18.01 SPRING 2005
MIDTERM 1 PREPARATION

There is nothing due next week in light of the midterm on Tuesday. A few of the problems below will appear on Problem Set 4 (so you can get a head start now), and all of them should be helpful in preparing for the exam.

Stuff for this week

Reading

Notes MVT, Simmons 2.6, 12.1–12.3.

Ungraded problems

Each problem is from the Notes unless stated otherwise:

- *Th 2/24/05:* 2G-2, 2G-3*, 2G-4
- *Fr 2/25/05:* Simmons 12.2 #5, 9, 13, 15, 25; 12.3 #9, 13, 17, 31, 41

See also the Chapter 12 additional problems listed below for much more practice with indeterminate forms and L'Hospital's rule.

Graded problems

The following will be included in Problem Set 4 (due March 8), but you might want to get a head start on them since they're relevant to material that may appear on Midterm 1.

1. A generalization of the mean value theorem says that if $f''(x)$ exists for all x and $a < b$, then there exists a number $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2. \tag{1}$$

   A useful application of this formula will be given in the next problem; the goal of this one is to prove the formula.

   (a) [2 pts] Show that there is a unique number k such that

$$f(b) = f(a) + f'(a)(b-a) + k(b-a)^2.$$

   (This is easy: just solve for k.) The goal from now on will thus be to show that $k = f''(c)/2$ for some $c \in (a, b)$.

   (b) [3 pts] Let $g(x) = f(x) - f(a) - f'(a)(x-a) - k(x-a)^2$. Use Rolle's theorem to show that there exists a number $c_1 \in (a, b)$ such that $g'(c_1) = 0$.

   (c) [2 pts] Compute $g'(a)$.

   (d) [3 pts] Show that there exists a number $c \in (a, c_1)$ such that $g''(c) = 0$.

   (e) [2 pts] Conclude that $k = f''(c)/2$.

2. (a) [2 pts] Write down the linear approximation to $\sqrt[3]{1+2x}$ for x near 0. (Recall that you can do this without differentiating anything!)

   (b) [5 pts] Use Formula 1 from the previous problem to determine an error estimate for your linear approximation from part (a), assuming $0 \leq x \leq 1/8$. That is, find a number $M > 0$ such that the magnitude of the difference between $\sqrt[3]{1+2x}$ and its linear approximation does not exceed M when $x \in [0, 1/8]$.

Stuff for Midterm 1

Exam info

The exam will occupy the whole 50 minute lecture period on Tuesday 3/1/05. There will be several straightforward problems to test your basic knowledge, and a few that require somewhat more thought. No notes, books or calculators are allowed, but you will be given scrap paper (which will not be graded). The material covered will include everything we've done so far, up to Friday's lecture.

General advice

Aside from trying the problems and questions below, the best thing you can do to study is to review the problem sets, especially the problems on which you didn't get full credit (now that you have the solutions). It is almost guaranteed that some things on the exam will be recycled directly from homework, and this applies to the "Part B" problems as well.

Office hours

In addition to my usual office hours next week, I'll be around for most of the afternoon on Monday (after recitation), at least until 6pm, so feel free to drop by for help.

Review problems

The following problems provide a good review of the topics we've covered so far; the majority of them are fairly simple and representative of the kinds of problems you'll encounter on the exam. In fact, a few of them will probably *be* problems on the exam. Note that the answers are in the back of the book.

From Simmons:

- Chapter 2 Additional problems (pp. 81–82): #13, 43, 45, 53, 61
- Chapter 3 Additional problems (pp. 111–114): #9, 11, 23, 31, 35, 39, 41, 43, 53
- Chapter 4 Additional problems (pp. 156–162): #25, 27, 33, 35, 41, 51, 59, 65
- Chapter 8 Additional problems (pp. 288–291): #23, 31, 33, 35
- Chapter 12 Additional problems (pp. 424–426): #3, 5, 17, 27, 39, 51, 57, 59, 67, 71, 73, 91, 93

Thinking questions

These are mostly not suitable for an exam, but they raise issues that are important to understand at this stage in the course—issues which may also reappear in different forms on the exam. We'll review these in recitation on Wednesday and Monday.

1. One statement in each of the following pairs is false. Determine which one, and think up a counterexample to illustrate why.
   (a)
   - Every continuous function is differentiable.
   - Every differentiable function is continuous.

   (b)
   - If $f(x)$ is a differentiable function, every local maximum or minimum is a critical point.
   - If $f(x)$ is a differentiable function, every critical point is a local maximum or minimum.

   (c)
   - If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a local minimum at x_0.
   - If $f''(x_0)$ exists and f has a local minimum at x_0, then $f'(x_0) = 0$ and $f''(x_0) > 0$.

   (d)
   - If $a < b$, every continuous function on the open interval (a, b) has a global maximum.
   - If $a < b$, every continuous function on the closed interval $[a, b]$ has a global maximum.

2. (a) Suppose a is a positive number and $\lim_{x \to 0} \dfrac{a^x - 1}{x} = 1$. What is a? (You can use L'Hospital's rule to get the answer if you must, but then try to prove it again *without* L'Hospital!)

(b) Along the same lines, compute $\lim_{x\to 0} \dfrac{\ln(1+x)}{x}$ by viewing it as the derivative of a certain function $f(x)$ at $x = 0$.

(c) Use part (b) to prove $\lim_{x\to 0}(1+x)^{1/x} = e$. (We did this in class, you may recall...)

3. Write down a function which is continuous and differentiable but not *twice* differentiable at 0, i.e. $f''(0)$ does not exist.

4. (a) Remind yourself how to compute the derivatives of $\tan x$, $\sec x$, $\sin^{-1} x$ and $\tan^{-1} x$. Be sure to write the last two derivatives without any reference to trigonometric functions.

   (b) Given a function f with $f(x_0) = y_0$, the inverse function f^{-1} satisfies $f^{-1}(y_0) = x_0$. What relation is there between the numbers $f'(x_0)$ and $(f^{-1})'(y_0)$?

   (c) For arbitrary constants $a > 0$, what is $\frac{d}{dx} \log_a x$? Compute it in two ways: (i) using implicit differentiation, and (ii) using what you know about logarithms and the special case $a = e$.

5. Draw a picture to compute the limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$$

(Without using L'Hospital's rule!) If you draw the picture right... well, it's not exactly a proof, but it's very convincing. (We did this in class too.)

6. (a) Show that for any constant $a > 1$ and any polynomial $P(x)$,

$$\lim_{x\to\infty} \frac{P(x)}{a^x} = 0.$$

   Informally, we say that "every exponential function grows faster than every polynomial."

   (b) Show similarly that every polynomial grows faster than $\ln x$.

   (c) Show that if $f(x)$ grows faster than $g(x)$ and $g(x)$ grows faster than $h(x)$, then $f(x)$ grows faster than $h(x)$. (These must all first be translated into precise mathematical statements—the fact that this is true can be seen as justification for our informal use of the term "grows faster".)

7. Under what circumstances is it *not* true that $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$?

8. We call 0/0 an *indeterminate form* because given any number c, one can find functions $f(x)$ and $g(x)$ with $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ and

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = c.$$

For instance, let $g(x) = x$ and $f(x) = cx$. The following expressions are also indeterminate forms—for each, demonstrate this by finding a simple pair of functions f and g such that the appropriate limit has the arbitrary value c.

(a) $0 \cdot \infty$ (b) ∞/∞ (c) 1^∞ (d) ∞^0 (e) 0^0 (f) $\infty - \infty$

Why are 0^∞ and $\infty + \infty$ not included here?