

$$(\nabla_{\dot{\mu}(t)} \bar{u})_p = (\nabla_{\bar{v}} \bar{u})_p = 0. \tag{3.7}$$

We lift $\bar{u}$, $\bar{v}$ to horizontal vector fields u, v, respectively, and compute at p

$$\begin{aligned}
\langle \nabla_t \dot{\mu}, \bar{u} \rangle = \partial_t \langle \dot{\mu}, \bar{u} \rangle = \partial_t \langle \dot{\gamma}, u \rangle
&= \langle \dot{\gamma}, \nabla_{\dot{\gamma}} u \rangle \\
&= \langle v + \underline{X}, \nabla_{v+\underline{X}} u \rangle \\
&= \langle \underline{X}, \nabla_{\underline{X}} u \rangle + \langle v, \nabla_{\underline{X}} u \rangle + \langle \underline{X}, \nabla_v u \rangle + \langle v, \nabla_v u \rangle \\
&= -\langle \nabla_{\underline{X}} \underline{X}, u \rangle + \langle v, \nabla_u \underline{X} \rangle - \langle \nabla_v \underline{X}, u \rangle + \langle v, \nabla_v u \rangle.
\end{aligned}$$

The last summand above vanishes. Indeed, by [**29**, Proposition 1.3], adapted to the locally free setting, and (3.7),

$$\begin{aligned}
\langle v, \nabla_v u \rangle &= \langle v, \nabla_u v \rangle + \langle v, [u, v] \rangle \\
&= \tfrac{1}{2} u(|v|^2) + \langle v, [u, v] \rangle \\
&= \tfrac{1}{2} \bar{u}(|\bar{v}|^2) + \langle \bar{v}, [\bar{u}, \bar{v}] \rangle \\
&= \langle \bar{v}, \nabla_{\bar{u}} \bar{v} \rangle + \langle \bar{v}, [\bar{u}, \bar{v}] \rangle \\
&= \langle \bar{v}, \nabla_{\bar{v}} \bar{u} \rangle = 0.
\end{aligned}$$

Using Lemma 3.2 we conclude that

$$\langle \nabla_t \dot{\mu}, \bar{u} \rangle = -2 \langle \nabla_v X, u \rangle = \Omega_X(u, v) = \sigma_X(\bar{u}, \dot{\mu}).$$

This shows that μ is a magnetic geodesic for (g_M, σ_X), as required. □

3.2. *Constructing the bundle.* In this section, we show how to construct, starting from a given orbifold M equipped with a closed two-form $\sigma \in \Omega^2(M)$, a locally free principal G-bundle Q and a connection form θ such that the quotient Q/G and M are isomorphic as orbifolds and σ is determined by the curvature of θ.

PROPOSITION 3.4. *Let M be a closed orbifold and let $\sigma \in \Omega^2(M)$ be a closed two-form. Then there exist:*

- *a locally free principal G-bundle Q, G compact Lie group;*
- *a connection form $\theta \in \Omega^1(Q, \mathfrak{g})$; and*
- *a central covector $Z \in \mathfrak{g}^\vee \setminus \{0\}$*

such that $M \cong Q/G$ as orbifolds and $\sigma = \sigma_Z$ is given by (3.4).

Before proving the proposition, we recall the definition of the Euler class for principal S^1-bundles using Čech cohomology. Let $\rho: Q \to P$ be a principal S^1-bundle and let P be a smooth manifold. Choose an open cover $\mathcal{U} = (U_i)_{i\in I}$ of P such that $Q|_{U_i}$ is trivial for all $i \in I$, and let $g_{ij} : U_i \cap U_j \to S^1$ be the corresponding trivialization change maps for all $i, j \in I$. The family $g = (g_{ij})$ is a Čech-cocycle and defines a Čech-cohomology class $[g] \in \check{H}^1(P, \mathcal{O}_{S^1})$, where by $\mathcal{O}_{S^1}$ we denote the sheaf of smooth maps with values in S^1. The short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$ induces a long exact sequence in Čech-cohomology

$$\cdots \to \check{H}^1(P; \mathcal{O}_{\mathbb{R}}) \to \check{H}^1(P; \mathcal{O}_{S^1}) \xrightarrow{\delta} \check{H}^2(P; \mathcal{O}_{\mathbb{Z}}) \to \check{H}^2(P; \mathcal{O}_{\mathbb{R}}) \to \cdots, \tag{3.8}$$

where by $\mathcal{O}_{\mathbb{R}}, \mathcal{O}_{\mathbb{Z}}$ we denote the sheaf of smooth maps with values in $\mathbb{R}$, $\mathbb{Z}$, respectively. Note that $\mathcal{O}_{\mathbb{Z}}$ is also the sheaf of locally constant maps denoted by $\underline{\mathbb{Z}}$. There is a canonical isomorphism $\check{H}^2(P; \underline{\mathbb{Z}}) \cong H^2(P; \mathbb{Z})$, where the right-hand side denotes the singular cohomology with integer coefficients (cf. [**16**, Theorem III.1.1]). We identify these groups without further mention and define the *Euler class*

$$\mathrm{eu}(Q) := \delta([g]) \in H^2(P; \mathbb{Z}).$$

It is a classical fact that the free part of the Euler class is represented by a curvature form: the Lie algebra of S^1 is canonically identified with $\mathbb{R}$, and a connection form on Q is simply a one-form $\theta \in \Omega^1(Q)$ that is S^1-invariant and satisfies $\theta(\underline{Z}) = 1$, where $\underline{Z}$ is the fundamental vector field of the fiber rotations. The two-form $\sigma = \sigma_Z$ defined in (3.4) is now given via $\rho^*\sigma = d\theta$. It is a closed form and after identifying the deRahm cohomology with $H^2(P; \mathbb{R})$ we denote by $[\sigma] \in H^2(P; \mathbb{R})$ the cohomology class represented by σ. Further, we denote by $\mathrm{eu}(Q)_{\mathbb{R}} \in H^2(P; \mathbb{R})$ the element $\mathrm{eu}(Q) \otimes 1$ under the canonical isomorphism $H^2(P; \mathbb{Z}) \otimes \mathbb{R} \cong H^2(P; \mathbb{R})$. A classical computation (cf. [**25**, p. 141]) shows that

$$\mathrm{eu}(Q)_{\mathbb{R}} = [\sigma] \in H^2(P; \mathbb{R}). \tag{3.9}$$

Conversely, given a cocycle $g = (g_{ij}) \in \check{H}^1(P; \mathcal{O}_{S^1})$, we obtain a circle bundle Q over P via the gluing construction

$$Q := \coprod_i U_i \times S^1/\sim \tag{3.10}$$

with identification $(p, \theta) \sim (p', \theta')$ if and only if $p = p'$ and $\theta = \theta' + g_{ij}(p)$, where $p \in U_i$ and $p' \in U_j$. Using the gluing construction and the exact sequence (3.8), we conclude that, for any $[\sigma] \in H^2(P; \mathbb{R})$ representing an integer cohomology class, there is a circle bundle $Q \to P$ such that $\mathrm{eu}(Q)_{\mathbb{R}} = [\sigma]$.

Now assume that on P we additionally have a locally free action $\varphi : G \times P \to P$, $\varphi(g, \cdot) = \varphi_g$ of a compact Lie group G. Then the quotient $M := P/G$ is canonically an orbifold; we denote by $\tau : P \to M$ the quotient map.

LEMMA 3.5. *For any $e \in H^2(M; \mathbb{Z})$, there exists a principal circle bundle $Q \to P$ equipped with a G-action $\tilde{\varphi} : G \times Q \to Q$, $\tilde{\varphi}(g, \cdot) = \tilde{\varphi}_g$ such that* $\mathrm{eu}(Q) = \tau^* e$ *and the G-action commutes with the S^1-action, i.e., $\tau \circ \tilde{\varphi}_g = \varphi_g \circ \tau$ for all $g \in G$.*

Proof. It suffices to find an open cover $\mathcal{U} = (U_i)_{i\in I}$ of P by G-invariant open subsets $U_i \subset P$ and a cocycle $g = (g_{ij} : U_i \cap U_j \to S^1)_{i,j\in I}$ such that:

- g_{ij} is G-invariant; and
- $\delta([g]) = e$.

Indeed, the product action of $G \times S^1$ on the patches $U_i \times S^1$ descends to an action of $G \times S^1$ on the quotient obtained by the gluing construction (3.10).

Since M admits a partition of unity, we have $\check{H}^i(M; \mathcal{O}_{\mathbb{R}}) = 0$ for $i \geq 1$, and replacing P with M in the sequence (3.8) we conclude that we have a canonical isomorphism $\bar{\delta} : \check{H}^1(M; S^1) \cong \check{H}^2(M; \underline{\mathbb{Z}})$. Also we have a canonical isomorphism $\check{H}^2(M; \underline{\mathbb{Z}}) \cong H^2(M; \mathbb{Z})$ (in fact, for any paracompact Hausdorff space M, cf. [**16**, Theorem III.1.1]) and we identify these groups via the isomorphism. Pick a good cover $\bar{U} = \{\bar{U}_i\}_{i \in I}$ of M and consider the pull-back cover $\mathcal{U} = \{U_i\}_{i \in I}$ of P, where $U_i = \tau^{-1}(\bar{U}_i)$. Let $[\bar{g}] \in \check{H}^1(\bar{\mathcal{U}}; S^1) \cong \check{H}^1(M; S^1)$ be a class such that $\bar{\delta}([\bar{g}]) = e$ and let $g = \tau^* \bar{g}$ be the pull-back. Recall that this means that $g = (g_{ij})$ is defined by $g_{ij} : U_i \cap U_j \to S^1$, $g_{ij} = \bar{g}_{ij} \circ \tau$. Obviously, g_{ij} is G-invariant, and since all involved isomorphisms are canonical we also have $\delta([g]) = \tau^* \bar{\delta}([\bar{g}]) = \tau^* e$. □

Proof of Proposition 3.4. By Proposition 2.6 we know that there exists a smooth manifold P equipped with a locally free $\mathrm{O}(n)$-action such that $M \cong P/\mathrm{O}(n)$. Let $\tau : P \to M$ be the quotient map.

If σ is exact with $d\eta = \sigma$, then we define $Q := S^1 \times P$ equipped with the product action of the Lie group $G = S^1 \times \mathrm{O}(n)$. Pick any Ad-invariant positive bilinear form on $\mathfrak{g}$ such that the splitting $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{o}(n)$ is orthogonal, where $\mathfrak{o}(n)$ denotes the Lie algebra of $\mathrm{O}(n)$, and identify $\mathfrak{g}^\vee$ with $\mathfrak{g}$ using the bilinear form. Then $Z = (1, 0) \in \mathbb{R} \oplus \mathfrak{o}(n)$ and the connection form is given by

$$\theta = \rho^* \theta_0 + (\hat{\tau}^* \eta + dt) \otimes Z,$$

where t denotes the variable in S^1, θ_0 is any connection for P, $\rho : S^1 \times P \to P$ is the projection to the second coordinate and $\hat{\tau} = \tau \circ \rho$. Moreover,

$$d\langle \theta, Z \rangle = \hat{\tau}^* d\eta = \hat{\tau}^* \sigma.$$

Now suppose that σ is not exact. Let $e_1, \ldots, e_m$ be a basis of the free part of $H^2(M; \mathbb{Z})$ which we identify under the isomorphism $H^2(M; \mathbb{Z}) \otimes \mathbb{R} \cong H^2(M; \mathbb{R})$ with a basis of $H^2(M; \mathbb{R})$. Let $Q_1, \ldots, Q_m$ be principal circle bundles over P with Euler classes $\tau^* e_1, \ldots, \tau^* e_m$, respectively, and which are equipped with lifted $\mathrm{O}(n)$-actions in the sense of Lemma 3.5. Consider the fiber-product

$$Q = \{(p_1, \ldots, p_m) \in Q_1 \times \cdots \times Q_m \mid \rho_1(p_1) = \cdots = \rho_m(p_m)\},$$

where $\rho_i : Q_i \to P$ denotes the quotient map. The space Q is a principal $\mathbb{T}^m$-bundle over P. Moreover the diagonal $\mathrm{O}(n)$-action on $Q_1 \times \cdots \times Q_m$ leaves Q invariant and commutes with the $\mathbb{T}^m$-action, so that the corresponding $G := \mathbb{T}^m \times \mathrm{O}(n)$-action restricted to Q has only finite stabilizers and the quotient is isomorphic to M. By Satake's theorem [**5**, p. 34], we find closed forms $\sigma_i \in \Omega^2(M)$ such that $[\sigma_i] = e_i$ for $i = 1, \ldots, m$. Further, we find constants $a_1, \ldots, a_m \in \mathbb{R}$ which are not all zero such that $[\sigma] = \sum_{i=1}^m a_i e_i$. Thus, up to changing representatives, if necessary, we have $\sigma = \sum_{i=1}^m a_i \sigma_i$. By the isomorphism (3.9), we can choose connection forms θ_i on Q_i

such that $d\theta_i = \tau_i^*\sigma_i$, where $\tau_i = \tau \circ \rho_i$. We identify the Lie algebra of $\mathbb{T}^m$ with $\mathbb{R}^m$, so that the quotient map is the exponential map. Further, we pick any Ad-invariant positive bilinear form on $\mathfrak{g}$ such that the splitting $\mathfrak{g} = \mathbb{R}^m \oplus \mathfrak{o}(n)$ is orthogonal and which is standard when restricted to $\mathbb{R}^m \oplus 0$. We identify $\mathfrak{g}^\vee$ with $\mathfrak{g}$ using the bilinear form. Then $Z = (a_1, a_2, \ldots, a_m, 0)$ and the connection form is

$$\theta = \rho^*\theta_0 + \sum_{i=1}^m \mathrm{pr}_i^*\theta_i \otimes Z_i,$$

where Z_i is the ith unit vector if $\mathbb{R}^m$, $\rho : Q \to P$ is the quotient map and $\mathrm{pr}_i : Q \to Q_i$ is the projection to the ith factor. By construction,

$$d\langle\theta, Z\rangle = \sum_{i=1}^m a_i \mathrm{pr}_i^* d\theta_i = \sum_{i=1}^m a_i \mathrm{pr}_i^* \tau_i^*\sigma_i = \hat{\tau}^* \sum_{i=1}^m a_i\sigma_i = \hat{\tau}^*\sigma,$$

where $\hat{\tau} : Q \to M$ denotes the quotient map. This completes the proof. □

4. *Symplectic reduction*

Here we give a brief account on the Hamiltonian formulation of the dynamical problem (for more details, we refer the reader to [**31**]). This chapter serves only as a motivation on how we will deduce the functional $\mathbb{S}_k$, and hence can be skipped by a reader who is only interested in the main argument.

It is easy to see that the action of a compact Lie group G on a manifold Q lifts to a Hamiltonian action of G on T^*Q

$$g \cdot (q, p) = (\varphi_g(q), \varphi_{g^{-1}}^* p)$$

with corresponding moment map given by

$$A : T^*Q \to \mathfrak{g}^\vee, \quad (q, p) \mapsto (X \mapsto \langle p, \underline{X}\rangle),$$

where $\langle\cdot\,, \cdot\rangle$ denotes the duality pairing. If G acts freely on $A^{-1}(Z)$ for a fixed central covector $Z \in \mathfrak{g}^\vee$, the *Marsden–Weinstein quotient* is defined by

$$((T^*Q)_Z, \omega_Z) \overset{\text{def}}{=} (A^{-1}(Z)/G, \omega_{\text{red}}),$$

where ω_{red} is uniquely determined by $\mathrm{pr}^*\omega_{\text{red}} = \iota^*\omega$. Here ω denotes the standard symplectic form on T^*Q, and $\iota : A^{-1}(Z) \to T^*Q$ and $\mathrm{pr} : A^{-1}(Z) \to A^{-1}(Z)/G$ are the canonical inclusion and the quotient projection, respectively. The construction carries over if G acts only locally freely: in this case, $(T^*Q)_Z$ is a symplectic orbifold.

As it turns out, $((T^*Q)_Z, \omega_Z)$ is symplectomorphic to a twisted cotangent bundle. To see this, we fix a principal connection form $\theta \in \Omega^1(Q, \mathfrak{g})$ and denote by $M = Q/G$, $\tau : Q \to M$ the base and the quotient map, respectively. We define the map $\Pi : A^{-1}(Z) \to T^*M$ implicitly via

$$\langle\Pi(q, p), d_q\tau(v)\rangle = \langle p, v\rangle - \langle Z, \theta_q(v)\rangle \quad \text{for all } \ v \in T_qQ.$$

Observe that Π is well defined since the kernel of $d_q\tau$ consists precisely of vectors on which the right-hand side vanishes. Moreover, Π is a G-invariant bundle map whose fibers are G-orbits. We conclude that Π induces a diffeomorphism $(T^*Q)_Z \cong T^*M$.

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LEMMA 4.1. *The map Π induces a symplectomorphism*

$$((T^*Q)_Z, \omega_Z) \cong (T^*M, \bar{\omega} + \bar{\pi}^*\bar{\sigma}_Z),$$

where $\bar{\omega}$ is the standard symplectic form on T^*M, $\bar{\pi} : T^*M \to M$ is the canonical projection, and $\sigma_Z \in \Omega^2(M)$ is given in (3.4).

Proof. See [**11**, Proposition 2.1]. □

In particular, we see that the magnetic flow on M given by the pair (g_M, σ_Z) lifts to the geodesic flow on Q defined by the metric g_Q, as constructed in §3.1. More precisely, let $\bar{H} : T^*M \to \mathbb{R}$ and $H : T^*Q \to \mathbb{R}$ be the kinetic Hamiltonians defined by g_M and g_Q, respectively, and let $X_{\bar{H}}$, X_H be the corresponding Hamiltonian vector fields. In the Hamiltonian formulation, Lemma 3.3 reads as follows.

LEMMA 4.2. *The geodesic flow on T^*Q with respect to g_Q leaves the subset $A^{-1}(Z) \cap H^{-1}(k)$ invariant and projects to the magnetic flow on $\bar{H}^{-1}(\bar{k}) \subset T^*M$ given by the pair (g_M, σ_Z), where $k = \bar{k} + \frac{1}{2}|Z|^2$.*

Proof. See [**11**, Lemma 2.2]. □

It follows that any curve $\bar{x} : \mathbb{R} \to T^*M$ satisfying

$$\dot{\bar{x}} = X_{\bar{H}}(\bar{x}), \quad x(T) = x(0), \quad \bar{H}(x) = \bar{k}, \tag{4.1}$$

for some $T > 0$, lifts to a curve $x : \mathbb{R} \to T^*Q$ such that

$$\dot{x} = X_H(x), \quad x(T) = g \cdot x(0), \quad H(x) = k, \quad A(x) = Z, \tag{4.2}$$

for some $g \in G$, that is, to a *constrained G-closed geodesic*. Conversely, every constrained G-closed geodesic projects to a curve satisfying (4.1).

Observe that, if $g = \exp(X)$ for some $X \in \mathfrak{g}$, then the rescaled curve

$$y : \mathbb{R} \to T^*Q, \quad y(t) = \exp(-tX) \cdot x(tT)$$

satisfies

$$\dot{y} = -X_{\langle A,X\rangle}(y) + TX_H(y), \quad y(1) = y(0), \quad H(y) = k, \quad A(y) = Z, \tag{4.3}$$

where by $X_{\langle A,X\rangle}$ we denote the Hamiltonian vector field associated with the Hamiltonian function $\langle A, X\rangle : T^*Q \to \mathbb{R}$ and the standard symplectic form. Conversely, every loop y satisfying (4.3) defines a curve x satisfying (4.2) by reversing the scaling. Following [**22**, §4.2], we see that such loops correspond to the critical points of the functional $\mathbb{A}_k : C^\infty(S^1, T^*Q \times \mathfrak{g}) \times \mathbb{R} \to \mathbb{R}$ given by

$$\mathbb{A}_k(y, \phi, T) = \int_0^1 y^*\lambda - \int_0^1 [T(H(y) - k) - \langle A(y) - Z, \phi\rangle]\, dt,$$

where λ denotes the Liouville one-form on T^*Q.

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5. *The functional $\mathbb{S}_k$*

Let M be a closed orbifold equipped with a Riemannian metric g_M and a closed two-form σ. In the previous chapters, we have reformulated the closed magnetic geodesics problem for (M, g_M, σ) as the problem of finding constrained G-closed geodesics on a suitable Riemannian locally free G-bundle (Q, g_Q), a G compact Lie group or, equivalently, critical points of the Rabinowitz-type action functional $\mathbb{A}_k$. Inspired by this, we now define a functional over a suitable space of loops in $Q \times \mathfrak{g}$, whose critical points correspond precisely to periodic magnetic geodesics in M of fixed energy. Thus, fix $k > \frac{1}{2}$ and, with the notation introduced in §3, define

$$\mathbb{S}_k : W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}) \times (0, \infty) \to \mathbb{R},$$
$$\mathbb{S}_k(\gamma, \phi, T) = \frac{1}{2T} \int_0^1 |\dot\gamma(t) + \underline{\phi(t)}(\gamma(t))|^2 \, dt - \int_0^1 \langle \phi(t), Z\rangle \, dt + kT,$$

where $\langle \cdot\,, \cdot\rangle$ is any Ad-invariant metric on $\mathfrak{g}$ such that $\langle Z, Z\rangle = 1$ and $\underline{\phi(t)}$ denotes the fundamental vector field associated with the Lie algebra element $\phi(t)$. For notational convenience we will hereafter omit the t-dependence everywhere. We set

$$\mathcal{M} := W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}) \times (0, +\infty),$$

and we observe that $\mathcal{M}$ has a natural structure of (non-complete) product Hilbert manifold; we denote the product metric by $g_{\mathcal{M}}$. We also notice that the functional $\mathbb{S}_k$ is smooth on $\mathcal{M}$ and that the connected components of $\mathcal{M}$ are in one-to-one correspondence with conjugacy classes in $\pi_1(Q)$.

Remark 5.1. The functional $\mathbb{S}_k$ can be thought of as the Legendre dual of the functional $\mathbb{A}_k$ introduced in §4. Indeed, computing for every fixed (ϕ, T) the Lagrangian $L_{\phi,T}$, which is the Fenchel dual of the Hamiltonian $H_{\phi,T} := T \cdot H - \langle A, \phi\rangle$, and then letting (ϕ, T) be free yields precisely the functional $\mathbb{S}_k$ above.

LEMMA 5.2. *If (γ, ϕ, T) is a critical point of $\mathbb{S}_k$, then the curve $t \mapsto \mu(t/T)$, $\mu := \tau \circ \gamma$ is a T-periodic magnetic geodesic for the pair (g_M, σ) with energy $\bar k = k - \frac{1}{2}$.*

Proof. By rescaling, it is enough to show that $\mu = \tau \circ \gamma$ is a magnetic geodesic for $(g_M, T\sigma)$ with energy $T^2(k - 1/2)$. For any $\psi \in L^2(S^1, \mathfrak{g})$,

$$\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=0} \mathbb{S}_k(\gamma, \phi + s\psi, T) \\
&= \frac{1}{T} \int_0^1 \langle \dot\gamma + \underline{\phi}, \underline{\psi}\rangle \, dt - \int_0^1 \langle \psi, Z\rangle \, dt \\
&= \frac{1}{T} \int_0^1 \langle \theta(\dot\gamma) + \phi - TZ, \psi\rangle \, dt.
\end{aligned}$$

Therefore

$$\theta_\gamma(\dot\gamma) + \phi = TZ \quad \text{for almost every } \ t \in S^1. \tag{5.1}$$

Now, for any interval $(a, b) \subset S^1$, define

$$\rho : (a, b) \to G, \qquad \rho := \exp \eta,$$

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where $\eta : (a, b) \to \mathfrak{g}$ is such that $\dot{\eta} = \phi$. Given any $\xi \in W^{1,2}(\gamma^* TQ)$ with compact support in (a, b) and s sufficiently small, we define

$$\gamma_s : (a, b) \to Q, \quad \gamma_s(t) := \exp_{\gamma(t)} s\xi(t),$$
$$\nu_s : (a, b) \to Q, \quad \nu_s(t) := \rho(t) \cdot \gamma_s(t),$$

and we compute using $\dot{\nu}_s = d\rho(\dot{\gamma}_s + \underline{\phi})$ and partial integration

$$\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=0} 2T \cdot \mathbb{S}_k(\gamma_s, \phi, T) \\
&= \frac{d}{ds}\bigg|_{s=0} \int_a^b |\dot{\gamma}_s + \underline{\phi}|^2 \, dt \\
&= \frac{d}{ds}\bigg|_{s=0} \int_a^b |\dot{\nu}_s|^2 \, dt \\
&= \int_a^b \langle \dot{\nu}, \nabla_t \zeta \rangle \, dt \\
&= -\int_a^b \langle \nabla_t \dot{\nu}, \zeta \rangle \, dt,
\end{aligned}$$

where $\zeta := (d/ds)|_{s=0}\nu_s = d\rho\, \xi$ and $\nu := \nu_0$. Since $d\rho$ is an isomorphism, we conclude that ν is a geodesic. Moreover, using (5.1), the equivariance property of the connection form θ and the centrality of Z we obtain

$$\theta(\dot{\nu}) = \theta(d\rho(\dot{\gamma} + \underline{\phi})) = TZ.$$

Lemma 3.3 implies therefore that $\mu = \tau \circ \nu = \tau \circ \gamma$ is a magnetic geodesic for the pair $(g_M, T\sigma)$. Finally, using (5.1) again we obtain

$$\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=0} \mathbb{S}_k(\gamma, \phi, T + s) \\
&= -\frac{1}{2T^2} \int_0^1 |\dot{\gamma} + \underline{\phi}|^2 \, dt + k \\
&= -\frac{1}{2T^2} \int_0^1 (|\dot{\mu}|^2 + |\theta(\dot{\gamma}) + \phi|^2) \, dt + k \\
&= -\frac{1}{2T^2} (|\dot{\mu}|^2 + T^2 - 2kT^2),
\end{aligned}$$

where with slight abuse of notation we denoted a local representative of μ in a chart with the same letter. This shows that the energy of μ is $T^2\bar{k}$, as required. □

Remark 5.3. If G is abelian, then the fundamental vector fields associated to (constant) Lie algebra elements give already enough symmetry to infer that critical points of $\mathbb{S}_k$ project to closed magnetic geodesics in M. In fact, in this setting, closed magnetic geodesics turn out to correspond to critical points of the functional $\mathbb{S}_k : W^{1,2}(S^1, Q) \times \mathfrak{g} \times (0, +\infty) \to \mathbb{R}$ given by

$$\mathbb{S}_k(\gamma, X, T) = \frac{1}{2T} \int_0^1 |\dot{\gamma} + \underline{X}(\gamma)|^2 \, dt - \langle X, Z \rangle + kT.$$

In the special case $G = S^1$, we retrieve precisely the functional considered in [**11**].

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5.1. *The gauge group.* The domain of $\mathbb{S}_k$ has many degrees of freedom which do not play any role for the magnetic geodesic obtained in the quotient. By the same token, different critical points of $\mathbb{S}_k$ might correspond to the same periodic magnetic geodesic in M. To remedy this fact, we introduce on $\mathcal{M}$ the action of a group $\mathcal{G}$ which leaves the critical set of $\mathbb{S}_k$ invariant. The group is the *loop group*

$$\mathcal{G} = W^{1,2}(S^1, G)$$

with group law given by pointwise multiplication and action on $\mathcal{M}$ given by

$$\rho \cdot (\gamma, \phi, T) = (\rho\gamma, \operatorname{Ad}_\rho \phi - \partial_t \rho\rho^{-1}, T) \quad \text{for all } (\gamma, \phi, T) \in \mathcal{M}.$$

LEMMA 5.4. *If (γ, ϕ, T) is a critical point of $\mathbb{S}_k$, then $\rho \cdot (\gamma, \phi, T)$ is also a critical point of $\mathbb{S}_k$ for any $\rho \in \mathcal{G}$.*

Proof. Set $\gamma^\rho := \rho\gamma$ and $\phi^\rho := \operatorname{Ad}_\rho \phi - \partial_t \rho\rho^{-1}$. We compute directly $\dot\gamma^\rho = d\rho\dot\gamma + \underline{\partial_t \rho\rho^{-1}}$, where $d\rho$ denotes the differential of the ρ-action. Using the identity $\underline{\operatorname{Ad}_\rho \phi} = d\rho\underline{\phi}$, we conclude that, for almost all $t \in S^1$,

$$\dot\gamma^\rho(t) + \underline{\phi^\rho(t)} = d\rho(t)(\dot\gamma(t) + \underline{\phi(t)}).$$

A straightforward computation shows now that $d\mathbb{S}_k(\rho \cdot (\gamma, \phi, T)) = 0$, as claimed. □

Although the set of critical points of $\mathbb{S}_k$ is, the functional $\mathbb{S}_k$ is not invariant under the action of $\mathcal{G}$. In fact,

$$\mathbb{S}_k(\rho \cdot (\gamma, \phi, T)) = \mathbb{S}_k(\gamma, \phi, T) + \Delta_\rho \quad \text{for all } \rho \in \mathcal{G} \text{ and for all } (\gamma, \phi, T) \in \mathcal{M}, \tag{5.2}$$

where Δ_ρ is given by

$$\Delta_\rho := \int_0^1 \langle \partial_t \rho\rho^{-1}, Z\rangle \, dt. \tag{5.3}$$

In the case $\rho = \exp(\eta)$ for some path $\eta : [0, 1] \to \mathfrak{g}$,

$$\Delta_\rho = \langle \eta(1) - \eta(0), Z\rangle. \tag{5.4}$$

Indeed, $\partial_t \rho\rho^{-1} = \operatorname{Ad}_\rho \partial_t \eta$ almost everywhere and hence

$$\int_0^1 \langle \partial_t \rho\rho^{-1}, Z\rangle \, dt = \int_0^1 \langle \partial_t \eta, Z\rangle \, dt$$

since Z is central. The claim then follows by the fundamental theorem of calculus. It is worth noticing that, if G is not connected, then not every $\rho \in \mathcal{G}$ can be written as $\exp(\eta)$ for some $\eta : [0, 1] \to \mathfrak{g}$. However, for our purposes it will always be sufficient to consider elements ρ which admit such a representation.

Also, $\mathbb{S}_k$ is not bounded from above nor from below. Indeed, consider $X \in \mathfrak{g}$ such that $\langle X, Z\rangle \neq 0$ and $\exp(X) = e$, where e is the neutral element of G. For every $m \in \mathbb{Z}$, define $\rho_m \in \mathcal{G}$ via $\rho_m(t) = \exp(mXt)$. Then, by (5.2),

$$\mathbb{S}_k(\rho_m \cdot (\gamma, \phi, T)) = \mathbb{S}_k(\gamma, \phi, T) + m\langle X, Z\rangle.$$

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5.2. *The Palais–Smale condition up to gauge transformations.* In infinite-dimensional Morse theory, the Palais–Smale condition plays the role of compactness since it, roughly speaking, allows one to find critical points of a functional on a Hilbert manifold from a sequence of approximately critical points. The lack of such a compactness property therefore poses major difficulties and one is forced to look for additional information in order to prove the existence of critical points. Evidence of this is represented precisely by the functional $\mathbb{S}_k : \mathcal{M} \to \mathbb{R}$. Indeed, in the case of S^1-actions treated in [**11**], the Palais–Smale condition for $\mathbb{S}_k$ does not hold on $\mathcal{M}$ but rather on subsets $\mathcal{M}_{[T_*,T^*]} \subset \mathcal{M}$ of triples (γ, X, T) with $0 < T_* \leq T \leq T^*$. As it turns out, this is enough to show the existence of critical points of $\mathbb{S}_k$—for almost every k—by means of a clever monotonicity argument, better known as the *Struwe monotonicity argument* [**36**] (for other applications, we refer the reader to, e.g.. [**1–4**, **6–9**, **17**]).

In the setting considered in this paper, the situation becomes even more delicate, since for $G \neq S^1$ the functional $\mathbb{S}_k$ fails to satisfy the Palais–Smale condition even on the subsets $\mathcal{M}_{[T_*,T^*]}$. Nevertheless, a suitable generalization of the Palais–Smale condition (namely, the Palais–Smale condition 'up to gauge transformations') for $\mathbb{S}_k$ on $\mathcal{M}_{[T_*,T^*]}$ turns out to hold true. Recall that a sequence $(\gamma_h, \phi_h, T_h)_{h\in\mathbb{N}} \subseteq \mathcal{M}$ is called a *Palais–Smale sequence for* $\mathbb{S}_k$ if there exists $c \in \mathbb{R}$ such that

$$\lim_{h\to+\infty} \mathbb{S}_k(\gamma_h, \phi_h, T_h) = c, \qquad \lim_{h\to+\infty} |d\mathbb{S}_k(\gamma_h, \phi_h, T_h)| = 0.$$

More precisely, we say that $(\gamma_h, \phi_h, T_h)_h$ is a *Palais–Smale sequence for* $\mathbb{S}_k$ *at* c. In the definition above, $|\cdot|$ denotes, with slight abuse of notation, the (dual) norm on $T^*\mathcal{M}$ induced by the metric $g_{\mathcal{M}}$. The functional $\mathbb{S}_k$ is said to *satisfy the Palais–Smale condition* if every Palais–Smale sequence admits a converging subsequence.

Definition 5.5. A sequence $(\rho_h) \subset \mathcal{G}$ is called *admissible* if the sequence $(\Delta_{\rho_h})_h \subset \mathbb{R}$, defined as in (5.3), is bounded from above and below.

Let $(\gamma_h, \phi_h, T_h)_h \subset \mathcal{M}$ be a Palais–Smale sequence for $\mathbb{S}_k$ and let $(\rho_h)_{h\in\mathbb{N}} \subset \mathcal{G}$ be an admissible sequence. Then, up to passing to a subsequence, if necessary, $(\rho_h \cdot (\gamma_h, \phi_h, T_h))_h$ is again a Palais–Smale sequence for $\mathbb{S}_k$. Indeed, after passing to a subsequence, we can assume that $\Delta_{\rho_h} \to \Delta$ converges. The claim follows now from Lemma 5.4 and Equation (5.2). If $G \neq S^1$, then it is easy to construct an admissible sequence $(\rho_h) \subset \mathcal{G}$ such that $(\rho_h \cdot (\gamma_h, \phi_h, T_h))$ does not contain a converging subsequence, even though $(\gamma_h, \phi_h, T_h) \subseteq \mathcal{M}_{[T_*,T^*]}$. Observe that the gauged sequence still belongs to $\mathcal{M}_{[T_*,T^*]}$ since $\mathcal{M}_{[T_*,T^*]}$ is obviously $\mathcal{G}$-invariant. Therefore, $\mathbb{S}_k$ does not satisfy the Palais–Smale condition on $\mathcal{M}_{[T_*,T^*]}$.

On the other hand, given a Palais–Smale sequence $(\gamma_h, \phi_h, T_h) \subseteq \mathcal{M}_{[T_*,T^*]}$, one might hope to find an admissible sequence $(\rho_h) \subseteq \mathcal{G}$ such that $(\rho_h \cdot (\gamma_h, \phi_h, T_h))$ has a converging subsequence. By this observation, we are naturally led to the following definition.

Definition 5.6. A $\mathcal{G}$-invariant subset $\mathcal{K} \subset \mathcal{M}$ is said to *satisfy the Palais–Smale condition up to gauge transformations* if, for any Palais–Smale sequence $(\gamma_h, \phi_h, T_h)_h \subset \mathcal{K}$, there exists an admissible sequence $(\rho_h) \subset \mathcal{G}$ such that $(\rho_h \cdot (\gamma_h, \phi_h, T_h))_h$ contains a converging subsequence.

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Before we come to the main result of this subsection, namely, that $\mathcal{M}_{[T_*,T^*]}$ does satisfy the Palais–Smale condition up to gauge transformations, we prove some elementary estimates for Palais–Smale sequences that will be useful later on.

LEMMA 5.7. *Suppose (γ_h, ϕ_h, T_h) is a Palais–Smale sequence for $\mathbb{S}_k$ at level c. Then*

$$\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\,dt = O(T_h^2), \qquad \left|\int_0^1 \langle \phi_h, Z\rangle\,dt\right| = O(T_h). \tag{5.5}$$

Moreover, $T_h \to 0$ if and only if $\int_0^1 \langle \phi_h, Z\rangle\,dt \to -c$. In this case, $\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\,dt \to 0$.

Proof. We have

$$c + o(1) = \frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\,dt - \int_0^1 \langle \phi_h, Z\rangle\,dt + kT_h, \tag{5.6}$$

$$o(1) = \frac{\partial \mathbb{S}_k}{\partial T}(\gamma_h, \phi_h, T_h) \;=\; k - \frac{1}{2T_h^2}\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\,dt. \tag{5.7}$$

From (5.7), it follows that

$$\frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\,dt = kT_h + T_h o(1).$$

The first equation in (5.5) follows. Replacing the last expression in (5.6) gives

$$-\int_0^1 \langle \phi_h, Z\rangle\,dt = c - 2kT_h - T_h o(1) + o(1). \qquad \square$$

LEMMA 5.8. *Suppose that $(\gamma_h, \phi_h, T_h)_h \subseteq \mathcal{M}$ is a Palais–Smale sequence for $\mathbb{S}_k$ such that $T_h = O(1)$. Then there exists an admissible sequence $(\rho_h) \subset \mathcal{G}$ such that the gauged sequence $(\rho_h \cdot \phi_h = \mathrm{Ad}_{\rho_h}\phi_h - \partial_t \rho_h \rho_h^{-1})_h$ admits a (strongly in L^2) converging subsequence.*

Proof. We divide the proof in two steps.

Step 1. Fix a maximal torus $T \subset G$ (that is, a connected, closed and abelian subgroup of maximal dimension) with Lie algebra $\mathfrak{t}$. For every $h \in \mathbb{N}$, we define

$$\overline{\phi}_h := \int_0^1 \phi_h(t)\,dt \in \mathfrak{g}.$$

By [**15**, Theorem 6.4], there exists $g_h \in G$ such that $\mathrm{Ad}_{g_h}\overline{\phi}_h \in \mathfrak{t}$. Set $\Lambda := \{X \in \mathfrak{t} \mid \exp(X) = e\}$, where, as usual, e denotes the neutral element in G. The subgroup $\Lambda \subset \mathfrak{t}$ is a lattice and so we find $X_h' \in \Lambda$ such that $|\mathrm{Ad}_{g_h}\overline{\phi}_h - X_h'|$ is uniformly bounded. With $X_h := \mathrm{Ad}_{g_h}^{-1} X_h'$, we now have

$$|\overline{\phi}_h - X_h| = O(1). \tag{5.8}$$

Now, for all $h \in \mathbb{N}$, define $\eta_h : [0, 1] \to \mathfrak{g}$ and $\rho_h : [0, 1] \to G$ by

$$\eta_h(t) := \int_0^t \phi_h(s)\,ds - t(\overline{\phi}_h - X_h), \qquad \rho_h(t) := \exp(\eta_h(t)).$$

Notice that $\eta_h(0) = 0$ and $\eta_h(1) = X_h$. Therefore $\rho_h \in \mathcal{G}$ for all $h \in \mathbb{N}$ as, by construction, ρ_h is of class $W^{1,2}$ and $\exp(X_h) = e$. Gauging ϕ_h by ρ_h yields

$$\rho_h \cdot \phi_h = \mathrm{Ad}_{\rho_h}\, \phi_h - \partial_t \rho_h \rho_h^{-1} = \mathrm{Ad}_{\rho_h}\, \phi_h - \mathrm{Ad}_{\rho_h}\, \partial_t \eta_h = \mathrm{Ad}_{\rho_h}(\overline{\phi}_h - X_h). \tag{5.9}$$

In particular, the gauged sequence $(\rho_h \cdot \phi_h)$ is contained in $W^{1,2}(S^1, \mathfrak{g})$ as Ad is smooth and $\overline{\phi}_h - X_h$ is constant. Moreover, we have the pointwise estimate

$$|\rho_h \cdot \phi_h| = |\mathrm{Ad}_{\rho_h} \phi_h - \partial_t \rho_h \rho_h^{-1}| = |\phi_h - \partial_t \eta_h| = |\overline{\phi}_h - X_h| = O(1).$$

This shows that the gauged sequence is uniformly bounded in L^∞ and hence also in L^2. To check that $(\rho_h)_h$ is admissible, we observe that, by Lemma 5.7,

$$\langle \overline{\phi}_h, Z\rangle = \int_0^1 \langle \phi_h(t), Z\rangle\, dt = O(T_h) = O(1),$$

and thus by (5.4) and (5.8),

$$|\Delta_{\rho_h}| = |\langle X_h, Z\rangle| \le |X_h - \overline{\phi}_h| + |\langle \overline{\phi}_h, Z\rangle| = O(1).$$

Step 2. By the first step, after gauging and passing to a subsequence, if necessary, we can assume that (γ_h, ϕ_h, T_h) is a Palais–Smale sequence for $\mathbb{S}_k$ with $(\phi_h) \subseteq W^{1,2}(S^1, \mathfrak{g})$ uniformly bounded in L^∞. We now repeat the gauge procedure as in Step 1, taking $X_h = 0$, and observe that the gauged sequence $(\rho_h \cdot \phi_h)$ is contained in $W^{2,2}(S^1, \mathfrak{g})$. Therefore, we can compute the derivative with respect to t of $\rho_h \cdot \phi_h$ and, using (5.9), we obtain

$$\partial_t(\rho_h \cdot \phi_h) = \mathrm{Ad}_{\rho_h}(\mathrm{ad}_{\partial_t \eta_h}(\overline{\phi}_h)).$$

In particular, we obtain the pointwise estimate

$$|\partial_t(\rho_h \cdot \phi_h)| = |\,\mathrm{ad}_{\partial_t \eta_h}\, \overline{\phi}_h| \le |\partial_t \eta_h||\overline{\phi}_h| \le (|\phi_h| + |\overline{\phi}_h|)|\overline{\phi}_h| = O(1),$$

which shows that $(\rho_h \cdot \phi_h)_h$ is uniformly bounded in $W^{1,\infty}$ and hence in $W^{1,2}$. The claim follows using the compactness of the embedding $W^{1,2} \hookrightarrow L^2$. □

Remark 5.9. As shown in the proof, the admissible sequence $(\rho_h) \subseteq \mathcal{G}$ can be chosen in such a way that the gauged sequence $(\rho_h \cdot \phi_h)$ is uniformly bounded in $W^{1,2}(S^1, \mathfrak{g})$ and hence, in particular, admits a weakly (in $W^{1,2}$) converging subsequence.

COROLLARY 5.10. *Suppose that (γ_h, ϕ_h, T_h) is a Palais–Smale sequence for $\mathbb{S}_k$ such that $0 < T_* \le T_h \le T^*$ for all $h \in \mathbb{N}$. Then there exists an admissible sequence $(\rho_h) \subset \mathcal{G}$ such that $(\rho_h \cdot (\gamma_h, \phi_h, T_h))$ contains a strongly converging subsequence.*

Proof. By Lemma 5.8, up to taking a subsequence and applying a gauge transformation, we can assume that ϕ_h converges weakly in $W^{1,2}$ (and strongly in L^2) to some $\phi \in W^{1,2}(S^1, \mathfrak{g})$ and that $T_h \to T \in [T_*, T^*]$ as $h \to \infty$. Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ and Lemma 5.7, we obtain

$$\int_0^1 |\dot\gamma_h|^2\, dt \le 2\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\, dt + 2\int_0^1 |\phi_h|^2\, dt = O(1). \tag{5.10}$$

This shows that $\|\dot\gamma_h\|_2$ is uniformly bounded and hence that the sequence $(\gamma_h)_h$ is $\frac{1}{2}$-Hölder equicontinuous. Up to taking a subsequence, the theorem of Arzela–Ascoli yields the existence of $\gamma \in C^0(S^1, E)$ such that $\gamma_h \to \gamma$ uniformly as $h \to \infty$. The fact that the convergence of γ_h to γ is actually strong in $W^{1,2}$ follows now by standard arguments (see, e.g., [**1**, Lemma 5.3]). □

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5.3. *A complete gradient vector field for $\mathbb{S}_k$.* Consider the bounded vector field

$$\mathcal{X}_k := \frac{-\mathrm{grad}\,\mathbb{S}_k}{\sqrt{1+|\mathrm{grad}\,\mathbb{S}_k|^2}} \tag{5.11}$$

conformally equivalent to $-\mathrm{grad}\,\mathbb{S}_k$, where the gradient of $\mathbb{S}_k$ is defined with respect to the Riemannian metric $g_{\mathcal{M}}$. Since $\mathbb{S}_k$ is smooth, the vector field $\mathcal{X}_k$ is locally Lipschitz continuous and hence its flow Φ_k is well defined. However, Φ_k is not complete since there are flow-lines on which the variable T approaches zero in finite time. On the other hand, the only source of non-completeness for Φ_k is represented by such flow-lines; and hence, 'stopping them' in a suitable fashion yields a complete flow. However, while doing this, we should be careful not to lose any geometric property of the functional $\mathbb{S}_k$. To this purpose, we need to know more about the behavior of the functional $\mathbb{S}_k$ in a neighborhood of elements that are approached by finite maximal flow-lines on which $T \to 0$.

We call an element $(\gamma, \phi, T) \in \mathcal{M}$ *vertical* if $\dot{\gamma} + \underline{\phi}(\gamma) = 0$ almost everywhere. We see that if (γ, ϕ, T) is vertical, then necessarily the vector $\dot{\gamma}(t)$ is vertical for almost every $t \in S^1$ and γ projects to a constant loop in the quotient M. Moreover, by (5.2), we see immediately that

$$\mathbb{S}_k(\gamma, \phi, T) = \int_0^1 \langle \phi, Z\rangle\, dt + kT. \tag{5.12}$$

We now examine neighborhoods of vertical elements in $\mathcal{M}$. For $\delta > 0$, define

$$\mathcal{V}_\delta := \left\{(\gamma, \phi, T) \in \mathcal{M} \;\middle|\; \int_0^1 |\dot{\gamma} + \underline{\phi}(\gamma)|^2\, dt < \delta\right\}.$$

Our first goal is to show that, for $\delta > 0$ sufficiently small, the space $\mathcal{V}_\delta$ is a disjoint union of neighborhoods of 'local minima' of $\mathbb{S}_k$. To this purpose, we need some notation: we set $Z(G) := \{g \in G \mid gh = hg,\ \forall\, h \in G\}$ to be the center of G, $\mathfrak{z}$ to be its Lie algebra and $p_{\mathfrak{z}} : \mathfrak{g} \to \mathfrak{z}$ to be the orthogonal projection. Note that $\mathfrak{z}$ is non-trivial because $Z \in \mathfrak{z}$ is non-trivial by Proposition 3.4. Further, we denote with $\Lambda_{\mathfrak{z}} := \{X \in \mathfrak{z} \mid \exp(X) = e\}$ the unit lattice in $\mathfrak{z}$, where $e \in G$ is the neutral element of G. Finally, for every $\phi \in L^2(S^1, \mathfrak{g})$, we define

$$\overline{\phi} := \int_0^1 \phi(t)\, dt \in \mathfrak{g}. \tag{5.13}$$

LEMMA 5.11. *There exists $N \in \mathbb{N}$ such that the following property holds. For all $(\gamma, \phi, T) \in \mathcal{V}_\delta$, there exists $X \in (1/N)\Lambda_{\mathfrak{z}}$ such that*

$$|X - p_{\mathfrak{z}}\overline{\phi}| < \sqrt{\delta}.$$

Proof. Define the path $\nu : [0, 1] \to Q$, $\nu(t) := \rho(t)\gamma(t)$, where

$$\rho : [0, 1] \to G, \quad \rho(t) = \exp\left(\int_0^t \phi(s)\, ds\right) \quad \text{for all } t \in [0, 1].$$

Since $\dot{\nu} = d\rho(\dot{\gamma} + \underline{\phi}(\gamma))$,

$$\int_0^1 |\dot{\nu}|^2\, dt = \int_0^1 |\dot{\gamma} + \underline{\phi}(\gamma)|\, dt < \delta.$$

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This shows, in particular, that

$$\operatorname{dist}(\gamma(0), \exp(\overline{\phi})\gamma(0)) = \operatorname{dist}(\nu(0), \nu(1)) < \sqrt{\delta}.$$

If $\Gamma_{\gamma(0)} \subset G$ denotes the stabilizer at $\gamma(0)$, then the previous inequality yields

$$\operatorname{dist}(g, h) < \sqrt{\delta}, \quad h = \exp(\overline{\phi}),$$

for some $g \in \Gamma_{\gamma(0)}$. Since Q is compact, there exists—up to conjugation—only finitely many different subgroups that appear as stabilizer groups. Let N be the product of their orders. It follows that $g^N = e$. Using the triangle inequality combined with the invariance of the distance on G with respect to right and left multiplication, we then obtain

$$\operatorname{dist}(e, h^N) < N\sqrt{\delta}, \quad h^N = \exp(N\overline{\phi}). \tag{5.14}$$

Now let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$ and integer lattice $\Lambda = \{X \in \mathfrak{t} \mid \exp(X) = e\}$. By [**15**, Theorem 6.4], there exists $\ell \in G$ such that $\operatorname{Ad}_\ell \overline{\phi} \in \mathfrak{t}$. Since $\exp : \mathfrak{t} \to T$ is a local isometry, we deduce from (5.14) that there exists $X \in \Lambda$ such that

$$|\operatorname{Ad}_\ell N\overline{\phi} - X| < N\sqrt{\delta}.$$

Moreover, since the splitting $\mathfrak{g} = \mathfrak{z} \oplus \ker p_{\mathfrak{z}}$ is left invariant by the adjoint action, we have $p_{\mathfrak{z}} \operatorname{Ad}_\ell \overline{\phi} = p_{\mathfrak{z}}\overline{\phi}$ and hence

$$|Np_{\mathfrak{z}}\overline{\phi} - p_{\mathfrak{z}}X| = |p_{\mathfrak{z}}(N \operatorname{Ad}_\ell \overline{\phi} - X)| \leq |\operatorname{Ad}_\ell N\overline{\phi} - X| < N\sqrt{\delta}.$$

The claim follows from the fact that $p_{\mathfrak{z}}(\Lambda) = \Lambda_{\mathfrak{z}}$. □

By the previous lemma, we see that for all $\delta > 0$ small enough,

$$\mathcal{V}_\delta = \bigsqcup_{X \in (1/N)\Lambda_{\mathfrak{z}}} \mathcal{V}_{\delta,X}, \quad \mathcal{V}_{\delta,X} := \{(\gamma, \phi, T) \in \mathcal{V}_\delta \mid |p_{\mathfrak{z}}\overline{\phi} - X| < \sqrt{\delta}\}.$$

Observe that $\mathcal{V}_{\delta,X}$ might be empty. However, if there exists $p \in Q$ such that $\exp(X) \in \Gamma_p$, where $\Gamma_p \subset G$ denotes the stabilizer subgroup at p, then $\mathcal{V}_{\delta,X}$ is non-empty and contains the vertical element (γ_X, ϕ_X, T) given by $\phi_X(t) = X$ and $\gamma_X(t) = \exp(tX)p$, for all $t \in [0, 1]$. For such a vertical element,

$$\mathbb{S}_k(\gamma_X, \phi_X, T) = -\langle X, Z\rangle + kT.$$

LEMMA 5.12. *For $\delta > 0$ small enough, there exists $\epsilon > 0$ such that for all $X \in (1/N)\Lambda_{\mathfrak{z}}$ with $\mathcal{V}_{\delta,X} \neq \emptyset$ we have*

$$\inf_{\mathcal{V}_{\delta,X}} \mathbb{S}_k = -\langle X, Z\rangle, \quad \inf_{\partial\mathcal{V}_{\delta,X}} \mathbb{S}_k > -\langle X, Z\rangle + \epsilon.$$

Proof. For every $(\gamma, \phi, T) \in \partial\mathcal{V}_{\delta,X}$ we compute

$$\mathbb{S}_k(\gamma, \phi, T) = \frac{\delta}{2T} - \langle\overline{\phi}, Z\rangle + kT \geq \sqrt{2\delta k} - \langle\overline{\phi}, Z\rangle \geq \sqrt{2\delta k} - \sqrt{\delta} - \langle X, Z\rangle,$$

where the first inequality is obtained by minimizing in T and the last is given by Lemma 5.11. Indeed, notice that, by construction, $Z \in \mathfrak{z}$ and hence

$$\langle\overline{\phi}, Z\rangle = \langle p_{\mathfrak{z}}\overline{\phi}, Z\rangle.$$

The second assertion follows, as $(\sqrt{2k} - 1)\sqrt{\delta} > 0$. The first assertion follows as well, since any $(\gamma, \phi, T) \in \mathcal{V}_{\delta,X}$ is contained $\partial\mathcal{V}_{\delta',X}$ for $\delta' := \int_0^1 |\dot{\gamma} + \underline{\phi}(\gamma)|^2$. □

COROLLARY 5.13. *Let (γ_h, ϕ_h, T_h) be a Palais–Smale sequence for $\mathbb{S}_k$ such that $T_h \to 0$. Then after gauging and taking a subsequence, if necessary, we have that*

$$\mathbb{S}_k(\gamma_h, \phi_h, T_h) \to -\langle X, Z\rangle$$

for some $X \in (1/N)\Lambda_3$ and, for any $\delta > 0$, the sequence (γ_h, ϕ_h, T_h) eventually enters the set $\mathcal{V}_{\delta,X}$.

Proof. Fix $\delta > 0$. By the first equation of (5.7), we see that

$$\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\, dt = 2T_h^2(k + o(1)) = 2T_h^2 k + o(T_h^2).$$

In particular, $(\gamma_h, T_h, Z_h) \in \mathcal{V}_\delta$ for h large enough. By Step 1 in the proof of Lemma 5.8, we can find an admissible sequence $(\rho_h)_h \subseteq \mathcal{G}$ such that the gauged sequence $(\rho_h \cdot \phi_h)$ is uniformly bounded in L^2. We pick one such admissible sequence and consider the gauged sequence $(\rho_h \cdot (\gamma_h, \phi_h, T_h))$. For sake of simplicity, we will denote the gauged sequence again by (γ_h, ϕ_h, T_h).

By Lemma 5.11, for every $h \in \mathbb{N}$ there exists $X_h \in (1/N)\Lambda_3$ such that

$$|p_3\overline{\phi}_h - X_h| < \sqrt{2(k + o(1))}T_h = \sqrt{2k}T_h + o(T_h). \tag{5.15}$$

It follows that

$$\begin{aligned}\mathbb{S}_k(\gamma_h, \phi_h, T_h) &= \frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\, dt - \langle\overline{\phi}_h, Z\rangle + kT_h\\ &\leq 2kT_h - \langle X_h, Z\rangle + \sqrt{2k}T_h + o(T_h)\\ &< -\langle X_h, Z\rangle + o(1)\end{aligned}$$

for h large enough. On the other hand,

$$\mathbb{S}_k(\gamma_h, \phi_h, T_h) \geq 2kT_h - \langle X_h, Z\rangle - \sqrt{2k}T_h + o(T_h) \geq -\langle X_h, Z\rangle$$

for h large enough, as $k > 1/2$. Since ϕ_h is uniformly bounded in L^2, $|\overline{\phi}_h|$ is uniformly bounded as well. Therefore, the set

$$\{\langle X_h, Z\rangle \mid h \in \mathbb{N}\} \subseteq \mathbb{R}$$

is discrete (actually finite). We conclude that there exists $X \in (1/N)\Lambda_3$ such that $X_h = X$ for every h large enough. In particular, $\mathbb{S}_k(\gamma_h, \phi_h, T_h) \to -\langle X, Z\rangle$ and, in virtue of (5.15), $(\gamma_h, \phi_h, T_h) \in \mathcal{V}_{\delta,X}$ for h large enough, as we wished to prove. □

The next lemma shows that maximal flow-lines of $\mathcal{X}_k$ that are defined on a finite interval have to approach vertical elements. The proof is analogous to that of [**11**, Lemma 4.9] and will be omitted.

LEMMA 5.14. *Suppose $u : [0, R) \to \mathcal{M}$ is a maximal flow-line of $\mathcal{X}_k$. Then there exist $X \in (1/N)\Lambda_3$ and a sequence $r_h \uparrow R$ such that $u(r_h) \in \mathcal{V}_{\delta,X}$ for all h large enough, and with $(\gamma_h, \phi_h, T_h) := u(r_h)$, we have*

$$\int_0^1 |\dot\gamma_h + \underline{\phi_h}(\gamma_h)|^2\, dt \to 0, \qquad \mathbb{S}_k(u(r_h)) \to -\langle X, Z\rangle.$$

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Using Lemma 5.14, it is now easy to get from Φ^k a complete flow by stopping flow-lines that enter the sets

$$\{\mathbb{S}_k < -\langle X, Z\rangle + \epsilon\} \cap \mathcal{V}_{\delta,X} \quad \text{for all } X \in \frac{1}{N}\Lambda_{\mathfrak{z}}.$$

With slight abuse of notation, we denote the complete flow also by Φ^k.

6. *Proof of Theorems 1.2 and 1.4*

In this section, building on the results of the previous ones, we prove Theorems 1.2 and 1.4. In order to show the existence of critical points of $\mathbb{S}_k$, we will use the topological assumption on M to find a suitable (non-trivial) minimax class on the Hilbert manifold $\mathcal{M}$ and a corresponding minimax function. We will then show that such a minimax function yields critical points of $\mathbb{S}_k$ for almost every $k > \frac{1}{2}$.

The proof for Theorem 1.2 follows closely [**11**]; however, some extra care is needed in the whole line of argument. Indeed, on the one hand, the functional $\mathbb{S}_k$ satisfies the Palais–Smale condition on $\mathcal{M}_{[T_*,T^*]}$ only up to gauge transformations and, on the other hand, the construction of the minimax class requires techniques coming from rational homotopy theory and orbifold theoretical homotopy theory.

6.1. *The minimax class for Theorem 1.2.* We call M *rationally aspherical* if $\pi_\ell^{\mathrm{orb}}(M) \otimes\mathbb{Q}$ is trivial for all $\ell \geq 2$, where by $\pi_\ell^{\mathrm{orb}}(M)$ we denote the orbifold-theoretic homotopy group as defined in [**5**, Definition 1.50]. Recall that, with the notation from Proposition 3.4 and [**5**, Proposition 1.51], the orbifold homotopy groups of M are the (classical) homotopy groups of the Borel quotient $BM := Q \times_G EG$, where EG denotes the universal G-bundle. For every $k \geq 1$, we obtain a homomorphism

$$\tau_k : \pi_k(Q) \to \pi_k^{\mathrm{orb}}(M), \tag{6.1}$$

which is induced by the quotient map $Q \times EG \to BM$.

LEMMA 6.1. *Assume that $\pi_k^{\mathrm{orb}}(M) \otimes \mathbb{Q} \neq 0$ for some $k \geq 2$. Then there exists a class $a \in \pi_\ell(Q)$ for some $\ell \geq 2$ such that $\tau_\ell(a)$ has infinite order.*

Proof. The fibration $G \hookrightarrow Q \times EG \to BM$ induces an exact homotopy sequence

$$\cdots \xrightarrow{\tau_{k+1}} \pi_{k+1}^{\mathrm{orb}}(M) \longrightarrow \pi_k(G) \longrightarrow \pi_k(Q) \xrightarrow{\tau_k} \pi_k^{\mathrm{orb}}(M) \longrightarrow \cdots . \tag{6.2}$$

If $\tau_k \otimes \mathbb{Q}$ is trivial for all $k \geq 2$, then the sequence splits into short exact sequences

$$0 \to \pi_{k+1}^{\mathrm{orb}}(M) \otimes \mathbb{Q} \to \pi_k(G) \otimes \mathbb{Q} \to \pi_k(Q) \otimes \mathbb{Q} \to 0 \quad \text{for all } k \geq 1.$$

In particular,

$$\dim \pi_{k+1}^{\mathrm{orb}}(M) \otimes \mathbb{Q} \leq \dim \pi_k(G) \otimes \mathbb{Q} \quad \text{for all } k \geq 1.$$

Let $\widetilde{BM}$ be the (classical) universal cover of BM. Since $\pi_k^{\mathrm{orb}}(M) = \pi_k(BM) \cong \pi_k(\widetilde{BM})$ for all $k \geq 2$, we obtain

$$\dim \pi_{k+1}(\widetilde{BM}) \otimes \mathbb{Q} \leq \dim \pi_k(G) \otimes \mathbb{Q} \quad \text{for all } k \geq 1.$$

Furthermore, since $\pi_{2j}(G) \otimes \mathbb{Q}$ is trivial for all $j \geq 1$ (cf. [**21**, §15(f)]), we see that $\pi_{2j+1}(\widetilde{BM}) \otimes \mathbb{Q}$ is trivial for all $j \geq 0$. By the same inequality, we conclude that

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$\pi_*(\widetilde{BM}) \otimes \mathbb{Q}$ has finite type and thus $H_*(\widetilde{BM}, \mathbb{Q})$ has as well, following the remark after [**21**, Theorem 15.11]. By [**21**, Theorem 15.11], we see that the minimal model (V, d) of $\widetilde{BM}$ has only even generators, which implies that the differential d is trivial. The same theorem implies that V^k is non-trivial for some $k \geq 2$ since, by assumption, $\pi_k^{\mathrm{orb}}(M) \otimes \mathbb{Q} \cong \pi_k(\widetilde{BM}) \otimes \mathbb{Q} \neq 0$. Let $x \in V^k$ be a non-trivial element. Since V is a free algebra, its powers $x^j \in V^{jk}$ are non-trivial for all $j \geq 1$. By the property of a minimal model and the vanishing of the differential of V, we have that $V^* \cong H^*(\widetilde{BM}, \mathbb{Q})$ and, in particular, that $H^{jk}(\widetilde{BM}, \mathbb{Q})$ is non-trivial for all $j \geq 1$.

Now let $\widetilde{M}$ be the universal orbifold cover of M in the sense of [**5**, Definition 2.16]. By [**5**, Proposition 2.17], the Borel quotient corresponding to $\widetilde{M}$ is $\widetilde{BM}$. The Vietoris–Begle theorem yields an isomorphism $H^*(\widetilde{BM}, \mathbb{Q}) \cong H^*(\widetilde{M}, \mathbb{Q})$, where the right-hand denotes the singular cohomology of the underlying topological space (cf. [**5**, Proposition 2.12]). It follows that $H^*(\widetilde{M}, \mathbb{Q})$ is non-trivial in arbitrary large degree, which is impossible for the finite-dimensional orbifold $\widetilde{M}$. □

For $\ell \in \mathbb{N}$, let $B^\ell \subset \mathbb{R}^\ell$ be the standard ball with boundary $S^{\ell-1}$. We identify the space Q with the subspace of constant loops in $W^{1,2}(S^1, Q)$. Further, we set $Q_{T_0} := Q \times \{0\} \times (0, T_0] \subset \mathcal{M}$ for $T_0 > 0$ fixed. By equation (5.12),

$$\max_{Q_{T_0}} \mathbb{S}_k = kT_0 \leq \varepsilon/2$$

if $T_0 > 0$ is chosen small enough, where $\varepsilon > 0$ is the constant from Lemma 5.12. It is well known that with any continuous map $u : (B^{\ell-1}, S^{\ell-2}) \to (\mathcal{M}, Q_{T_0})$ we can associate a continuous map $v = v_u : S^\ell \to Q \times \mathfrak{g} \times (0, \infty)$ and, conversely, with every smooth map $v : S^\ell \to Q \times \mathfrak{g} \times (0, T_0]$ we can associate a continuous map of pairs of spaces $u = u_v : (B^{\ell-1}, S^{\ell-2}) \to (\mathcal{M}, Q_{T_0})$ such that v_{u_v} is homotopic to v. Moreover, a homotopy of u induces a homotopy of u_v and *vice versa* (cf. [**28**, Proof of Theorem 2.4.20] for more details). By abuse of notation, we denote by $[u] \in \pi_\ell(Q)$ the homotopy class associated to v_u, where we have additionally identified $\pi_\ell(Q \times \mathfrak{g} \times (0, \infty)) \cong \pi_\ell(Q)$ canonically.

LEMMA 6.2. *There exists $\delta > 0$ such that, for any $u : (B^{\ell-1}, S^{\ell-2}) \to (\mathcal{M}, Q_{T_0})$ satisfying $u(x) \in \mathcal{V}_\delta$ for all $x \in B^{\ell-1}$, we have that $\tau_\ell([u]) = 0$ in $\pi_\ell^{\mathrm{orb}}(M)$.*

Proof. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$ and unit lattice $\Lambda \subset \mathfrak{t}$. For $x \in B^{\ell-1}$, write $u(x) = (\gamma_x, \phi_x, T_x)$ and let $\overline{\phi}_x \in \mathfrak{g}$ be as defined in (5.13). As explained in the proof of Lemma 5.11, for any $x \in B^{\ell-1}$ we find $g = g_x \in G$ and $X = X_x \in (1/N)\Lambda$ such that $|\mathrm{Ad}_g \overline{\phi}_x - X| < \sqrt{\delta}$. Hence, either $|\overline{\phi}_x| < \sqrt{\delta}$ or

$$|\overline{\phi}_x| > \Delta - \sqrt{\delta}, \quad \Delta := \min_{X \in (1/N)\Lambda \setminus \{0\}} |X| > 0.$$

For δ small enough, the statements are mutually exclusive and, since $x \mapsto |\overline{\phi}_x|$ is continuous and vanishes for all $x \in S^{\ell-2}$, we conclude that $|\overline{\phi}_x| < \sqrt{\delta}$ for all $x \in B^{\ell-1}$. Now set $(\gamma_x^{\rho_x}, \phi_x^{\rho_x}, T_x) := \rho_x \cdot (\gamma_x, \phi_x, T_x)$, where

$$\rho_x : S^1 \to G, \quad t \mapsto \exp\biggl(\int_0^t \phi_x(t')\, dt' - \overline{\phi}_x t'\biggr).$$

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As in the proof of Lemma 5.8, we see that $|\phi_x^{\rho_x}| \leq |\overline{\phi}_x| < \sqrt{\delta}$ and hence, using (5.10),

$$\int_0^1 |\dot{\gamma}_x^{\rho_x}|^2 \, dt < 4\delta.$$

In particular, for $\delta > 0$ small enough, the map $x \mapsto \gamma_x^{\rho_x}$ defines the trivial homotopy class in $\pi_\ell(Q)$ (cf. [**27**, Theorem 1.4.15]). We conclude that $[u]$ is in the image of $\pi_\ell(G) \to \pi_\ell(Q)$ and, in particular, is mapped to the trivial class to $\pi_\ell^{\mathrm{orb}}(M)$. $\square$

Given the homotopy class $a \in \pi_\ell(Q)$ as in Lemma 6.1, we now define

$$\mathcal{P} := \{u : (B^{\ell-1}, S^{\ell-2}) \to (\mathcal{M}, Q_{T_0}) \mid [u] = a\}.$$

We readily see that $\mathcal{P} \neq \emptyset$ since $u_v \in \mathcal{P}$ for any $v : S^\ell \to Q \times \{0\} \times (0, T_0]$ smooth such that $[v] = a$. Obviously, $\mathcal{P}$ is invariant under the complete flow defined in §5.3, provided that $T_0 > 0$ is small enough. The last property of $\mathcal{P}$ that we need is that every element $u \in \mathcal{P}$ has to intersect $\partial\mathcal{V}_\delta$ (more precisely, $\partial\mathcal{V}_{\delta,0}$). This follows trivially from Lemma 6.2.

6.2. *The minimax class for Theorem 1.4.* We adapt the argument in [**26**] to our setting. Consider a tube $G \times_\Gamma U$, where $\Gamma \subset G$ is a stabilizer group and $U \subset Q$ is a contractible slice (i.e., a submanifold which is Γ invariant). From the G-equivariant embedding $G \times_\Gamma U \hookrightarrow Q$, we obtain an embedding

$$U \times_\Gamma EG \cong (G \times_\Gamma U) \times_G EG \hookrightarrow Q \times_G EG = BM,$$

which induces a group homomorphism

$$\rho : \Gamma \cong \pi_1(U \times_\Gamma EG) \to \pi_1(BM) \cong \pi_1^{\mathrm{orb}}(M). \tag{6.3}$$

Such a homomorphism is precisely the homomorphism defined in [**5**, Lemma 2.22]. We deduce that, if M is not developable, we can find a tube $U \times_\Gamma EG$ and a non-trivial element $[\bar{\gamma}]$ in $\pi_1^{\mathrm{orb}}(U)$ that is trivial in $\pi_1^{\mathrm{orb}}(M)$. Consider the diagram with exact rows

$$\begin{tikzcd}
\pi_1(G) \arrow[r] \arrow[d, "="] & \pi_1(G \times_\Gamma U) \arrow[r] \arrow[d] & \pi_1^{\mathrm{orb}}(U) \arrow[r] \arrow[d] & \pi_0(G) \arrow[d, "="] \\
\pi_1(G) \arrow[r] & \pi_1(Q) \arrow[r] & \pi_1^{\mathrm{orb}}(M) \arrow[r] & \pi_0(G)
\end{tikzcd}$$

A simple diagram chase shows that there exists also a class $[\gamma]$ in $\pi_1(G \times_\Gamma U)$ that is trivial in $\pi_1(Q)$. Without loss of generality, we assume that the representative γ is vertical and $X = \theta(\dot{\gamma})$ constant in t. Using the homotopy of γ to a constant loop in Q, we now define a non-trivial minimax class for the functional $\mathbb{S}_k$. More precisely, consider the space of continuous maps

$$\mathcal{P} := \{u : [0, 1] \to \mathcal{M} \mid u(0) = (\gamma, -X, T_0),\ u(1) \in Q_{T_0}\}.$$

Clearly, $\mathcal{P}$ is non-empty and invariant under the flow Φ^k defined in §5.3, provided that $T_0 > 0$ is chosen small enough. The last thing we need to check is that every $u \in \mathcal{P}$ has to intersect $\partial\mathcal{V}_\delta$ for all $\delta > 0$ sufficiently small. For any $x \in [0, 1]$, we write $u(x) = (\gamma_x, \phi_x, T_x)$, set $\overline{\phi}_x$ as in (5.13) and assume by contradiction that $u(x) \in \mathcal{V}_\delta$ for all $x \in [0, 1]$. Exactly as in the proof of Lemma 6.2, we see that, for all $x \in [0, 1]$, we have either $|\overline{\phi}_x| < \sqrt{\delta}$ or $|\overline{\phi}_x| > \Delta - \sqrt{\delta}$. Since $|\overline{\phi}_0| = |X| > \Delta - \sqrt{\delta}$, $|\overline{\phi}_1| = 0$, and the two conditions above are mutually exclusive if δ is small enough, we obtain a contraction to the continuity of the map $x \mapsto |\overline{\phi}_x|$.

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6.3. *End of proofs.* We define the minimax function

$$c : (1/2, +\infty) \to (0, +\infty), \quad c(k) := \inf_{u \in \mathcal{P}} \max \mathbb{S}_k \circ u,$$

where $\mathcal{P}$ is the minimax class defined in §§6.1 and 6.2. By Lemma 5.12, we have $c(k) \geq \epsilon$ for all $k > 1/2$, for every $u \in \mathcal{P}$ has to intersect $\partial \mathcal{V}_\delta$. However, this is not enough to exclude that T_h converges to zero as $h \to +\infty$ for some Palais–Smale sequence (γ_h, Φ_h, T_h) for $\mathbb{S}_k$ at level $c(k)$, as it might be that $c(k) = \langle X, Z \rangle$ for some $X \in (1/N)\Lambda_{\mathfrak{z}}$. For that, we will need the piece of additional information given by the following lemma. For the proof we refer to [**11**, Lemma 5.3].

LEMMA 6.3. *Let u be any element of $\mathcal{P}$. Suppose that $x^* \in B^{\ell-1}$ (respectively, $[0, 1]$) is such that*

$$\mathbb{S}_k(u(x^*)) \geq \max \mathbb{S}_k \circ u - \epsilon/2. \tag{6.4}$$

Then $u(x^*) \notin \bigcup_{X \in (1/N)\Lambda_{\mathfrak{z}}} \{\mathbb{S}_k < -\langle X, Z \rangle + \epsilon/2\} \cap \mathcal{V}_{\delta, X}$.

The function $c(\cdot)$ is monotonically increasing in k and hence almost everywhere differentiable. The next proposition shows there exist Palais–Smale sequences $(\gamma_h, \phi_h, T_h) \subseteq \mathcal{M}$ for $\mathbb{S}_k$ with the T_h bounded away from zero and uniformly bounded, provided k is a point of differentiability for $c(\cdot)$.

PROPOSITION 6.4. *Let k^* be a point of differentiability for $c(\cdot)$. Then there exists a Palais–Smale sequence $(\gamma_h, \phi_h, T_h) \subseteq \mathcal{M}$ for $\mathbb{S}_{k^*}$ such that the T_h are uniformly bounded and bounded away from zero.*

The proof relies on the celebrated *Struwe monotonicity argument* [**36**] and will be omitted since it is a plain adaptation of the proof of [**11**, Proposition 5.4] (see also [**1**, Lemma 8.1], [**17**, Proposition 7.1] and [**8**, Proposition 4.1]), taking into account the fact that $\mathbb{S}_k$ satisfies the Palais–Smale condition on $\mathcal{M}_{[T_*, T^*]}$ only up to gauge transformations.

Proof of Theorems 1.2 and 1.4. Combine Proposition 6.4 with Lemma 5.10. □

Proof of Corollary 1.3. We prove (i). Up to passing to the compact orbifold universal cover, we can assume that $\pi_1^{\mathrm{orb}}(M) = 0$. Notice that, in this case, M must be orientable. Clearly, it suffices to show that M is not rationally aspherical. Suppose, by contradiction, that $\pi_*(BM) \otimes \mathbb{Q}$ is trivial for all $* \geq 2$, where BM is the classifying space of M. By assumption, $\pi_1(BM) = 0$. In particular, BM is simply connected and its minimal model is trivial (cf. [**21**, Theorem 15.11]). This implies that $H^*(BM; \mathbb{Q}) \cong 0$ for all $* \geq 1$. By [**5**, Proposition 2.12], we have $H^n(M; \mathbb{Q}) \cong H^n(BM; \mathbb{Q}) \cong 0$ with $n = \dim M$, which is impossible by Poincaré duality.

Now we prove (ii). Assume, by contradiction, that the universal orbicover $\widetilde{M}$ is rationally aspherical. As in the proof of Lemma 6.1, we see it must have a minimal model (V^*, d) with vanishing differential. Further, by assumption, $V^2 \cong H^2(\widetilde{M}, \mathbb{Q})$ is non-trivial, because the pull-back of σ to $\widetilde{M}$ yields a non-trivial cohomology class. This shows that $V^* \cong H^*(\widetilde{M}, \mathbb{Q})$ is infinite dimensional, which is in contradiction to the assumption that M is a finite-dimensional orbifold. □

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7. *Proof of Theorem 1.5*

Let (M, g_M) be a closed Riemannian orbifold. By Proposition 2.6, there exists a smooth manifold Q equipped with a locally free action of a compact group G such that $Q/G \cong M$ as orbifolds. Without loss of generality, we assume that G is connected. Indeed, if $G_0 \subset G$ denotes the connected component of the neutral element, the quotient Q/G_0 is a finite cover of Q/G and a closed geodesic in the former yields one in the latter. Now consider the metric g_Q on Q associated with g, as explained in §2, and define the functional

$$\mathbb{E} : W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}) \to \mathbb{R}, \quad \mathbb{E}(\gamma, \phi) := \int_0^1 |\dot{\gamma} + \underline{\phi}(\gamma)|^2 \, dt,$$

where $\mathfrak{g}$ denotes the Lie algebra of G and $|\cdot|$ denotes the norm on TQ induced by g_Q. From the discussion in §5, it follows immediately that the functional $\mathbb{E}$ is smooth and bounded from below (by zero). Moreover, it is invariant under gauge transformations, satisfies the Palais–Smale condition up to gauge transformations and its critical points project to a closed geodesic in (M, g_M). In particular, critical points contained in $\mathbb{E}^{-1}(0)$ project to point curves.

Proof of Theorem 1.5. If M is not developable, then we obtain a non-constant closed geodesic on M by literally repeating the proof of Theorem 1.4. Indeed, consider a non-zero class $[\gamma]$ in $\pi_1(G \times_\Gamma U)$ that is trivial in $\pi_1(Q)$. We assume that the representative γ is vertical and that $X = \theta(\dot{\gamma})$ constant in t and we define the following minimax class for the functional $\mathbb{E}$.

$$\mathcal{P}_0 := \{u : [0, 1] \to W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}) \mid u(0) = (\gamma, -X), \ u(1) = (q_0, 0)\}.$$

Clearly, $\mathcal{P}_0$ is non-empty and invariant under the negative gradient flow of $\mathbb{E}$. Moreover, every $u \in \mathcal{P}_0$ has to intersect $\partial\mathcal{V}_\delta$ for all $\delta > 0$ sufficiently small, where

$$\mathcal{V}_\delta := \{(\gamma, \phi) \in W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}) \mid \mathbb{E}(\gamma, \phi) < \delta\}.$$

Therefore, the minimax value

$$c := \inf_{u \in \mathcal{P}_0} \max_{x \in [0,1]} \mathbb{E}(u(x))$$

is strictly larger than δ. This yields the existence of a critical point for $\mathbb{E}$ at level c, and thus of a non-constant closed geodesic in M, as the functional $\mathbb{E}$ satisfies the Palais–Smale condition up to gauge transformations.

If $\pi_1^{\mathrm{orb}}(M)$ is finite, then up to passing to the orbifold universal cover, we see that M is not rationally aspherical and hence the proof follows by the same argument used for Theorem 1.2. Indeed, consider $a \in \pi_\ell(Q) \otimes \mathbb{Q}$ such that $\tau_\ell(a) \neq 0 \in \pi_\ell^{\mathrm{orb}}(M) \otimes \mathbb{Q}$ for some $\ell \geq 2$. Denote by $\mathcal{P}$ the space of continuous maps

$$u : (B^{\ell-1}, S^{\ell-2}) \to (W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g}), Q \times \{0\})$$

representing the homotopy class a. Since every $u \in \mathcal{P}$ has to intersect the boundary of the set $\mathcal{V}_\delta$ for $\delta > 0$ sufficiently small, the existence of the desired non-constant closed geodesic follows by minimax over the class $\mathcal{P}$.

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Finally, assume that $\pi_1^{\mathrm{orb}}(M)$ contains an element of infinite order, say, $\tilde{a}$. Consider the exact homotopy sequence

$$\cdots \to \pi_1(G) \to \pi_1(Q) \to \pi_1^{\mathrm{orb}}(M) \to \pi_0(G) = 1. \tag{7.1}$$

From the surjectivity of the map $\tau_1 : \pi_1(Q) \to \pi_1^{\mathrm{orb}}(M)$, we deduce that there is an element $a \in \pi_1(Q)$ such that $\tau_1(a) = \tilde{a}$. In particular, a has infinite order. Now consider the connected component $\mathcal{C}_a$ of $W^{1,2}(S^1, Q) \times L^2(S^1, \mathfrak{g})$ associated to a. We claim that there exists $\delta > 0$ sufficiently small such that

$$\mathbb{E}(\gamma, \phi) \geq \delta \quad \text{for all } (\gamma, \phi) \in \mathcal{C}_a.$$

Indeed, assume by contradiction that we can find a sequence $(\gamma_h, \phi_h) \subset \mathcal{C}_a$ such that $\mathbb{E}(\gamma_h, \phi_h) \to 0$; then $\gamma = \gamma_h$ lies in some tube $G \times_\Gamma U$, with U contractible, for some h large enough. But this shows that, up to conjugation, a lies in the image of $\pi_1(G \times_\Gamma U) \to \pi_1(Q)$. This shows, in particular, that, up to conjugation, $\tilde{a}$ lies in the image of $\Gamma \cong \pi_1^{\mathrm{orb}}(U) \to \pi_1^{\mathrm{orb}}(M)$ and hence must have finite order, which is in contradiction with our assumption. The existence of the required closed geodesic in M follows now by minimizing the functional $\mathbb{E}$ over the connected component $\mathcal{C}_a$. $\square$

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REFERENCES

[1] A. Abbondandolo. Lectures on the free period Lagrangian action functional. *J. Fixed Point Theory Appl.* **13**(2) (2013), 397–430.

[2] A. Abbondandolo, L. Asselle, G. Benedetti, M. Mazzucchelli and I. A. Taimanov. The multiplicity problem for periodic orbits of magnetic flows on the 2-sphere. *Adv. Nonlinear Stud.* **1** (2017), 1–17.

[3] A. Abbondandolo, L. Macarini, M. Mazzucchelli and G. P. Paternain. Infinitely many periodic orbits of exact magnetic flows on surfaces for almost every subcritical energy level. *J. Eur. Math. Soc. (JEMS)* **19**(2) (2017), 551–579.

[4] A. Abbondandolo, L. Macarini and G. P. Paternain. On the existence of three closed magnetic geodesics for subcritical energies. *Comment. Math. Helv.* **90**(1) (2015), 155–193.

[5] A. Adem, J. Leida and Y. Ruan. *Orbifolds and Stringy Topology*. Cambridge University Press, New York, 2007.

[6] L. Asselle. On the existence of Euler–Lagrange orbits satisfying the conormal boundary conditions. *J. Funct. Anal.* **271** (2016), 3513–3553.

[7] L. Asselle and G. Benedetti. Infinitely many periodic orbits in non-exact oscillating magnetic fields on surfaces with genus at least two for almost every low energy level. *Calc. Var. Partial Differential Equations* **54**(2) (2015), 1525–1545.

[8] L. Asselle and G. Benedetti. The Lusternik-Fet theorem for autonomous Tonelli Hamiltonian systems on twisted cotangent bundles. *J. Topol. Anal.* **8**(3) (2016), 545–570.

[9] L. Asselle and G. Benedetti. On the periodic motions of a charged particle in an oscillating magnetic flows on the two-torus. *Math. Z.* **286**(3–4) (2016), 843–859.

[10] L. Asselle and M. Mazzucchelli. On Tonelli periodic orbits with low energy on surfaces. *Trans. Amer. Math. Soc.* (2017), https://doi.org/10.1090/tran/7185.

[**11**] L. Asselle and F. Schmäschke. On the existence of closed magnetic geodesics via symplectic reduction. *J. Fixed Point Theory Appl.* **20** (2018), article 49.

[**12**] G. Benedetti. The contact property for symplectic magnetic fields on s^2. *Ergod. Th. & Dynam. Sys.* **36**(3) (2016), 682–713.

[**13**] G. Benedetti and K. Zehmisch. On the existence of periodic orbits for magnetic systems on the two-sphere. *J. Mod. Dyn.* **9** (2015), 141–146.

[**14**] A. M. Bloch, J. E. Marsden, P. S. Krishnaprasad and Richard M. Murray. Nonholonomic mechanical systems with symmetry. *Arch. Ration. Mech. Anal.* **136** (1996), 21–99.

[**15**] G. E. Bredon. *Introduction to Compact Transformation Groups (Pure and Applied Mathematics, 46)*. Academic Press, New York, 1972.

[**16**] G. E. Bredon. *Sheaf Theory*, second edn. Springer, New York, 1997.

[**17**] G. Contreras. The Palais–Smale condition on contact type energy levels for convex Lagrangian systems. *Calc. Var. Partial Differential Equations* **27**(3) (2006), 321–395.

[**18**] C. M. Davison and H. R. Dullin. Geodesic flow on three-dimensional ellipsoids with equal semi-axes. *Regul. Chaotic Dyn.* **12**(2) (2007), 172–197.

[**19**] G. C. Dragomir. Closed geodesics on orbifolds of nonpositive or nonnegative curvature. *Geom. Dedicata* **172**(1) (2014), 199–211.

[**20**] Y. N. Fedorov and B. Jovanović. Integrable nonholonomic geodesic flows on compact Lie groups. *Topological Methods in the Theory of Integrable Systems*. Camb. Sci. Publ., Cambridge, 2006, pp. 115–152.

[**21**] Y. Félix, S. Halperin and J.-C. Thomas. *Rational Homotopy Theory*. Springer, New York, 2001.

[**22**] U. Frauenfelder. The Arnold-Givental conjecture and moment Floer homology. *Int. Math. Res. Not. IMRN* **42** (2008), 2179–2269.

[**23**] V. L. Ginzburg. *On Closed Trajectories of a Charge in a Magnetic Field (An Application of Symplectic Geometry, 8)*. Cambridge University Press, Cambridge, 1996, pp. 131–148.

[**24**] V. L. Ginzburg and B. Z. Gürel. Periodic orbits of twisted geodesic flows and the Weinstein–Moser theorem. *Comment. Math. Helv.* **84** (2009), 865–907.

[**25**] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley, New York, 1978.

[**26**] K. Guruprasad and A. Haefliger. Closed geodesics on orbifolds. *Topology* **45**(3) (2006), 611–641.

[**27**] W. Klingenberg. *Lectures on Closed Geodesics*. Springer, Berlin, 1978.

[**28**] W. Klingenberg. *Riemannian Geometry*. De Gruyter, Berlin, New York, 1995.

[**29**] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Wiley Classics Library, New York, 1996.

[**30**] C. Lange. On the existence of closed geodesics on two-orbifolds. *Pacific J. Math.* **294**(2) (2018), 453–472.

[**31**] J. Mardsen and T. Ratiu. *Introduction to Mechanics and Symmetry*. Springer, New York, 1998.

[**32**] A. Yu. Ol'shanskii and M. V. Sapir. Non-amenable finitely presented torsion-by-cyclic groups. *Publ. Math. Inst. Hautes Études Sci.* **96** (2002), 43–169.

[**33**] M. Schneider. Closed magnetic geodesics on S^2. *J. Differential Geom.* **87**(2) (2011), 343–388.

[**34**] M. Schneider. Alexandrov-embedded closed magnetic geodesics on S^2. *Ergod. Th. & Dynam. Sys.* **32**(4) (2012), 1471–1480.

[**35**] M. Schneider. Closed magnetic geodesics on closed hyperbolic Riemann surfaces. *Proc. Lond. Math. Soc. (3)* **105**(2) (2012), 424–446.

[**36**] M. Struwe. Existence of periodic solutions of Hamiltonian systems on almost every energy surface. *Bol. Soc. Brasil. Mat. (N.S.)* **20**(2) (1990), 49–58.

[**37**] I. A. Taimanov. Non-self-intersecting closed extremals of multivalued or not everywhere positive functionals. *Izv. Akad. Nauk SSSR Ser. Mat.* **55**(2) (1991), 367–383.

[**38**] I. A. Taimanov. Closed extremals on two-dimensional manifolds. *Uspekhi Mat. Nauk* **47**(2(284)) (1992), 143–185 223.

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