18.01 SPRING 2005
PROBLEM SET 1
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a) Figure 1 shows a graph of $G(x) = \sin\left(\frac{1}{x}\right)$. What must be observed is that as x approaches zero, $G(x)$ oscillates back and forth between -1 and 1 infinitely many times. To see that this must be so, set $z = 1/x$ and consider $\sin z$. As $x \to 0^+$, $z \to \infty$, which allows for infinitely many oscillations of $\sin z$. The same thing happens when $x \to 0^-$. We see that $G(x)$ attains every value between -1 and 1 infinitely many times as $x \to 0$, but it cannot be said to "approach" any one value in particular. Thus $\lim_{x\to 0} G(x)$ does not exist.

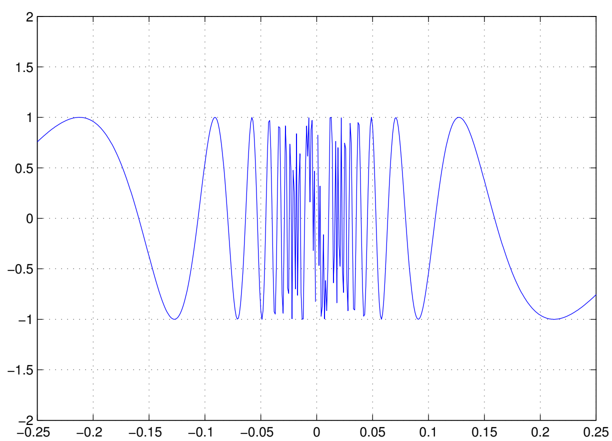

Figure 1: The function $G(x) = \sin(1/x)$

(b) Choose $f(x) = -|x|$ and $h(x) = |x|$. Then since $|\sin(1/x)| \leq 1$ for all x, we have $f(x) \leq x\sin(1/x) \leq h(x)$ for all x (see Figure 2), and therefore

$$\lim_{x\to 0} F(x) = 0.$$

This is also the value of $F(0)$ by definition, thus F is continuous at 0.

(c) If $F(x)$ were differentiable at 0, its derivative there would be

$$F'(0) = \lim_{h\to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h\to 0} \frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0} \sin(1/h).$$

But we saw in part (a) that this limit does not exist, thus F is not differentiable at 0.

(d) Using again the fact that $|\sin(1/x)| \leq 1$, we have

$$-x^2 \leq x^2\sin(1/x) \leq x^2,$$

and since $-x^2$ and x^2 both equal zero at $x = 0$, $\lim_{x\to 0} x^2\sin(1/x) = 0$. By the definition of the derivative,

$$H'(0) = \lim_{h\to 0} \frac{H(0+h) - H(0)}{h} = \lim_{h\to 0} \frac{h^2\sin(1/h) - 0}{h} = \lim_{h\to 0} h\sin(1/h) = 0,$$

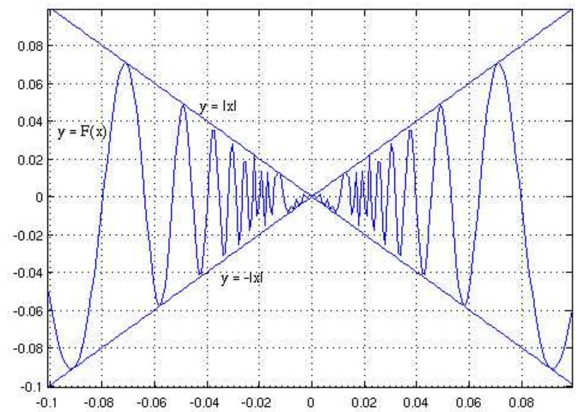

Figure 2: $F(x) = x\sin(1/x)$ along with two "squeezing functions," $f(x) = -|x|$ and $h(x) = |x|$.

by the result of part (b).

2. (a) $\displaystyle\lim_{x\to 0}\frac{\sin ax}{x} = \lim_{x\to 0}\left(a\frac{\sin ax}{ax}\right) = a\lim_{ax\to 0}\frac{\sin ax}{ax} = a\cdot 1 = a$, since as $x \to 0$, so does ax.

(b) The polygon is made up of n identical isosceles triangles, each with two sides of length r. For any one of these triangles, the angle between the two identical sides is $2\pi/n$ radians. Divide this triangle symmetrically into two right triangles, both with height h and base a. Then

$$\frac{h}{r} = \cos\left(\frac{\pi}{n}\right) \quad \text{and} \quad \frac{a}{r} = \sin\left(\frac{\pi}{n}\right).$$

The area of the full isosceles triangle is then

$$\frac{1}{2}2ah = ah = r^2\cos(\pi/n)\sin(\pi/n) = \frac{r^2}{2}\sin(2\pi/n),$$

where we've used the double angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Multiplying by n, the polygon has area

$$A_n = \frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right).$$

(c) Using the result of part (a),

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right) = \frac{r^2}{2}\lim_{1/n\to 0}\frac{\sin\left(2\pi\frac{1}{n}\right)}{1/n} = \frac{r^2}{2}2\pi = \pi r^2.$$