18.01 SPRING 2005
PROBLEM SET 12
SOLUTIONS

Ungraded problems, Part B

See attached photocopies.

Ungraded problems, Part C

1. (a) The power series is $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{2^k k}$. To find the interval of convergence, we first check whether the series converges *absolutely* for any given value of x, i.e. we test the convergence of

$$\sum_{k=1}^{\infty} \left| \frac{x^k}{2^k k} \right| = \sum_{k=1}^{\infty} \frac{|x|^k}{2^k k}.$$

With power series, the best method for this is usually the ratio test. Let $a_k = \frac{|x|^k}{2^k k}$, then

$$\frac{a_{k+1}}{a_k} = \frac{|x|^{k+1}/2^{k+1}(k+1)}{|x|^k/2^k k} = \frac{|x|}{2} \frac{k}{k+1} \to \frac{|x|}{2} \qquad \text{as } k \to \infty.$$

Thus by the ratio test, $\sum_k a_k$ converges if $|x|/2 < 1$ and diverges if $|x|/2 > 1$; for $|x|/2 = 1$ we cannot deduce anything at this stage. This does at least show that the radius of convergence is 2, for we deduce that the power series converges absolutely whenever $|x| < 2$ but diverges when $x > 2$. For $|x| = 2$ we must explicitly check convergence in the two cases $x = \pm 2$. Plugging in $x = 2$, the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges (recall that we proved this in class, using the integral test; a different proof may be found in Simmons as well). For $x = -2$ on the other hand, we have

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{2^k k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

and this converges by the alternating series test, but only conditionally since $\sum_k 1/k$ diverges.

To summarize, the power series $\sum_{k=1}^{\infty} \frac{x^k}{2^k k}$ converges if and only if $x \in [-2, 2)$; that's the *interval of convergence*. Moreover, it converges *absolutely* for $x \in (-2, 2)$, and *conditionally* at $x = -2$.

(b) Within the radius of convergence, we're allowed to assume that the sum rule applies to infinite sums as well as finite ones, i.e. we can differentiate *term by term*. Thus

$$f'(x) = \frac{d}{dx} \sum_{k=1}^{\infty} \frac{1}{2^k k} x^k = \sum_{k=1}^{\infty} \frac{d}{dx} \frac{1}{2^k k} x^k = \sum_{k=1}^{\infty} \frac{1}{2^k} x^{k-1}$$

$$= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \ldots = \frac{1}{2} \left[1 + \frac{x}{2} + \left(\frac{x}{2} \right)^2 + \ldots \right].$$

We could use the same methods as above to find the interval of convergence, but it's easier simply to observe that this is a geometric series, and thus it converges if and only if $\left|\frac{x}{2}\right| < 1$, that is, $|x| < 2$. The same statement applies for *absolute convergence*, since the series

$$\sum_k \left(\frac{|x|}{2} \right)^k$$

converges iff $|x| < 2$. So the interval of convergence is $(-2, 2)$, and in this case convergence is always absolute. (Notice that the series for $f(x)$ converges at $x = -2$, but the one for $f'(x)$ does not. This often happens at points on the boundary of the interval of convergence.)

(c) By the usual formula for geometric series,

$$f'(x) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{2} \frac{1}{1 - x/2} = \boxed{\frac{1}{2 - x}}$$

for all x in the interval of convergence.

(d) Since we now have an explicit formula for $f'(x)$, we integrate to find

$$f(x) = \int \frac{dx}{2 - x} = -\ln(2 - x) + C.$$

The constant can be found by comparing the above expression with the original infinite series at one particular choice of x. The only x for which we can easily compute the series is $x = 0$: we should have

$$f(0) = -\ln 2 + C = \sum_{k=1}^{\infty} \frac{0^k}{2^k k} = 0,$$

thus $C = \ln 2$, and

$$f(x) = -\ln(2 - x) + \ln 2 = \boxed{\ln\left(\frac{2}{2 - x}\right)}.$$

(e) Using the explicit formula for $f(x)$ above,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k} = f(-1) = \ln\left(\frac{2}{2 - 1}\right) = \boxed{\ln 2}.$$

2. (a) Integrating by parts with $u = \frac{1}{x}$ and $dv = \cos 2\pi x \, dx$, we have $du = -\frac{1}{x^2}$, $v = \frac{\sin 2\pi x}{2\pi}$, and

$$\int_1^{\infty} \frac{\cos 2\pi x}{x} \, dx = \left.\frac{\sin 2\pi x}{2\pi x}\right|_1^{\infty} + \int_1^{\infty} \frac{\sin 2\pi x}{2\pi x^2} \, dx.$$

We need to verify that both terms on the right make sense. The first one is really the limit,

$$\left.\frac{\sin 2\pi x}{2\pi x}\right|_1^{\infty} = \lim_{x \to \infty} \frac{\sin 2\pi x}{2\pi x} - \frac{\sin 2\pi}{2\pi},$$

which clearly exists since $\sin 2\pi x$ remains bounded as $x \to \infty$, thus sending $\frac{\sin 2\pi x}{2\pi x}$ to 0. For the integral on the right, it will suffice to show that

$$\int_1^{\infty} \left|\frac{\sin 2\pi x}{2\pi x^2}\right| \, dx$$

converges. This follows immediately from the observation that $|\sin 2\pi x| \le 1$, thus

$$\int_1^{\infty} \left|\frac{\sin 2\pi x}{2\pi x^2}\right| \, dx \le \frac{1}{2\pi} \int_1^{\infty} \frac{1}{x^2} < \infty.$$

(b) For all integers n, $\cos 2\pi n = 1$, thus

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

a fact which should by now be quite familiar. The integral test tells us nothing in this case because the function $\frac{\cos 2\pi x}{x}$ is not what we call “eventually decreasing”, rather it oscillates back and forth, contributing negative area which balances the positive area just enough to make the integral converge. The same phenomenon doesn't happen with the sum, which only pays attention to the peaks of the oscillating curve: every contribution is positive, and this contributions are just enough to push the sum to infinity.

3. (a) For any fixed positive integer q, we can write $e - S_q$ as an infinite series in the form

$$e - S_q = \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{q} \frac{1}{n!} = \sum_{n=q+1}^{\infty} \frac{1}{n!}.$$

Thus

$$\begin{aligned} q!(e - S_q) &= \sum_{n=q+1}^{\infty} \frac{q!}{n!} = \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \dots \\ &= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \dots \\ &< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \dots = \frac{1}{q+1} \sum_{n=0}^{\infty} \left(\frac{1}{q+1} \right)^n \\ &= \frac{1}{q+1} \left(\frac{1}{1 - \frac{1}{q+1}} \right) = \frac{1}{q+1} \left(\frac{q+1}{q} \right) = \frac{1}{q}. \end{aligned}$$

In the last line we used the fact that $\frac{1}{q+1} < 1$ for any positive integer q, thus the geometric series converges.

(b) Suppose that e is a rational number after all, so there are positive integers p and q with $e = p/q$. We first observe that

$$q!S_q = q! \sum_{n=0}^{q} \frac{1}{n!} = \frac{q!}{1} + \frac{q!}{2!} + \dots + \frac{q!}{(q-1)!} + \frac{q!}{q!}$$

is an integer since $q!$ is divisible by the denominator in each term. Thus if $e = p/q$,

$$q!(e - S_q) = q!\frac{p}{q} - q!S_q = (q-1)!p - q!S_q$$

is also an integer, and it is positive since $q!$ and $e - S_q$ are both positive. Combining this with the result of part (a), we conclude that $q!(e - S_q)$ is a positive integer less than $1/q$. But there is no such integer, as $1/q$ is always less than or equal to 1. Since this cannot be true, our assumption that $e = p/q$ must be false, and we're forced to conclude that e never equals p/q for any positive integers p and q.

For more about irrational numbers, including a proof that π is also irrational, see Simmons Pp. 478–479. (In case your wondering: no, this will not be on the final.)