18.01 SPRING 2005
PROBLEM SET 11
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a)

$$\frac{1}{n^2+4n+3} = \frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3} = \frac{1/2}{n+1} - \frac{1/2}{n+3} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right).$$

(b)

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n^2+4n+3} = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) \\ &= \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \ldots + \left(\frac{1}{N+1} - \frac{1}{N+3}\right)\right] \\ &= \boxed{\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{N+2}-\frac{1}{N+3}\right)} \end{aligned}$$

since all other terms in the sum cancel.

(c) As $N \to \infty$, the terms in S_N that have N in the denominator disappear, leaving $\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \boxed{\frac{5}{12}}$.

2. (a) In the first step one interval of length $1/3$ is removed. Next we remove two intervals, each a third of the previous length, i.e. $1/9$. Then four intervals of length $\frac{1}{3}\frac{1}{9} = \frac{1}{27}$, eight intervals of length $\frac{1}{3}\frac{1}{27}$ and so forth. This produces the sum

$$\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \ldots = \sum_{n=0}^{\infty}\frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\frac{1}{1-2/3} = \frac{1}{3}3 = \boxed{1},$$

using the formula for geometric series. Informally, we can rephrase this result by saying that "almost all" numbers in $[0,1]$ are not in the Cantor set.

(b) It's easy to find isolated numbers in $[0,1]$ that can never be removed in the recursive process: this is true for instance of 0, $1/3$, $2/3$, 1, $1/9$, $2/9$ etc., indeed any number that is an endpoint of one of the removed intervals. Recall that the intervals are all *open*, so the actual endpoints are not removed, and all subsequent removals take place in the interior, away from the endpoints.