**Humboldt-Universität zu Berlin**
Institut für Mathematik
**Advanced Topics in Optimization**
Mathematics Image Processing
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# Exercise Sheet 2

**1)** Let $X$ be a real topological vector space. Recall the definition of the *epigraph* of a function $f : X \to \overline{\mathbb{R}}$:

$$\mathrm{epi}(F) = \{(x, \alpha) \in V \times \mathbb{R} : \ f(x) \leq \alpha\}.$$

Prove the following:

(i) A function $F$ is convex if and only if its epigraph is convex.
(ii) A function $F$ is lower semicontinuous if and only if its epigraph is closed.
(iii) If $X$ is a Banach space and the function $F : X \to \overline{\mathbb{R}}$ is lower semicontinuous with respect to the norm then it is also lower semicontinuous with respect to the weak topology.

**2)** For a function $F : X \to \overline{\mathbb{R}}$ we define $\overline{F}$ to be the lower semicontinuous envelope of $F$, that is, the pointwise supremum of all lower semicontinuous functions that are everywhere less than $F$:

$$\overline{F}(x) = \sup \{G(x) : \ G \leq F \text{ and } G \text{ is lsc}\}.$$

Prove the following:

(i) $\mathrm{epi}(\overline{F}) = \overline{\mathrm{epi}(F)}$, (the closure of $\mathrm{epi}(F)$).
(1) For every $x \in X$ and $x_n \to x$, $\overline{F}(x) \leq \liminf_{n\to\infty} F(x_n)$.

**3)** Let $X$ be a real topological vector space. The indicator function $\mathcal{I}_A$ of a set $A \subseteq X$ is defined as

$$\mathcal{I}_X(u) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

Show that:

(i) $\mathcal{I}_A$ is convex if and only if $A$ is convex.
(ii) $\mathcal{I}_A$ is lower semicontinuous if and only if $A$ is closed.
(iii) $\overline{\mathcal{I}_A} = \mathcal{I}_{\overline{A}}$

**4)** Let $X$ be a reflexive Banach space and $F : X \to \overline{\mathbb{R}}$ be a weakly lower semicontinuous function with the property that $F(x) \to \infty$ when $\|x\| \to \infty$ (in particular it is coercive). Let $A \subset X$ be a convex set, closed with respect to the norm. Prove that the constrained minimization problem $\inf_{x\in A} F(x)$, has a solution.

**5)** Let $X$ be a Banach space and $F : X \to \overline{\mathbb{R}}$ a convex function. We define $F^* : X^* \to \overline{\mathbb{R}}$ to be the *polar* or *convex conjugate* of $F$ as

$$F^*(x^*) = \sup_{x\in X} \langle x^*, x\rangle_{X^*,X} - F(x).$$

(i) Calculate the convex conjugate of : $F : L^2(\Omega) \to \mathbb{R}$ with $F(u) = \frac{1}{2}\|u\|^2_{L^2(\Omega)}$.
(ii) Show that $F : L^2(\Omega) \to \mathbb{R}$ with $F(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}$, is lower semicontinuous (where $F(u) := \infty$ if $\nabla u \notin L^2(\Omega, \mathbb{R}^d)$). Revisit problem (6) of Exercise Sheet 1.