J. Fixed Point Theory Appl. (2018) 20:49
https://doi.org/10.1007/s11784-018-0528-3
Published online February 20, 2018

Journal of Fixed Point Theory and Applications

On the existence of closed magnetic geodesics via symplectic reduction

Luca Asselle and Felix Schmäschke

Abstract. Let (M, g) be a closed Riemannian manifold and σ be a closed 2-form on M representing an integer cohomology class. In this paper, using symplectic reduction, we show how the problem of existence of closed magnetic geodesics for the magnetic flow of the pair (g, σ) can be interpreted as a critical point problem for a Rabinowitz-type action functional defined on the cotangent bundle T^*E of a suitable S^1-bundle E over M or, equivalently, as a critical point problem for a Lagrangian-type action functional defined on the free loopspace of E. We then study the relation between the stability property of energy hypersurfaces in $(T^*M, dp \wedge dq + \pi^*\sigma)$ and of the corresponding codimension 2 coisotropic submanifolds in $(T^*E, dp \wedge dq)$ arising via symplectic reduction. Finally, we reprove the main result of Asselle and Benedetti (J Topol Anal 8(3):545–570, 2016) in this setting.

Mathematics Subject Classification. 37J45, 58E05.

Keywords. Magnetic flows, periodic orbits, mañé critical values, rabinowitz action functional, symplectic reduction.

Contents

Birkhäuser

1. Introduction

Let (M, g) be a closed Riemannian manifold and let σ be a closed 2-form on M. Up to passing to the orientable double cover of M we can suppose without loss of generality that M is orientable. Consider the kinetic Hamiltonian

$$\bar{H} : T^*M \to \mathbb{R}\,, \qquad \bar{H}(\bar{q}, \bar{p}) = \frac{1}{2}|\bar{p}|^2_{\bar{q}}\,,$$

whereas usual $|\cdot|$ denotes the (dual) norm on T^*M induced by the metric g. Consider also the twisted symplectic form $\bar{\omega}_\sigma = \bar{\omega} + \bar{\pi}^*\sigma$, where $\bar{\omega} = d\bar{p} \wedge d\bar{q}$ is the canonical symplectic form on T^*M and $\bar{\pi} : T^*M \to M$ the canonical projection. The pair $(\bar{H}, \bar{\omega}_\sigma)$ defines a vector field $X^\sigma_{\bar{H}}$ on T^*M by

$$\bar{\omega}_\sigma\big(X^\sigma_{\bar{H}}, \cdot\big) = -d\bar{H},$$

called the Hamiltonian vector field of $\bar{H}$ with respect to $\bar{\omega}_\sigma$. Its flow $\Phi^\sigma_{\bar{H}} : T^*M \to T^*M$ is the *magnetic flow of the pair* (g, σ). The reason of this terminology is that it models the motion of a charged particle in M under the effect of a magnetic field represented by σ. In fact, if $x : I \to T^*M$ is a flow line of $X^\sigma_{\bar{H}}$, then the curve $\mu = \pi \circ x$ satisfies the second-order ordinary differential equation

$$\nabla_t \dot{\mu} = Y_\mu(\dot{\mu})\,, \tag{1.1}$$

where ∇_t denotes the covariant derivative associated with g and $Y : TM \to TM$ is the linear bundle map (known as *Lorentz force*) given by

$$g_q(u, Y_{\bar{q}}(v)) = \sigma_{\bar{q}}(u, v), \qquad \forall\, u, v \in T_{\bar{q}}M,\ \forall \bar{q} \in M.$$

Conversely, given a solution $\mu : I \to M$ of (1.1), the lift $x = (\mu, p_\mu) : I \to T^*M$ is a flow line of $X^\sigma_{\bar{H}}$, where p_μ is the g-dual of $\dot{\mu}$.

Periodic orbits of such a flow are usually called *closed magnetic geodesics*. The magnetic flow preserves $\bar{H}$, since it is the Hamiltonian of the system; therefore, it makes sense to look at periodic orbits on a given level set. In this paper, we will be interested in the following problem: given $\bar{k} > 0$, does there exist a period $T > 0$ and a curve $x : \mathbb{R} \to T^*M$ which satisfies the following conditions?

$$\begin{cases} \dot{x}(t) = X^\sigma_{\bar{H}}(x(t))\,; \\ x(T) = x(0)\,; \\ \bar{H}(x) = \bar{k}\,. \end{cases} \tag{1.2}$$

A particular case of magnetic flow is given by the choice $\sigma = 0$, in which case we retrieve the *geodesic flow* of (M, g). The problem of the existence of closed geodesics has received in the last century the attention of many outstanding mathematicians as Birkhoff, Lyusternik, Gromoll and Meyer, just to mention few of them. The existence of periodic orbits for magnetic flows represents a natural generalization of the closed geodesic problem. However, unlike the geodesic case, the dynamics in the magnetic setting turns out to depend essentially on the kinetic energy of the particle. This is one of the reasons why existence results for closed geodesics cannot be straightforward generalized to the magnetic setting. In fact, Hedlund [2] provided an example of a “critical” energy level without closed magnetic geodesics on any surface with genus at least two. On the other hand, almost every energy level contains at least one closed magnetic geodesic (cf. [1] and references therein).

In the literature, various approaches and techniques, coming for instance from the classical calculus of variations [3–8], symplectic geometry [9–16], symplectic homology [17] and contact homology [18], are used to tackle the problem of existence of closed magnetic geodesics. See also [19–22] for existence results based on a minimization procedure in case the configuration space is two dimensional. In particular, for magnetic flows defined by an exact 2-form $\sigma = d\theta$ the existence of closed magnetic geodesics can be shown using a variational characterization of periodic orbits as critical points of the free-period Lagrangian action functional (see, e.g., [3,7]). If one tries to generalize this approach dropping the exactness assumption, then one has to overcome the difficulty given by the fact that the action functional is not well defined but rather "multi-valued". Nevertheless, following ideas contained in [23–25], progresses in this direction have been recently made in [1,26] by studying the existence of zeros of the action 1-form.

In this paper, we use another approach to study the existence of solutions to (1.2) based on the following remark: the twisted cotangent bundle arises naturally via symplectic reduction (cf. [27, Ex. 5.2] or [28, Sect. 6.6]). If σ represents an integer cohomology class, then this allows to interpret the magnetic flow as a geodesic flow on the cotangent bundle of a suitable S^1-bundle E over M, at the cost of introducing a symmetry group. In particular, closed magnetic geodesics with energy $\bar{k}$ turn out to correspond to the critical points of a Rabinowitz-type action functional

$$\mathbb{A}_k : C^\infty(S^1, T^*E) \times (0, +\infty) \times \mathbb{R} \to \mathbb{R}$$

or equivalently, using the Legendre transform, to the critical points of a Lagrangian-type action functional

$$\mathbb{S}_k : H^1(S^1, E) \times (0, +\infty) \times \mathbb{R} \to \mathbb{R}.$$

Here, $k = \bar{k} + \frac{1}{2}$ and $H^1(S^1, E)$ denotes the Hilbert manifold of absolutely continuous loops in E with square-integrable derivative. Notice that the correspondence between closed magnetic geodesics and critical points of $\mathbb{A}_k$ would allow to use a version of Rabinowitz-Floer homology for contact type (or, at least, stable) coisotropic submanifolds—as developed by Kang [29]—to infer existence on a given energy level. To this purpose, it is important to study the stability property of such coisotropic submanifolds, also in relation with the stability property of the corresponding hypersurfaces in T^*M. This will be carried over in Sect. 3, where we also provide some concrete examples. In the last part of the paper, building on the latter correspondence, we reprove the main theorem of [1] in the setting of magnetic flows given by closed 2-forms representing an integer cohomology class.

Theorem 1.1. *Let (M, g) be a closed non-aspherical Riemannian manifold, i.e., $\pi_\ell(M) \neq 0$ for some $\ell \geq 2$, and σ be a closed 2-form on M representing an integer cohomology class. Then for almost every $\bar{k} > 0$, there exists a contractible closed magnetic geodesic with energy $\bar{k}$.*

We end this introduction by giving a summary of the contents of this paper: In Sect. 2, we recall how the magnetic flow can be seen as a projected geodesic flow and introduce the functional $\mathbb{A}_k$. In Sect. 3, we discuss the relation between stability and contact property of energy hypersurfaces and of the corresponding coistropic submanifolds arising via symplectic reduction. In Sect. 4, we introduce the functional $\mathbb{S}_k$ and study its properties. In Sect. 5, we prove Theorem 1.1.

2. Symplectic reduction

2.1. The magnetic flow as a projected geodesic flow

Let (M,g) be a closed orientable Riemannian manifold and let σ be a closed 2-form on M. We call the pair $(T^*M, \bar{\omega}_\sigma := d\bar{p} \wedge d\bar{q} + \bar{\pi}^*\sigma)$ the *twisted cotangent bundle*. It has been known for a long time that twisted cotangent bundles arise via symplectic reduction (cf. for example [27, Ex. 5.2]). Here, we quickly recall this construction.

Throughout this paper, we assume that the deRahm cohomology class represented by σ is integral, i.e., $[\sigma] \in H^2(M;\mathbb{Z})$. Let $S^1 = \{e^{it} \in \mathbb{C} \mid t \in \mathbb{R}\}$ be the Lie group of complex numbers of norm one. If σ represents an integral cohomology class, then there is a principal S^1-bundle $\tau : E \to M$ with Euler class $e(E) = [\sigma] \in H^2(M;\mathbb{Z})$.

Recall that the Euler class is defined as follows: choose a *connection 1-form* $\theta \in \Omega^1(E)$, which is an S^1-invariant 1-form satisfying $\theta(Z) = 1$, where Z denotes the fundamental vector field of the S^1-action

$$Z_q = \frac{d}{dt}\, e^{it} q\Big|_{t=0} \in T_qE, \qquad \forall q \in E\,.$$

The form θ induces a splitting of the tangent bundle

$$TE = \ker\theta \oplus \mathbb{R}\cdot Z, \tag{2.1}$$

(vectors in $\ker\theta$ are called *horizontal*), and uniquely defines a *curvature form* $\tilde{\sigma} \in \Omega^2(M)$ by

$$\tilde{\sigma}_{\bar{q}}(u,v) = (d\theta)_q(u^{\mathrm{hor}}, v^{\mathrm{hor}})\,,$$

where $u,v \in T_{\bar{q}}M$, $\bar{q} \in M$, $q \in \tau^{-1}(\bar{q})$ and $u^{\mathrm{hor}}, v^{\mathrm{hor}} \in T_qE$ are horizontal vectors that project to u,v via $d_q\tau$, respectively (called *horizontal lift*). Obviously, $d\tilde{\sigma} = 0$. The *Euler class* is defined as the cohomology class represented by $\tilde{\sigma}$. To see that $[\tilde{\sigma}]$ does not depend on the choice of θ, one shows that any another connection form θ' must satisfy $\theta' = \theta + \tau^*\beta$ for some $\beta \in \Omega^1(M)$. The curvature of θ' is, therefore, $\tilde{\sigma} + d\beta$ and hence defines the same cohomology class. Notice that this also shows that the map $\theta \mapsto \tilde{\sigma}$ from the space of connection 1-forms to the space of closed forms on M representing the cohomology class $e(E)$ is surjective. In particular, for a given closed 2-form σ on M representing an integer cohomology class we can always find a connection 1-form θ such that $d\theta = \tau^*\sigma$.

By push-forward the S^1-action on E lifts canonically to an S^1-action on T^*E

$$T^*E \to T^*E, \quad (q,p) \mapsto \left(\mathrm{e}^{it}q, p\cdot (d_q e^{it})^{-1}\right).$$

It is a classical fact (see for instance [30]) that this action on T^*E is the Hamiltonian flow with respect to the standard symplectic structure of the Hamiltonian

$$A : T^*E \longrightarrow \mathbb{R}\,, \qquad (q,p) \longmapsto \langle p, Z_q\rangle\,.$$

Since the action is free, for every $c\in\mathbb{R}$ the symplectic quotient is well defined

$$T^*E/\!\!/_c\, S^1 := A^{-1}(c)/S^1\,.$$

This quotient manifold is naturally endowed with a symplectic form $\bar\omega_c$, which is defined as the unique form such that $\mathrm{pr}^*\bar\omega_c = \imath^*\omega$, where $\imath : A^{-1}(c) \hookrightarrow T^*E$, $\mathrm{pr} : A^{-1}(c) \to T^*E/\!\!/_c\, S^1$ and ω denote, respectively, the natural inclusion, the projection map and the standard symplectic form on T^*E. Fix a connection form θ and define a map $\Pi_c : A^{-1}(c) \to T^*M$ implicitly via

$$\langle \Pi_c(q,p), d_q\tau\, v\rangle = \langle p, v\rangle - c\,\theta(v), \qquad \forall\; v\in T_qE\,. \tag{2.2}$$

Note that Π_c is well defined because the kernel of $d_q\tau$ is spanned precisely by the fundamental vector field, on which the right-hand side vanishes. Moreover, it is not hard to see that Π_c is a bundle map with fibres consisting of S^1-orbits for the lifted S^1-action. We conclude that Π_c induces a diffeomorphism $T^*E/\!\!/_c S^1 \cong T^*M$.

Proposition 2.1. *For all $c\in\mathbb{R}$, the map Π_c induces a symplectomorphism*

$$(T^*E/\!\!/_c S^1, \bar\omega_c) \cong (T^*M, \bar\omega + c\bar\pi^*\sigma)\,.$$

Proof. We need to show that $\Pi_c^*(\bar\omega + c\bar\pi^*\sigma) = i^*\omega$. Since we have $\bar\pi\circ\Pi_c = \tau\circ\pi$, we conclude that

$$\Pi_c^*\bar\pi^*\sigma = \pi^*\tau^*\sigma = \pi^* d\theta = d\pi^*\theta\,.$$

Hence, it suffices to see that $\Pi_c^*\bar\lambda + c\pi^*\theta = i^*\lambda$, where $\bar\lambda$, λ are the Liouville forms in T^*M and T^*E, respectively. For any $v\in T_{(q,p)}A^{-1}(c)$, we denote $(\bar q,\bar p) = \Pi_c(q,p)$ and compute

$$(\Pi_c^*\bar\lambda)_{(q,p)}(v) = \langle \bar p, d\bar\pi d\Pi_\theta v\rangle = \langle \bar p, d\tau d\pi v\rangle\,,$$

and using the definition (2.2) we continue the computation

$$(\Pi_c^*\bar\lambda)_{(q,p)}(v) = \langle p, d\pi v\rangle - c\theta(d\pi v) = \lambda_{q,p}(v) - c(\pi^*\theta)_{q,p}(v)\,.$$

This shows the claim. $\square$

Fix a connection form θ for σ and lift the metric on M to a metric on E via $g^\theta := \tau^*g + \theta\otimes\theta$. In other words, consider the unique metric on E such that:

- $d_q\tau : \ker\theta_q \to T_{\tau q}M$ is an isometry for all $q\in E$.
- $g^\theta(X,X) = 1$,
- the splitting (2.1) is orthogonal.

By abuse of notation, we denote again the (dual) norm on T^*E induced by g^θ with $|\cdot|$ and also the kinetic Hamiltonian with

$$H : T^*E \longrightarrow \mathbb{R}, \qquad H(q,p) = \frac{1}{2}|p|_q^2.$$

Since by construction the metric g^θ is S^1-invariant, the Hamiltonian flow of H commutes with the Hamiltonian flow of A. In particular, the flow of H preserves the levels of A and via Proposition 2.1 projects to a Hamiltonian flow on $(T^*M, \bar{\omega}_\sigma)$. We show now that this reduced flow is precisely the magnetic flow.

Lemma 2.2. *We have $H = \bar{H} \circ \Pi_1 + \frac{1}{2}$ and $d\Pi_1 X_H = X^\sigma_{\bar{H}}$. In particular, a curve $\bar{x} : \mathbb{R} \to T^*M$ that satisfies (1.2) for some $T > 0$ lifts to a curve $x : \mathbb{R} \to T^*E$ with*

$$\begin{cases} \dot{x}(t) = X_H(x(t))\,; \\ x(T) = e^{i\varphi}x(0)\,; \\ H(x) = \bar{k} + \frac{1}{2}\,; \\ A(x) = 1\,, \end{cases} \tag{2.3}$$

for some $\varphi \in \mathbb{R}$. Conversely, a curve $x : \mathbb{R} \to T^*E$ satisfying (2.3) projects to a closed magnetic geodesic with energy $\bar{k}$.

Proof. Given any $(q,p) \in A^{-1}(1)$ and $v \in T_qE$. Set $(\bar{q},\bar{p}) := \Pi_1(q,p)$ and $\bar{v} := d_q\tau v$. Splitting into horizontal and vertical components, we conclude by (2.2)

$$\langle p, v\rangle = \langle p, v^{\mathrm{hor}}\rangle + \langle Z, v\rangle, \quad \langle p, v^{\mathrm{hor}}\rangle = \langle \bar{p}, \bar{v}\rangle.$$

Hence by definition of the dual norm

$$|p| = \max_{|v|^2=1} \langle p, v\rangle = \max_{x\in[-1,1]} \max_{|v^{\mathrm{hor}}|=\sqrt{1-x^2}} \langle p, v^{\mathrm{hor}}\rangle + x = \max_x \sqrt{1-x^2}|\bar{p}| + x\,.$$

By maximization in the x variable we verify $|p| = \sqrt{|\bar{p}|^2+1}$. This shows $H = \bar{H} \circ \Pi_1 + \frac{1}{2}$. The rest follows since by Proposition 2.1 we have $\Pi_1^*\bar{\omega}_\sigma = i^*\omega$. □

2.2. A Rabinowitz-type action functional

Lemma 2.2 above shows that, in order to find closed magnetic geodesics with energy $\bar{k}$, it suffices to look for geodesics in T^*E with kinetic energy $\bar{k} + \frac{1}{2}$ that are closed up to S^1-action and which lie on the level set $A^{-1}(1)$. For our variational approach, we reformulate (2.3) into a problem of closed curves with period 1. More precisely, if (x, T, φ) is a solution of (2.3), then the curve $y : [0,1] \to T^*E$ defined by $y(t) := \mathrm{e}^{-it\varphi}x(tT)$ satisfies

$$\begin{cases} \dot{y}(t) = -\varphi X_A(y(t)) + TX_H(y(t))\,; \\ y(1) = y(0)\,; \\ H(y) = \bar{k} + \frac{1}{2}\,; \\ A(y) = 1\,. \end{cases} \tag{2.4}$$

Conversely, every solution of (2.4) gives a solution of (2.3) by reversing the rescaling.

Lemma 2.3. *Set* $k := \bar{k} + \frac{1}{2}$. *A triple* (y, T, φ) *satisfies* (2.4) *if and only if it is a critical point of the functional* $\mathbb{A}_k : C^\infty(S^1, T^*E) \times (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *given by*

$$\mathbb{A}_k(y, T, \varphi) = \int_0^1 y^*\lambda \; - \int_0^1 \big(TH_k(y) - \varphi A_1(y)\big)\, dt\,, \tag{2.5}$$

where λ *is the Liouville* 1*-form,* $H_k(q, p) := H(q, p) - k$ *and* $A_1(q, p) := A(q, p) - 1$.

Proof. Let $s \mapsto u_s \in C^\infty(S^1, T^*E)$ be a differentiable curve with $u_0 = y$ and

$$\xi := \left.\frac{d}{\mathrm{d}s}\right|_{s=0} u_s.$$

Abbreviate the Hamiltonian $\widehat{H} := TH_k - \varphi A_1$ and use

$$\omega(\partial_s u, \partial_t u) = (d\lambda)(\partial_s u, \partial_t u) = \partial_s \lambda(\partial_t u) - \partial_t \lambda(\partial_s u)$$

to conclude that

$$\begin{aligned} \mathrm{d}\mathbb{A}_k(y)[\xi] &= \int_0^1 \omega(\xi, \dot{y}) - \int_0^1 \mathrm{d}\widehat{H}(\xi)\, \mathrm{d}t \\ &= \int_0^1 \omega(\xi, \dot{y}) + \int_0^1 \omega(X_{\widehat{H}}(y), \xi)\, \mathrm{d}t = \int_0^1 \omega(\xi, \dot{y} - X_{\widehat{H}}(y))\, \mathrm{d}t\,. \end{aligned}$$

If (y, T, φ) solves (2.4), then clearly $\mathrm{d}\mathbb{A}_k(y)[\xi] = 0$ for all ξ. On the other hand, if $\mathrm{d}\mathbb{A}_k(y)[\xi] = 0$ for all ξ, then by the fundamental lemma of calculus of variations and by non-degeneracy of ω the curve y has to solve the first equation in (2.4). Differentiating $\mathbb{A}_k$ in direction T and φ shows that

$$\frac{\partial \mathbb{A}_k}{\partial T}(y, T, \varphi) = -\int_0^1 H_k(y), \qquad \frac{\partial \mathbb{A}_k}{\partial \varphi}(y, T, \varphi) = \int_0^1 A_1(y)\,.$$

Now it is clear that $\partial\mathbb{A}_k/\partial T(y, T, \varphi) = \partial\mathbb{A}_k/\partial\varphi(y, T, \varphi) = 0$ if (y, T, φ) is a solution of (2.4). On the other hand, if (y, T, φ) is a critical point of $\mathbb{A}_k$, then H and A are constant along y (since they Poisson commute) and hence $H(y) = k$ and $A(y) = 1$ as required. □

The functional $\mathbb{A}_k$ in (2.5) can be thought of as the classical Rabinowitz action functional (cf. [31–33]) with two Lagrange multipliers instead of only one and fits precisely in the setting considered in [29], where Rabinowitz-Floer homology for contact coisotropic submanifolds is defined. Notice indeed that, in the setting of the lemma above, $\Sigma := H^{-1}(k) \cap A^{-1}(1)$ is a coisotropic submanifold of T^*E of codimension 2, for the Hamiltonians H and A Poisson-commute. Therefore, it is not unreasonable to try to use Rabinowitz-Floer homology to infer existence results of critical points of the functional $\mathbb{A}_k$. However, this is very far from being a straightforward application of the results in [29]. Indeed, the coisotropic submanifold Σ is in general not of contact type (cf. Sect. 3), even though all energy level sets of H are trivially of contact type on (T^*E, ω). Notice that the latter fact is in sharp contrast with what happens on $(T^*M, \bar{\omega}_\sigma)$, where very little is known about the contact property for energy level sets of the kinetic Hamiltonian. In fact, low energy levels on surfaces different from the two-torus are known to be not of contact type, in case for instance σ is an exact form (cf. [34, Theorem 1.1]); it is, however, an open problem to determine whether such energy levels are stable or not. We refer to [35] for the definition of stability and (for instance) to [3, Corollary 8.4] for the relation between the stability property and the existence of periodic orbits. Analogously, one could ask whether the coisotropic submanifold Σ is stable or not. This will be done in the next section.

We finish this section noticing that we might not expect the existence of critical points of $\mathbb{A}_k$ for every k, as the example of the horocycle flow [2] shows.

3. Stability and contact property of coisotropic submanifolds

In the previous section, we showed that, in order to prove the existence of solutions to (1.2), it suffices to show the existence of 1-periodic orbits for the Hamiltonian flow defined by the Hamiltonian $T\cdot H - \varphi\cdot A : T^*E \to \mathbb{R}$, for some $T > 0$, $\varphi \in \mathbb{R}$, and the standard symplectic form on T^*E which are contained in the coisotropic submanifold $\Sigma := H^{-1}(k) \cap A^{-1}(1)$ or, equivalently, to show the existence of critical points of the Rabinowitz-type action functional $\mathbb{A}_k$ given by (2.5). In order to potentially apply the techniques developed in [29], we first need to know that Σ is of contact type or, at least, stable.

Let us first recall the notions of contact type, resp. stable coisotropic submanifold, which were introduced by Bolle [36,37]. For examples of stable resp. contact type coisotropic submanifolds, we refer to [29]. Other examples in the setting considered in the present paper will be discussed in the next subsections.

Definition 3.1. Let (Y^{2m}, ω) be a symplectic manifold and let $H_0, \dots, H_{k-1} : Y \to \mathbb{R}$ be Poisson-commuting Hamiltonians such that zero is a regular value for each function and such that the intersection of the zero-energy level sets of $H_0, \dots, H_{k-1}$

$$\Sigma := \bigcap_{j=0}^{k-1} H_j^{-1}(0)$$

is cut-out transversely. Then, Σ is a $(2m-k)$-dimensional coisotropic submanifold. The coisotropic submanifold Σ is called stable if there exist one-forms $\alpha_0, \dots, \alpha_{k-1}$ such that $\ker \omega_\Sigma \subseteq \ker d\alpha_j$, for all $j = 0, \dots, k-1$, and

$$\alpha_0 \wedge \cdots \wedge \alpha_{k-1} \wedge \omega_\Sigma^{2(m-k)} \neq 0$$

everywhere on Σ, where ω_Σ denotes the restriction of ω to Σ. We say that Σ is of contact type if the stabilizing forms $\alpha_0, \dots, \alpha_{k-1}$ can be chosen within the set of primitives of ω_Σ.

Obviously a necessary condition for Σ to be of contact type is that the restricted symplectic form ω_Σ is exact. Furthermore, being of contact type for closed coisotropic submanifolds of codimension higher than one is also topologically obstructed, as the next lemma shows.

Lemma 3.2. *Let Σ be a closed k-codimensional coisotropic submanifold of (Y^{2m}, ω). If Σ is of contact type, then* $\dim H^1(\Sigma, \mathbb{R}) \geq k-1$.

Proof. Suppose by contradiction that Σ is contact and $\dim H^1(\Sigma, \mathbb{R}) < k-1$. Let $\alpha_0, \dots, \alpha_{k-1}$ be primitives of ω_Σ satisfying the requirements of the definition above. The differences $\alpha_1 - \alpha_0, \dots, \alpha_{k-1} - \alpha_0$ are closed one-forms and the corresponding cohomology classes are, by assumption, linearly dependent. Consequently, there exist coefficients $\lambda_1, \dots, \lambda_{k-1} \in \mathbb{R}$ not all equal to zero such that

$$\lambda_1(\alpha_1 - \alpha_0) + \cdots + \lambda_{k-1}(\alpha_{k-1} - \alpha_0) = \mathrm{d}f,$$

for some function $f : \Sigma \to \mathbb{R}$. Assume without loss of generality that $\lambda_1 = 1$. We rewrite the last equation as:

$$\alpha_1 = \mathrm{d}f + \alpha_0 + \sum_{j=2}^{k-1} \lambda_j (\alpha_0 - \alpha_j)\,.$$

Plugging the last equation into the wedge product yields

$$\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \omega_\Sigma^{2(m-k)} = \alpha_0 \wedge \mathrm{d}f \wedge \alpha_2 \wedge \cdots \wedge \alpha_{k-1} \wedge \omega_\Sigma^{2(m-k)}\,.$$

Since Σ is closed, f has at least one critical point and hence

$$\alpha_0 \wedge \cdots \wedge \alpha_{k-1} \wedge \omega_\Sigma^{2(m-k)}$$

cannot be a volume form on Σ. □

3.1. Coisotropic submanifolds arising via symplectic reduction

Let us return to the case we are interested in, i.e., when $\Sigma_k = H^{-1}(k) \cap A^{-1}(1)$, for $k > 1/2$. In Lemma 3.5 below, we relate the stability and contact type condition for Σ_k to the corresponding conditions for the hypersurface $\bar{\Sigma} = H^{-1}(\bar{k}) \subset T^*M$. To this purpose, we first need to verify that Σ_k is cut-out transversely.

Lemma 3.3. *If $k > 1/2$, then Σ_k is cut-out transversely.*

Proof. Assume by contradiction that there exist $p \in \Sigma_k$ where d_pH_k and d_pA_1 are linearly dependent, i.e., $\lambda_1 d_pH_k + \lambda_2 d_pA = 0$ for some coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$ not both equal to zero. Let $\pi : T^*E \to E$ be the projection. Vectors in $\ker d_p\pi$ are canonically identified with T_q^*E where $q = \pi(p)$ and with that identification in mind we conclude that for any vector $\xi \in \ker d_p\pi$ we have $d_pH_k(\xi) = \langle p, \xi \rangle$ and $d_pA_1(\xi) = \langle Z_q, \xi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes in the first equation the duality pairing and in the second equation the dual metric. Thus, $0 = \lambda_1 \langle p, \xi \rangle + \lambda_2 \langle Z_q, \xi \rangle$ for any vertical vector ξ, which implies that $0 = \lambda_1 p + \lambda_2 \zeta$, where ζ is the dual of Z_q. Hence

$$\begin{aligned} 0 = |\lambda_1 p + \lambda_2 \zeta|^2 &= \lambda_1^2 |p|^2 + 2\lambda_1\lambda_2 \langle p, \zeta \rangle + \lambda_2^2 |\zeta|^2 \\ &= \lambda_1^2 (2k) + 2\lambda_1 \lambda_2 + \lambda_2^2 \\ &= (\lambda_1 + \lambda_2)^2 + (2k-1)\lambda_1^2\,. \end{aligned}$$

Since $2k-1$ is positive, both summands in the last expression must vanish. This shows that $\lambda_1 + \lambda_2 = \lambda_1 = 0$ in contradiction with the assumption. □

We now fix $k > 1/2$ and set $\Sigma = \Sigma_k$ for the rest of the section. Recall that Σ is stable if there exist one-forms α_0, α_1 on Σ such that $\ker \omega_\Sigma \subseteq \ker d\alpha_i$, $i = 0, 1$, and

$$\alpha_0 \wedge \alpha_1 \wedge \omega_\Sigma^{n-1} \neq 0\,, \tag{3.1}$$

where n denotes the dimension of M. Moreover, Σ is of contact type if the stabilizing forms α_0 and α_1 are additionally primitives of ω_Σ. Note that by Lemma 3.2, if Σ is of contact type then we must have $H^1(\Sigma, \mathbb{R}) \neq 0$.

The next lemma provides a criterion for the contact property of Σ in terms of the Hamiltonian vector fields of H and A, denoted X_0 and X_1, respectively. A similar statement holds clearly also for the stability condition.

Lemma 3.4. *The submanifold Σ is contact if and only if there exist primitives α_0, α_1 of ω_Σ such that the following matrix is non-singular on Σ:*

$$\begin{pmatrix} \alpha_0(X_0) & \alpha_0(X_1) \\ \alpha_1(X_0) & \alpha_1(X_1) \end{pmatrix}. \tag{3.2}$$

Proof. The two-form ω_Σ has kernel on Σ generated exactly by the Hamiltonian vector fields X_0 and X_1. In particular, the matrix (3.2) is non-singular everywhere on Σ if and only if the contraction of the form in (3.1) by X_0 and X_1 is non-zero on the complement of $\ker \omega_\Sigma$. □

We now come the main result of the section. We denote by $\bar\Sigma = \bar H^{-1}(\bar k)$ the sphere bundle in T^*M.

Lemma 3.5. *The following statements hold:*

(i) *If $\bar\Sigma$ is of contact type in $(T^*M, \bar\omega_\sigma)$, then Σ is of contact type in (T^*E, ω).*

(ii) *The space $\bar\Sigma$ is stable in $(T^*M, \bar\omega_\sigma)$ if and only if Σ is stable in (T^*E, ω).*

Proof. (i) Let $\bar\alpha$ be a contact form for $\bar\Sigma$ and consider $\alpha_1 := \pi^*\bar\alpha$, $\alpha_0 = \lambda_\Sigma$ restriction to Σ of the Liouville 1-form on T^*E. By definition, we have

$$\omega_\Sigma = \pi^*\bar\omega_\sigma|_{\bar\Sigma} = \pi^* d\bar\alpha = d\alpha_1 = d\alpha_0.$$

By construction, we have $d\pi X_0 = \bar X$, $d\pi X_1 = 0$, where $\bar X$ denotes the Hamiltonian vector field defined by the kinetic Hamiltonian and the twisted symplectic form on T^*M. It follows by the contact condition that

$$\alpha_0(X_1) \equiv 1, \quad \alpha_1(X_0) = \bar\alpha(\bar X) \neq 0, \quad \alpha_1(X_1) = \bar\alpha(0) = 0,$$

and hence the matrix (3.2) is nowhere singular on Σ.

(ii) Suppose now that $\bar\Sigma$ is stable with stabilizing form $\bar\alpha$ and consider the one-forms α_0, α_1 on Σ as above. It suffices to show that $\ker \omega_\Sigma \subseteq \ker d\alpha_1$. By the stability property of $\bar\Sigma$, we know that any vector $v \in \ker \omega_\Sigma$ projects to a vector in $\ker d\bar\alpha$, since $\bar v := d\pi v \in \ker \bar\omega_\sigma|_{\bar\Sigma} \subseteq \ker d\bar\alpha$. It follows that for all $w \in T\Sigma$ we have

$$(d\alpha_1)(v, w) = d\pi^*\bar\alpha(v, w) = \pi^* d\bar\alpha(v, w) = d\bar\alpha(\bar v, \bar w) = 0$$

and hence $v \in \ker d\alpha_1$. Conversely, suppose that Σ is stable and let β_0, β_1 be a stabilizing pair for Σ. Starting from β_0, β_1, we define a new stabilizing pair β_0', β_1' for Σ which is invariant under the flow of X_1 (denoted by ϕ_1^t) by

$$\beta_i'(v) := \int_0^1 (\phi_1^t)^* \beta_i[v]\, \mathrm{d}t, \quad \forall v \in T_p\Sigma,\ p \in \Sigma,\ i = 0, 1.$$

Since

$$\mathrm{d}\beta_i' = \int_0^1 (\phi_1^t)^* d\beta_i \, \mathrm{d}t$$

and ϕ_1 preserves $\ker \omega_\Sigma$ (since it preserves ω_Σ), we have that $\ker \omega_\Sigma \subseteq \ker \mathrm{d}\beta_i'$, for $i = 0, 1$. Moreover, since by assumption $\beta_0 \wedge \beta_1 \wedge \omega_\Sigma^{n-2} \neq 0$, we can conclude that $\beta_0' \wedge \beta_1' \wedge \omega_\Sigma^{n-2} \neq 0$. By construction, we have $(\phi_1^t)^* \beta_i' = \beta_i'$ for all $t \in \mathbb{R}$, for $i = 0, 1$. Deriving in t and evaluating at $t = 0$ yields

$$0 = \frac{d}{\mathrm{d}t} (\phi_1^t)^* \beta_i' \Big|_{t=0} = \mathcal{L}_{X_1} \beta_i' = d(\imath_{X_1} \beta_i') + \imath_{X_1} \mathrm{d}\beta_i' = d(\imath_{X_1} \beta_i'). \tag{3.3}$$

This shows that the functions $\beta_0'(X_1)$ and $\beta_1'(X_1)$ are constant along Σ. We set $b_0 := \beta_0'(X_1)$, $b_1 := \beta_1'(X_1)$, and denote by $\Pi : \Sigma \to \bar{\Sigma}$ the quotient map. Finally, we define a 1-form $\bar{\beta}$ implicitly via $\Pi^* \bar{\beta} = b_1 \beta_0' - b_0 \beta_1'$, i.e.,

$$\bar{\beta}_{\bar{p}}(\bar{v}) := b_1 \cdot (\beta_0')_p(v) - b_0 \cdot (\beta_1')_p(v),$$

for all $p \in \Sigma$ in the fibre over $\bar{p}$ and $v \in T_p\Sigma$ such that $d_p\Pi v = \bar{v}$. Notice that this is a good definition since β_0' and β_1' are ϕ_1^t-invariant and by construction the right-hand side vanishes on the kernel of $d\Pi$, which is spanned by the vector field X_1. Since $d\Pi X_0 = \bar{X}$ we conclude

$$\bar{\beta}(\bar{X}) = b_1 \beta_0'(X_0) - b_0 \beta_1'(X_0) = \det(\beta_i'(X_j)) \neq 0\,,$$

which implies that that $\ker \bar{\omega}_\sigma|_{\bar{\Sigma}} \subseteq \ker d\bar{\beta}$. □

Remark 3.6. The contact condition for Σ is in general weaker than the contact condition for $\bar{\Sigma}$ as the following example shows. Consider the flat torus $(\mathbb{T}^2, g)$ and let σ be the area form induced by g. Then, energy levels $\bar{H}^{-1}(\bar{k})$ are stable in $(T^*\mathbb{T}^2, \bar{\omega}_\sigma)$ for every $\bar{k} > 0$ with stabilizing form given by the angular form $d\theta$ but never of contact type, for the 2-form $\pi^*\sigma|_{\bar{H}^{-1}(\bar{k})}$ is never exact (in fact, the map $\pi^* : H^2(\mathbb{T}^2) \to H^2(\bar{H}^{-1}(\bar{k}))$ is injective). However, the associated coisotropic submanifold Σ in T^*E is of contact type with contact forms given by α_0 and $\alpha_1 := \alpha_0 + \tau^* d\theta$, where α_0 denotes the restriction of the Liouville 1-form to Σ.

Arguing as in the proof of Statement ii in Lemma 3.5, we see that $\bar{\Sigma}$ is of contact type in $(T^*M, \bar{\omega}_\sigma)$, provided that Σ is of contact type in (T^*E, ω) and the constants b_0, b_1 satisfy $b_1 - b_0 = 1$.

3.2. Examples

From Lemma 3.5, we deduce that all examples of stable resp. contact type hypersurfaces in $(T^*M, \bar{\omega}_\sigma)$ discussed in [9] give rise to examples of stable, resp. contact type coisotropic submanifolds in (T^*E, ω). From [9], we also get examples of non-stable coisotropic submanifolds. We now explain another class of positive examples arising from compact coadjoint orbits. Before stating the result, we need to recall some basic facts about coadjoint orbits.

Let G be a compact Lie group acting on the dual $\mathfrak{g}^\vee$ of its Lie algebra $\mathfrak{g}$ via the coadjoint action. Fix any $\zeta \in \mathfrak{g}^\vee$ and denote the *coadjoint orbit*

$$M := G \cdot \zeta := \{Ad_g^* \zeta \mid g \in G\} \subset \mathfrak{g}^\vee .$$

The space M is an embedded submanifold diffeomorphic to G/G_ζ where $G_\zeta := \{g \in G \mid Ad_g^* \zeta = \zeta\}$ is the isotropy group. Let $\mathfrak{g}_\zeta = \{X \in \mathfrak{g} \mid ad_X^* \zeta = 0\}$ its Lie-algebra. We fix an $Ad(G)$-invariant positive bilinear form B on $\mathfrak{g}$, which is possible because G is assumed to be compact. Taking the orthogonal complement with respect to B, we obtain an $Ad(G_\zeta)$-invariant splitting $\mathfrak{g} = \mathfrak{g}_\zeta \oplus \mathfrak{m}$ which induces the isomorphism

$$TM \cong G \times_{G_\zeta} \mathfrak{m} . \tag{3.4}$$

We identify the tangent bundle via this isomorphism and denote tangent vectors as their G_ζ-equivalence classes $[g, X]$ with $g \in G$ and $X \in \mathfrak{m}$. The *canonical symplectic form* on M is defined as:

$$\sigma_{[g]}([g, X], [g, Y]) = \langle \zeta, [X, Y] \rangle ,$$

where $[\cdot,\cdot]$ and $\langle \cdot,\cdot \rangle$ on the right-hand side denotes the Lie-bracket and the duality pairing, respectively. The quotient $M = G/G_\zeta$ is taken with respect to the action of G_ζ on G by right-multiplication. We have a remaining G-action on M by left-multiplication. The bilinear form B induces a G-invariant metric ℓ on M via

$$\ell_{[g]}([g, X], [g, Y]) = B(X, Y) .$$

We denote the dual metric on T^*M still by ℓ and the corresponding kinetic Hamiltonian by $\bar{H}$. The Levi-Civita connection induces a splitting of the tangent bundle of T^*M into horizontal and vertical bundle, both of which are canonically identified with the right-hand side of (3.4). We identify further $\mathfrak{g}^\vee$ with $\mathfrak{g}$ via the bilinear form B and denote by Z the vector corresponding to ζ under the identification.

Lemma 3.7. *In the splitting and identification above, the Hamiltonian vector field of $\bar{H}$ at $(q, p) \in T^*M$ with respect to the twisted symplectic form $\omega_\sigma = d\lambda + \pi^*\sigma$ is*

$$X_{(q,p)} = (p, -ad_Z p) .$$

Proof. Let $X = (X_h, X_v)$ be the Hamiltonian vector field and $\xi = (\xi_h, \xi_v)$ be any vector at (q, p). For convenience, we omit the foot-point in the following computation. By definition, X solves

$$-d\bar{H}(\xi) = (d\lambda + \pi^*\sigma)(X, \xi) .$$

Computing the left-hand side we obtain

$$-d\bar{H}(\xi) = -\langle p, \xi_v \rangle .$$

The right-hand side reads

$$(d\lambda + \pi^*\sigma)(X, \xi) = \langle X_v, \xi_h \rangle - \langle X_h, \xi_v \rangle + \sigma(X_h, \xi_h) .$$

By definition of the symplectic form the last summand is

$$\sigma(X_h, \xi_h) = \langle \zeta, [X_h, \xi_h] \rangle = \langle Z, ad_{X_h} \xi_h \rangle = -\langle ad_{X_h} Z, \xi_h \rangle \,.$$

The assertion follows. □

Lemma 3.8. *Let $M = G/G_\zeta$, σ the canonical symplectic form on M and $\bar{H}$ be the kinetic Hamiltonian with respect to ℓ, then $\bar{\Sigma}_{\bar{k}} = \bar{H}^{-1}(\bar{k}) \subset T^*M$ is stable with respect to ω_σ for all $\bar{k} > 0$.*

Proof. We first note that ad_Z is invertible on $\mathfrak{m}$ and define the differential form $\alpha \in \Omega^1(T^*M)$ by

$$\alpha_{(q,p)}(\xi_h, \xi_v) = \langle ad_Z^{-1} p, \xi_v \rangle \,.$$

It remains to show that α is a stabilizing form. First we need to see that $\alpha(X) \neq 0$ restricted to $\bar{\Sigma}_k$. By Lemma 3.7, we have for $(q, p) \in \bar{\Sigma}_k$

$$\alpha(X)_{(q,p)} = -\langle ad_Z^{-1} p, ad_Z p \rangle = |p|^2 = 2\bar{k} \neq 0 \,.$$

One checks that the differential of α is given by

$$d\alpha_{(q,p)}((\xi_h, \xi_v), (\xi_h', \xi_v')) = \langle ad_Z^{-1} \xi_v, \xi_v' \rangle \,.$$

Any vector $\xi = (\xi_h, \xi_v)$ at (q, p) tangent to $\bar{\Sigma}_k$ satisfies $\xi \in \ker d\bar{H}$ which is equivalent to $\langle p, \xi_v \rangle = 0$. We compute

$$d\alpha_{(q,p)}(X, (\xi_h, \xi_v)) = -\langle ad_Z^{-1} ad_Z p, \xi_v \rangle = \langle p, \xi_v \rangle = 0 \,.$$

This shows that $\ker \omega_\Sigma \subset \ker d\alpha$. □

4. The Lagrangian action functional $\mathbb{S}_k$

Unfortunately, the functional $\mathbb{A}_k$ defined in (2.5) is not well suited for finding critical points using classical Morse theory. In fact, the natural space over which it is defined—namely $H^{1/2}(S^1, T^*E)$—does not have a good structure of an infinite dimensional manifold due to the fact that curves of class $H^{1/2}$ might have discontinuities. Furthermore, the functional $\mathbb{A}_k$ turns out to be strongly indefinite, meaning that all its critical points have infinite Morse index and coindex. Therefore, using the Legendre transform $\mathcal{L} : TE \to T^*E$, we introduce a related Lagrangian action functional $\mathbb{S}_k$ defined on the product Hilbert manifold $H^1(S^1, E) \times (0, +\infty) \times \mathbb{R}$, whose critical points correspond to those of $\mathbb{A}_k$. Here, $H^1(S^1, E)$ denotes the space of absolutely continuous loops $\gamma : S^1 \to E$ with square-integrable first derivative; it is well known that $H^1(S^1, E)$ has a natural structure of Hilbert manifold (cf. [38]) with Riemannian metric g_{H^1} naturally induced by the metric g^α. On $\mathcal{M} := H^1(S^1, E) \times (0, +\infty) \times \mathbb{R}$, we then consider the product metric $g_{\mathcal{M}} = g_{H^1} + dT^2 + d\varphi^2$. Observe that $(\mathcal{M}, g_{\mathcal{M}})$ is not complete.

In the following, we will prove the existence of critical points of $\mathbb{S}_k$ using variational methods, even though the functional $\mathbb{S}_k$ might fail to satisfy a crucial compactness property (namely the *Palais–Smale condition*). To overcome this difficulty, we will use a monotonicity argument, better known as the *Struwe monotonicity argument*, which is originally due to Struwe [39] and has been already successfully applied [1,3,6,7,19,26] to the existence of closed magnetic geodesics.

We recall that the connected components of $\mathcal{M}$ are in one to one correspondence with the set of conjugacy classes in $\pi_1(E)$, for the canonical inclusions

$$C^\infty(S^1,E) \hookrightarrow H^1(S^1,E) \hookrightarrow C^0(S^1,E)$$

are dense homotopy equivalences. Finally, we denote with $\mathcal{M}_0$ the connected component of $\mathcal{M}$ given by the contractible loops.

4.1. The variational principle

As in the previous sections we denote with Z the fundamental vector field of the S^1-action on E. For fixed values of T and φ the Legendre transform $\mathcal{L} : TE \to T^*E$ of the Tonelli Hamiltonian $\widehat{H} := TH_k - \varphi A_1$ yields the following Tonelli Lagrangian

$$L_{T,\varphi} : TE \to \mathbb{R}, \qquad L_{T,\varphi}(q,v) = \frac{1}{2T}|v+\varphi Z(q)|^2 - \varphi + kT,$$

where $k := \bar{k} + \frac{1}{2}$, and an associated Lagrangian action functional $H^1(S^1,E) \to \mathbb{R}$,

$$\gamma \longmapsto \frac{1}{2T}\int_0^1 |\dot\gamma(t) + \varphi Z(\gamma(t))|^2 \,\mathrm{d}t - \varphi + kT.$$

By letting the values of T and φ free, we thus get a functional $\mathbb{S}_k : \mathcal{M} \to \mathbb{R}$,

$$\mathbb{S}_k(\gamma,T,\varphi) = \frac{1}{2T}\int_0^1 |\dot\gamma(t) + \varphi Z(\gamma(t))|^2 \,\mathrm{d}t - \varphi + kT. \tag{4.1}$$

For sake of completeness, we now verify that critical points of $\mathbb{S}_k$ project to T-periodic magnetic geodesics with energy $\bar{k}$. In order to do that we need an auxiliary lemma. In what follows we denote with $\langle\cdot,\cdot\rangle$ the metric g^θ on E as constructed in Sect. 2 and with ∇ the associated Levi-Civita connection.

Lemma 4.1. *For all $u,v \in TE$ we have*

$$d\theta(u,v) = 2\langle \nabla_u Z, v\rangle.$$

Proof. We denote by Φ the flow of Z. Consider $c(s,t) = \Phi^s\gamma(t)$ for some path γ in E with $\partial_t\gamma(0) = u$. Since by construction Φ^s is an isometry for each s, we have

$$|\partial_t c(s,t)| = |d\Phi^s \partial_t\gamma(t)| = |\partial_t\gamma(t)|, \quad \forall s \in \mathbb{R}.$$

In particular

$$0 = \frac{1}{2}\partial_s|\partial_t c|^2 = \langle \nabla_s\partial_t c, \partial_t c\rangle = \langle \nabla_t\partial_s c, \partial_t c\rangle = \langle \nabla_t Z, \partial_t c\rangle\,.$$

Thus, $\langle \nabla_u Z, u\rangle = 0$, for all u. This shows that the tensor $(u,v) \mapsto \langle \nabla_u Z, v\rangle$ is skewsymmetric. Now let $(s,t) \mapsto c(s,t)$ be any map such that $\partial_s c(0) = u$ and $\partial_t c(0) = v$. We have $\theta(\partial_s c) = \langle \partial_s c, Z\rangle$. Deriving by ∂_t gives

$$\partial_t\theta(\partial_s c) = \langle \nabla_t\partial_s c, Z\rangle + \langle \partial_s c, \nabla_t Z\rangle\,.$$

Interchanging the role of ∂_s and ∂_t gives

$$\partial_s\theta(\partial_t c) = \langle \nabla_s\partial_t c, Z\rangle + \langle \partial_t c, \nabla_s Z\rangle.$$

Finally, subtracting the two equations we get by skewsymmetry

$$\begin{aligned} d\theta(\partial_s c, \partial_t c) &= \partial_s\theta(\partial_t c) - \partial_t\theta(\partial_s c) \\ &= \langle \nabla_s\partial_t c, Z\rangle + \langle \partial_t c, \nabla_s Z\rangle - \langle \nabla_t\partial_s c, Z\rangle - \langle \partial_s c, \nabla_t Z\rangle \\ &= 2\langle \partial_t c, \nabla_s Z\rangle. \end{aligned}$$

□

Proposition 4.2. *If $(\gamma, T, \varphi) \in \mathcal{M}$ is a critical point of $\mathbb{S}_k$, then the periodic curve $\mu : [0, T] \to M$ defined by*

$$\mu(t) := \tau \circ \gamma(t/T)\,, \tag{4.2}$$

for all $t \in [0, T]$ is a closed magnetic geodesic with energy $\bar{k}$.

Proof. Consider a variation $s \mapsto \gamma_s \in H^1(S^1, E)$ with $\gamma := \gamma_0$ and

$$\xi := \frac{d}{\mathrm{d}s}\Big|_{s=0}\gamma_s.$$

Differentiating $\mathbb{S}_k$ in the γ-variable and evaluating at ξ yields

$$\begin{aligned} d_\gamma\mathbb{S}_k(\gamma)[\xi] &= \frac{1}{T}\int_0^1 \langle \dot\gamma + \varphi Z, \nabla_t\xi + \varphi\nabla_\xi Z\rangle\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^1 \Big[\langle \dot\gamma, \nabla_t\xi\rangle + \varphi\langle Z, \nabla_t\xi\rangle + \varphi\langle \dot\gamma, \nabla_\xi Z\rangle + \varphi^2\langle Z, \nabla_\xi Z\rangle\Big]\mathrm{d}t \\ &= \frac{1}{T}\int_0^1 \Big[-\langle \nabla_t\dot\gamma, \xi\rangle + \varphi\big(\langle \dot\gamma, \nabla_\xi Z\rangle - \langle \nabla_t Z, \xi\rangle\big)\Big]\mathrm{d}t \\ &= \frac{1}{T}\int_0^1 \Big[-\langle \nabla_t\dot\gamma, \xi\rangle + 2\varphi\langle \dot\gamma, \nabla_\xi Z\rangle\Big]\mathrm{d}t \\ &= \frac{1}{T}\int_0^1 \Big[-\langle \nabla_t\dot\gamma, \xi\rangle + \varphi d\theta(\xi, \dot\gamma)\Big]\mathrm{d}t, \end{aligned}$$

where in the third equality we have used integration by parts and the fact that

$$\langle Z, \nabla_\xi Z\rangle = \frac{1}{2}\frac{d}{\mathrm{d}s}\Big|_{s=0}|Z|^2 = 0,$$

in the penultimate one the skewsymmetry of the tensor $\langle \nabla_u Z, v\rangle$ and in the last one Lemma 4.1. As (γ, T, φ) is a critical point of $\mathbb{S}_k$, the above quantity has to vanish for every choice of ξ and hence we conclude that

$$\langle \nabla_t\dot\gamma, \cdot\rangle = \varphi d\theta(\cdot, \dot\gamma). \tag{4.3}$$

Differentiating $\mathbb{S}_k$ in the T-direction yields

$$0 = -\frac{1}{2T^2}\int_0^1 |\dot\gamma + \varphi Z|^2\,\mathrm{d}t + k\,, \quad \Rightarrow \quad \int_0^1 |\dot\gamma + \varphi Z|^2\,\mathrm{d}t = 2T^2 k, \tag{4.4}$$

whilst differentiating $\mathbb{S}_k$ in the φ-direction gives

$$0 = \frac{1}{T}\int_0^1 \langle \dot\gamma + \varphi Z, Z\rangle\,\mathrm{d}t - 1\,, \quad \Rightarrow \quad \int_0^1 \langle \dot\gamma, Z\rangle\,\mathrm{d}t = T - \varphi. \tag{4.5}$$

Now observe that by Lemma 4.1 and (4.3)

$$\frac{d}{\mathrm{d}t}\langle\dot\gamma, Z\rangle = \langle\nabla_t\dot\gamma, Z\rangle + \langle\dot\gamma, \nabla_t Z\rangle = 0;$$

therefore, $\langle\dot\gamma, Z\rangle$ is constant and hence by (4.5) we have

$$\langle\dot\gamma, Z\rangle = T - \varphi. \tag{4.6}$$

Similarly using Lemma 4.1 and equation (4.3), we conclude that

$$\frac{d}{\mathrm{d}t}\langle\dot\gamma+\varphi Z, \dot\gamma+\varphi Z\rangle = 2\langle\nabla_t\dot\gamma, \dot\gamma\rangle + 2\varphi\langle\nabla_t\dot\gamma, Z\rangle + 2\varphi\langle\dot\gamma, \nabla_t Z\rangle = 0\,.$$

This together with (4.4) shows that

$$|\dot\gamma+\varphi Z|^2 = 2T^2 k. \tag{4.7}$$

Now set $\mu(t) := \tau(\gamma(t/T))$ and use the splitting

$$\dot\gamma = \xi + \langle\dot\gamma, Z\rangle Z = \xi + (T-\varphi)Z,$$

with $\xi \in \ker\theta$. By construction, we have $d\tau(\dot\gamma) = d\tau(\xi) = T\dot\mu$; therefore

$$2T^2 k = |\dot\gamma+\varphi Z|^2 = |\xi + TZ|^2 = T^2|\dot\mu|^2 + T^2$$

and hence $\frac{1}{2}|\dot\mu|^2 = \bar k = k - \frac{1}{2}$. Now, by definition we have $\nabla_t\dot\gamma = \nabla_{\dot\gamma}\dot\gamma$; inserting the splitting $\dot\gamma = \xi + (T-\varphi)Z$ in both arguments yields by (4.3) for any $u \in T_\gamma E$

$$\varphi d\theta(u, \dot\gamma) = \langle\nabla_t\dot\gamma, u\rangle = T^2\langle\nabla_{\dot\mu}\dot\mu, \bar u\rangle + 2(T-\varphi)\langle\nabla_\xi Z, u\rangle\,,$$

where $\bar u = d\tau(u)$. Since $\tau^*\sigma = d\theta$ by Lemma 4.1, we get

$$\varphi T\sigma(\bar u, \dot\mu) = T^2\langle\nabla_t\dot\mu, \bar u\rangle + T(\varphi - T)\sigma(\bar u, \dot\mu),$$

for all $\bar u \in T_\mu M$ which implies that

$$\langle\nabla_t\dot\mu, \bar u\rangle = \sigma(\bar u, \dot\mu) = \langle\bar u, Y_\mu(\dot\mu)\rangle \;\Rightarrow\; \nabla_t\dot\mu = Y_\mu(\dot\mu),$$

as we wished to prove. □

Conversely, one shows that every closed magnetic geodesic in M is obtained as a projection of a critical point of $\mathbb{S}_k$ via (4.2). Moreover, we want to emphasize that this correspondence of critical points of $\mathbb{S}_k$ to closed magnetic geodesics is far from bijective. We do not prove these facts, as they are irrelevant to our arguments, since we are only interested in the existence of a single closed magnetic geodesic.

4.2. The Palais–Smale condition for $\mathbb{S}_k$

As already explained in the introduction to this section, we will prove the existence of critical points for $\mathbb{S}_k$ using variational methods. To this purpose, we will need the following definition.

Definition 4.3. A sequence $(\gamma_h, T_h, \varphi_h)$ contained in a given connected component of $\mathcal{M}$ is called a **Palais–Smale sequence at level c for** $\mathbb{S}_k$ if

$$\lim_{h\to+\infty} \mathbb{S}_k(\gamma_h, T_h, \varphi_h) = c\,, \qquad \lim_{h\to+\infty} |d\mathbb{S}_k(\gamma_h, T_h, \varphi_h)| = 0\,.$$

In the definition above, $|\cdot|$ denotes, with slight abuse of notation, the (dual) norm on $T^*\mathcal{M}$ induced by the Riemannian metric $g_{\mathcal{M}}$. Observe that a limit point of a Palais–Smale sequence for $\mathbb{S}_k$ is trivially a critical point of $\mathbb{S}_k$. Therefore, we need to look for necessary and sufficient conditions for a Palais–Smale sequence to admit converging subsequences. Before doing that we need a lemma comparing the behavior of T_h and φ_h on a Palais–Smale sequence. In the following, we will denote with $e(\gamma) := \int_0^1 |\dot\gamma|^2\,\mathrm{d}t$ the kinetic energy of a loop $\gamma : S^1 \to E$.

Lemma 4.4. *Suppose $(\gamma_h, T_h, \varphi_h)$ is a Palais–Smale sequence for $\mathbb{S}_k$ at level c, then:*

1. *$T_h \to 0$ if and only if $\varphi_h \to -c$.*
2. *The T_h's are uniformly bounded from above if and only if the φ_h's are uniformly bounded.*
3. *$T_h \to +\infty$ if and only if $\varphi_h \to +\infty$.*

Proof. If $(\gamma_h, T_h, \varphi_h)$ is a Palais–Smale sequence, then we have

$$c + o(1) = \mathbb{S}_k(\gamma_h, T_h, \varphi_h) = \frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \varphi_h Z(\gamma_h)|^2\,\mathrm{d}t - \varphi_h + kT_h\,; \tag{4.8}$$

$$o(1) = \frac{\partial \mathbb{S}_k}{\partial T}(\gamma_h, T_h, \varphi_h) = k - \frac{1}{2T_h^2}\int_0^1 |\dot\gamma_h + \varphi_h Z(\gamma_h)|^2\,\mathrm{d}t\,; \tag{4.9}$$

$$o(1) = \frac{\partial \mathbb{S}_k}{\partial \varphi}(\gamma_h, T_h, \varphi_h) = \frac{1}{T_h}\int_0^1 \langle\dot\gamma_h, Z(\gamma_h)\rangle\,\mathrm{d}t + \frac{\varphi_h}{T_h} - 1\,. \tag{4.10}$$

From (4.9), it follows that

$$\frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \varphi_h Z(\gamma_h)|^2\,\mathrm{d}t = kT_h + T_h o(1)$$

and then by replacing in (4.8) we get

$$kT_h + T_h o(1) - \varphi_h + kT_h = c + o(1)$$

from which it follows that

$$\varphi_h = 2kT_h + T_h o(1) - c + o(1)\,.$$

This shows at once (1), (2) and (3). $\square$

Lemma 4.5. *Suppose $(\gamma_h, T_h, \varphi_h)$ is a Palais–Smale sequence for $\mathbb{S}_k$ at level c in a given connected component of $\mathcal{M}$. Then, the following hold:*

1. *Set $\mu_h := \tau(\gamma_h)$ for every $h \in \mathbb{N}$. If $T_h \to 0$, then*
$$\int_0^1 |\dot\gamma_h + \varphi_h Z(\gamma_h)|^2\,dt \to 0, \qquad e(\mu_h) \to 0.$$
2. *If $0 < T_* \le T_h \le T^* < +\infty$ for every $h \in \mathbb{N}$, then $(\gamma_h, T_h, \varphi_h)$ admits a converging subsequence.*

Proof. We start proving (1). The first assertion follows trivially from (4.9). We now show that $e(\mu_h) \to 0$. For every $h \in \mathbb{N}$, we consider the splitting

$$\dot\gamma_h = \zeta_h + \langle \dot\gamma_h, Z(\gamma_h)\rangle Z(\gamma_h),$$

with $\zeta_h \in \ker\theta$, and using again (4.9) we get

$$\begin{aligned} 2kT_h^2 + o(T_h^2) &= \int_0^1 \Big|\zeta_h + \big(\langle \dot\gamma_h, Z(\gamma_h)\rangle + \varphi_h\big) Z(\gamma_h)\Big|^2 \,\mathrm{d}t \\ &= \int_0^1 |\zeta_h|^2 \,\mathrm{d}t + \int_0^1 \big(\langle \dot\gamma_h, Z(\gamma_h)\rangle + \varphi_h\big)^2 \,\mathrm{d}t. \end{aligned}$$

In particular,

$$e(\zeta_h) = \int_0^1 |\zeta_h|^2 \,\mathrm{d}t = o(1).$$

This shows the claim, as by construction $d\tau$ is an isometry on $\ker\theta$.

We now prove (2). Since the T_h's are uniformly bounded and bounded away from zero, by Lemma 4.4 we have that also the φ_h's are uniformly bounded, i.e., there exists $b \in \mathbb{R}$ such that $|\varphi_h| \le b$ for every $h \in \mathbb{N}$. Therefore, up to passing to a subsequence, we can assume that $T_h \to \bar T$ and $\varphi_h \to \bar\varphi$ for $h \to +\infty$. Moreover, using (4.8) and (4.10), we get

$$\begin{aligned} c+1 &\ge \frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \varphi_h Z(\gamma_h)|^2 \,\mathrm{d}t - \varphi_h + kT_h \\ &= \frac{1}{2T_h}\int_0^1 |\dot\gamma_h|^2 \,\mathrm{d}t + \frac{\varphi_h}{T_h}\int_0^1 \langle \dot\gamma_h, Z(\gamma_h)\rangle \,\mathrm{d}t + \frac{\varphi_h^2}{2T_h} - \varphi_h + kT_h \\ &= \frac{1}{2T_h}\int_0^1 |\dot\gamma_h|^2 \,\mathrm{d}t - \frac{\varphi_h^2}{2T_h} + kT_h + o(1), \end{aligned}$$

from which we deduce that, up to neglecting finitely many $h \in \mathbb{N}$,

$$\int_0^1 |\dot\gamma_h|^2 \,\mathrm{d}t \le 2T_h\left(c + 2 + \frac{\varphi_h^2}{2T_h} - kT_h\right) \le 2T^*\left(c + 2 + \frac{b^2}{2T_*}\right).$$

It follows that the family $\{\gamma_h\} \subseteq H^1(S^1, E)$ is $\frac{1}{2}$-Hölder-equicontinuous and hence by the Ascoli–Arzelá theorem it converges (up to a subsequence) uniformly to an element $\gamma \in C^0(S^1, E)$. Now one argues exactly as in [3, Lemma 5.3] to conclude that actually $\gamma_h \to \gamma$ strongly in H^1. □

4.3. Properties of $\mathbb{S}_k$ close to fiberwise rotations

In this subsection, we study the properties of the functional $\mathbb{S}_k$ close to rotations on the fibers; in particular, we show that fiberwise rotations are in some sense local minima of $\mathbb{S}_k$. This generalizes to our setting a similar well-known statement in the classical Lagrangian setting (see for instance [3,7]) saying that constant loops are "local minima" for the free-period Lagrangian action functional. The contents of this section will be then used in the next one to associate with the functional $\mathbb{S}_k$ a complete negative gradient flow by truncating gradient flow lines which approach fiberwise rotations.

Thus, suppose that the loop $\gamma_f : S^1 \to E$ satisfies $\dot\gamma_f = -\varphi Z(\gamma_f)$. Clearly, $\varphi \in 2\pi\mathbb{Z}$. Assume that $\varphi = 2\pi a$, for some $a \in \mathbb{Z}$, and notice that

$$\mathbb{S}_k(\gamma_f, T, 2\pi a) = -2\pi a + kT > -2\pi a \tag{4.11}$$

converges to $-2\pi a$ for $T \to 0$. For $\delta > 0$, we now define the set

$$\mathcal{V}_\delta := \Big\{(\gamma, T, \varphi) \in \mathcal{M} \;\Big|\; \int_0^1 |\dot\gamma + \varphi Z(\gamma)|^2 \,\mathrm{d}t < \delta\Big\}.$$

Our first goal is to show that, for $\delta > 0$ sufficiently small, the value of φ has to be close to $2\pi\mathbb{Z}$ for every element in $\mathcal{V}_\delta$.

Lemma 4.6. *If $(\gamma, T, \varphi) \in \mathcal{V}_\delta$, then $\varphi \in (2\pi a - \sqrt{\delta}, 2\pi a + \sqrt{\delta})$ for some $a \in \mathbb{Z}$.*

Proof. If $(\gamma, T\varphi) \in \mathcal{V}_\delta$, then γ satisfies $\dot\gamma = -\varphi Z(\gamma) + \eta$, for some η such that

$$\left(\int_0^1 |\eta| \,\mathrm{d}t\right)^2 \le \int_0^1 |\eta|^2 \,\mathrm{d}t = \int_0^1 |\dot\gamma + \varphi Z(\gamma)|^2 \,\mathrm{d}t < \delta.$$

We now consider $\mu(t) := \mathrm{e}^{i\varphi t}\gamma(t)$ and compute

$$\dot\mu = \varphi Z(\mu) + \mathrm{e}^{i\varphi t}\dot\gamma = \varphi Z(\mu) + \mathrm{e}^{i\varphi t}\big(-\varphi Z(\gamma) + \eta\big) = \mathrm{e}^{i\varphi t}\eta.$$

If we denote with $d(\cdot,\cdot)$ the distance on E induced by the Riemannian metric g^θ, then from the computation above it follows that

$$d(\mu(1), \mu(0)) \le \int_0^1 |\mathrm{e}^{i\varphi t}\eta| \,\mathrm{d}t < \sqrt{\delta};$$

moreover, $\mu(0) = \gamma(0) = \gamma(1) = \mathrm{e}^{-i\varphi}\mu(1)$. This implies that

$$d(\mu(0), \mathrm{e}^{-i\varphi}\mu(0)) = d(\mathrm{e}^{-i\varphi}\mu(1), \mathrm{e}^{-i\varphi}\mu(0)) = d(\mu(1), \mu(0)) < \sqrt{\delta}$$

and hence trivially $\varphi \in (2\pi a - \sqrt{\delta}, 2\pi a + \sqrt{\delta})$ for some $a \in \mathbb{Z}$. $\square$

By the lemma above, we easily get that $\mathcal{V}_\delta$ is the disjoint union of the sets $\mathcal{V}_\delta^a := \mathcal{V}_\delta \cap \{\varphi \in (2\pi a - \sqrt{\delta}, 2\pi a + \sqrt{\delta})\}$. Furthermore, each set $\mathcal{V}_\delta^a$ contains only the fiberwise rotations given by $(\gamma_f, T, 2\pi a)$. Our next step will be to show that the value of $\mathbb{S}_k$ on $\partial\mathcal{V}_\delta^a$ is bounded away from $-2\pi a$ by a positive constant.

Lemma 4.7. *For $\delta > 0$ small enough, there exists $\epsilon > 0$ such that, for all $a \in \mathbb{Z}$,*

$$\inf_{\mathcal{V}_\delta^a} \mathbb{S}_k = -2\pi a, \qquad \inf_{\partial\mathcal{V}_\delta^a} \mathbb{S}_k > -2\pi a + \epsilon.$$

Proof. For every $(\gamma, T, \varphi) \in \partial\mathcal{V}_\delta^a$, we readily compute

$$\mathbb{S}_k(\gamma, T, \varphi) = \frac{\delta}{2T} - \varphi + kT \ge \sqrt{2k}\sqrt{\delta} - \varphi \ge \sqrt{2k}\sqrt{\delta} - 2\pi a - \sqrt{\delta},$$

where in the penultimate inequality we have used minimization in the variable T, whilst in the last one we have used Lemma 4.6. The thesis follows as $\epsilon := (\sqrt{2k} - 1)\sqrt{\delta}$ is positive for $k > 1/2$. $\square$

By Equation (4.11), we can easily find T_0 such that

$$\mathbb{S}_k(\gamma_f, T, 2\pi a) \in (-2\pi a, -2\pi a + \epsilon/4) \tag{4.12}$$

for every $T \in (0, T_0]$ and every $a \in \mathbb{Z}$. Observe that the fibers of E might be contractible, as the example of the Hopf fibration $S^3 \to S^2$ shows. However, fiberwise rotations with different winding number, say $(\gamma_f, T, 2\pi a)$ and $(\gamma'_f, T', 2\pi a')$ with $a \neq a' \in \mathbb{Z}$ and $T, T' \leq T_0$, are not contained in the same connected component of

$$\big\{\mathbb{S}_k < \max\{-2\pi a + \epsilon, -2\pi a' + \epsilon\}\big\}$$

as every path from $(\gamma_f, T, 2\pi a)$ to $(\gamma'_f, T', 2\pi a')$ has to intersect $\partial\mathcal{V}_\delta$, being the two fiberwise rotations in different connected components of $\mathcal{V}_\delta$.

Finally we notice that, combining the discussion above with Lemma 4.5,(i) we obtain the following statement for Palais–Smale sequences with T_h going to zero.

Corollary 4.8. *Let $(\gamma_h, T_h, \varphi_h)$ be a Palais–Smale sequence for $\mathbb{S}_k$ at level c in a given connected component of $\mathcal{M}$ such that $T_h \to 0$. Then, $c = 2\pi a$ for some $a \in \mathbb{Z}$ and $(\gamma_h, T_h, \varphi_h)$ eventually enters the set $\{\mathbb{S}_k < -2\pi a + \epsilon/4\} \cap \mathcal{V}^a_\delta$.*

Proof. Fix $\delta > 0$. By (4.9), we have that $(\gamma_h, T_h, \varphi_h) \in \mathcal{V}_{2kT_h^2 + o(T_h^2)}$ for every h. In particular $(\gamma_h, T_h, \varphi_h) \in \mathcal{V}_\delta$ for h large enough. Furthermore, by Lemma 4.6,

$$\varphi_h \in \big(2\pi a_h - \sqrt{2k}T_h + o(T_h), 2\pi a_h + \sqrt{2k}T_h + o(T_h)\big),$$

for some $a_h \in \mathbb{Z}$. It follows that

$$\begin{aligned}\mathbb{S}_k(\gamma_h, T_h, \varphi_h) &= \frac{1}{2T_h}\int_0^1 |\dot\gamma_h + \varphi_h Z(\gamma_h)|^2\,\mathrm{d}t - \varphi_h + kT_h \\ &\leq 2kT_h - 2\pi a_h + \sqrt{2k}T_h + o(T_h) < -2\pi a_h + \epsilon/4\end{aligned}$$

for h large enough. On the other hand

$$\mathbb{S}_k(\gamma_h, T_h, \varphi_h) \geq 2kT_h - 2\pi a_h - \sqrt{2k}T_h + o(T_h) \geq -2\pi a_h$$

for h large enough, as $k > 1/2$. Since $\mathbb{S}_k(\gamma_h, T_h, \varphi_h) \to c$, we conclude that there exists some $a \in \mathbb{Z}$ such that $a_h = a$ for every h large enough. In particular $c = -2\pi a$, $\varphi_h \to 2\pi a$ and, combining the estimates above,

$$(\gamma_h, T_h, \varphi_h) \in \big\{\mathbb{S}_k \in [-2\pi a, -2\pi a + \epsilon/4)\big\} \cap \mathcal{V}^a_\delta$$

for every h large enough, as we wished to prove. □

4.4. A truncated negative gradient flow for $\mathbb{S}_k$

Consider the bounded vector field

$$X_k := \frac{-\nabla\mathbb{S}_k}{\sqrt{1 + |\nabla\mathbb{S}_k|^2}} \tag{4.13}$$

conformally equivalent to $-\nabla\mathbb{S}_k$, where the gradient of $\mathbb{S}_k$ is made with respect to the Riemannian metric $g_\mathcal{M}$ on $\mathcal{M}$ and $|\cdot|$ is the norm induced by $g_\mathcal{M}$.

Clearly, the only source of non-completeness for the flow Φ_k induced by X_k is given by flow lines on which the variable T goes to zero. With the next lemma, we show that such flow lines have to approach fiberwise rotations.

Lemma 4.9. *Suppose $u : [0,R) \to \mathcal{M}, u(r) = (\gamma(r), T(r), \varphi(r))$ is a maximal flow line of Φ^k. Then, there exist $a \in \mathbb{Z}$ and $\{r_h\}_{h\in\mathbb{N}}$ such that $r_h \uparrow R$ and*

$$\int_0^1 |\dot{\gamma}(r_h) + \varphi(r_h)Z(\gamma(r_h))|^2\, dt \to 0, \quad \varphi(r_h) \to 2\pi a, \quad \mathbb{S}_k(u(r_h)) \to -2\pi a.$$

Proof. Since $\liminf_{r\to R} T(r) = 0$, we can find a sequence $\{r_h\}_{h\in\mathbb{N}}$ such that $r_h \uparrow R$, $T(r_h) \to 0$ and $T'(r_h) \le 0$ for every $h \in \mathbb{N}$. Using (4.4), we get that

$$0 \ge \rho_h T'(r_h) = -\frac{\partial \mathbb{S}_k}{\partial T}(u(r_h)) = \frac{1}{2T(r_h)^2}\int_0^1 |\dot{\gamma(r_h)} + \varphi(r_h)Z(\gamma(r_h))|^2\,\mathrm{d}t - k,$$

where $\rho_h := \sqrt{1 + |\nabla \mathbb{S}_k(\gamma_h)|^2}$, and hence

$$\int_0^1 |\dot{\gamma(r_h)} + \varphi(r_h)Z(\gamma(r_h))|^2\,\mathrm{d}t \le 2kT(r_h)^2. \tag{4.14}$$

This proves the first assertion. We now use Lemma 4.6 to infer that

$$\varphi(r_h) \in \big(2\pi a(r_h) - \sqrt{2k}T(r_h), 2\pi a(r_h) + \sqrt{2k}T(r_h)\big)$$

for some $a(r_h) \in \mathbb{Z}$ and compute

$$\begin{aligned}\mathbb{S}_k(u(r_h)) &= \frac{1}{2T(r_h)}\int_0^1 |\dot{\gamma(r_h)} + \varphi(r_h)Z(\gamma(r_h))|^2\,\mathrm{d}t - \varphi(r_h) + kT(r_h)\\ &\le 2kT(r_h) - 2\pi a(r_h) + \sqrt{2k}T(r_h)\\ &< -2\pi a(r_h) + \epsilon\end{aligned}$$

for h large enough, where ϵ is the constant given by Lemma 4.7. On the other hand

$$\mathbb{S}_k(u(r_h)) \ge -2\pi a(r_h),$$

for the infimum of $\mathbb{S}_k$ on $\mathcal{V}^{a(r_h)}_{2kT(r_h)^2}$ is $-2\pi a(r_h)$. This shows that

$$(\gamma(r_h), T(r_h), \varphi(r_h)) \in \big\{\mathbb{S}_k < -2\pi a(r_h) + \epsilon)\big\} \cap \mathcal{V}^{a(r_h)}_{2kT(r_h)^2}$$

for every h large enough. Since $r \mapsto S_k \circ u(r)$ is non-increasing, we conclude that there exist $\delta > 0$, $a \in \mathbb{Z}$ and $\bar{h} \in \mathbb{N}$ such that $u(r) \in \mathcal{V}^a_\delta$ for every $r \ge r_{\bar{h}}$. In particular, $a(r_h) = a$ for every h large enough and hence $\varphi(r_h) \to 2\pi a$, $\mathbb{S}_k(u(r_h)) \to -2\pi a$, as we wished to show. □

Using Lemma 4.9, it is now easy to get from Φ^k a complete flow. Namely, we stop flow lines which enter the connected component of the sublevel set $\{\mathbb{S}_k < -2\pi a + \epsilon/2\}$ containing the fiberwise rotations $(\gamma_f, T, 2\pi a)$, $T \le T_0$. With slight abuse of notation, we denote the complete flow also with Φ^k.

5. Proof of Theorem 1.1

In this section, building on the results of the previous ones, we prove Theorem 1.1. In order to show the existence of critical points of $\mathbb{S}_k$, we will use the topological assumption on M to build a suitable (non-trivial) minimax class on the Hilbert manifold $\mathcal{M}$ and a corresponding minimax function. We will then show that such a minimax function yields critical points of $\mathbb{S}_k$ for almost every $k > \frac{1}{2}$.

The first step in this direction is, therefore, to show that the assumption on the topology of M is preserved when passing to the S^1-bundle. As a precursor, we recall the relation between the homotopy groups of E and the ones of M.

Lemma 5.1. *The maps $\pi_\ell(\tau) : \pi_\ell(E) \to \pi_\ell(M)$, $\ell \in \mathbb{N}$, of homotopy groups induced by the S^1-bundle $\tau : E \to M$ satisfy*

- *$\pi_\ell(\tau)$ is an isomorphism for $\ell \geq 3$.*
- *$\pi_1(\tau)$ is surjective and its kernel is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.*
- *$\pi_2(\tau)$ is injective and $\pi_2(M) \cong m\mathbb{Z} \oplus \operatorname{im} \pi_2(\tau)$.*

Here, m is defined by the relation $\{\langle e, A\rangle \mid A \in H_2^S(M)\} = m\mathbb{Z}$, where $H_2^S(M) \subset H_2(M;\mathbb{Z})$ denotes the image of the Hurewicz map $\pi_2(M) \to H_2(M;\mathbb{Z})$, $e \in H^2(M)$ the Euler class of $E \to M$ and $\langle e, A\rangle$ the dual pairing.

Proof. Consider the long exact homotopy sequence

$$\cdots \to \pi_\ell(S^1) \to \pi_\ell(E) \xrightarrow{\pi_\ell(\tau)} \pi_\ell(M) \to \pi_{\ell-1}(S^1) \to \cdots,$$

This readily shows the first assertion. For $\ell = 2$, the connecting homomorphism fits into the commuting square

$$\begin{tikzcd} 0 \arrow[r] & \pi_2(E) \arrow[r] & \pi_2(M) \arrow[r] \arrow[d] & \pi_1(S^1) \arrow[r] \arrow[d] & \pi_1(E) \arrow[r] & \pi_1(M) \arrow[r] & 0 \\ & & H_2^S(M;\mathbb{Z}) \arrow[r] & \mathbb{Z} & & & \end{tikzcd}$$

where the vertical arrows are the Hurewicz map and the canonical isomorphism and the horizontal maps are the connecting homomorphism and the pairing with the Euler class. This readily implies the other two statements. □

Lemma 5.2. *If M is non-aspherical, then E is non-aspherical.*

Proof. Recall that, by Lemma 5.1, $\pi_\ell(M)$ is isomorphic to $\pi_\ell(E)$ for every $\ell \geq 3$. In particular, if $\pi_\ell(M) \neq \{0\}$ for some $\ell \geq 3$, then also $\pi_\ell(E) \neq \{0\}$. Thus, we are left with the case $\pi_2(M) \neq \{0\}$ and $\pi_\ell(M) = \{0\}$, for every $\ell \geq 3$. Assume by contradiction that E is aspherical, i.e., $\pi_\ell(E) = 0$ for all $\ell \geq 2$. But then again by Lemma 5.1, we conclude that $\pi_2(M) \cong \mathbb{Z}$ and thus the universal cover of M satisfies

$$\pi_2(\widetilde{M}) \cong \mathbb{Z}, \qquad \pi_\ell(\widetilde{M}) = \{0\}, \qquad \forall\ \ell \neq 2.$$

In particular, $\widetilde{M}$ is homotopy equivalent (cf. [40,41]) to the Eilenberg–Maclane space $K(\mathbb{Z}, 2) \cong \mathbb{CP}^\infty$. This is, however, not possible for a finite-dimensional manifold, since $H_{2j}(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}$ for every $j \in \mathbb{N}$. □

By the lemma above, there exists a non-zero element $\mathfrak{u} \in \pi_\ell(E)$ for some $\ell \geq 2$. Notice that, by Lemma 5.1, $\pi_\ell(\tau)(\mathfrak{u}) \neq 0 \in \pi_\ell(M)$. Recall that $\mathcal{M}_0 \subset \mathcal{M}$ denotes the connected component of loops which are contractible. With $\mathfrak{u}$, we now associate a suitable class of paths in $\mathcal{M}_0$ over which we will perform the minimax procedure.

We start observing that any continuous map

$$f : (B^{\ell-1}, S^{\ell-2}) \to (H^1(S^1, E), E),$$

defines a continuous map $v(f) : S^\ell \to E$ (cf. for instance [42, Proof of Theorem 2.4.20]); here, with slight abuse of notation, we have denoted with E the set of constant loops in $H^1(S^1, E)$. Conversely, every regular map $v : S^\ell \to E$ defines a continuous map

$$f(v) : (B^{\ell-1}, S^{\ell-2}) \to (H^1(S^1, E), E).$$

Notice furthermore that, by (4.12) we can find a positive constant $T_0 > 0$ such that $\max \mathbb{S}_k|_{E_{T_0,0}} \leq \epsilon/4$, where $\epsilon > 0$ is the constant given by Lemma 4.7 and

$$E_{T_0,0} := \bigcup_{T \leq T_0} E \times \{T\} \times \{0\}.$$

Now set

$$\mathcal{P} := \Big\{ u = (f, T, \varphi) : (B^{\ell-1}, S^{\ell-2}) \to (\mathcal{M}_0, E_{T_0,0}) \Big| \; [v(f)] = \mathfrak{u} \Big\}.$$

We readily see that $\mathcal{P} \neq \emptyset$, since $(f(v), T, \varphi) \in \mathcal{P}$ for any smooth map $v : S^\ell \to E$ such that $[v] = \mathfrak{u}$ and $T \leq T_0$. Moreover, $\mathcal{P}$ is by construction invariant under the complete flow Φ^k defined in Sect. 4.4. The last property of $\mathcal{P}$ we will need is that every element $u \in \mathcal{P}$ has to intersect $\partial \mathcal{V}_\delta$ (more precisely, $\partial \mathcal{V}_\delta^0$). Indeed, if $u(\cdot) \subseteq \mathcal{V}_\delta$, then $u(\cdot)$ would have to be entirely contained in $\mathcal{V}_\delta^0$ (simply because $\varphi(S^{l-2}) = 0$ and $\mathcal{V}_\delta$ is the disjoint union of the sets $\mathcal{V}_\delta^a$, $a \in 2\pi\mathbb{Z}$) and hence, using the splitting

$$\dot{f(s)} = \zeta(s) + \langle \dot{f(s)}, Z(f(s)) \rangle Z(f(s))$$

with $\zeta(s) \in \ker \theta$, we would get $e(\zeta(s)) < \delta$ for every $s \in [0,1]$. In particular, since by construction $d\tau$ is an isometry on $\ker \theta$, we would have that $e(\tau \circ f(s)) < \delta$, for all $s \in [0,1]$. This would imply that $[\tau \circ f] = 0 \in \pi_\ell(M)$ (see, for instance [42, Sect. 2.4]), in contradiction with our assumption (recall indeed that $\pi_\ell(\tau)(\mathfrak{u}) \neq 0$).

We now define the minimax function

$$c : \left(\frac{1}{2}, +\infty\right) \to (0, +\infty), \qquad c(k) := \inf_{u \in \mathcal{P}} \max_{\zeta \in B^{\ell-1}} \mathbb{S}_k(u(\zeta)).$$

Observe that $c(k) \geq \epsilon$, for every $u \in \mathcal{P}$ has to intersect $\partial \mathcal{V}_\delta^0$. However, this is not enough to exclude that the periods of a Palais–Smale sequence for $\mathbb{S}_k$ converge to zero as $h \to +\infty$, as it might well be that $c(k) = 2\pi a$ for some $a \in \mathbb{Z}$. For that we will need the piece of additional information given by the following lemma.

Lemma 5.3. *Let u be any element of $\mathcal{P}$. Suppose that $\zeta^* \in B^{\ell-1}$ is such that*

$$\mathbb{S}_k(u(\zeta^*)) \geq \max_{B^{\ell-1}} \mathbb{S}_k \circ u - \epsilon/2. \tag{5.1}$$

Then, $u(\zeta^*) \notin \cup_{a\in\mathbb{Z}}(\{\mathbb{S}_k < -2\pi a + \epsilon/4\} \cap \mathcal{V}^a_\delta)$.

Proof. Suppose by contradiction that there exists $a \in \mathbb{Z}$ such that $u(\zeta^*) \in \{\mathbb{S}_k < -2\pi a + \epsilon/4\} \cap \mathcal{V}^a_\delta$. Since $u \in \mathcal{P}$, there exists $\zeta \in B^{\ell-1}$ such that $u(\zeta) \in \partial\mathcal{V}^a_\delta$. Using Lemma 4.7, we now readily see that

$$\begin{aligned}\max_{B^{\ell-1}} \mathbb{S}_k \circ u - \mathbb{S}_k(u(\zeta^*)) &\geq \mathbb{S}_k(u(\zeta)) - \mathbb{S}_k(u(\zeta^*)) \\ &> -2\pi a + \epsilon + 2\pi a - \epsilon/2 = \epsilon/2,\end{aligned}$$

in contradiction with (5.1). □

Clearly, the function $c(\cdot)$ is monotonically increasing in k and hence almost everywhere differentiable. With the next proposition we show that we can find Palais–Smale sequences $(\gamma_h, T_h, \varphi_h) \subseteq \mathcal{M}_0$ for $\mathbb{S}_k$ with T_h's bounded away from zero and uniformly bounded, as soon as k is a point of differentiability for $c(\cdot)$. The proof goes along the line of [3, Lemma 8.1] (see also [7, Proposition 7.1]) and [1, Proposition 4.1] and relies on the celebrated *Struwe monotonicity argument* [39]. This concludes the proof of Theorem 1.1 in virtue of Lemma 4.5,(2).

Proposition 5.4. *Let k^* be a point of differentiability for $c(\cdot)$. Then, there exists a Palais–Smale sequence $(\gamma_h, T_h, \varphi_h) \subseteq \mathcal{M}_0$ for $\mathbb{S}_{k^*}$ with T_h bounded and bounded away from zero.*

Proof. Let M be a right linear modulus of continuity for $c(\cdot)$ at k^*. This means that for all $k \geq k^*$ sufficiently close to k^* we have

$$c(k) - c(k^*) \leq M(k - k^*). \tag{5.2}$$

Consider a sequence $k_n \downarrow k^*$ and set $b_n := k_n - k^* \downarrow 0$. Without loss of generality, we suppose that (5.2) holds for $k = k_n$ and every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ pick an element $u_n \in \mathcal{P}$ such that

$$\max_{\zeta\in B^{\ell-1}} \mathbb{S}_{k_n}(u_n(\zeta)) < c(k_n) + b_n \leq c(k^*) + (M+1)b_n\,.$$

If $\zeta \in B^{\ell-1}$ is such that $\mathbb{S}_{k^*}(u_n(\zeta)) \geq c(k^*) - b_n$, then using (5.2) we get

$$T_n(\zeta) = \frac{\mathbb{S}_{k_n}(u_n(\zeta)) - \mathbb{S}_{k^*}(u_n(\zeta))}{b_n} \leq M + 2.$$

It follows that, for all $n \in \mathbb{N}$, u_n is contained in

$$\{\mathbb{S}_{k^*} \leq c(k^*) - b_n\} \cup \Big\{\mathbb{S}_{k^*} \in \big(c(k^*) - b_n, c(k^*) + (M+1)b_n\big),\ T \leq M + 2\Big\}.$$

For every $r \in [0, 1]$ and every $n \in \mathbb{N}$ we now define $u^r_n \in \mathcal{P}$ by

$$u^r_n(\zeta) := \Phi^{k^*}_r(u_n(\zeta)), \quad \forall \zeta \in B^{\ell-1},$$

where $\Phi^{k^*}_r$ is the complete flow defined in Sect. 4.4. Namely, for $\zeta \in B^{\ell-1}$ fixed, $r \mapsto u^r_n(\zeta)$ is the flow line of Φ^{k^*} starting at $u_n(\zeta)$. Since $\mathbb{S}_{k^*}$ is non-increasing along flow lines of Φ^{k^*} and the vector-field generating Φ^{k^*} has norm less than or equal to one we have that, for all $r \in [0,1]$ and every $n \in \mathbb{N}$,

$$u_n^r \in \{\mathbb{S}_{k^*} \leq c(k^*) - b_n\} \cup \Big\{\mathbb{S}_{k^*} \in \big(c(k^*) - b_n, c(k^*) + (M+1)b_n\big),\ T \leq M+3\Big\}.$$

For any $\zeta \in B^{\ell-1}$, we now have two possibilities:

(i) $\mathbb{S}_{k^*}(u_n^1(\zeta)) \leq c(k^*) - b_n$.

(ii) $\mathbb{S}_{k^*}(u_n^r(\zeta)) \in (c(k^*) - b_n, c(k^*) + (M+1)b_n)$, for every $r \in [0,1]$.

If $\zeta \in B^{\ell-1}$ satisfies the second alternative, then we have

$$\begin{aligned}\mathbb{S}_{k^*}(u_n^r(\zeta)) > c(k^*) - b_n &> \max_{B^{\ell-1}} \mathbb{S}_{k^*} \circ u_n^r - (M+2)b_n \\ &> \max_{B^{\ell-1}} \mathbb{S}_{k^*} \circ u_n^r - \epsilon/2\end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Therefore, by Lemma 5.3, $u_n^r(\zeta) \notin \cup_{a\in\mathbb{Z}}(\{\mathbb{S}_{k^*} < -2\pi a + \epsilon/2\} \cap \mathcal{V}_\delta^a)$ for every $r \in [0,1]$ and every $n \in \mathbb{N}$ large enough. In other words, $r \mapsto u_n^r(\zeta)$ is a genuine flow line for the flow of the vector field X_{k^*} in (4.13). We now claim that there exists a Palais–Smale sequence for $\mathbb{S}_{k^*}$ contained in

$$\mathfrak{K} := \{T \leq M+3\} \setminus \bigcup_{a\in\mathbb{Z}} (\{\mathbb{S}_{k^*} < -2\pi a + \epsilon/2\} \cap \mathcal{V}_\delta^a).$$

Notice that this completes the proof, since such a Palais–Smale sequence has T_h trivially uniformly bounded and bounded away from zero by Corollary 4.8.

Thus, suppose by contradiction that $\mathbb{S}_{k^*}$ does not have Palais–Smale sequences contained in $\mathfrak{K}$. Set $\mathfrak{K}' := \mathfrak{K} \cap \{S_{k^*} \in (c(k^*) - 1, c(k^*) + M + 1)\}$. Assume without loss of generality that $b_n \leq 1$ for all $n \in \mathbb{N}$. Since $\mathfrak{K}'$ does not contain a Palais–Smale sequence as well and S_{k^*} is bounded on $\mathfrak{K}'$ we find $\rho > 0$ such that $|X_{k^*}| \geq \rho$ on $\mathfrak{K}'$. If $\zeta \in B^{\ell-1}$ satisfies the alternative ii) above, then $u_n^r(\zeta) \in \mathfrak{K}'$ for all $r \in [0,1]$ and we compute

$$(M+2)b_n > \mathbb{S}_{k^*}(u_n(\zeta)) - \mathbb{S}_{k^*}(u_n^1(\zeta)) = \int_0^1 |X_{k^*}|^2 \mathrm{d}r \geq \rho^2,$$

which is impossible for n large. It follows that, for n large enough, every $\zeta \in B^{\ell-1}$ satisfies the alternative i), that is

$$\max_{B^{\ell-1}} \mathbb{S}_{k^*} \circ u_n^1 \leq c(k^*) - b_n,$$

in contradiction with the definition of $c(k^*)$. □

Acknowledgements

We warmly thank Prof. Felix Schlenk and Prof. Alberto Abbondandolo for helpful discussions. We are also in debt to Gabriele Benedetti for his precious suggestions about a previous version of the draft. Finally, we want to thank the anonymous referee for his careful reading of the draft and his constructive comments. L. A. is partially supported by the DFG Grants AB 360/2-1 "Periodic orbits of conservative systems below the Mañé critical energy value" and AS 546/1-1 "Morse theoretical methods in Hamiltonian dynamics".

References

[1] Asselle, L., Benedetti, G.: The Lusternik–Fet theorem for autonomous Tonelli Hamiltonian systems on twisted cotangent bundles. J. Topol. Anal. **8**(3), 545–570 (2016)

[2] Hedlund, G.A.: Geodesics on a two-dimensional Riemannian manifold with periodic coefficients. Ann. Math. **33**(4), 719–739 (1932)

[3] Abbondandolo, A.: Lectures on the free period Lagrangian action functional. J. Fixed Point Theory Appl. **13**(2), 397–430 (2013)

[4] Abbondandolo, A., Macarini, L., Mazzucchelli, M., Paternain, G.P.: Infinitely many periodic orbits of exact magnetic flows on surfaces for almost every subcritical energy level. J. Eur. Math. Soc. **19**(2), 551–579 (2017)

[5] Abbondandolo, A., Macarini, L., Paternain, G.P.: On the existence of three closed magnetic geodesics for subcritical energies. Comment. Math. Helv. **90**(1), 155–193 (2015)

[6] Asselle, L., Benedetti, G.: Infinitely many periodic orbits in non-exact oscillating magnetic fields on surfaces with genus at least two for almost every low energy level. Calc. Var. Partial Differ. Equ. **54**(2), 1525–1545 (2015)

[7] Contreras, G.: The Palais–Smale condition on contact type energy levels for convex Lagrangian systems. Calc. Var. Partial Differ. Equ. **27**(3), 321–395 (2006)

[8] Merry, W.: Closed orbits of a charge in a weakly exact magnetic field. Pac. J. Math. **47**(1), 189–212 (2010)

[9] Cieliebak, K., Frauenfelder, U., Paternain, G.P.: Symplectic topology of Mañé's critical values. Geom. Topol. **14**(3), 1765–1870 (2010)

[10] Frauenfelder, U., Schlenk, F.: Hamiltonian dynamics on convex symplectic manifolds. Isr. J. Math. **159**, 1–56 (2007)

[11] Ginzburg, V.L.: New generalizations of Poincaré's geometric theorem. Funktsional. Anal. i Prilozhen. **21**(2), 16–22 (1987)

[12] Ginzburg, V.L.: On closed trajectories of a charge in a magnetic field. An application of symplectic geometry, vol. 8, pp. 131–148. Cambridge University Press, Cambridge (1996)

[13] Ginzburg, V.L., Gürel, B.Z.: Periodic orbits of twisted geodesic flows and the Weinstein-Moser theorem. Comment. Math. Helv., pp. 865–907, (2009)

[14] Paternain, G.P.: Helicity and the Mañé critical value. Algebr. Geom. Topol. **9**, 1413–1422 (2009)

[15] Schlenk, F.: Applications of Hofer's geometry to Hamiltonian dynamics. Comment. Math. Helv. **81**(1), 105–121 (2006)

[16] Usher, M.: Floer homology in disk bundles and symplectically twisted geodesic flows. J. Mod. Dyn. **3**, 61–101 (2009)

[17] Benedetti, G.: The contact property for symplectic magnetic fields on s^2. Ergodic Theory Dyn. Syst. 1–32 (2014)

[18] Ginzburg, V.L., Gürel, B.Z., Macarini, L.: On the Conley conjecture for Reeb flows. Int. J. Math. **26**, 1550047 (2015)

[19] Asselle, L., Mazzucchelli, M.: On Tonelli periodic orbits with low energy on surfaces. Trans. Am. Math. Soc. (2017)

[20] Taimanov, I.A.: Non-self-itersecting closed extremals of multivalued or not everywhere positive functionals. Izv. Akad. Nauk SSSR Ser. Mat. **55**(2), 367–383 (1991)

[21] Taimanov, I.A.: Closed extremals on two-dimensional manifolds. *Uspekhi Mat. Nauk*, 47(2(284)):143–185, 223, (1992)

[22] Taimanov, I.A.: Closed non-self-intersecting extremals of multivalued functionals. *Sibirsk. Mat. Zh.*, 33(4):155–162, 223, (1992)

[23] Novikov, S.P.: The Hamiltonian formalism and a multivalued analogue of Morse theory. Usp. Mat. Nauk **37**, 3–49 (1982)

[24] Novikov, S.P., Taĭmanov, I.A.: Periodic extremals of multivalued or not everywhere positive functionals. Dokl. Akad. Nauk SSSR **274**(1), 26–28 (1984)

[25] Taimanov, I.A.: The principle of throwing out cycles in Morse–Novikov theory. Dokl. Akad. Nauk SSSR **268**(1), 46–50 (1983)

[26] Asselle, L., Benedetti, G.: On the periodic motions of a charged particle in an oscillating magnetic flows on the two-torus. Math. Z. (2016) **(online first)**

[27] Reyman, A.G., Semenov-Tian-Shansky, M.A.: Group-theoretical methods in the theory of finite-dimensional integrable systems. In: Arnold, V.I., Novikov, S.P. (eds.) Dynamical Systems. VII, Encyclopaedia of Mathematical Sciences, vol. 16, pp. 116–223. Springer, Berlin (1994)

[28] Ortega, J.P., Ratiu, T.: Momentum maps and Hamiltonian reduction. Birkhäuser Verlag, (2004)

[29] Kang, J.: Generalized Rabinowitz Floer homology and coisotropic intersections. Int. Math. Res. Not. **10**, 2271–2322 (2013)

[30] Abraham, R., Marsden, J.E.: Foundations of Mechanics. Advanced Book Program, 2nd edn. Benjamin/Cummings Publishing Co. Inc., San Francisco (1978)

[31] Albers, P., Frauenfelder, U.: Leaf-wise intersections and Rabinowitz Floer homology. J. Topol. Anal. **2**(1), 77–98 (2010)

[32] Rabinowitz, P.H.: Periodic solutions of Hamiltonian systems. Commun. Pure Appl. Math. **31**(2), 157–184 (1978)

[33] Rabinowitz, P.H.: Periodic solutions of a Hamiltonian system on a prescribed energy surface. J. Differ. Equ. **33**(3), 336–352 (1979)

[34] Contreras, G., Macarini, L., Paternain, G.P.: Periodic orbits for exact magnetic flows on surfaces. Int. Math. Res. Not. **8**, 361–387 (2004)

[35] Hofer, H., Zehnder, E.: Symplectic invariants and Hamiltonian dynamics. Basler Lehrbücher. Birkhäuser Verlag, Basel, Birkhäuser Advanced Texts (1994)

[36] Bolle, P.: Une condition de contact pour les sous-variétés coisotropes d'une variété symplectique. Comput. R. Acad. Sci. Paris **322**, 83–86 (1996)

[37] Bolle, P.: A contact condition for p-dimensional submanifolds of a symplectic manifold. Math. Z. **227**, 211–230 (1998)

[38] Abbondandolo, A., Schwarz, M.: A smooth pseudo-gradient for the Lagrangian action functional. Adv. Nonlinear Stud. **9**(4), 597–623 (2009)

[39] Struwe, M.: Existence of periodic solutions of Hamiltonian systems on almost every energy surface. Bol. Soc. Brasil. Mat. N.S. **20**(2), 49–58 (1990)

[40] Eilenberg, S., MacLane, S.: Relations between homology and homotopy groups of spaces. Ann. Math. **46**, 480–509 (1945)

[41] Eilenberg, S., MacLane, S.: Relations between homology and homotopy groups of spaces ii. Ann. Math. **51**, 514–533 (1950)

[42] Klingenberg, W.: Riemannian geometry. De Gruyter, (1995)

Luca Asselle
Mathematisches Institut
Justus Liebig Universität
Gießen 35392
Germany
e-mail: luca.asselle@ruhr-uni-bochum.de

Felix Schmäschke
Mathematisch-Naturwissenschaftliche Fakultät
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
Germany
e-mail: felix.schmaeschke@math.hu-berlin.de