Summer semester 2018/19

Advanced Topics in Optimization: Mathematical Image Processing

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Website:
https://www2.mathematik.hu-berlin.de/~teaching_hint/ato/main.htm

Rooms:
0.307 RUD26 (lectures)
1.304 RUD26 (exercise classes)

Dates:

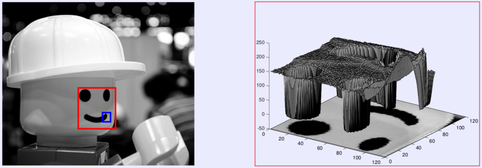
- 11.04.2019: 11:00-13:00 (lecture), 15:00-17:00 (exer.)
- 25.04.2019: 11:00-13:00 (lecture), 15:00-17:00 (exer.)
- 09.05.2019: 11:00-13:00 (lecture), 15:00-17:00 (exer.)
- 23.05.2019: 11:00-13:00 (lecture), 15:00-17:00 (exer.)
- 06.06.2019: 11:00-13:00 (lecture), 15:00-17:00 (exer.)
- **20.06.2019:** 11:00-13:00 (lecture), 15:00-17:00 (exer.) $\rightarrow$ **16.05?**
- 04.07.2019: 11:00-13:00 (lecture), 15:00-17:00 (exer.)

What is this course about?

Advanced and rigorous mathematical techniques for reconstructing/restoring digital images from given data.
Modelling is done in the continuous/function space world and implementation/programming in the discrete setting (pixels).

Mathematical tools:

- Functional and convex analysis
- Partial differential equations
- Measure theory, Calculus of Variations
- Sobolev spaces and functions of bounded variation
- Nonlinear optimization
- Implementation/Programming



157	157	158	155	155	153	157	160	155	154	160	161	162	162
155	158	156	156	150	164	149	126	148	159	157	160	161	165
158	156	152	152	166	106	11	2	17	122	167	157	160	162
155	154	154	166	92	1	3	5	0	13	148	165	162	164
152	154	167	92	1	9	1	1	7	1	85	172	159	163
159	160	76	2	2	1	1	2	3	2	32	172	160	165
144	48	1	5	2	1	1	1	3	0	15	165	166	167
17	1	5	2	1	1	0	2	0	3	6	155	167	166
0	5	1	3	2	3	1	1	2	3	2	160	171	170
4	2	3	1	1	0	1	1	2	0	14	167	178	172
1	1	0	3	1	2	1	0	3	1	34	182	171	174
1	1	1	1	1	1	1	1	5	1	76	190	170	176
1	0	3	0	2	2	1	3	2	1	138	184	175	177
1	1	0	1	3	1	0	7	0	43	185	174	176	177
0	1	2	1	1	1	5	1	14	165	184	173	178	178
1	0	0	1	2	7	0	1	152	182	171	178	179	178
0	1	0	0	4	2	4	142	186	172	175	180	175	177

Figure take from: `http://kostaspapafitsoros.weebly.com/uploads/2/3/7/9/23790650/poster.pdf`

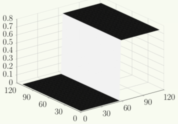
Obtained from natural images through sampling and quantisation. A finite grid Ω of pixels is superimposed, each element being associated to a number describing visual properties (e.g. average brightness).

$$u : \Omega \to \{0, \dots, 255\}, \quad \text{(Black \& White image)}$$
$$u : \Omega \to \{0, \dots, 255\}^3, \quad \text{(RGB (coloured) image)}$$

In the continuous world we will model an image as

$$u : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \;\; \text{(or } \mathbb{R}^3 \text{ for coloured images)}$$

Edges as prominent features



Edges are the structures to preserve (sharp images)!

Discontinuous versus smooth functions

Question: What is the right mathematical space to model discontinuities?

Images and Random Noise

- Measurements are often **non-accurate/noisy** due to errors/limitations during the acquisition process.
- Such types of interferences cannot be predicted and eliminated through machine calibrations (**random noise**).

Random noise typically adversely effects any **reconstruction** process.

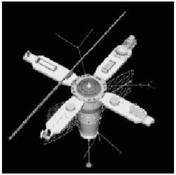
Image Deblurring: A case study.

Image Reconstruction and Random Noise

Blur occurs due to...
Camera/patient movement, light refraction, atmospheric turbulence,...

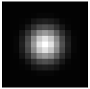
Blurring may be the result of **weighted averaging** of the data via a **point spread function (PSF)**:

Blurry Data = (PSF) $*$ (Ideal Data) + Random Noise

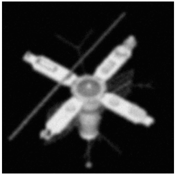


satellite

averaged by



Gaussian PSF



blur satellite

Image Reconstruction and Random Noise

Blurry Data = (PSF) $*$ (Ideal Data) + Random Noise

Image Reconstruction and Random Noise

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(PSF)$^{-1}$ (Blurry Data) $\approx$ Real Data ?

Image Reconstruction and Random Noise

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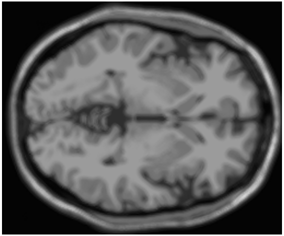
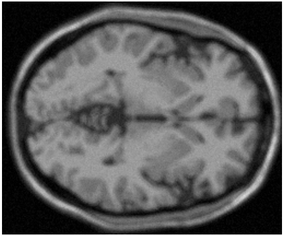
(PSF)$^{-1}$ (Blurry Data) $\approx$ Real Data ?

BAD IDEA

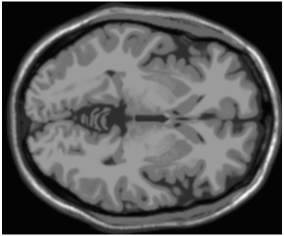
Image Reconstruction and Random Noise



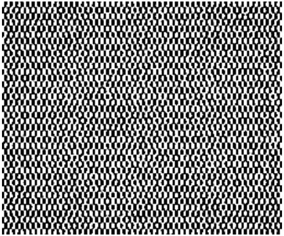
Blur without noise



Blur with noise



Reconstruction



Reconstruction (fails)

This is an ill-posed problem (as are many imaging problems)!

The inverse problems setting-classical imaging tasks

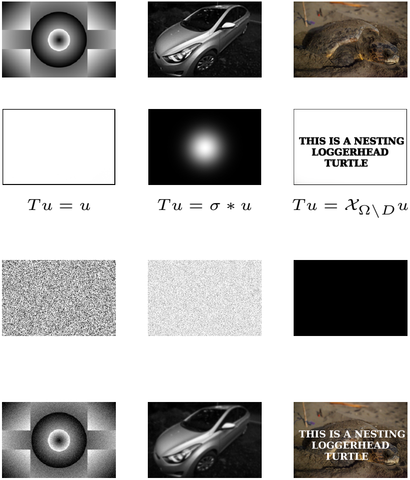
$$f = Tu + \eta$$

$\boldsymbol{u}$: The original image, defined in $\Omega \subseteq \mathbb{R}^2$.

$\boldsymbol{T}$: The type of degradation, modelled by a bounded, linear operator.

$\boldsymbol{\eta}$: Random noise, e.g. Gaussian, impulse, Poisson...

$\boldsymbol{f}$: The given, corrupted data.



$$f = Tu + \eta$$

Magnetic Resonance Imaging (MRI)

The measurement...

...an **incomplete** sample of Fourier (frequency) coefficients of an emitted signal which contains information of the measured object.

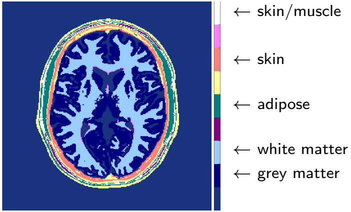
Wikipedia, CC BY-SA 3.0

Qualitative MRI

Relative tissue contrast

Image by G. Dong

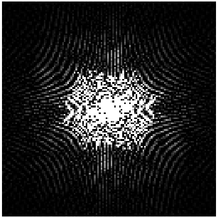
Quantitative MRI



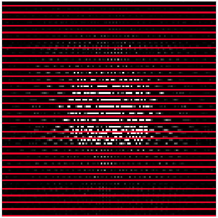
Precise tissue parameters

Image by G. Dong

$$f = Tu + \eta$$

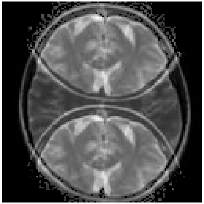


Highly sampled data



Low sampled data

Reconstruction



Reconstruction

Images by G. Dong

The inverse problems setting-medical imaging tasks

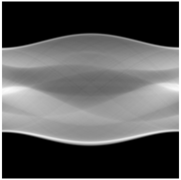
$$f = Tu + \eta$$

Tomography (CT scan, PET scan...).

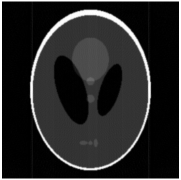
The measurement...

...the attenuation of wave energy gone through the tissue under a specific angle. As the number of angles is limited the data collection is inevitably **incomplete**.

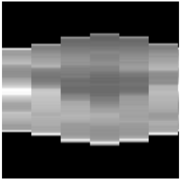
Wikipedia, CC BY 2.0



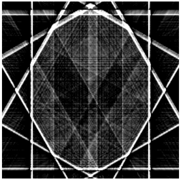
Highly sampled data



Reconstruction



Low sampled data



Reconstruction

Images by G. Dong

Regularization

- The **inversion process** of the data formation (forward) operation T needs to be **regularized**.
- **Regularization** ensures **stable** image recovery and **noise elimination** (filtering).

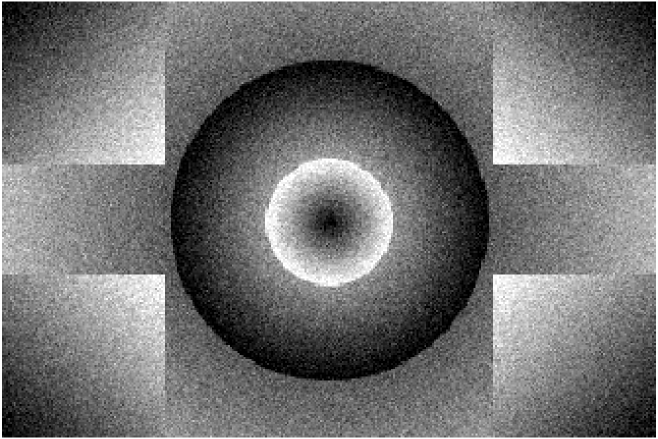
- Typically it takes the form of an optimization problem:

Minimize

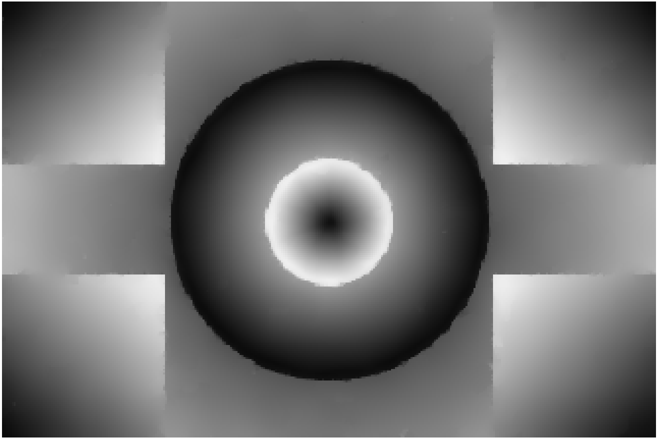
$$\text{distance}(T(u), f) + \text{regularization}(u)$$

over u.

Denoising



Denoising



Deblurring

Deblurring

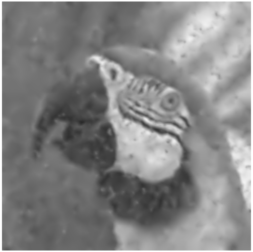

THIS IS A NESTING
LOGGERHEAD
TURTLE

Inpainting

$$\underset{\text{all possible reconstructions } u}{\text{minimize}} \quad \text{distance}(T(u), f) + \text{regularization}(u)$$

How to measure distance matters!

Salt & Pepper noise



Sum of squares

Sum of absolute values

Challenges

$$\underset{\text{all possible reconstructions } u}{\text{minimize}} \quad \mathrm{distance}(T(u), f) + \mathrm{regularization}(u)$$

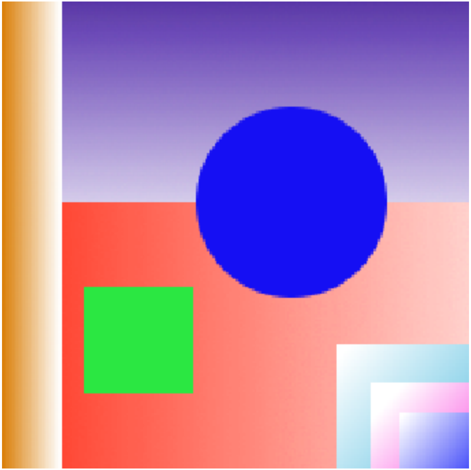
How to regularize matters!

Gaussian noise

Smooth regularization

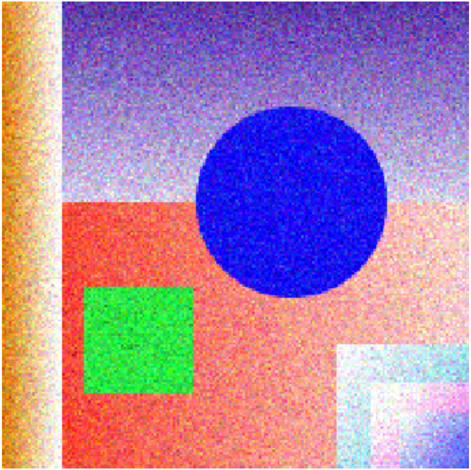
Non-smooth regularization

Smooth vs Non-smooth regularization



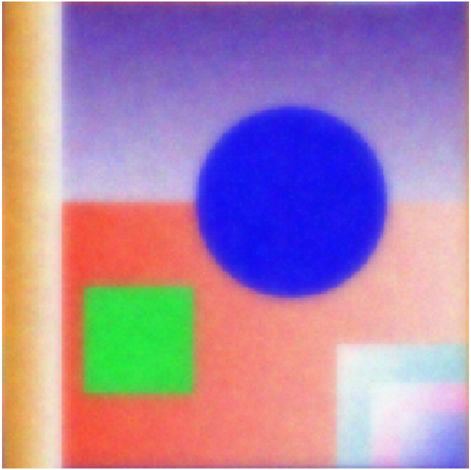
Original image

Smooth vs Non-smooth regularization



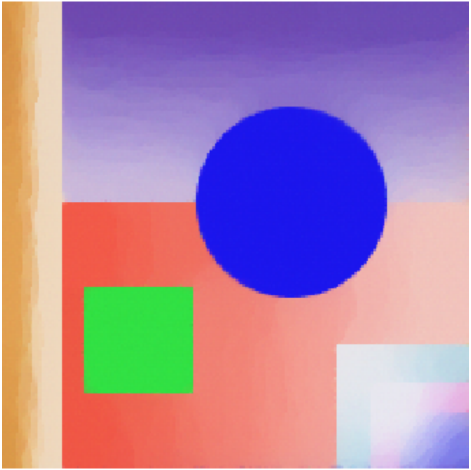
Noisy image

Smooth vs Non-smooth regularization



Smooth

Smooth vs Non-smooth regularization



non-smooth

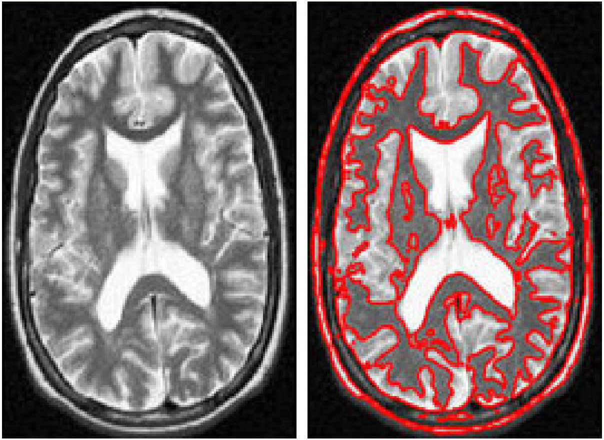
Smooth vs Non-smooth regularization

- Easy and fast to optimize
- No edge preservation

- Harder to optimize
- Edge and detail preservation

Requires: Advanced mathematical theories

Image Segmentation



Summarizing...

$$\min_{u \in X} \mathrm{D}(T(u), f) + R(u)$$

- Existence of solutions
  (**Functional analysis, Calculus of Variations**)

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- Characterization and properties of minimizers
  (**Convex analysis, Duality theory**)
- How to compute the minimizers
  (**Numerical Analysis, Optimization**)

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$$\min_{u \in X} \mathrm{D}(T(u), f) + R(u)$$

- Existence of solutions
  (**Functional analysis, Calculus of Variations**)
- Characterization and properties of minimizers
  (**Convex analysis, Duality theory**)
- How to compute the minimizers
  (**Numerical Analysis, Optimization**)
- Implementation of the minimization
  (**Computing, Programming**)

Learning targets of the course

At the end of the course you shall be able to understand the mathematical theory and implement an algorithm that does the following:

Total variation denoising

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