18.01 SPRING 2005
PROBLEM SET 3
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a) The difference in areas $A(r + \Delta r) - A(r)$ is precisely the area of the shaded annulus in Figure 1. If Δr is very small, then the inner and outer circumferences of this annulus are almost the same, so one can imagine cutting through the annulus and "unwrapping" it into a rectangle with width Δr and height approximately $C(r) \approx C(r + \Delta)$. Thus

$$A(r + \Delta r) - A(r) \approx C(r)\Delta r.$$

This matches the linear approximation formula $A(r + \Delta r) \approx A(r) + A'(r)\Delta r$ if $C(r) = A'(r)$, so we conclude (without necessarily knowing the formulas for $C(r)$ and $A(r)$) that $C(r)$ is indeed the derivative of $A(r)$.

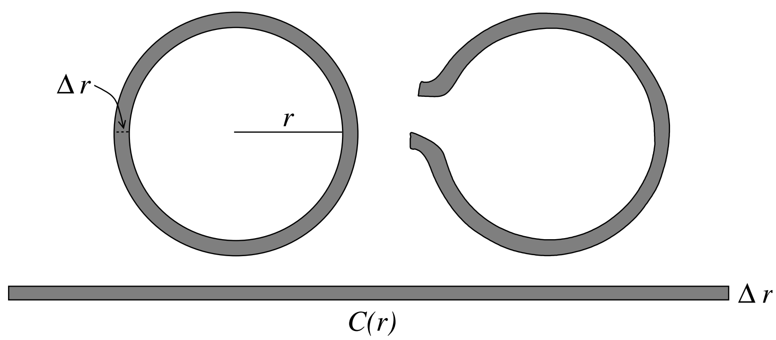

Figure 1: Cutting and unwrapping a thin annulus

(b) Now the difference in volumes $V(r + \Delta r) - V(r)$ is the volume of a thin spherical shell, whose inner and outer surface areas are nearly the same if Δr is small. Thus one can imagine cutting this shell open and "unwrapping" it to form a thickened surface with area $S(r) \approx S(r + \Delta r)$ and thickness Δr, hence volume

$$V(r + \Delta r) - V(r) \approx S(r)\Delta r.$$

This gives the linear approximation $V(r + \Delta r) \approx V(r) + V'(r)\Delta r$ if $V'(r) = S(r)$.

2. (a) If $F(r) = GmM/r^2$, we are assuming that this has the value mg when $r = r_0$. Solving for g yields

$$g = \frac{GM}{r_0^2}.$$

(b) We have $F'(r_0) = -2GmM/r_0^3$, $F(r_0) = mg = GmM/r_0^2$ and $r - r_0 = h$, thus the linear approximation $F(r) \approx F(r_0) + F'(r_0)(r - r_0)$ takes the form

$$F \approx mg - \frac{2GmM}{r_0^3}h = mg - \frac{2mg}{r_0}h = mg\left(1 - 2\frac{h}{r_0}\right),$$

in particular, $C = 2$. Notice that the term $-2h/r_0$ is insignificant unless $2h$ and r_0 are of comparable magnitude—thus the constant approximation is sufficient as long as the height h is much less than half the radius of the Earth (certainly a safe assumption for most Earth-based experiments).

3. The function has a vertical asymptote at $x = 0$, going to $-\infty$ on the right side of the axis, and (using the fact that f is an odd function) $+\infty$ on the left side. There's also a horizontal asymptote at $y = 0$, on both the right and (since f is odd) also the left. From problem (1F-6) in the Notes, we conclude that f' is even and f'' is also odd; in particular, whenever some point x_0 is a critical point or inflection point, so is $-x_0$. Thus the only critical points are -3, -1, 1 and 3, and there are inflection points at -4, -2, 2 and 4. Figure 2 shows the graph, with all crucial details included.

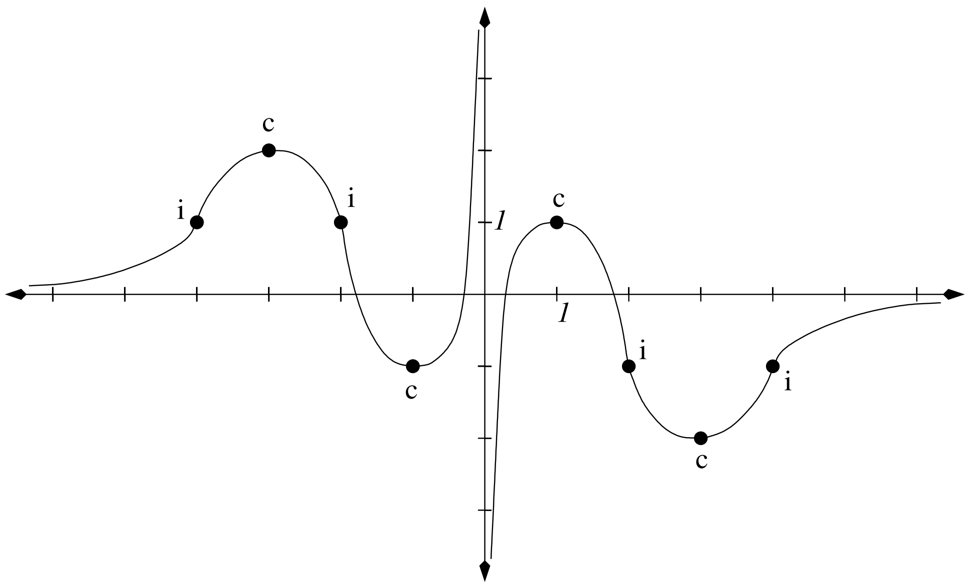

Figure 2: The graph of $f(x)$, with critical points labeled "c" and inflection points "i".