18.01 SPRING 2005
PROBLEM SET 4
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a) The equation $f(b) = f(a) + f'(a)(b-a) + k(b-a)^2$ is satisfied if

$$k = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2},$$

which is possible since $b - a \neq 0$ by assumption.

(b) Rolle's theorem will guarantee that g has a critical point $c_1 \in (a, b)$ if $g(a) = g(b)$. This is indeed the case since $g(a) = f(a) - f(a) - f'(a)(a-a) - k(a-a)^2 = 0$ and, choosing k as in part (a), $g(b) = f(b) - f(a) - f'(a)(b-a) - k(b-a)^2 = 0$.

(c) To differentiate $g(x)$, we treat a as a constant and find

$$g'(x) = f'(x) - f'(a)\frac{d}{dx}(x-a) - k\frac{d}{dx}(x-a)^2 = f'(x) - f'(a) - 2k(x-a),$$

thus $g'(a) = f'(a) - f'(a) - 2k(a-a) = 0$.

(d) Since $g'(a) = g'(c_1) = 0$, Rolle's theorem implies that g' has a critical point $c \in (a, c_1)$, hence $g''(c) = 0$.

(e) As we saw above, $g'(x) = f'(x) - f'(a) - 2k(x-a)$, thus $g''(x) = f''(x) - 2k$ for any x, and by the result of part (d),

$$0 = g''(c) = f''(c) - 2k \quad \Longrightarrow \quad k = \frac{f''(c)}{2}.$$

2. (a) You can always use the general formula $f(x) \approx f(a) + f'(a)(x-a)$, but in this case it's simpler to apply the special case $(1+x)^p \approx 1 + px$, thus

$$\sqrt[3]{1+2x} = [1 + (2x)]^{1/3} \approx 1 + \frac{1}{3}(2x) = \boxed{1 + \frac{2}{3}x}.$$

(b) The idea of an error estimate is to find an upper bound for the quantity

$$|f(x) - L(x)|,$$

where $L(x)$ is the linear approximation $L(x) = f(a) + f'(a)(x-a)$ and x varies over a specified interval. From the formula in the previous problem, we know that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$ for some $c \in (a, x)$, hence

$$\Big| f(x) - [f(a) + f'(a)(x-a)] \Big| = \left| \frac{f''(c)}{2}(x-a)^2 \right|.$$

The goal is thus to find out how large $\left|\frac{f''(c)}{2}(x-a)^2\right|$ can get. In our case, $a = 0$ and x is being allowed to vary over the interval $[0, 1/8]$, thus $0 < c < x \leq 1/8$ implies $c \in (0, 1/8)$. For $f(x) = \sqrt[3]{1+2x}$, we compute

$$f''(x) = -\frac{8}{9}(1+2x)^{-5/3}.$$

This function is negative and has no critical points on the interval $[0, 1/8]$, so its maximum absolute value must be attained at either 0 or 1/8. By checking both, we find that the maximum is at $x = 0$,* where $|f''(0)| = 8/9$. Thus

$$\left| \sqrt[3]{1+2x} - \left(1 + \frac{2}{3}x\right) \right| = \left| \frac{f''(c)}{2} x^2 \right| \leq \frac{1}{2}\frac{8}{9}\left(\frac{1}{8}\right)^2 \cong \boxed{0.0069},$$

where we're using the fact that $x \leq 1/8$.

*The original version of these solutions stated that the maximum was at $x = 1/8$, which is clearly wrong. Sorry about the confusion.