18.01 SPRING 2005
FINAL EXAM PREPARATION

Material covered

The final exam will take place Tuesday, May 17 from 9:00 to noon in 2-143 (note that this is not our usual classroom). It will be designed to be doable in two hours, even though you have three. You can expect to spend the first hour on a number of straightforward problems dealing with basic concepts. For the second hour, there will be a few more serious problems that require careful thought—don't rush through these, you'll have plenty of time. There may also be a bonus question.

With a few exceptions listed below, the exam may include any topics we've covered this semester, in particular anything that has appeared on any problem or that is listed on the current version of the syllabus (see the website—it's changed a little since the beginning). There may be a slight bias toward topics from the second half of the semester, including at most two questions on sequences and series (drawn directly from Problem Sets 11 and 12).

Here are some topics we've covered (or at least mentioned) that you're free to avoid worrying about as you study:

- Rolle's theorem and the mean value theorem
- hyperbolic functions (i.e. sinh, cosh and tanh)
- conic sections (focus, directrix, Kepler and all that)
- slopes of polar curves

As usual, the best way to study is by reviewing the problem sets, and in fact most of the exam questions will be taken directly from them. The sets of review questions that were recommended for the first two exams are still worthwhile, and a new set for the more recent topics is listed below.

Formula sheet

The exam will include a formula sheet, with the following information:

1. Some useful antiderivatives:

$$\int \sec^2 x\ dx = \tan x + C \qquad \int \csc^2 x\ dx = -\cot x + C$$

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\ dx = \ln|\csc x - \cot x| + C$$

$$\int \sec x \tan x\ dx = \sec x + C \qquad \int \csc x \cot x\ dx = -\csc x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

2. Triogonometry:
   - $\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta$
   - $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
   - $\sin(2\theta) = 2\sin\theta\cos\theta, \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

3. Area in polar coordinates: $dA = \frac{1}{2} r^2 d\theta$

4. Arc length:

   - For the graph of a function $y(x)$: $d\ell = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
   - For a polar graph $r(\theta)$: $d\ell = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
   - For a parametric curve $(x(t), y(t))$: this is easy, figure it out for yourself!

5. Geometric series: $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$ if $|r| < 1$

6. Limit comparison test: Suppose $a_k > 0$, $b_k > 0$ and $a_k/b_k \to L > 0$: then $\sum a_k$ converges if and only if $\sum b_k$ converges.

7. Alternating series test: Suppose $\{a_k\}$ is a decreasing sequence of positive numbers, converging to 0: then $\sum (-1)^k a_k$ converges.

8. Integral test, comparison test: you should know these off the top of your head.

9. Ratio test: suppose $a_k > 0$ and $a_{k+1}/a_k \to r$. Then $\sum a_k$ converges if $r < 1$ and diverges if $r > 1$. No conclusion if $r = 1$. (Compare with what you know about geometric series.)

10. Taylor expansions:

    - $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

      or if you prefer...
    - $f(a+h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} h^k = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots$

11. Some favorite Taylor series:

    - $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
    - $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
    - $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
    - $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Review problems

If you get bored with the usual problem sets, here are some more problems to practice with for recent topics (since Midterm 2). Note that these are not meant as a *comprehensive* review—for that purpose the problem sets are better.

From Simmons:

- Chapter 7 Additional problems (pp. 254–257): #43, 45, 53, 55

- Chapter 12 Additional problems (pp. 424–426): #101, 103, 107, 117
- Chapter 13 Additional problems (pp. 470–476): #35, 51, 53
- Chapter 16 Additional problems (pp. 583–585): #17, 21, 25
- Chapter 17 Additional problems (pp. 627–629): #3

Thinking questions

1. Draw a graph of a function that isn't continuous. Now draw one that is continuous but not differentiable. What does it mean for a function to be differentiable?

2. Remind yourself what the definition of the derivative is. Now go back to Problem Set 1 and remind yourself why you actually need to use it once in a while.

3. What's the difference between a definite integral and an indefinite integral? What do these have to do with antiderivatives? State the fundamental theorem of calculus.

4. Given a function $f(x)$, how do you go about trying to graph it (intelligently)? Which points should you plot first? Now assuming $f(x)$ is well understood, consider the function

$$g(x) = \int_1^x f(t)\ dt.$$

   If you're able to compute the integral explicitly, then great, but often this may not be the case. Then how can you graph $g(x)$?

5. How would you apply the techniques of the previous question to graph a *polar* equation $r = f(\theta)$?

6. Given a parametric curve $(x(t), y(t))$, how can you determine when the slope is horizontal or vertical? What can happen at a point where both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero? (Remember the cycloid; see Simmons Sec. 17.2).

7. Use integration to compute the area and circumference of a circle, and then the volume and surface area of a sphere.

8. Compute the area and circumference of a circle *again*, but in polar coordinates: consider the circle $r = 2R\cos\theta$. (Verify first that it really is a circle; what's its radius and where is its center?)

9. What does the fact that $\int_1^\infty \frac{dx}{x^p}$ converges for $p > 1$ have to do with the fact that $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$?

10. For what values of p does $\int_0^1 \frac{dx}{x^p}$ converge?

11. Write down an infinite series that converges absolutely. Write down another that converges conditionally. How about one that diverges?

12. If $\sum_n a_n$ converges, what do you immediately know about the sequence of terms a_n? Why is it blindingly obvious that $\displaystyle\sum_n \frac{n}{n+1}$ doesn't converge?