18.01 SPRING 2005
PROBLEM SET 10
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a) With $L = r^2 \frac{d\theta}{dt}$ and $r = \frac{a}{1+\varepsilon\cos\theta}$ we have

$$\frac{d\theta}{dt} = \frac{L}{r^2} = \frac{L}{a^2}(1+\varepsilon\cos\theta)^2.$$

(b) Differentiating the expression $t = t(\theta(t))$ with respect to t yields

$$1 = \frac{d}{dt}t(\theta(t)) = \frac{dt}{d\theta}\frac{d\theta}{dt}$$

by the chain rule, showing that these two derivatives are reciprocals of each other. The notation here is admittedly a little confusing because the symbol t is being used in two roles: both as a variable and as the inverse function. To avoid this confusion we could denote by $F(\theta)$ function which is inverse to $\theta(t)$, thus by definition

$$\theta(t) = c \quad\Longleftrightarrow\quad F(c) = t,$$

and thus $F(\theta(t)) = t$ for all t. So what we're really doing is differentiating *this* expression with respect to t:

$$1 = \frac{d}{dt}F(\theta(t)) = F'(\theta(t))\frac{d\theta}{dt}.$$

This shows that the reciprocal of $\frac{d\theta}{dt}$ at time t is $F'(\theta)$ when $\theta = \theta(t)$; this $F'(\theta)$ is exactly what we previously referred to as $\frac{dt}{d\theta}$.
Note that this argument didn't use any specifics of the problem at hand: it is a general property of inverse functions in any context that their derivatives are reciprocals.

(c) Taking the reciprocal of the expression from part (a) gives

$$\frac{dt}{d\theta} = \frac{1}{d\theta/dt} = \frac{a^2}{L(1+\varepsilon\cos\theta)^2}.$$

If $\theta(0) = \theta_0$ then $t(\theta_0) = 0$, so given θ_0 we now seek the unique function $t(\theta)$ such that $t(\theta_0) = 0$ and $\frac{dt}{d\theta}$ is the function above. The answer is

$$\boxed{t(\theta) = \int_{\theta_0}^{\theta} \frac{a^2}{L(1+\varepsilon\cos\varphi)^2}\, d\varphi}$$

as is easy to check: clearly $t(\theta_0) = 0$ since the integral of anything from θ_0 to θ_0 is zero, and the fundamental theorem of calculus implies

$$t'(\theta) = \frac{a^2}{L(1+\varepsilon\cos\theta)^2}.$$

Note the importance of using a dummy variable (φ) inside the integral rather than θ, which now appears only in the upper limit of integration.

As for the question "when will Mars get to position $\theta = \pi/4$", let's say we set our clock so that right now $t = 0$, and we simultaneously observe the current angular position of Mars to be θ_0. Then the answer to the question is the value of the integral

$$\int_{\theta_0}^{\pi/4} \frac{a^2}{L(1+\varepsilon\cos\varphi)^2}\, d\varphi,$$

which we can't compute exactly, but we can approximate it by adding up areas of thin rectangles, or better, having a computer (or graphing calculator) do this for us, to any desired accuracy. That's more or less what they do at NASA.

(d) The question is: how long does it take Mars to get from position θ_0 to position $\theta_0 + 2\pi$? The answer is

$$\boxed{\int_{\theta_0}^{\theta_0+2\pi} \frac{a^2}{L(1+\varepsilon\cos\varphi)^2}\, d\varphi},$$

and you should take a moment to convince yourself that this answer does not actually depend on θ_0 (because $\cos\varphi$ is periodic). Indeed, the year lasts the same amount of time whether you start counting on January 1 or April 19.

2. We use two important formulas from Problem 1:

$$r = \frac{a}{1+\varepsilon\cos\theta}, \qquad \text{and} \qquad \frac{d\theta}{dt} = \frac{L}{a^2}(1+\varepsilon\cos\theta)^2.$$

The idea then is to think of $r(t)$ as a *composite* function $r(\theta(t)) = \dfrac{a}{1+\varepsilon\cos[\theta(t)]}$, so

$$r_{\text{avg}} = \frac{1}{T}\int_0^T \frac{a}{1+\varepsilon\cos[\theta(t)]}\, dt.$$

We now treat $\theta(t)$ as a substitution, with

$$d\theta = \frac{d\theta}{dt}\, dt = \frac{L}{a^2}(1+\varepsilon\cos\theta)^2\, dt \quad \Longrightarrow \quad dt = \frac{a^2}{L(1+\varepsilon\cos\theta)^2}\, d\theta,$$

thus

$$\frac{1}{T}\int_0^T \frac{a}{1+\varepsilon\cos[\theta(t)]}\, dt = \frac{1}{T}\int_{\theta(0)}^{\theta(T)} \frac{a}{1+\varepsilon\cos\theta}\,\frac{a^2\, d\theta}{L(1+\varepsilon\cos\theta)^2} = \frac{a^3}{LT}\int_0^{2\pi} \frac{d\theta}{(1+\varepsilon\cos\theta)^3},$$

where we've assumed $\theta(0) = 0$. This assumption is harmless, as we can always reset our clocks so that it's true.