Humboldt-Universität zu Berlin
Institut für Mathematik

Advanced Topics in Optimization
Mathematical Image Processing
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Exercise Sheet 4

Recall the general duality theorem:
Let X, Y be Banach spaces, $\Lambda \in \mathcal{L}(X,Y)$, and let $\mathcal{F} : X \to \overline{\mathbb{R}}$, $\mathcal{G} : Y \to \overline{\mathbb{R}}$ be two proper, convex lower semicontinuous functions. Suppose that there exists a $p_0 \in X$ such that $\mathcal{F}(p_0) < \infty$, $\mathcal{G}(\Lambda p_0) < \infty$ and $\mathcal{G}$ is continuous at Λp_0. Then

$$\underbrace{\inf_{p \in X} \mathcal{F}(p) + \mathcal{G}(\Lambda p)}_{\textit{primal problem}} = \underbrace{\sup_{u \in Y^*} -\mathcal{F}^*(\Lambda^\star u) - \mathcal{G}^*(-u)}_{\textit{dual problem}}, \qquad \text{(zero duality gap)}$$

and the dual problem above admits a solution. Further $\hat{p}$ and $\hat{u}$ are solutions of the primal and the dual problems respectively if and only if

$$-\hat{u} \in \partial \mathcal{G}(\Lambda \hat{p}), \tag{1}$$

$$\Lambda^* u \in \partial \mathcal{F}(\hat{p}). \tag{2}$$

Recall also the definition of the spaces $H(\mathrm{div};\Omega)$ and $H_0(\mathrm{div};\Omega)$:

$$H(\mathrm{div};\Omega) = \left\{ p \in L^2(\Omega, \mathbb{R}^d) : \ \mathrm{div} p \in L^2 \right\},$$

where $\mathrm{div}p$ is the weak divergence of p, i.e., it satisfies

$$\int_\Omega \nabla\phi \cdot p \, dx = -\int_\Omega \phi \, \mathrm{div} p \, dx, \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

1) Use the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$ to show that $\mathrm{div}p$ is unique.

2) Show that the space $H(\mathrm{div};\Omega)$ is Banach, when equipped with the norm

$$\|p\|^2_{H(\mathrm{div};\Omega)} = \|p\|^2_{L^2(\Omega,\mathbb{R}^d)} + \|\mathrm{div} p\|^2_{L^2(\Omega)}.$$

3) Define $H_0(\mathrm{div};\Omega)$ as

$$H_0(\mathrm{div};\Omega) = \overline{C_c^\infty(\Omega, \mathbb{R}^d)}^{\|\cdot\|_{H(\mathrm{div};\Omega)}},$$

and show that

$$\int_\Omega \nabla\phi \cdot p \, dx = -\int_\Omega \phi \, \mathrm{div} p \, dx, \qquad \text{for all } p \in H_0(\mathrm{div};\Omega), \ \phi \in C^\infty(\overline{\Omega}).$$

4) Let $f \in L^2(\Omega)$, $T : L^2(\Omega) \to L^2(\Omega)$ bounded, linear, with $T^\star T$ being invertible and let $\alpha > 0$. Consider the problem

$$\text{(Primal)} \qquad \begin{cases} \displaystyle\min_{p \in H_0(\mathrm{div};\Omega)} \frac{1}{2} \|\mathrm{div} p + T^\star f\|_B^2 \\ \text{such that } -\alpha \le p(x) \le \alpha \text{ for almost every } x \in \Omega \end{cases}$$

meaning that $-\alpha \le p_i(x) \le \alpha$ for all $i = 1, \dots, d$ where $p = (p_1, \dots, p_d)$. Show that this problem has a solution. Here $\| \cdot \|_B^2$ is defined as in the previous exercise sheet.

5) By using the general duality theorem, with appropriately defined $X, Y, \mathcal{F}, \mathcal{G}$ and Λ (*Hint*: $\Lambda p = -\text{div} p$), show that the dual problem of (Primal) is equivalent (up to a constant $\frac{1}{2}\|f\|^2_{L^2(\Omega)}$) to

$$\text{(Dual)} \qquad \inf_{u \in L^2(\Omega)} \frac{1}{2}\|Tu - f\|_{L^2(\Omega)} + \alpha \text{TV}(u),$$

where

$$\text{TV}(u) := \sup\left\{ \int_\Omega u \,\text{div} p \, dx : \; p \in H_0(\text{div}; \Omega), \;\; -\alpha \leq p(x) \leq \alpha \text{ for almost every } x \in \Omega \right\}.$$

6) Show that there is zero duality gap for the the problems (Primal) and (Dual) and show that $\hat{p}$ and $\hat{u}$ are solutions of the (Primal) and (Dual) respectively if and only if

$$B\hat{u} = \text{div}\hat{p} + T^\star f, \tag{3}$$

$$\int_\Omega \hat{u} \,\text{div}\hat{p} = \text{TV}(\hat{u}) \quad \text{and } -\alpha \leq p(x) \leq \alpha \text{ for almost every } x \in \Omega. \tag{4}$$