The Usual Instructions

- Write up your solutions neatly, preferably with all pages stapled. You need not show every arithmetic calculation, but must always show enough work to demonstrate the process by which the answer is reached. Without this, the grader can't be sure that you didn't just copy the answers from someone else, and there's no way to give partial credit.

- You're free to work together in groups, but you must write up the solutions independently. Plagiarism is easy to detect.

- You can either hand in your solutions in class by the due date, or slip them through the slot in my office door (2-172) by 11:59pm that night.

Reading

Simmons 16.4–16.5, Notes AV.

Ungraded problems

Do the following exercises for practice—preferably after the corresponding lecture—but *do not hand them in.* The solutions are available to you, so you should check your work. Starred problems are especially recommended.

Each problem is from the Notes unless stated otherwise:

- *Th 4/21/05:* 4I-1*, 4I-2*, 4I-3
- *Fr 4/22/05:* 4D-1, 4D-3*, 4D-9, 4I-5*

Graded problems, Part A [40 pts total]

From Simmons:

- 16.3 #10 [3 pts], 18 [6 pts]
- 16.4 #4 [4 pts], 6 [5 pts], 8 [4 pts], 12, 14 [3 pts each]
- 16.5 #4, 6, 10 [4 pts each],

Graded problems, Part B [20 pts total]

1. Johannes Kepler showed by analyzing Tycho Brahe's celestial data that the planet Mars orbits along an ellipse, a curve which can be regarded as the graph of a polar equation

$$r = \frac{a}{1 + \varepsilon \cos\theta}.$$

Here a and ε are positive constants, and ε, known as the *eccentricity*, is less than one. (The equation still makes sense for $\varepsilon \geq 1$, but then its graph is a parabola or hyperbola—this would be the path of a much faster celestial object that does not orbit the sun but rather approaches, curves around it slightly and then escapes.)

Kepler also derived from this data what has come to be known as his *second law*, that planets trace out "equal areas in equal times". If Mars has polar coordinates $r(t)$ and $\theta(t)$ as functions of time, we can use our knowledge of area elements in polar coordinates to rewrite Kepler's second law as the equation

$$\frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}.$$

We'll now see how this can be used to learn more about the functions $r(t)$ and $\theta(t)$: in particular we'd like to be able to predict where Mars will be at a given time. To start with, let $L = r^2 \frac{d\theta}{dt}$; Kepler's second law tells us that this is constant in time, so we can measure it by observation and assume it is a known quantity.

(a) [3 pts] Show that the function $\theta(t)$ satisfies the differential equation

$$\frac{d\theta}{dt} = \frac{L}{a^2} \left(1 + \varepsilon \cos \theta\right)^2.$$

(b) [4 pts] Since $\theta(t)$ is always increasing with t, the function can be inverted: we can regard t as a function of θ, with derivative $\frac{dt}{d\theta}$, and $t(\theta(t)) = t$. Use implicit differentiation to demonstrate the (seemingly obvious) fact that $\frac{d\theta}{dt}$ and $\frac{dt}{d\theta}$ are reciprocals. (Remember the expression $\frac{d\theta}{dt}$ is not literally a quotient of "numbers" $d\theta$ and dt, so the statement about its reciprocal is not as obvious as it looks. There are cases, e.g. in multivariable calculus, where such seemingly obvious statements are not true.)

(c) [5 pts] Write down $\frac{dt}{d\theta}$ as a function of θ. Now suppose $\theta(0) = \theta_0$, and write down a definite integral that gives the value of $t(\theta)$: this is the amount of time it takes the planet to travel between positions with angular coordinates θ_0 and θ. Do not attempt to evaluate the integral, but say a few words about how you might find the answer if someone put a gun to your head and said "tell me when Mars will get to position $\theta = \pi/4$!"

(d) [3 pts] Using the result of part (c), write down (but do not try to evaluate) a definite integral that computes the length of the Martian year.

2. [8 pts] This problem continues the discussion of the orbit of Mars, so answering Problem 1 first would be helpful (but not essential). If r is regarded as a function of θ, then the average distance of Mars from the sun with respect to angular position along its orbit is clearly

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{a}{1 + \varepsilon \cos \theta} \, d\theta.$$

This however is not the same as the average distance *with respect to time*, which would be

$$\frac{1}{T} \int_0^T r(t) \, dt$$

where T is the length of the Martian year. Perform a change of variable and use the relations derived between r, θ and t in Problem 1 to show that this average can be rewritten as the integral

$$\frac{a^3}{LT} \int_0^{2\pi} \frac{d\theta}{\left(1 + \varepsilon \cos \theta\right)^3}.$$

The moral is that the average value of a function is highly dependent on precisely *which variable* we regard it to be a function of.