18.01 SPRING 2005
PROBLEM SET 3
DUE TUESDAY, FEBRUARY 22

Note: This assignment is due on Tuesday as usual, although classes on the 22nd will be running on a Monday schedule (so we have recitation that day instead of lecture).

The Usual Instructions

- Write up your solutions neatly, preferably with all pages stapled. You need not show every arithmetic calculation, but must always show enough work to demonstrate the process by which the answer is reached. Without this, the grader can't be sure that you didn't just copy the answers from someone else, and there's no way to give partial credit.
- You're free to work together in groups, but you must write up the solutions independently. Plagiarism is easy to detect.
- You can either hand in your solutions in class by the due date, or slip them through the slot in my office door (2-172) by 11:59pm that night.

Reading

Notes A, Simmons 5.2 (read casually with linear approximation in mind, but don't worry about the finer points of differentials), 4.1–4.4.

Ungraded problems

Do the following exercises for practice—preferably after the corresponding lecture—but *do not hand them in*. The solutions are available to you, so you should check your work. Starred problems are especially recommended.

Each problem is from the Notes unless stated otherwise:

- *Tu 2/15/05:* 2A-1, 2A-2, 2A-3*, 2A-12b, 2A-13 (just linear approximations)
- *Th 2/17/05:* 2B-1*, 2B-2, 2B-3*, 2B-4; Simmons 4.1 #21
- *Fr 2/18/05:* 2C-1, 2C-2, 2C-10, 2C-13, 2C-15

Graded problems, Part A [58 pts total]

From Simmons:

- 4.1 #20 [3 pts], 22 [3 pts], 26 [6 pts], 30 [6 pts]
- 4.2 #4, 8, 12, 16 [4 pts each]
- 4.3 #2, 4, 18, 32 [4 pts each]
- 4.4 #8, 16 [4 pts each]

Graded problems, Part B [30 pts total]

1. You're familiar with the formulas for the area and circumference of a circle in terms of its radius; if you also know the corresponding formulas for the volume and surface area of a sphere, then you may have noticed an interesting pattern:

$$\text{Circle:} \qquad A(r) = \pi r^2, \qquad C(r) = 2\pi r,$$
$$\text{Sphere:} \qquad V(r) = \frac{4}{3}\pi r^3, \qquad S(r) = 4\pi r^2.$$

Indeed, $C(r) = A'(r)$ and $S(r) = V'(r)$.

(a) [4 pts] To see why this should be true for the circle, consider the difference in areas between a circle of radius r and one of radius $r + \Delta r$. Argue geometrically that if Δr is very small, the difference in areas is approximately $C(r) \cdot \Delta r$ (draw a picture). Thinking in terms of linear approximation, what does this tell you about $A'(r)$? (You could answer this without even knowing the formulas.)

(b) [4 pts] Adapt the discussion from part (a) for the sphere in order to explain why it should be true that $V'(r) = S(r)$. (Again, you shouldn't need to know any of the actual formulas in order to make this argument.)

2. Galileo's observation that "all things fall with the same constant acceleration" can be restated via the equation

$$F = mg. \tag{1}$$

Here F is the force exerted by gravity on an object near the surface of the Earth, m is its mass, and g is the constant acceleration, equal to approximately 32 feet per second per second. This is only an approximation to Newton's *universal law of gravitation*, which replaces Equation (1) with

$$F = \frac{GmM}{r^2}, \tag{2}$$

where m is still the mass of the falling object, r is its distance from the center of the Earth, M is the mass of the Earth, and G is a fixed number known as the *gravitational constant.* This seems at first sight quite different from the constant expression $F = mg$. The purpose of this problem is to show that, for everyday situations, Equation (1) is a perfectly reasonable approximation to Equation (2).

(a) [4 pts] Let r_0 denote the radius of the Earth, and call h the height of the object above sea level, so by definition $r = r_0 + h$. Think of both equations (1) and (2) as defining the force F as a function of r. In the case of (1), this is a constant function, thus we wish to interpret $F(r) = mg$ as the *constant approximation* to $F(r) = GmM/r^2$ for objects near sea level, i.e. for r close to r_0. In particular, we can assume both formulas match exactly when $r = r_0$ if all the constants have the right values. Use this assumption to derive a formula for g in terms of G, M and r_0.

(b) [8 pts] The next step is to show that when r deviates within a reasonable range from r_0, Equation (1) remains a good approximation to Equation (2). Derive the linear approximation to $F(r) = GmM/r^2$ for r near r_0, and show that it can be rewritten in the form

$$F \cong mg\left(1 - C\frac{h}{r_0}\right)$$

for some constant C. What is C? The answer determines whether mg is a suitable approximation to GmM/r^2 for h small. Indeed, about how large does h have to get before the linear approximation differs substantially from the constant approximation? (You needn't be precise; just think in terms of order of magnitude.)

3. [10 pts] Sketch the graph of a function $f(x)$ that satisfies all of the following properties. Label all local maxima and minima, inflection points and asymptotes.

- f is defined, continuous and differentiable for all $x \neq 0$.
- f is an *odd* function, i.e. $f(-x) = -f(x)$ for all x.
- $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = 0$.
- $f'(1) = f'(3) = 0$, $f'(x) \neq 0$ for $0 < x < 1$, $1 < x < 3$ and $x > 3$.
- $f''(-2) = f''(-4) = 0$, $f''(x) \neq 0$ for $x < -4$, $-4 < x < -2$ and $-2 < x < 0$.
- $f(-1) = -1$, $f(3) = -2$, $f(-2) = f(-4) = 1$.

Note: you may find the result of problem 1F-6 from the Notes helpful.