18.01 SPRING 2005
PROBLEM SET 12

This last problem set will not be collected or graded since it's so late in the semester, but it's still strongly recommended that you work through it. It covers material from the final two weeks of class which will likely also appear on the final exam. Solution sets will be made available during the last week of class.

Reading

Simmons 13.5, 13.6 (through Example 2), 13.8, 13.7 (to middle of P. 463), 14.1–14.4

Ungraded problems, Part A

Do the following exercises for practice—preferably after the corresponding lecture. The solutions are available to you, so you should check your work. Starred problems are especially recommended.

Each problem is from the Notes unless stated otherwise:

- *Tu 5/03/05:* 7B-1*
- *Th 5/05/05:* 7B-2*
- *Fr 5/06/05:* 7B-6*
- *Tu 5/10/05:* 7C-1*, 7D-1
- *Th 5/12/05:* 7D-3

Ungraded problems, Part B [92 pts total]

Note: the point values in this case have nothing to do with grading but indicate the relative difficulty and/or length of the problems.
From Simmons:

- 13.4 #10 [5 pts]
- 13.5 #4, 6, 8, 12 [3 pts each]
- 13.6 #8 [2 pts], 9 [5 pts]
- 13.8 #2, 4, 14, 24 [4 pts each], 32 [9 pts]
- 14.2 #4, 6, 16 [4 pts each]
- 14.3 #5 [4 pts]
- 14.4 #2 [3 pts], #4abcefi [18 pts], 5 [2 pts], 12 [4 pts]

Ungraded problems, Part C [38 pts total]

1. This problem demonstrates a clever way to compute the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k}$.

   (a) [4 pts] Let $f(x)$ be the function defined as the power series

   $$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k k} x^k.$$

   What is the interval of convergence for this series, i.e. for what range of x does the series converge? For what x is the convergence *absolute*, and when is it *conditional*?

   (b) [4 pts] Write down $f'(x)$ as a power series. What is the interval of convergence for *this* series? For what x is the convergence absolute?

(c) [3 pts] If you did part (b) correctly, you might notice that the series for $f'(x)$ is geometric. Use this fact to write down an explicit formula for $f'(x)$. (In this context, "explicit" means "not involving any infinite series".)

(d) [3 pts] Integrate the expression from part (c) to obtain an explicit formula for $f(x)$. Be careful with the constant (it's not arbitrary!).

(e) [2 pts] Use the above results to compute $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k}$.

2. Consider the improper integral

$$\int_1^{\infty} \frac{\cos 2\pi x}{x}\, dx.$$

(a) [5 pts] Rewrite it using integration by parts, with $u = \frac{1}{x}$ and $dv = \cos 2\pi x\, dx$. Do not attempt to compute the resulting integral, but show that it converges. You can do this by comparison with the convergent integral $\int_1^{\infty} \frac{dx}{x^2}$, using also the following general fact:

$$\text{If } \int_a^{\infty} |f(x)|\, dx \text{ converges, then so does } \int_a^{\infty} f(x)\, dx.$$

This is analogous to the similar statement about absolutely convergent series.

(b) [5 pts] A careless application of the integral test would now use the result of part (a) to argue that

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi n}{n}$$

converges, but it isn't true. Show that this sum actually diverges. (It's easy—what is this sum, really?) Why is it wrong to use the integral test here?

3. What do you really *know* about the number e?

From a mathematical perspective, one only needs to know the basic ideas of differential calculus in order to see that there is a real number (call it a for the moment) such that the function $f(x) = a^x$ is its own derivative—and indeed, there is only one. We define e to be that number. One can do this without having any idea that the number in question is approximately 2.718. Knowing a little more calculus, one can then apply Taylor's formula and derive the expansion $e^x = 1 + x + x^2/2! + \ldots$, which produces a formula for e in the form

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This can be used to approximate e, but it offers little obvious insight beyond that.

The goal of this problem is to prove that e is an *irrational* number. Recall that a number is called *rational* if it can be expressed as a fraction p/q, where both p and q are integers; otherwise, it's called irrational. So we want to prove that any attempt to write e precisely as a fraction is doomed.

In order to do this, we look at the partial sums of the series above:

$$S_q = \sum_{n=0}^{q} \frac{1}{n!}.$$

Since the series is known to converge, we know that the sequence of numbers S_q gets closer to e as $q \to \infty$; put another way, we know that the positive numbers $e - S_q$ shrink toward 0 as q gets larger. We can make this more precise in the following way.

(a) [6 pts] Let's write $e - S_q$ as an infinite series in the form $e - S_q = \sum_{n=q+1}^{\infty} \frac{1}{n!}$. Use this to prove that $q!(e - S_q) < 1/q$ for all positive integers q. Hint: it may help you to remove the sigma-notation and write out the first several terms of the series above, each term multiplied by $q!$. Then think about how you might compare this to some geometric series.

(b) [6 pts] We now prove that if $e = p/q$, with p and q both positive integers, then something impossible is true. Show first that $q!S_q$ is always a positive integer. Conclude from this that if $e = p/q$, then $q!(e - S_q)$ is also a positive integer. Why does this contradict the result of part (a)?

Conclusion: there is no pair of integers p and q such that $p/q = e$. Irrational numbers are here to stay!