18.01 SPRING 2005
MIDTERM 1 TAKE-HOME ADDENDUM
DUE TUESDAY, MARCH 8 (IN LECTURE)

Note: *This and Problem Set 4 are both due on the same day, though not at the same time.*

Instructions

The problems below are closely related to some of the problems on Midterm 1 that most people had trouble with.

- Write up your answers the same way that you would for a problem set. You must show work and explain your answers—err on the side of using more paper rather than less.
- The assignment is due **IN LECTURE** on Tuesday. (By contrast, Problem Set 4 is due on the same day but at 11:59pm as usual.)
- Grading will be on a pass/fail basis: you need to answer about 80% of the assignment correctly to pass. Also, **you must pass this assignment in order to pass the class.** I'll be available for help on Monday, and extensions may be granted if absolutely necessary.

Problems

1. Parts (a)–(d) ask you to find examples demonstrating that 0^0, 1^∞, ∞^0 and $\infty - \infty$ are indeterminate forms. Don't make things harder than they need to be—try to think of *simple* functions.

   (a) Find functions f and g such that $f(0) = g(0) = 0$ and
   $$\lim_{x\to 0}[f(x)]^{g(x)} = 4.$$

   (b) Find functions f and g such that $f(0) = 1$, $\lim_{x\to 0} g(x) = +\infty$ and
   $$\lim_{x\to 0}[f(x)]^{g(x)} = 3.$$

   (c) Find functions f and g such that $\lim_{x\to 0} f(x) = +\infty$, $g(x) = 0$ and
   $$\lim_{x\to 0}[f(x)]^{g(x)} = \frac{1}{2}.$$

   (d) Find functions f and g such that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = +\infty$ and
   $$\lim_{x\to 0}\,[f(x) - g(x)] = -2.$$

   (e) Suppose $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$. What are $\lim_{x\to 0}[f(x)]^{g(x)}$ and $\lim_{x\to 0} \frac{f(x)}{g(x)}$? (This shows that 0^∞ and $\frac{0}{\infty}$ are *not* indeterminate forms.)

2. Compute each limit. Note that not all of these are indeterminate forms. Feel free to use a calculator to check your answers for plausibility.
   (a) $\displaystyle\lim_{x\to 0} x^{\ln(1+x)}$ (b) $\displaystyle\lim_{x\to 0^+} (1+x)^{1/\cos x}$ (c) $\displaystyle\lim_{x\to\infty} x^3 e^{-x}$
   (d) $\displaystyle\lim_{x\to\infty} (x+e^x)^{2/x}$ (e) $\displaystyle\lim_{x\to 0^+}\left(\frac{\sin 2x}{x^2} - \frac{2+3x}{x}\right)$

3. The *hyperbolic functions* $\cosh x$ and $\sinh x$ are defined by the formulas
   $$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

   As you'll see, they have many properties analogous to the sine and cosine functions.

(a) Show that $\cosh^2 x - \sinh^2 x = 1$ for all x. (This is analogous to the identity $\cos^2 x + \sin^2 x = 1$, and is the reason we call them "hyperbolic" functions. They are related to the geometry of the hyperbola $x^2 - y^2 = 1$, just as sin and cos are related to the unit circle $x^2 + y^2 = 1$; though the relation is slightly more complicated.)

(b) Show that $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$. *Notice: no minus signs!*

(c) Sketch the graphs of $\sinh x$ and $\cosh x$, paying attention to vital details such as critical points, inflection points and intersections with the x-axis.

(d) The *inverse* sinh function is called $\sinh^{-1}$, so by definition,

$$y = \sinh^{-1} x \qquad \Longleftrightarrow \qquad \sinh y = x.$$

Sketch the graph of $\sinh^{-1} x$. What is its domain?

(e) Find the derivative of $\sinh^{-1} x$, and be sure to write it as a function of x. In fact, use the identity from part (a) to write the derivative in the simplest possible way, without any reference to hyperbolic functions. (There isn't any "right triangle trick" in this case as with $\sin^{-1}$, $\cot^{-1}$ etc., since the sinh function has nothing directly to do with the geometry of a triangle. But the basic principle is still analogous: convenient identities allow you to write the derivative in a nice way in terms of x.)

(f) Sketch the derivative and make sure it looks right.

4. Define

$$f(x) = \begin{cases} e^{-1/x^4} + x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This function is continuous and differentiable for all x.

(a) Compute $f'(x)$ for $x \neq 0$.

(b) Compute $f'(0)$.

(c) [BONUS] Compute $f''(0)$.