

18.01 SPRING 2005
PROBLEM SET 1
DUE TUESDAY, FEBRUARY 8

Instructions

- Write up your solutions neatly, preferably with all pages stapled. You need not show every arithmetic calculation, but must always show enough work to demonstrate the process by which the answer is reached. Without this, the grader can't be sure that you didn't just copy the answers from someone else, and there's no way to give partial credit.
- You're free to work together in groups, but you must write up the solutions independently. Plagiarism is easy to detect.
- You can either hand in your solutions in class by the due date, or slip them through the slot in my office door (2-172) by 11:59pm that night.

Reading

Notes G and C, Simmons 2.1–2.5, 3.1–3.3.

Ungraded problems

Do the following exercises for practice—preferably after the corresponding lecture—but *do not hand them in*. The solutions are available to you, so you should check your work. Starred problems are especially recommended.

From the Notes:

- *Precalculus review:* 1A-1a, 1A-2a, 1A-3ce, 1A-5, 1A-7*, 1A-9*
- *Th 2/3/05:* 1B-2, 1C-4a,c, 1C-6*, 1D-1bcdff*, 1D-2, 1D-3, 1D-5*
- *Fr 2/4/05:* 1E-1, 1E-5, 1F-1, 1F-2, 1F-6*

Graded problems, Part A [56 pts total]

From Simmons: 2.3 #41 [5 pts]; 2.4 #14 [10 pts]; 2.5 #18 [12 pts] and 20 [14 pts]; 3.2 #4 [2 pts], 30 [4 pts] and 40 [3 pts]; 3.3 #4, 16 and 26 [2 pts each]

Graded problems, Part B [32 pts total]

1. Recall the *squeeze theorem*, which says that if we have three functions f , g and h defined in some interval around the number a , with

$$f(x) \leq g(x) \leq h(x)$$

for all x in that interval and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$, then the limit $\lim_{x \rightarrow a} g(x)$ exists, and is equal to the other two. A similar statement can be made about the one-sided limits. This is particularly useful if f and h are relatively simple functions but g is not.

- (a) [5 pts] Let $G(x) = \sin\left(\frac{1}{x}\right)$, defined on the domain $\{x \neq 0\}$. Draw a rough sketch of the graph (try to do it without using a calculator!), and use it to argue that neither $\lim_{x \rightarrow 0^+} G(x)$ nor $\lim_{x \rightarrow 0^-} G(x)$ exists.
- (b) [5 pts] Use the squeeze theorem to show that the function

$$F(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$. Draw a rough sketch of F together with the two “squeezing” functions. *Hint:* remember that $-1 \leq \sin \theta \leq 1$ for all θ .

- (c) [5 pts] Use part (a) to show that the function $F(x)$ from part (b) is *not* differentiable at $x = 0$. You will need the definition of the derivative.
- (d) [5 pts] Consider the function

$$H(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use the squeeze theorem again to show that H is continuous at $x = 0$. Now use the definition of the derivative to show that, unlike in part (c), H is differentiable at $x = 0$. Compute $H'(0)$. (*Please note:* the usual differentiation rules will not help; you only need the definition.)

2. As was mentioned in the first lecture, one can argue on intuitive grounds that a circle's circumference should be proportional to its diameter, and *define* the number π to be the proportionality constant. Thus by definition,

$$C = \pi D.$$

Calculating π is then a matter of empirical observation: one only has to take a wheel and use a flexible tape measure to find its circumference and diameter as accurately as possible, then divide. Having invented π , it now makes sense to measure angles in *radians*, defining $360^\circ = 2\pi$ radians since 2π is the circumference of a unit circle. The sin and cos functions are then defined so that the input angle is always measured in radians. The point is that all this can be done without having any idea what the *area* of a circle is.

So how do we get from here to the popular formula $A = \pi r^2$? The following steps outline one method, which is similar to the idea behind integral calculus.

- (a) [2 pts] Use the formula $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ to compute

$$\lim_{x \rightarrow 0} \frac{\sin ax}{x}$$

when a is any constant. (The solution is very simple once you see it; if you know L'Hospital's rule, *do not use it!*)

- (b) [6 pts] Consider the regular polygon with n sides inscribed in a circle of radius r (Figure 1 shows two examples). Use some basic geometry and trigonometry to derive a formula for the area A_n of this polygon in terms of r and n . *Hint:* show first that each of the n isosceles triangles inside the polygon has area $r^2 \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$, then use a trigonometric identity to put this in a nicer form.

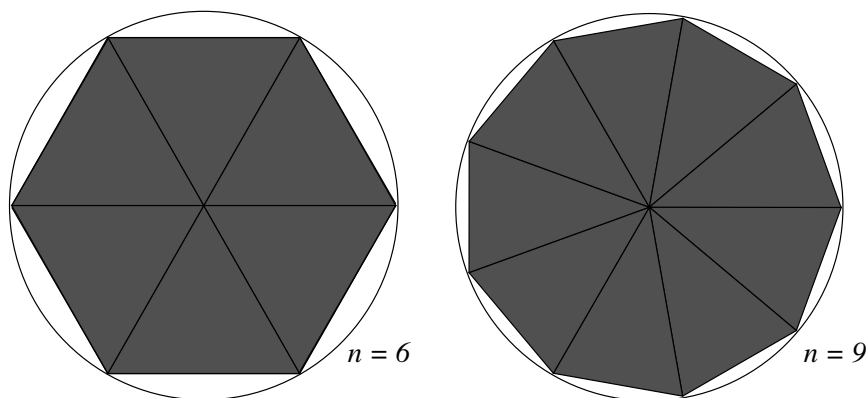


Figure 1: regular polygons inscribed in circles

- (c) [4 pts] Show that $\lim_{n \rightarrow \infty} A_n = \pi r^2$.