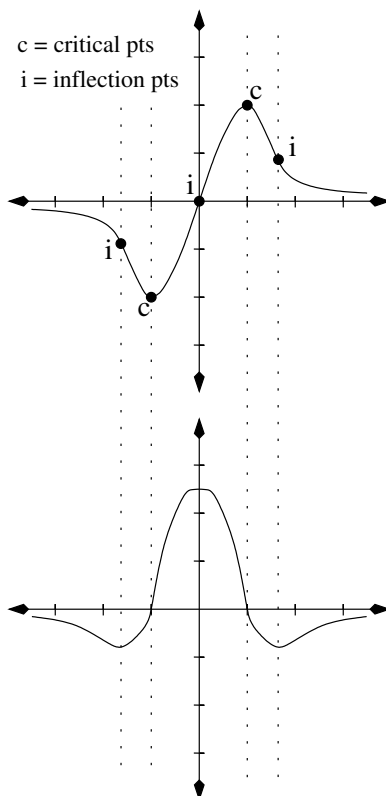


**18.01 SPRING 2005
MIDTERM 1 SOLUTIONS**

1. [15 points] A graph of a differentiable function is shown below. Copy the graph into your exam booklet with any critical points or inflection points labeled, and then sketch underneath it the graph of the function's derivative.



The top graph shows the approximate locations of two critical points and three inflection points. The critical points are lined up with zeros of the derivative in the bottom graph, while inflection points are lined up with maxima and minima of the derivative.

2. [20 points] Compute each limit:

- (a) $\lim_{x \rightarrow 0^+} x^{\sin x}$. This is an indeterminate form of type 0^0 , so we simplify it by taking the natural logarithm:

$$\lim_{x \rightarrow 0^+} \ln(x^{\sin x}) = \lim_{x \rightarrow 0^+} (\sin x)(\ln x).$$

This is now an indeterminate form of type $0 \cdot \infty$ since $\ln x \rightarrow -\infty$ as $x \rightarrow 0$. We turn this into $\frac{\infty}{\infty}$ so that L'Hospital's rule can be applied:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sin x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\cos x/\sin^2 x} = - \lim_{x \rightarrow 0^+} \frac{\sin^2 x}{x \cos x} = - \lim_{x \rightarrow 0^+} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x} = 1 \cdot 0 = 0,$$

where we've used the fact that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. The answer is therefore e^0 , also known as 1.

- (b) $\lim_{x \rightarrow \infty} \frac{e^{-x}}{x^2}$. This is easy: the numerator approaches 0 and the denominator approaches ∞ , so the answer is 0. (Recall that $0/\infty$ is *not* an indeterminate form.)

3. [15 points] Differentiate each function with respect to x . (Wherever possible, try to do it *the easy way*.)

(a) $\frac{x^3 + x + 1}{x^2}$. Rewrite this as $x + \frac{1}{x} + \frac{1}{x^2} = x + x^{-1} + x^{-2}$, so the derivative is $\boxed{1 - \frac{1}{x^2} - \frac{2}{x^3}}$.

(b) $3^{\cos(x^2)}$. Here are two possible methods:

i. Use the fact that $3 = e^{\ln 3}$ to rewrite the function as $(e^{\ln 3})^{\cos(x^2)} = e^{(\ln 3) \cos(x^2)}$, so by the chain rule, the derivative is

$$\begin{aligned} e^{(\ln 3) \cos(x^2)} \frac{d}{dx} [(\ln 3) \cos(x^2)] &= e^{(\ln 3) \cos(x^2)} (\ln 3) [-\sin(x^2)] (2x) \\ &= -2(\ln 3)x \sin(x^2) e^{(\ln 3) \cos(x^2)} = \boxed{-2(\ln 3)x \sin(x^2) 3^{\cos(x^2)}}. \end{aligned}$$

ii. Use *logarithmic differentiation*: write $y = 3^{\cos(x^2)}$, so $\ln y = \ln [3^{\cos(x^2)}] = \cos(x^2) \cdot \ln 3$. Differentiating both sides of this last equation with respect to x ,

$$\frac{1}{y} \frac{dy}{dx} = (\ln 3) [-\sin(x^2)] (2x) = -2(\ln 3)x \sin(x^2),$$

so we solve for dy/dx and find

$$\frac{dy}{dx} = -y \cdot 2(\ln 3)x \sin(x^2) = \boxed{-2(\ln 3)x \sin(x^2) 3^{\cos(x^2)}}.$$

(c) $\frac{1}{\ln x}$. The chain rule yields $-(\ln x)^{-2} \frac{d}{dx} \ln x = \boxed{-\frac{1}{x(\ln x)^2}}$.

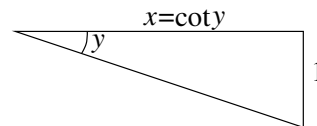
4. [15 points] Differentiate $\cot^{-1} x$, and write the derivative as a function of x with no reference to trigonometric functions.

Write $y = \cot^{-1} x$, which means $x = \cot y = \frac{\cos y}{\sin y}$. Then by implicit differentiation,

$$\begin{aligned} 1 &= \frac{d}{dx} x = \frac{d \cos y}{dx \sin y} = \frac{d}{dy} \left(\frac{\cos y}{\sin y} \right) \cdot \frac{dy}{dx} = \frac{\left(\frac{d}{dy} \cos y \right) \sin y - (\cos y) \frac{d}{dy} \sin y}{\sin^2 y} \cdot \frac{dy}{dx} \\ &= \frac{-\sin^2 y - \cos^2 y}{\sin^2 y} \cdot \frac{dy}{dx} = -\frac{1}{\sin^2 y} \cdot \frac{dy}{dx}, \end{aligned}$$

thus $\frac{dy}{dx} = -\sin^2 y$. To write this as a function of x in a nice way, we need to find a simple relation between $\sin^2 y$ and $x = \cot y$. Imagine a right triangle in which y is one of the acute angles (in radians), the near side has length x and the far side has length 1 (figure below). Then $\cot y = x/1 = x$, as desired. The hypotenuse must have length $\sqrt{1+x^2}$, by Pythagoras. Thus

$$\sin y = \frac{1}{\sqrt{1+x^2}},$$



and we conclude $\frac{dy}{dx} = -\sin^2 y = \boxed{-\frac{1}{1+x^2}}$.

5. [15 points] Find the positive number that exceeds its cube by the largest amount.

We're looking for the absolute maximum of $f(x) = x - x^3$ on the interval $x > 0$. Observe that this function is positive for any positive x close enough to zero, while $f(0) = 0$ and $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$. Thus $f(x)$ does have a maximum, which must be a critical point since the function is differentiable. We have $f'(x) = 1 - 3x^2 = 0$ when $x = 1/\sqrt{3}$; there is also a negative solution, but this is irrelevant to the problem. Thus the only possible answer is $\boxed{1/\sqrt{3}}$.

6. Consider the function $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(a) [5 points] Show that $f(x)$ is continuous at $x = 0$.

The function will be continuous at 0 if $f(0) = \lim_{x \rightarrow 0} f(x)$. This is true since $f(0) = 0$ by definition, and as $x \rightarrow 0$, $-1/x^2 \rightarrow -\infty$, thus $e^{-1/x^2} \rightarrow 0$.

(b) [8 points] Compute the limit $\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$. *Hint:* it may help to rewrite the fraction as $\frac{1/x}{e^{1/x^2}}$.

The fraction in the hint is an indeterminate form of type ∞/∞ , so by L'Hospital,

$$\lim_{x \rightarrow 0} \frac{1/x}{e^{1/x^2}} = \lim_{x \rightarrow 0} \frac{-1/x^2}{e^{1/x^2} \left(-\frac{2}{x^3}\right)} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x^3/x^2}{e^{1/x^2}} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x}{e^{1/x^2}} = \boxed{0}$$

since the numerator approaches 0 while the denominator approaches ∞ .

Note: you may sensibly ask, why can't we just apply L'Hospital's rule to the original fraction $\frac{e^{-1/x^2}}{x}$? Actually we can—this is also an indeterminate form—but it doesn't help, because the expression we get by differentiating the top and bottom is more complicated than the one we started with. Try it, you'll see what I mean.

(c) [7 points] Compute $f'(0)$. (You will need the definition of the derivative!)

What you need to recognize here is that, though we can use the usual rules to differentiate e^{-1/x^2} , this only computes the derivative for $x \neq 0$, and is thus *irrelevant* to the question being asked. We must resort to the definition of the derivative:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} = \boxed{0},$$

by the result of part (b).

(d) [BONUS: 5 points] Write down an expression for the function $f'(x)$ and use it to compute $f''(0)$. We've already computed $f'(0) = 0$, and as mentioned above, we can compute $f'(x)$ for $x \neq 0$ by the usual methods, thus:

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

With this we can use the definition of the derivative again to compute

$$f''(0) = \lim_{h \rightarrow 0} \frac{f'(0+h) - f'(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2}{h^3} e^{-1/h^2} - 0}{h} = 2 \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^4}.$$

We can apply to this limit the same trick as suggested in the hint for part (b):

$$\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^4} = \lim_{x \rightarrow 0} \frac{1/x^4}{e^{1/x^2}} = \lim_{x \rightarrow 0} \frac{-4/x^5}{-\frac{2}{x^3} e^{1/x^2}} = \frac{4}{2} \lim_{x \rightarrow 0} \frac{1/x^2}{e^{1/x^2}}.$$

Since $1/x^2 \rightarrow \infty$ as $x \rightarrow 0$, we can substitute $u = 1/x^2$ and rewrite this last limit as

$$\lim_{u \rightarrow \infty} \frac{u}{e^u},$$

which is zero, by another application of L'Hospital, or simply by quoting the fact that " e^u grows faster than any polynomial". We conclude $\boxed{f''(0) = 0}$.

Remark. This function is a rather interesting example: not only does it turn out to be infinitely differentiable, but all of its higher order derivatives at $x = 0$ are *zero*. Proving this is a bit tricky, but in principle it's just an extension of the methods used above (try it if you're up for a challenge). The result is a bit surprising: one might have expected that a function whose derivatives all vanish at a particular point should equal zero everywhere in some interval around that point—and for most functions one would think of writing down, that's true. The example shows therefore that the set of all differentiable functions is quite a bit larger and more varied than the set of functions that one would usually think of writing down! It's a jungle out there.