Humboldt-Universität zu Berlin
Institut für Mathematik
Advanced Topics in Optimization
Mathematical Image Processing
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Exercise Sheet 3

1) Let X be a Banach space. We define $\Gamma(X)$ to be the set of all functions $F : X \to \overline{\mathbb{R}}$ which are pointwise supremum of a family of continuous affine functions. Show that the following are equivalent:
(i) $F \in \Gamma(X)$ and proper.
(ii) F is proper, convex and lower semicontinuous.

2) Let $F, G : X \to \overline{\mathbb{R}}$. Then show that the following are equivalent:
(i) G is the pointwise supremum of the continuous affine functions which are everywhere less than F.
(ii) G is the largest function in $\Gamma(X)$ which is everywhere less than F.
If the above holds then we say that F is the Γ-*regularization* of F.

3) Let $F : X \to \mathbb{R}$. Show that the convex bi-conjugate F^{**} of F is equal to its Γ-regularization.

4) Let $F : X \to \overline{\mathbb{R}}$ and G its Γ-regularization. Show that

$$\operatorname{epi} G = \overline{\operatorname{co}(\operatorname{epi} F)}$$

where "co" denotes the convex hull. Conclude that if $A \subset X$, then

$$\mathcal{I}_A^{**} = \mathcal{I}_{\overline{\operatorname{co}(A)}}$$

(5) Let $T : L^2(\Omega) \to L^2(\Omega)$ be a bounded linear operator, with the property that $B := T^\star T$ is invertible ($T^\star$ denotes here the adjoint operator). Let $f \in L^2(\Omega)$ and define the functional $F : L^2(\Omega) \to \mathbb{R}$, $F(u) = \frac{1}{2}\|Tu - f\|^2_{L^2(\Omega)}$. Show that the convex conjugate of F is

$$F^*(u^*) = \frac{1}{2}\|u^* + T^* f\|_B^2 - \frac{1}{2}\|f\|^2_{L^2(\Omega)}$$

where

$$\|v\|_B^2 := \int_\Omega v B^{-1} v \, dx$$

(6) Let $F : X \to \overline{\mathbb{R}}$. Show that
(i) $x^* \in \partial F(x)$ if and only if

$$F(x) + F^*(x^*) = \langle x^*, x \rangle_{X^*, X}$$

(ii) $x^* \in \partial F(x) \Rightarrow x \in \partial F^*(x^*)$. Show that if $F \in \Gamma(X)$ then "$\Leftarrow$" holds as well.

(7) Let X, Y Banach spaces, $F \in \Gamma(Y)$ and let $\Lambda : X \to Y$ be linear and continuous. Assume that there exists a point Λx_0 where F is continuous and finite. Then for all points $x \in X$, it holds

$$\partial(F \circ \Lambda)(x) = \Lambda^\star \partial F(\Lambda x)$$