18.01 SPRING 2005
MIDTERM 2 SOLUTIONS

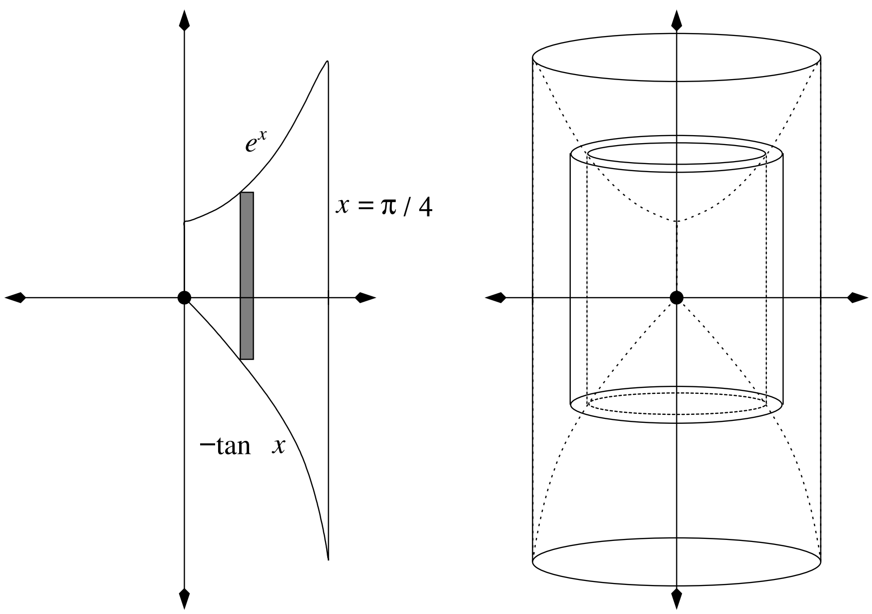
1. [20 pts] For each of the following, set up but **do not evaluate** a definite integral for computing the requested quantity.

   (a) The area enclosed by the curves $y = e^x$, $y = -\tan x$, $x = 0$ and $x = \pi/4$.
   Dividing the region into narrow rectangles with thickness Δx (figure below, left) produces the integral

   $$\boxed{\int_0^{\pi/4} (e^x + \tan x)\ dx}.$$

   (b) The volume of the solid obtained by rotating the region of part (a) about the y-axis.
   We divide the solid into cylindrical shells of thickness Δx (figure below, right), starting from radius $x = 0$ and extend out to radius $x = \pi/4$. Each shell then has volume $\Delta V = 2\pi x \Delta x \cdot (e^x + \tan x)$, giving the integral

   $$\boxed{\int_0^{\pi/4} 2\pi x(e^x + \tan x)\ dx}.$$



2. [20 pts] Compute each integral:

   (a) $\displaystyle\int \frac{dx}{x^2 + 2x - 3}$. Partial fractions: we can find constants A and B such that

   $$\frac{1}{x^2 + 2x - 3} = \frac{1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

   for all x. Multiplying both sides by $(x+3)(x-1)$, we have

   $$1 = A(x-1) + B(x+3).$$

   Now plugging in $x = 1$ gives $1 = 4B$ and plugging in $x = -3$ gives $1 = -4A$, thus $B = 1/4$ and $A = -1/4$. Therefore,

   $$\int \frac{dx}{x^2 + 2x - 3} = \int \left(\frac{-1/4}{x+3} + \frac{1/4}{x-1} \right)\ dx = \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+3| = \boxed{\frac{1}{4} \ln\left|\frac{x-1}{x+3}\right| + C}.$$

Note that you could also do this by completing the square and using trigonometric substitution, but it's harder.

(b) $\displaystyle\int_1^{e^\pi} \frac{\sin(\ln x)\ dx}{x}$. Substitute $u = \ln x$, so $du = \frac{1}{x}dx$ and

$$\int_1^{e^\pi} \frac{\sin(\ln x)}{x}\ dx = \int_{\ln(1)}^{\ln(e^\pi)} \sin u\ du = \int_0^\pi \sin u\ du = -\cos u|_0^\pi = -(-1-1) = \boxed{2}.$$

3. (a) [10 pts] Compute $\displaystyle\int \frac{\sec\theta\ d\theta}{\tan^2\theta}$. *Hint:* Write everything in terms of $\sin\theta$ and $\cos\theta$.

$$\int \frac{\sec\theta}{\tan^2\theta}\ d\theta = \int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\ d\theta = \int \frac{\cos\theta\ d\theta}{\sin^2\theta} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\sin\theta} = \boxed{-\csc\theta + C}$$

using the substitution $u = \sin\theta$.

(b) [20 pts] Compute $\displaystyle\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+5}}$. For full credit, you should write an answer that doesn't involve any trigonometric functions.

We need to complete the square and then use a trigonometric substitution: first,

$$\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+5}} = \int \frac{dx}{(x+1)^2\sqrt{x^2+2x+1+4}} = \int \frac{dx}{(x+1)^2\sqrt{(x+1)^2+4}}.$$

Now substitute $x+1 = 2\tan\theta$, so $x = 2\tan\theta - 1$, $dx = 2\sec^2\theta\ d\theta$ and $\sqrt{(x+1)^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\tan^2\theta+1} = 2\sec\theta$, thus

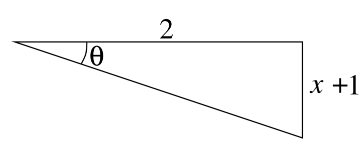
$$\int \frac{dx}{(x+1)^2\sqrt{(x+1)^2+4}} = \int \frac{2\sec^2\theta\ d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta\ d\theta}{\tan^2\theta} = -\frac{1}{4}\csc\theta$$

by the result of part (a). We could rewrite this in terms of x as

$$-\frac{1}{4}\csc\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right],$$

which is correct, but not pretty enough for full credit. It can be rewritten in a nicer form by constructing a right triangle with θ as one of its acute angles (figure below). Since $\tan\theta = \frac{x+1}{2}$, we can assume the far side has length $x+1$ and the near side has length 2. Then the Pythagorean theorem gives $\sqrt{(x+1)^2+4}$ (or $\sqrt{x^2+2x+5}$) for the length of the hypotenuse, so $\csc\theta = \frac{\sqrt{x^2+2x+5}}{x+1}$. We conclude

$$\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+5}} = \boxed{-\frac{1}{4}\frac{\sqrt{x^2+2x+5}}{x+1} + C}.$$



4. [10 pts] Let $F(x) = \displaystyle\int_{\cos x}^2 e^{t^2}\ dt$. What is $F'(x)$?

The integral can't be computed exactly, but we can still use the fundamental theorem of calculus to compute $F'(x)$, i.e. using

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x).$$

To apply this, reverse the order of the limits, substitute $u = \cos x$ and use the chain rule:

$$F'(x) = \frac{d}{dx}\int_{\cos x}^2 e^{t^2}\ dt = -\frac{d}{dx}\int_2^u e^{t^2}\ dt = -\left(\frac{d}{du}\int_2^u e^{t^2}\ dt\right)\frac{du}{dx} = -e^{u^2}(-\sin x) = \boxed{e^{\cos^2 x}\sin x}.$$

5. [20 pts] Find the unique function $x(t)$ that satisfies the differential equation

$$\frac{dx}{dt} = \sqrt{(t-1)(x-4)}$$

and the initial condition $x(5) = 8$.

Separating x and t terms in the differential equation gives

$$\frac{dx}{\sqrt{x-4}} = \sqrt{t-1}\,dt \quad \Longrightarrow \quad \int \frac{dx}{\sqrt{x-4}} = \int \sqrt{t-1}\,dt \quad \Longrightarrow \quad 2\sqrt{x-4} = \frac{2}{3}(t-1)^{3/2} + C$$

for some constant C. At this stage we can determine the constant by plugging in the initial condition $x(5) = 8$:

$$2\sqrt{8-4} = \frac{2}{3}(5-1)^{3/2} + C \quad \Longrightarrow \quad 4 = \frac{16}{3} + C \quad \Longrightarrow \quad C = -\frac{4}{3}.$$

Now solving for x gives the function

$$\boxed{x(t) = \left[\frac{1}{3}(t-1)^{3/2} - \frac{2}{3}\right]^2 + 4}.$$

As an alternative way to do things, one could first determine the general solution and then plug in the initial condition. Solving for x before determining C gives the function

$$x(t) = \left[\frac{1}{3}(t-1)^{3/2} + c\right]^2 + 4,$$

where c is another constant (not the same one as before). This method gets a little bit tricky when we now solve for c: plugging in $x(5) = 8$ leads to the result $c = -\frac{8}{3} \pm 2$. This is not one answer but two, only one of which matches the one we obtained above. In fact, that one is right, and the other choice $c = -\frac{14}{3}$ is wrong, though this is not so easy to see at first. The reason is that if we plug our general solution back into the differential equation, we find

$$\frac{dx}{dt} = \left[\frac{1}{3}(t-1)^{3/2} + c\right]\sqrt{t-1},$$

which needs to match $\sqrt{x-4}\sqrt{t-1}$. It does—but only if the expression $\left[\frac{1}{3}(t-1)^{3/2} + c\right]$ is *greater than or equal to zero*, since $\sqrt{x-4}$ is always assumed to be the *positive* square root. Thus our general solution only technically satisfies the equation for a given range of t when c is chosen so that the quantity in brackets is nonnegative; in the present case this is true at and near $t = 5$ for $c = -2/3$, but not for $c = -14/3$.