Humboldt-Universität zu Berlin
Institut für Mathematik

Advanced Topics in Optimization
Mathematics Image Processing
Summer semester 2018/19

Exercise Sheet 1

1) Let X be a normed vector space. Prove the following:

(i) The function defined by the norm $\|\cdot\|_X : X \to \mathbb{R}$ is weakly lower semicontinuous.
(ii) The function defined by the dual norm $\|\cdot\|_{X^*} : X^* \to \mathbb{R}$ is weakly* lower semicontinuous.

2) The *Uniform Boundedness Principle* theorem states that if X is a Banach space, Y is a normed space and $(T_i)_{i\in I}$ is a familly of bounded linear operators from X to Y that are pointwise bounded (i.e., for every $x \in X$ the set $\{T_i(x)\}_{i\in I}$ is bounded) then there exists an $M > 0$ such that $\|T_i\| \leq M$ for every $i \in I$.

Use the theorem above to prove that if K is a weakly compact subset of X then K is bounded with respect to the norm of X.

3) Let $\Omega = (0, 1)$ and $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ with $u_n(x) = \sin(2\pi n x)$ for every $n \in \mathbb{N}$. Find the weak limit of this sequence. Does this sequence converge strongly?

Hint: Use the fact that $C^1(0, 1)$ is dense in $L^2(\Omega)$ (why?).

4) Let X be a topological vector space and let $(F_i)_{i\in I}$ be a family of lower semicontinuous functions where $F_i : X \to \overline{\mathbb{R}}$ for every $i \in I$. Show that the function $F : X \to \overline{\mathbb{R}}$ is also lower semicontinuous where

$$F(x) := \sup_{i\in I} F_i(x).$$

5) Let $\Omega \subset \mathbb{R}^d$ an open bounded domain. Define the function $F : L^1(\Omega) \to \overline{\mathbb{R}}$ as

$$F(u) = \begin{cases} \|u\|_{L^2(\Omega)} & \text{if } u \in L^2(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus L^2(\Omega). \end{cases}$$

Show that F is lower semicontinuous with respect to the norm topology of $L^1(\Omega)$.

6) Does the problem

$$\inf_{\substack{u\in W^{1,1}(\Omega) \\ u(0)=0 \\ u(1)=1}} \int_0^1 \sqrt{u^2 + (u')^2}\, dx$$

have a solution?

7) Show that the problem

$$\inf_{u\in W^{1,2}(\Omega)} \frac{1}{2}\|u - f\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u|^2\, dx$$

has a solution. Is the solution unique?